STATIONARY POINTS OF THE YANG-MILLS ACTION

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Abstract. We examine the structure of a recently discovered set of non-self-dual solutions of the Yang-Mills equations. These solutions have a symmetry that reduces the YM equations to a set of ODE's. The distinct solutions are indexed by two postive odd integers. We develop a scheme to approximate on a computer the solutions for small values of the indexing integers, and present some numerical results. We then analyze the asymptotic behavior of the solutions as the indexing integers become large.

1. Introduction

In this paper we consider stationary points of the $SU(2)$ Yang-Mills action on Euclidean $\mathbb{R}^4$. These are solutions to the variational equations

$$0 = \frac{\delta S}{\delta A_\mu} \equiv \partial^\nu F_{\mu\nu} + i[A^\nu, F_{\mu\nu}], \tag{1.1}$$

often called the *Yang-Mills equations.* Local minima of the action have been known for some time; these include the vacuum ($A_\mu = 0$) and various instanton, anti-instanton, and multi-instanton solutions [BPST, ADHM]. All these solutions are characterized by having an (anti)self-dual curvature tensor. That is,

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}, \tag{1.2}$$

where the dual is

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{1.3}$$

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Recall that gauge fields are divided into topological sectors by the instanton number k, which is a gauge-invariant integer,

$$k = \frac{1}{16\pi^2} \int d^4x \mathrm{Tr}\ (F^{\mu\nu} \tilde{F}_{\mu\nu}). \tag{1.4}$$

Defining the self-dual and anti-self-dual parts of the curvature,

$$F_\pm = \frac{1}{2}\left(F \pm \tilde{F}\right), \tag{1.5}$$

we can rewrite the instanton number and the action as

$$\begin{aligned} k =& \frac{1}{16\pi^2} \int d^4x(|F_-|^2 - |F_+|^2) \\ S \equiv& \frac{1}{2} \int d^4x |F|^2 = \frac{1}{2} \int d^4x(|F_+|^2 + |F_-|^2). \end{aligned} \tag{1.6}$$

It follows that in each topological sector the action is at least $8\pi^2|k|$, and the fields that attain this minimum are precisely the (anti)self-dual fields. No local minima exist at higher values of the action, as was shown by Bourguignon and Lawson [BL] and Taubes [T], so a non-self-dual solution to (1.1) must be a saddle point of the action.

We are interested in non-self-dual solutions on Euclidean space with group $SU(2)$, which were only recently discovered. Other examples of non-self-dual solutions are known, namely solutions with larger groups [I,Ma], and solutions on manifolds that are more complicated topologically [Ur,P2] or geometrically [P1]. For many years, the preponderance of evidence suggested that non-self-dual solutions of the $SU(2)$ Yang-Mills equations on Euclidean space should not exist. However, in 1989, Sibner, Sibner, and Uhlenbeck [SSU] demonstrated the existence of non-self-dual solutions with zero instanton number. We [SS2, SS3] have found a family of non-self-dual solutions, including solutions with nonzero instanton number. This family of solutions is parametrized by two odd integers $n_\pm \geq 3$. The instanton number of the solution corresponding to the pair (n_+, n_-) is $k = (n_-^2 - n_+^2)/8$, as explained in [SS1,ASSS]. Every integer k except ± 1 can be obtained from this formula, so we have solutions with *arbitrary* instanton number (except $k = \pm 1$). For some k there are several distinct solutions. For example, $k = 5$ can arise in two ways, as $(7^2 - 3^2)/8$ or as $(11^2 - 9^2)/8$. There are an infinite number of examples with instanton number zero; just take $n_+ = n_-$.

In Section 2 we review the general properties of solutions in this family; the details are in [SS2, SS3]. In Section 3 we describe a scheme for numerically approximating these solutions, and we give results for some solutions with small $n_\pm$. In Section 4 we develop an asymptotic description of the solutions as the integers $n_\pm$ become large.

2. Symmetric Gauge Fields

The Yang-Mills action and the (anti) self-dual equations are conformally invariant, so solutions on Euclidean space correspond to solutions on the four-sphere S^4, and vice-versa. We work on S^4, because our solutions have a symmetry group that acts by isometries (orthogonal transformations) on S^4, while the corresponding group action on $\mathbb{R}^4$ is more complicated.

Let $V \simeq \mathbb{R}^5$ be the space of symmetric, traceless, real 3×3 matrices Q, with inner product $\langle Q, Q' \rangle = \frac{1}{2} Tr(QQ')$. The matrices

$$Q_0 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \qquad Q_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \qquad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$$Q_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad Q_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

are an orthonormal basis. A matrix g in $SO(3)$ acts on V by conjugation, $g(Q) \to gQg^{-1}$. $SU(2)$ is the double cover of $SO(3)$, so there is an action of $SU(2)$ on V, which is just the spin-2 representation. The unit sphere S^4 in V inherits this $SU(2)$ action.

Since all matrices in V are diagonalizable, it is not hard to check that every $Q \in S^4$ is related by the group action to a unique $Q_\theta = \cos(\theta) Q_0 + \sin(\theta) Q_3$ with $0 \leq \theta \leq \pi/3$. The orbits of Q_0 and $Q_{\pi/3}$ are two-dimensional, while all other orbits are three-dimensional.

We define a coordinate chart $(y^0 = \theta, y^1, y^2, y^3)$ by the correspondence

$$(\theta, \vec{y}) \mapsto \exp(-\vec{y} \cdot \vec{\sigma}) \, (Q_\theta) \, , \tag{2.2}$$

where σ_1, σ_2, and σ_3 are the (antihermitian) generators of $SU(2)$, σ_i corresponding to the generators of the $SO(3)$ rotation about the i-th axis of $\mathbb{R}^3$. The $SO(3)$ matrix corresponding to $\exp(-\vec{y} \cdot \vec{\sigma})$ acts on Q_θ by conjugation as above. As θ ranges from 0 to $\pi/3$, Q_θ moves along a geodesic segment on the sphere. On this segment the tangent vectors $\partial_\theta \equiv \partial/\partial\theta$ and $\partial_i \equiv \partial/\partial y^i$ are orthogonal but not orthonormal. The vector ∂_θ is normalized, but the length of the vector ∂_i at Q_θ is $f_i(\theta)$, where

$$f_1(\theta) = 2\sin(\pi/3 + \theta); \qquad f_2(\theta) = 2\sin(\pi/3 - \theta); \qquad f_3(\theta) = 2\sin(\theta). \tag{2.3}$$

Note that f_3 vanishes at $\theta = 0$, as Q_0 is invariant under rotations about the 3-axis of $\mathbb{R}^3$. Similarly, f_2 vanishes at $\pi/3$.

¿From this it is easy to see how the duality operator acts on 2-forms:

$$\tilde{F}_{0i} = \frac{f_j f_k}{f_i} F_{jk}; \qquad \tilde{F}_{jk} = \frac{f_i}{f_j f_k} F_{0i} \tag{2.4}$$

where (i, j, k) are cyclic permutations of $(1, 2, 3)$. To simplify the notation we define the functions

$$G_1 = \frac{f_2 f_3}{f_1}; \qquad G_2 = \frac{f_1 f_3}{f_2}; \qquad G_3 = \frac{f_1 f_2}{f_3}. \tag{2.5}$$

G_1 and G_2 have zeroes at $\theta = 0$, while G_3 has a pole. Similarly, G_1 and G_3 have zeroes at $\theta = \pi/3$, while G_2 has a pole.

A symmetric gauge potential is determined by its values on the segment $\{Q_\theta\}$ where $0 \leq \theta \leq \pi/3$. This potential must also be invariant under rotations by π about the 1, 2, and 3 axes, as these rotations take the point Q_θ to itself. The most general potential that is invariant under these rotations is

$$A = -\sum_{i=1}^{3} a_i(\theta) dy^i \otimes \sigma^i, \tag{2.6}$$

where a_1, a_2, and a_3 are real-valued functions of θ. We call such a triplet of functions $a = (a_1, a_2, a_3)$ a *reduced potential* or a *reduced connection.*

Given a symmetric potential A, the curvature F is easily computed:

$$F = \left((a_1 + a_2 a_3) dy^2 \wedge dy^3 - a_1' d\theta \wedge dy^1\right) \otimes \sigma^1 + (cyclic), \tag{2.7}$$

where $'$ denotes $d/d\theta$, and $(cyclic)$ denotes the other cyclic permutations of the indices $(1, 2, 3)$. By equation (2.4) the (anti) self-duality equations are then

$$-a_1' = \pm\frac{(a_1 + a_2 a_3)}{G_1}, \qquad -a_2' = \pm\frac{(a_2 + a_1 a_3)}{G_2}, \qquad -a_3' = \pm\frac{(a_3 + a_1 a_2)}{G_3}, \tag{2.8}$$

where $+$ denotes self-duality and $-$ denotes anti-self-duality.

¿From F we compute the action, with the result

$$\begin{aligned} S(A) = \pi^2 \int_0^{\pi/3} d\theta \Big[&(a_1')^2 G_1 + (a_2')^2 G_2 + (a_3')^2 G_3 \\ &+ \frac{(a_1 + a_2 a_3)^2}{G_1} + \frac{(a_2 + a_1 a_3)^2}{G_2} + \frac{(a_3 + a_1 a_2)^2}{G_3} \Big]. \end{aligned} \tag{2.9}$$

Finite action reduced potentials must have well-defined boundary values $r = a_3(0)$ and $t = a_2(\pi/3)$. Also, since G_1 and G_2 have zeroes at $\theta = 0$, finite-action reduced potentials must have

$$a_1(0) + a_2(0) a_3(0) = a_2(0) + a_1(0) a_3(0) = 0. \tag{2.10}$$

If $r \neq \pm 1$, these conditions imply that both $a_1(0)$ and $a_2(0)$ equal zero. Similarly, if $t \neq \pm 1$ then $a_1(\pi/3) = a_3(\pi/3) = 0$.

Not all finite-action reduced potentials correspond to smooth potentials on all of S^4. For smooth potentials, $r+1$ and $t+1$ must both be integers divisible by 4 [BoMo], in which case $n_+ \equiv |r|$ and $n_- \equiv |t|$ are positive odd integers. Otherwise our reduced potential corresponds to a singular potential with non-integer Chern number [FHP, SiSi], with the singularities occurring at the orbits of Q_0 and $Q_{\pi/3}$.

In [SS2, SS3] we showed that for every pair (n_+, n_-) with $n_+ > 1$ and $n_- > 1$, there exists a stationary point of the action that is neither self-dual nor anti-self-dual. This result was proven by finding minima of the action within each symmetry class. By the principle of symmetric criticality [Pal], these minima are solutions of the full Yang-Mills equations. We then proved that self-dual or anti-self-dual solutions do not exist in these symmetry classes, so these Yang-Mills solutions must be non-self-dual. This existence proof is nonconstructive, and does not give the exact form of the solutions. The remainder of the present paper examines the form of these solutions.

3. Fourier Analysis

The functions $a_i(\theta)$ can be extended to functions on the entire circle $0 \leq \theta \leq 2\pi$ via equation (2.6), since the points Q_θ are well-defined for all θ. The points Q_θ, $Q_{-\theta}$, and $Q_{\theta \pm 2\pi/3}$ are related by rotations that permute the coordinates $\{y^i\}$ and also permute the $\{\sigma^i\}$'s. As a result, the functions a_i satisfy

$$a_3(-\theta) = a_3(\theta), \qquad a_2(\theta) = a_3(\theta + 2\pi/3), \qquad a_1(\theta) = a_3(\theta - 2\pi/3). \tag{3.1}$$

Thus the behavior of the three functions a_i on $[0, \pi/3]$ is expressed by the single even function a_3 on $[0, 2\pi]$. The potential A is smooth on all of S^4 if and only if a_3 is smooth on the entire circle [BoMo].

Since the three functions $G_i(\theta)$ also satisfy the symmetry relations

$$G_2(\theta) = G_3(\theta + 2\pi/3), \qquad G_1(\theta) = G_3(\theta - 2\pi/3), \tag{3.3}$$

we can rewrite the action functional (2.9) as a function of a_3 alone:

$$S = \pi^2 \left[\int_0^{\pi/3} + \int_{2\pi/3}^{\pi} + \int_{4\pi/3}^{5\pi/3} \right] d\theta \left[(a_3')^2 G_3 + \frac{(a_3 + a_1 a_2)^2}{G_3} \right] \tag{3.3}$$

$$= \pi^2 \left[\int_0^{\pi/3} + \int_{2\pi/3}^{\pi} + \int_{4\pi/3}^{5\pi/3} \right] d\theta \left[(a_3'(\theta))^2 G_3 + \frac{(a_3(\theta) + a_3(\theta+2\pi/3) a_3(\theta - 2\pi/3))^2}{G_3} \right]$$

To simplify the notation, we let $\mathcal{I}$ denote the region of integration $[0, \pi/3] \cup [2\pi/3, \pi] \cup [4\pi/3, 5\pi/3]$, and let

$$b(\theta) = a_3(\theta) + a_3(\theta - 2\pi/3)a_3(\theta + 2\pi/3). \tag{3.4}$$

This lets us write the action as

$$S = \pi^2 \int_{\mathcal{I}} (a_3')^2 G_3 + b^2/G_3, \tag{3.5}$$

and the first variation of the action as

$$\delta S = 2\pi^2 \int_{\mathcal{I}} (a_3')(\delta a_3')G_3 + (\delta b)b/G_3. \tag{3.6}$$

The Euler-Lagrange equations $\delta S = 0$ are

$$(G_1 a_1')' = \frac{a_1 + a_2 a_3}{G_1} + \frac{a_3(a_2 + a_1 a_3)}{G_2} + \frac{a_2(a_3 + a_1 a_2)}{G_3} \tag{3.7a}$$

$$(G_2 a_2')' = \frac{a_2 + a_1 a_3}{G_2} + \frac{a_1(a_3 + a_1 a_2)}{G_3} + \frac{a_3(a_1 + a_2 a_3)}{G_1} \tag{3.7b}$$

$$(G_3 a_3')' = \frac{a_3 + a_1 a_2}{G_3} + \frac{a_2(a_1 + a_2 a_3)}{G_1} + \frac{a_1(a_2 + a_1 a_3)}{G_2} \tag{3.7c}$$

on the interval $[0, \pi/3]$, or equivalently any *one* of these equations (say, 3.7c) on the entire circle.

Fourier Decomposition of a_3

¿From now on we take $n_\pm \geq 3$, and so have $a_3(\pi/3) = a_3(2\pi/3) = 0$. We express $a_3(\theta)$ as a Fourier series

$$a_3(\theta) = \sum_{n=-\infty}^{\infty} A_n \alpha^n, \tag{3.8}$$

where $\alpha \equiv e^{i\theta}$. Since a_3 is real and even, we have

$$A_n = A_n^* = A_{-n} \tag{3.9}$$

for all n. The function a_3 is described by the sequence A_0, $A_1, \ldots$. These coordinates are not independent, as a_3 satisfies the boundary conditions

$$a_3(0) = r, \qquad a_3(\pi/3) = a_3(2\pi/3) = 0, \qquad a_3(\pi) = t. \tag{3.10}$$

To see the effect of these conditions, we define, for $m = 0, 1, 2, \ldots, 5$

$$c_m = \sum_{l=0}^{\infty} A_{6l+m}, \tag{3.11}$$

and let $C_0 = 2c_0 - A_0$, $C_1 = c_1 + c_5$, $C_2 = c_2 + c_4$, and $C_3 = c_3$. Our boundary conditions are

$$\begin{aligned} C_0 + 2C_1 + 2C_2 + 2C_3 &= r, \qquad & C_0 + C_1 - C_2 - 2C_3 &= 0, \\ C_0 - C_1 - C_2 + 2C_3 &= 0, \qquad & C_0 - 2C_1 + 2C_2 - 2C_3 &= t. \end{aligned} \tag{3.12}$$

Solving these equations gives

$$C_0 = C_2 = (r+t)/6, \quad C_1 = 2C_3 = (r-t)/6, \tag{3.13}$$

fixing the values of A_0, A_1, A_2, and A_3 as a function of the remaining coordinates.

Our allowed variations are therefore as follows. Every change in A_{6l+m}, $l \geq 1$, $m = 0, 1, 2, \ldots, 5$, must be compensated by the opposite change in A_m, thereby keeping c_m fixed. Also, any change in A_4 and A_5 must be compensated for by a change in A_2 and A_1, respectively, keeping C_2 and C_1 fixed. That is, the allowed variations in a_3 are spanned by the functions

$$h_{l,m}(\theta) = \alpha^{6l+m} + \alpha^{-6l-m} - \alpha^m - \alpha^{-m} \tag{3.14}$$

with $l \geq 1$ and $0 \leq m \leq 5$, or with $l = 1$ and $m = -1$ or -2. These last two possibilities correspond to variations of A_5 and A_4, respectively.

Finite Mode Approximations

We are now ready to look for solutions to the variational (Yang-Mills) equations (3.7). As a first try we look for solutions that have only a finite number of nonzero Fourier coefficients. Unfortunately, there are only three such solutions, and they have (anti) self-dual curvature. They are the flat potential (gauge equivalent to the vacuum) with the zeroth Fourier coefficient $A_0 = -1$ and all other coefficients zero, the round BPST instanton with $A_0 = A_1 = A_{-1} = 1$ and all other coefficients zero, and the round BPST anti-instanton with $A_0 = 1$, $A_1 = A_{-1} = -1$ and all other coefficients zero. These solutions have $(n_+, n_-) = (1,1)$, $(3,1)$ and $(1,3)$ respectively.

To see that no other finite-term solutions exist, we look at equation (3.7c). If A_k is the highest order non-zero coefficient, then the right-hand-side of (3.7c) has a (nonzero) term proportional to α^{3k-1}, while the left hand side has terms only up to α^{k+1}. Thus there are no solutions with $3k - 1 > k + 1$, i.e. $k > 1$. The equations for $k \le 1$ are easy to solve, and give the three solutions above.

All remaining solutions must have infinite Fourier expansions. However, since solutions of the Yang-Mills equations are known to be smooth in an appropriate gauge, we expect a_3 to be smooth, and so expect the Fourier coefficients A_n to decrease exponentially with n. As a result, we can get very good approximate solutions by considering only the first few coefficients. Specifically, we pick a small integer l_{max}, and consider functions a_3 with only the first $6l_{max} + 6$ components nonzero. This is a $6l_{max} + 2$ dimensional space ($6l_{\max} + 6$ variables with 4 boundary conditions). In this space we minimize the action numerically, using Newton's method.

The results for $n_+ = 5$, $n_- = 3$ are listed in Table 1 for several values of l_{max}. In this example even taking $l_{max} = 1$ is enough to give a good estimate, and taking $l_{max} = 4$ gives 14-digit accuracy.

For larger values of $n_\pm$ the coefficients A_n do not decrease quite so rapidly, so larger and larger values of l_{max} are needed. As a result, this method of computing approximate solutions is only practical for $(n_+ + n_-)$ up to 100 or so. For larger values of $n_\pm$ we must use the asymptotic analysis of the next section.

The actions of 19 low-lying non-self-dual stationary points of the Yang-Mills actions are listed in Table 2, as computed with $l_{max} = 5$. In each case, the last significant digit is the last digit listed. We display $S/8\pi^2$ rather than S, as (anti)self-dual solutions have $S/8\pi^2 = |k|$, and in particular since the BPST instanton has $S/8\pi^2 = 1$.

4. Asymptotic analysis: a succession of models

In this section we study the one-dimensional variational problem for large $|r|$

and $|t|$, which will yield information about Yang-Mills solutions corresponding to large odd integers n_+ and n_-. We work with the three functions a_i on the interval $[0, \pi/3]$ rather than the single function a_3 on the circle. Notice that the action is invariant under the change of sign of any two of the a_i's. We can therefore assume, without loss of generality, that both r and t are positive.

To get the asymptotic form of the Yang-Mills solutions and their actions as r and t become large, we look at a succession of models. We start by solving a simple toy model, and add more and more features until we reach the reduced Yang-Mills action functional. Models 1–4 are symmetric models, in the sense that $r = t$. The Yang-Mills problem is Model 4, with the key results being equations (4.4.2) and (4.4.4). The Yang-Mills problem with $r \neq t$ is Model 6, with the key results tabulated in Table 4.

We will make repeated use of the following useful fact, which applies in the absence of catastrophes.

Useful Fact. *Let $f(p) = \{f_1, \ldots, f_k\}$ be a collection of functions on the interval $[a, b]$, satisfying boundary conditions that depend on a parameter $p \in \mathbb{R}$. Let $S^*(p)$ be the minimum value of the functional $S(f) = \int_a^b \sum_i [(f_i')^2 + V(f)]dx$ subject to the boundary conditions specified by p, and let $f^* = \{f_1^*, \ldots, f_k^*\}$ be the corresponding minimizing functions. Then*

$$dS^*/dp = 2 \sum_i \left(f_i^{*\prime}(b) \dot{f}_i^*(b) - f_i^{*\prime}(a) \dot{f}_i^*(a) \right), \tag{4.1}$$

where dot $(\dot{\ })$ denotes d/dp.

The proof is just integration by parts:

$$dS^*/dp = \sum_i \int 2 f_i^{*\prime} \dot{f}_i^{*\prime} + \dot{f}_i^* \partial_i V(f^*) = \sum_i 2 \dot{f}_i^* f_i^{*\prime} \big|_a^b + \sum_i \int \dot{f}_i^* (\partial_i V(f^*) - 2 f_i^{*\prime\prime}). \tag{4.2}$$

The terms in parentheses are all zero by the variational equations. When clear from context, we will not distinguish between the functional S and its minimum value S^*, or between f_i and f_i^*.

Toy Model Number 1

In this model we have two functions, f and g, on the interval $[-1, 1]$, and we want to minimize the functional

$$S_1(f, g) = \int_{-1}^{1} (f')^2 + (g')^2 + b^2 f^2 g^2, \tag{4.1.1}$$

with the boundary conditions $g(1) = f(-1) = 0$, $f(1) = g(-1) = n$. Here b is a parameter independent of n. By the Useful Fact this means that $dS_1/dn = 2(f'(1) - g'(-1)) = 4f'(1)$, which we can integrate to get S_1.

We first consider $b = 1$. The variational equations for this system are

$$f'' = fg^2; \qquad g'' = gf^2. \tag{4.1.2}$$

These equations are invariant under translation $x \to x + a$, and under the scale change $f(x), g(x) \to \lambda f(\lambda x), \lambda g(\lambda x)$.

It is clear that f and g cannot be large at the same places, or else the f^2g^2 term would get huge. Therefore, for large n we expect f to be small for x significantly negative, and expect g to be small for x significantly positive, and for there to be a small overlap region near zero. By the variational equations, for x much bigger than 0, f will approach a straight line, and g will decay rapidly. By the WKB approximation, this decay goes as $\exp(-\sqrt{n}x^2/2)$.

We now look for solutions to (4.1.2) with g going to zero as $x \to \infty$, with f going to zero as $x \to -\infty$, and symmetric about the origin ($f(x) = g(-x)$). These will not be exact solutions to our variational problem, since $g(1)$ will not be exactly zero, but the error introduced is exponentially small in $\sqrt{n}$. By rescaling we can make the asymptotic slope of f equal to 1, so that, for x much bigger than 1,

$$f(x) \approx x + c, \tag{4.1.3}$$

for some yet-to-be determined constant c.

To get the boundary condition $f(1) = g(-1) = n$, we must rescale again, this time by a factor λ. For x much bigger than $1/\lambda$, our solution is now

$$f(x) = \lambda^2 x + \lambda c, \tag{4.1.4}$$

with the same constant c as before. We identify n with $f(1) = \lambda^2 + \lambda c$, and get

$$\lambda = \sqrt{n + c^2/4} - c/2, \tag{4.1.5}$$

and so

$$dS_1/dn = 4f'(1) = 4\lambda^2 = 4n + 2c^2 - 4c\sqrt{n + c^2/4}. \tag{4.1.6}$$

Integrating we get

$$S_1(n) = 2n^2 + 2c^2 n - \frac{8c}{3}(n + c^2/4)^{3/2} + K, \tag{4.1.7}$$

plus exponentially small terms. Here K is a constant of integration. (Note, by the way, that the overlap region is of size $O(1/\lambda) = O(n^{-1/2})$).

All that remains is to find c (and K). We put this model on the computer, dividing the interval [-1,1] into 50, 100, and 200 slices, for $n = 10$, 20, 30, 40, and 50. c is best estimated not from the limiting actions, but from the limiting solutions, since $c = (n - f'(1))/\sqrt{f'(1)}$. From these studies we get $c = 0.64847 \pm 0.00002$. This fits the numerical actions as well, to within 0.2 or so, with $K = 0$, from which we deduce that K is not larger than 0.5 or so, and may in fact be zero.

We next consider $b \neq 1$. The variational equations are then

$$f'' = b^2 f g^2, \qquad g'' = b^2 g f^2. \tag{4.1.8}$$

Under the (vertical) rescaling $f(x) \to bf(x)$, $g(x) \to bg(x)$ these equations become the same as (4.1.2). As a result we have

$$S_1(b,n) = b^{-2} S_1(1,bn) = 2n^2 - \frac{8c}{3\sqrt{b}} n^{3/2} + O(n). \tag{4.1.9}$$

Toy Model Number 2

Next we add a mass term, and do our calculation on the interval $[-T, T]$, for some constant T, rather than on $[-1, 1]$.

$$S_2 = \int_{-T}^{T} (f')^2 + (g')^2 + f^2 + g^2 + b^2 f^2 g^2. \tag{4.2.1}$$

For now we take $b = 1$; other values of b are treated as in Model 1.

Note that the mass term $f^2 + g^2$, restricted to the overlap region, contributes only $O(\sqrt{n})$ to the action, and so can be ignored in looking for the $O(n^{3/2})$ behavior of the system. This is therefore Model 1 in the overlap region. Outside the overlap region, either f or g is near zero, and the system evolves into hyperbolic functions. So instead of $f(x)$ asymptotically approaching $\lambda^2 x + \lambda c$, $f(x)$ is asymptotically $\lambda^2 \sinh(x) + \lambda c \cosh(x)$. We identify n with $f(T)$, and so

$$n = \lambda^2 \sinh(T) + \lambda c \cosh(T). \tag{4.2.2}$$

It is convenient to define $\mathcal{N} = n/\sinh(T)$, and have

$$\lambda = \sqrt{\mathcal{N} + c^2 \coth^2(T)/4} - c\coth(T)/2, \tag{4.2.3}$$

so

$$\begin{aligned} f'(T) =& \lambda^2 \cosh(T) + \lambda c \sinh(T) \\ =& \mathcal{N} \cosh(T) - \lambda c/\sinh(T) \\ =& \mathcal{N} \cosh(T) - c\sqrt{\mathcal{N}}/\sinh(T) + O(1) \end{aligned} \tag{4.2.4}$$

Now we use our Useful Fact and integrate to get $S_2(n)$. (If $b \neq 1$ the subleading term must then be divided by $\sqrt{b}$, as in Model 1.) Since

$$\frac{dS_2}{d\mathcal{N}} = \sinh(T)\frac{dS_2}{dn} = 4\sinh(T) f'(T), \tag{4.2.5}$$

we get

$$S_2 = 2\mathcal{N}^2 \cosh(T)\sinh(T) - \frac{8c}{3\sqrt{b}}\mathcal{N}^{3/2} + O(\mathcal{N}). \tag{4.2.6}$$

In terms of n this is

$$S_2 = 2n^2 \coth(T) - \frac{8cn^{3/2}}{3\sqrt{b}\sinh(T)^{3/2}} + O(n). \tag{4.2.7}$$

Toy Model Number 3

Next we consider a functional involving our Yang-Mills fields a_i:

$$S_3(a_2, a_3) = \pi^2 \int_0^{\pi/3} d\theta \ \left[(a_3')^2 G_3 + (a_2')^2 G_2 + a_3^2/G_3 + a_2^2/G_2 + a_2^2 a_3^2/G_1\right]. \tag{4.3.1}$$

This is our true Yang-Mills action functional, only with a_1 identically zero. It is also just a small perturbation of Model 2. We define the new coordinate

$$x(\theta) = -\int_\theta^{\pi/6} dy/G_3(y), \tag{4.3.2}$$

for $\theta < \pi/6$, and

$$x(\theta) = \int_{\pi/6}^{\theta} dy/G_2(y), \tag{4.3.3}$$

for $\theta > \pi/6$. We also let $f(x)=a_2(\theta)$, $g(x) = a_3(\theta)$, and $T = \int_0^{\pi/6} dy/G_3(y)$. Our action functional is then

$$\begin{aligned} S_3 =&\pi^2 \int_{-T}^0 dx[(g')^2 + g^2 + f^2 G_3/G_2 + (f')^2 G_2/G_3 + f^2 g^2 G_3/G_1] \\ &+ \pi^2 \int_0^T dx[(f')^2 + f^2 + g^2 G_2/G_3 + (g')^2 G_3/G_2 + f^2 g^2 G_2/G_1], \end{aligned} \tag{4.3.4}$$

which we approximate as

$$\tilde{S}_3 = \pi^2 \int_{-T}^T (f')^2 + (g')^2 + f^2 + g^2 + 4 f^2 g^2. \tag{4.3.5}$$

To the left of the overlap region, f is exponentially small, so the difference between G_3/G_2 and 1 contributes only $o(1)$ to the action, and can be ignored. Similarly to the right of the overlap region, where g is small. In the overlap region $G_3/G_2 = 1 + O(x)$, and so the difference between G_3/G_2 and 1 contributes $O(1)$ to the action (since f^2 and g^2 are $O(n)$ and the width of the region is $O(n^{-1/2})$). Finally, in the overlap region the coefficient of $f^2 g^2$ is $4 + O(x)$. The difference between this and 4 contributes $O(n)$ to the action.

So $S_3(n)$ and $\tilde{S}_3(n)$ differ only by $O(n)$, and we have

$$S_3(n) = \tilde{S}_3(n) + O(n) = 2\pi^2 n^2 \coth(T) - \frac{8\pi^2 c n^{3/2}}{3\sqrt{2}\sinh(T)^{3/2}} + O(n). \tag{4.3.6}$$

If we do the integral for T we find that

$$T = \log(\sqrt{3}+1) - \log\sqrt{6}; \qquad \coth(T) = 4 + 3\sqrt{3}$$
$$1/\sinh(T) = \sqrt{\coth^2(T) - 1} = \sqrt{42 + 24\sqrt{2}}. \tag{4.3.7}$$

Plugging this back into (4.3.6) we get

$$S_3(n) = 8\pi^2(2.2990381n^2 - 4.2246n^{3/2}) + O(n). \tag{4.3.8}$$

Model 4. The Full System

Finally we consider the full Yang-Mills action:

$$S_4 = \pi^2 \int_0^{\pi/3} d\theta \ \left[(a_1')^2 G_1 + (a_2')^2 G_2 + (a_3')^2 G_3 \right. \\ \left. + \ (a_1 + a_2 a_3)^2/G_1 + (a_2 + a_1 a_3)^2/G_2 + (a_3 + a_1 a_2)^2/G_3\right]. \tag{4.4.1}$$

We claim that this differs from S_3 only by $O(\sqrt{n})$, and thus that eq.(4.3.8) applies to the full action. To see this we consider the effect of a nonzero a_1.

First we look at the potential energy (nonderivative) terms. To the left of the overlap region a_2 is small, as is $a_2 a_3$, so a nonzero a_1 can only add to the potential energy. In effect, the potential energy is changed from a_3^2/G_3 to $a_3^2(G_3^{-1} + a_1^2/G_2) + a_1^2/G_1$. Clearly, then, a_1 will be small in this region, and will similarly be small to the right of the overlap region. In the overlap region, however, a nonzero a_1 can help reduce the large $(a_1 + a_2 a_3)^2$ potential energy term. If a_1 is of size $O(A)$ in this region, $(a_1 + a_2 a_3)^2$ is reduced by $O(nA)$ (since a_2 and a_3 are $O(\sqrt{n})$), which, integrated over the region of size $O(1/\sqrt{n})$, gives a savings of $O(A\sqrt{n})$.

This must be balanced against the increase in kinetic energy, which goes as $O(A^2\sqrt{n})$, since the change from 0 to A occurs in a region of size $O(1/\sqrt{n})$. The difference between the savings and cost is maximized when A is $O(1)$, making the net savings $O(\sqrt{n})$. In short,

$$\begin{aligned} S_4(n) = S_3(n) + O(\sqrt{n}) =& 2\pi^2 n^2 \coth(T) - \frac{8\pi^2 c n^{3/2}}{3\sqrt{2}\sinh(T)^{3/2}} + O(n) \\ =& 8\pi^2(2.2990381\, n^2 - 4.2246\, n^{3/2}) + O(n). \end{aligned} \tag{4.4.2}$$

We remark that $S_4(n)$ is bounded by the quadratic term, $2\pi^2 n^2 \coth(T)$, for all values of n, not just for n sufficiently large. This is because $2\pi^2 n^2 \coth(T)$ is precisely the action of the trial wavefunction

$$a_1 = 0, \quad a_2 = \begin{cases} 0, & x \leq 0 \\ n \sinh(x)/\sinh(T), & x > 0. \end{cases} \quad a_3 = \begin{cases} -n \sinh(x)/\sinh(T), & x < 0 \\ 0, & x \geq 0. \end{cases} \tag{4.4.3}$$

All that remains is a numerical fit of the data for $S_4(n) - 8\pi^2(2.2990381n^2 + 4.2246n^{3/2})$. The leading behavior should be $O(n)$, and there should also be a $\sqrt{n}$ term, a constant term, and an $o(1)$ remainder. There might also be a $\log(n)$ term, from the integral of n^{-1} terms in the expansion of f'. (There were no such terms in the basic Model 1, but we can't rule them out for the more complicated models). Fitting data for $n = 1, 3, 5, \ldots, 19$ (gotten by the Fourier expansion of Section 3), we find that

$$S(n) \approx 8\pi^2(2.2990381n^2 - 4.2246n^{3/2} + 2.76n - 1.0\sqrt{n} + 0.16) \tag{4.4.4}$$

Although we cannot rule it out, there is no evidence of a $\log(n)$ term.

Table 3 shows the $S(n)$ vs. the right hand side of (4.4.4) for various values of n. Note that the estimated action (4.4.4) is almost as accurate for $n = 29$, which was not used in making the fit to the data, as for n up to 19.

Toy Model Number 5: $r \neq t$

We now look at asymmetric models, with $r \neq t$. We define the new variables

$$N = r + t; \qquad \alpha = (r - t)/N.$$

We are interested in the $O(N^2)$ and $O(N^{3/2})$ behavior of the action for large N, with α fixed.

We consider a toy model on the interval $[-T_2, T_1]$, with action

$$S_5 = \int_{-T_2}^{T_1} (f')^2 + a^2 f^2 + (g')^2 + b^2 g^2 + f^2 g^2, \tag{4.5.1}$$

where a and b are constants, with the boundary conditions $f(T_1) = t$, $g(-T_2) = r$. We also require that

$$\frac{r}{t} = \frac{\sinh(bT_2)/b}{\sinh(aT_1)/a}. \tag{4.5.2}$$

This insures that asymptotically the center of the overlap region, which we call the *crossover point*, will be at the origin. Eq. (4.5.2) is not really an extra condition, since given a value of α and an interval of size T_1+T_2 we can always choose the origin in such a way that (4.5.2) holds. Note that the coefficient of f^2g^2 is 1; other values of that coefficient can be treated as in Models 1 and 2.

Unfortunately, for finite N the crossover point is not exactly at the origin, but at a point $\delta(N)$, to be computed below. Although in the end δ does not contribute to our result, it does greatly complicate the algebra. We shall see that $\delta=O(N^{-1/2})$.

We proceed as in Model 2. Away from the overlap region there is a constant λ such that

$$f(x)\approx\frac{\lambda^2}{a}\sinh(a(x-\delta))+\lambda c\cosh(a(x-\delta)),\tag{4.5.3}$$

$$f'(x)\approx\lambda^2\cosh(a(x-\delta))+\lambda ca\sinh(a(x-\delta)),\tag{4.5.4}$$

for x much bigger than δ, and

$$g(x)\approx\frac{-\lambda^2}{b}\sinh(b(x-\delta))+\lambda c\cosh(b(x-\delta)),\tag{4.5.5}$$

$$g'(x)\approx-\lambda^2\cosh(b(x-\delta))+\lambda cb\sinh(b(x-\delta)),\tag{4.5.6}$$

for x much less than δ.

Evaluating at the endpoints we get

$$\begin{aligned}t=f(T_1)=&\frac{\lambda^2}{a}\sinh(a(T_1-\delta))+\lambda c\cosh(a(T_1-\delta))\\=&\frac{\lambda^2}{a}\sinh(aT_1)+\lambda c\cosh(aT_1)-\delta[\lambda^2\cosh(aT_1)+\lambda ac\sinh(aT_1)]\end{aligned}\tag{4.5.7}$$

plus $O(\delta^2)$ corrections. Similarly

$$\begin{aligned}r=g(-T_2)=&\frac{\lambda^2}{b}\sinh(b(T_2+\delta))+\lambda c\cosh(b(T_2+\delta))\\=&\frac{\lambda^2}{b}\sinh(bT_2)+\lambda c\cosh(bT_2)+\delta[\lambda^2\cosh(bT_2)+\lambda bc\sinh(bT_2)]\end{aligned}\tag{4.5.8}$$

plus $O(\delta^2)$ corrections. Looking at the derivatives we find

$$\begin{aligned}f'(T_1)=&\lambda^2\cosh(a(T_1-\delta))+\lambda ac\sinh(a(T_1-\delta))\\=&\lambda^2\cosh(aT_1)+\lambda ac\sinh(aT_1)-a^2\delta t,\end{aligned}\tag{4.5.9}$$

$$\begin{aligned}g'(-T_2)=&-\lambda^2\cosh(b(T_2+\delta))-\lambda bc\sinh(b(T_2+\delta))\\=&-\lambda^2\cosh(bT_2)-\lambda bc\sinh(bT_2)-b^2\delta r,\end{aligned}\tag{4.5.10}$$

where we have used the fact that $f'(x+\epsilon) \approx f'(x) + \epsilon f''(x) = f'(x) + \epsilon a^2 f(x)$.

We now define the rescaled boundary value

$$M = \frac{at}{\sinh(aT_1)} = \frac{br}{\sinh(bT_2)}. \tag{4.5.11}$$

Equations (4.5.7) and (4.5.8) then become

$$M = \lambda^2(1 - a\delta\coth(aT_1)) + \lambda ac(\coth(aT_1) - a\delta) \tag{4.5.12a}$$

$$= \lambda^2(1 + b\delta\coth(bT_2)) + \lambda bc(\coth(bT_2) + b\delta). \tag{4.5.12b}$$

Setting these two expressions equal we can compute δ:

$$\delta = \frac{c(a\coth(aT_1) - b\coth(bT_2))}{\lambda(a\coth(aT_1) + b\coth(bT_2)) + c(a^2+b^2)} = O(\lambda^{-1}) = O(M^{-1/2}). \tag{4.5.13}$$

Solving eq. (4.5.12a) for λ and expanding we find

$$\lambda = \sqrt{M} + \sqrt{M}a\delta\coth(aT_1)/2 - ca\coth(aT_1)/2 + o(1) \tag{4.5.14a}$$

$$\lambda^2 = M + a\sqrt{M}\coth(aT_1)(\delta\sqrt{M} - c) + O(1), \tag{4.5.14b}$$

and so

$$\begin{aligned} f'(T_1) = &(M - ac\sqrt{M}\coth(aT_1))\cosh(aT_1) + a\delta M\coth(aT_1)\cosh(aT_1) \\ &+ ac\sqrt{M}\sinh(aT_1) - a^2\delta n_- + O(1) \\ = &at\coth(aT_1) - c\sqrt{n_-}\left(\frac{a}{\sinh(aT_1)}\right)^{3/2} + \frac{a^2\delta t}{\sinh^2(aT_1)} + O(1) \end{aligned} \tag{4.5.15}$$

Similarly solving eq. (4.5.13b) gives

$$\lambda = \sqrt{M} - \sqrt{M}b\delta\coth(bT_2)/2 - cb\coth(bT_2)/2 + o(1) \tag{4.5.16a}$$

$$\lambda^2 = M - b\sqrt{M}\coth(bT_2)(\delta\sqrt{M} + c) + O(1), \tag{4.5.16b}$$

$$\begin{aligned} -g'(-T_2) = &(M - bc\sqrt{M}\coth(bT_2))\cosh(bT_2) - b\delta M\coth(bT_2)\cosh(bT_2) \\ &+ bc\sqrt{M}\sinh(bT_2) - b^2\delta r + O(1) \\ = &br\coth(bT_2) - c\sqrt{r}\left(\frac{b}{\sinh(bT_2)}\right)^{3/2} - \frac{b^2\delta r}{\sinh^2(bT_2)} + O(1) \end{aligned} \tag{4.5.17}$$

We're finally ready to compute the action. By the Useful Fact we know

$$\begin{aligned} dS_5 =& 2f'(T_1)dt - 2g'(-T_2)dr \\ =& 2\left[at\coth(aT_1) - c\sqrt{t}\left(\frac{a}{\sinh(aT_1)}\right)^{\frac{3}{2}} + \frac{\delta a^2 t}{\sinh^2(aT_1)}\right]dt \\ &+ 2\left[br\coth(bT_2) - c\sqrt{r}\left(\frac{b}{\sinh(bT_2)}\right)^{\frac{3}{2}} - \frac{\delta b^2 r}{\sinh^2(bT_2)}\right]dr. \end{aligned} \tag{4.5.18}$$

By eq. (4.5.2) the terms involving δ cancel. Integrating (4.5.18) we get at last our action

$$S_5 = at^2\coth(aT_1) + br^2\coth(bT_2) - \frac{4c}{3}\left(\frac{at}{\sinh(aT_1)}\right)^{\frac{3}{2}} - \frac{4c}{3}\left(\frac{br}{\sinh(bT_2)}\right)^{\frac{3}{2}} + O(n). \tag{4.5.19}$$

Note that by (4.5.2) the last two terms are equal. Also note that for $a = b = 1$, $r = t = n$, $T_1 = T_2 = T$, this result reduces to (4.2.7), the result of Model 2.

Model 6: The Full System with $r \neq t$

Here we look at the full Yang-Mills functional S_4, with $r \neq t$. As we have seen, setting $a_1 = 0$ only introduces an $O(\sqrt{N})$ error in the action, which makes no difference to the $O(N^2)$ and $O(N^{3/2})$ terms. We therefore take $a_1 = 0$, reducing the full action S_4 to S_3.

Now for large N, the system will have a crossover point θ_0, which will be calculated below. We define the constants

$$T = \int_0^{\theta_0} \frac{dy}{G_3(y)}; \qquad T' = \int_{\theta_0}^{\pi/3} \frac{dy}{G_2(y)}; \qquad k = \frac{G_2(\theta_0)}{G_3(\theta_0)}; \qquad K = \frac{G_1(\theta_0)}{G_3(\theta_0)}, \tag{4.6.1}$$

and make the change of coordinates

$$x(\theta) = -\int_\theta^{\theta_0} dy/G_3(y), \tag{4.6.2}$$

for $\theta < \theta_0$, and

$$x(\theta) = \int_{\theta_0}^{\theta} k\,dy/G_2(y), \tag{4.6.3}$$

for $\theta \geq \theta_0$. The factor of k in (4.6.3) is to make $dx/d\theta$ continuous at θ_0.

In terms of x the action is

$$S = \pi^2 \int_{-T}^{0} a_3^2 + (a_3')^2 + a_2^2 G_3/G_2 + (a_2')^2 G_2/G_3 + (a_2 a_3)^2 G_3/G_1 dx \quad (4.6.4)$$
$$+ \pi^2 \int_0^{kT'} a_2^2/k + k(a_2')^2 + a_3^2 G_2/(kG_3) + (a_3')^2 kG_3/G_2 + (a_2 a_3)^2 G_2/(kG_1) dx$$

Letting $g = a_3$, $f = \sqrt{k} a_2$, we can approximate this as

$$S \approx \pi^2 \int_{-T}^{T'} \frac{g^2}{k^2} + (g')^2 + f^2 + (f')^2 + \frac{f^2 g^2}{kK} dx. \quad (4.6.5)$$

This is precisely Model 5, with $T_1 = kT'$, $a = 1/k$, so $aT_1 = T'$, and with $T_2 = T$, $b = 1$, and so $bT_2 = T$, and with the factor $1/(kK)$ multiplying the $f^2 g^2$ term. Also, the boundary value at T' is $r\sqrt{k}$, not r.

Plugging this all into (4.5.19) we get

$$S = \pi^2 \Big(r^2 b \coth(bT_2) + t^2 ka \coth(aT_1) - \frac{4c(kK)^{\frac{1}{4}}}{3} \left[\left(\frac{at\sqrt{k}}{\sinh(aT_1)} \right)^{\frac{3}{2}} + \left(\frac{br}{\sinh(bT_2)} \right)^{\frac{3}{2}} \right] \Big)$$
$$= \pi^2 (r^2 \coth(T) + t^2 \coth(T')) \quad (4.6.6)$$
$$- \frac{4c\pi^2}{3} \left(\left(\frac{t}{\sinh(T')} \right)^{\frac{3}{2}} \left(\frac{G_1(\theta_0) G_3(\theta_0)}{G_2(\theta_0)^2} \right)^{\frac{1}{4}} + \left(\frac{r}{\sinh(T)} \right)^{\frac{3}{2}} \left(\frac{G_1(\theta_0) G_2(\theta_0)}{G_3(\theta_0)^2} \right)^{\frac{1}{4}} \right).$$

So all that remains to compute $S(r,t)$ through order $N^{3/2}$ is to compute θ_0 (and hence T, T', and the hyperbolic functions of T and T'). θ_0 is determined by setting the derivative of the $O(N^2)$ term with respect to θ_0 equal to zero, or equivalently setting the two $O(N^{3/2})$ terms equal to each other. As with Model 4, the quadratic term gives an upper bound to the action for all values of N.

The integrals (4.6.1) are not difficult, and give

$$\coth(T) = \frac{4\mu - 1}{2\mu - 2} = \frac{3}{2(1-\mu)} - 2, \qquad (\text{where } \mu \equiv \cos(\theta_0)), \quad (4.6.7)$$

with a similar result for $\coth(T')$. The derivative of the $O(N^2)$ term is then easily taken, and setting it equal to zero gives the condition

$$\frac{r^2 \sin(\theta_0)}{(1 - \cos(\theta_0))^2} = \frac{t^2 \sin(\pi/3 - \theta_0)}{(1 - \cos(\pi/3 - \theta_0))^2}. \quad (4.6.8)$$

Given a value of α this equation can be solved numerically, and plugged back into (3.8.6) to give the N^2 and $N^{3/2}$ coefficients. This data is compiled in Table 4.

5. Conclusions

We have approximated certain non-self-dual solutions of the Yang-Mills equations on the sphere. For small (n_+, n_-), the approximate solutions were obtained by truncating the Fourier expansion of the reduced connection $a_i(\theta)$ and minimizing on the computer. For large values of (n_+, n_+), we derived asymptotic formulas, and fitted the constants using the computer-generated solutions for small (n_+, n_-). No solutions on Euclidean space (equivalently the sphere) are known explicitly, either of this symmetry type or of any other type.

In the case of self-dual Yang-Mills solutions, t'Hooft constructed a class of 'multi-instanton' solutions (see [JNR], [AHS] sec. 7). In a certain limit, these solutions have curvature localized near a finite set of points, with approximately one unit of 'topological charge' concentrated near each point. Although the Yang-Mills equations are nonlinear, these solutions can be interpreted as some sort of superposition of single instanton solutions. This procedure does not produce all self-dual solutions, but only a subset.

It is a possibility that some non-self-dual solutions admit a similar interpretation, as a superposition of instantons and anti-instantons. The idea is that in one region of S^4, the solution would be almost self-dual, and in another region, the solution would be almost anti-self-dual. These regions would be connected by an overlap region. Sibner, Sibner, and Uhlenbeck [SSU] suggest that their non-self-dual solutions may correspond to a superposition of an m-instanton and an m-anti-instanton, with $m \geq 2$. Parker's solution on S^4 with a modified metric [P1,P2] can be interpreted as a superposition of an instanton and an anti-instanton at antipodal points.

However, our non-self-dual solutions do not admit such an interpretation. They do not approximately satisfy the (anti)self-dual equations (2.8) anywhere on the segment $0 \leq \theta \leq \pi/3$. To the left of the overlap region, we have $a_1 \approx a_2 \approx 0$, so the (anti)self-duality equations reduce to $-da_3/dx = \pm a_3$, where x is given by (4.6.2) and (4.6.3). Similarly, to the right the equations are $-da_2/dx = \pm a_2$. In either region, e^{-x} is self-dual and e^x is anti-self-dual. As we have seen, our solutions go as $\sinh(x)$, which means that near the overlap region we have an equal mixture of self-dual and anti-self-dual. As we move away from the overlap region towards $\theta = 0$, the amplitude of the self-dual component (e^{-x}) increases somewhat, while the anti-self-dual amplitude decreases, but the ratio of the amplitudes, namely e^{-2x}, is finite even at $\theta = 0$ ($x = -T$). When the asymmetry parameter $\alpha = (r-t)/(r+t)$ is equal to zero, this ratio at $\theta = 0$ is only $(2+\sqrt{3})/3 \approx 1.243$.

In the asymmetric case, when α is positive, T will be somewhat larger, so on the left side of the overlap region, the solution will be more self-dual and less anti-self-dual, while on the right side of the overlap, the proportions will differ less. However, as long as α is less than 1, the ratio of self-dual to anti-self-dual amplitudes at $\theta = 0$

will remain finite.

The solutions become approximately (anti)self-dual locally only as α approaches 1. In that limit, the crossover point moves to $\theta = \pi/3$, T becomes infinite, and the solution (which is supported entirely on the left side of the crossover point) becomes entirely self-dual, taking the form

$$a_1(\theta) = a_2(\theta) = 0, \qquad a_3(\theta) = r \exp(-\int_0^\theta dy/G_3(y)). \tag{5.1}$$

In the case $(r,t) = (n,1)$ with n large, our formula (4.6.6) gives an action $S = \pi^2 n^2 + O(n)$, which asymptotically matches the topological lower bound $\pi^2(n^2-1)$.

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Table 1. Finite-Mode Approximations for $(n_+, n_-) = (5, 3)$.

n	A_n for $l = 1$	$l = 2$	$l = 3$
0	-0.3245945605	-0.3245713584	-0.3245713855
1	-1.4142100298	-1.4142096286	-1.4142096235
2	-0.5362985122	-0.5363137348	-0.5361317255
3	-0.6687388537	-0.6687367686	-0.6687367756
4	0.2040423555	0.2040428056	0.2040428225
5	0.0885205177	0.0885274511	0.0885274545
6	-0.0043693864	-0.0043523523	-0.0043523755
7	-0.0074408414	-0.0074473554	-0.0074473502
8	-0.0016187954	-0.0016410762	-0.0016410774
9	0.0020721870	0.0020742051	0.0020741999
10	0.0005416187	0.0005755070	0.0005755227
11	-0.0002029798	-0.0001648856	-0.0001648778
12	0	-0.0000286352	-0.0000286438
13	0	-0.0000394540	-0.0000394588
14	0	0.0000014885	0.0000014772
15	0	-0.0000041032	-0.0000041040
16	0	0.0000016766	0.0000016566
17	0	0.0000005393	0.0000005864
18	0	0	0.0000000454
19	0	0	-0.0000000624
20	0	0	-0.0000000139
21	0	0	0.0000000130
22	0	0	0.0000000045
23	0	0	-0.0000000014
$S/8\pi^2$	11.4882182679	11.4882177866	11.4882177866

Table 2. Actions of Some Low-Lying Solutions

(n_+, n_-)	$S/8\pi^2$
(3,3)	5.43281507547525
(5,3)	11.4882177865800
(7,3)	18.9925969277396
(5,5)	21.9519857528157
(9,3)	27.8614519776604
(7,5)	34.1521844389385
(11,3)	38.04236576029
(9,5)	47.96544358790
(13,3)	49.498844315
(7,7)	51.23396533677
(15,3)	62.20367167
(11,5)	63.3120814979
(9,7)	70.1032953685
(17,3)	76.1355939
(13,5)	80.135023031
(11,7)	90.668949991
(19,3)	91.277452
(9,9)	94.14892253
(15,5)	98.3908046

Table 3. Numerical vs. Asymptotic Actions

n	Numerical $S(n,n)/8\pi^2$	Asymptotic Formula (4.4.4)	Difference
1	0	-0.00059	0.00059
3	5.43282	5.45249	-0.01967
5	21.95199	21.97212	-0.02013
7	51.23397	51.25092	-0.01695
9	94.14892	94.16215	-0.01323
11	151.25581	151.26545	-0.00964
13	222.95467	222.95946	-0.00479
15	309.54623	309.54693	-0.00070
17	411.26764	411.26797	-0.00033
19	528.32139	528.31912	0.00227
29	1348.57662	1348.55178	0.02484

Table 4. Coefficients for $S/8\pi^2$

α	N^2 coefficient	$N^{3/2}$ coefficient	θ_0
0	0.5747595264	-1.4936	0.5235987756
0.1	0.5713916984	-1.4878	0.5585551849
0.2	0.5612469955	-1.4703	0.5939076369
0.3	0.5441954125	-1.4405	0.6300877011
0.4	0.5199966585	-1.3976	0.6676074578
0.5	0.4882542493	-1.3401	0.7071288703
0.6	0.4483208416	-1.2657	0.7495894773
0.7	0.3990863929	-1.1701	0.7964690692
0.8	0.3384135611	-1.0444	0.8504824968
0.9	0.2610245328	-0.8648	0.9181185734
0.91	0.2518690752	-0.8414	0.9262302578
0.92	0.2423379341	-0.8163	0.9347270475
0.93	0.2323739184	-0.7892	0.9436758704
0.94	0.2218998821	-0.7597	0.9531664337
0.95	0.2108070566	-0.7268	0.9633242998
0.96	0.1989326069	-0.6896	0.9743358721
0.97	0.1860107315	-0.6459	0.9865024134
0.98	0.1715437937	-0.5912	1.0003811278
0.99	0.1543234661	-0.5129	1.0173034421
0.992	0.1502760714	-0.4910	1.0213497358
0.994	0.1458738872	-0.4646	1.0257817480
0.996	0.1409426774	-0.4306	1.0307849275
0.998	0.1350610333	-0.3797	1.0368043702
0.999	0.1313507206	-0.3360	1.0406281236
0.9995	0.1290087378	-0.2980	1.0430500586
0.9999	0.1263765530	-0.2268	1.0457755838

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