SOME MAXIMUM PRINCIPLES
FOR PARABOLIC MIXED LOCAL/NONLOCAL OPERATORS

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Abstract. The goal of this paper is to establish new Maximum Principles for parabolic equations in the framework of mixed local/nonlocal operators.

In particular, these results apply to the case of mixed local/nonlocal Neumann boundary conditions, as introduced in [DPLV23].

Moreover, they play an important role in the analysis of population dynamics involving the so-called Allee effect, which is performed in [DPLV]. This is particularly relevant when studying biological populations, since the Allee effect detects a critical density below which the population is severely endangered and at risk of extinction.

1. Introduction

In this paper we deal with some Maximum Principles for parabolic equations involving operators of mixed order. In particular, we take into account diffusive operators in which the classical and the fractional Laplacians coexist.

More precisely, the elliptic operator that we consider has the form

$$-\alpha\Delta u+\beta(-\Delta)^s, \tag{1.1}$$

with α, $\beta\in[0,+\infty)$, $\alpha+\beta\neq 0$, and $s\in(0,1)$. Here, the fractional Laplacian is defined as

$$(-\Delta)^s u(x):=\frac{1}{2}\int_{\mathbb{R}^n}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy,$$

up to a normalization constant that we neglect since it does not play a role in our analysis (see e.g. [DNPV12, AV19]).

Mixed order operators of the form (1.1) are a very natural subject of investigation, since they arise in many theoretical and applied branches of mathematics. For instance, they correspond to the superposition of different types of stochastic processes such as a classical random walk and a Lévy flight, having also applications to the study of optimal animal foraging strategies, see e.g. [MPV13, PV18, DPLV23, DV21].

Furthermore, these operators offer several challenges from the technical point of view, in light of the combination of nonlocal difficulties and the lack of invariance under scaling. Several papers in the recent literature have focused on existence and regularity type results (see [JK05, BI08, BJK10, BCCI12, CKSV12, AC21, BDVV22, CDV22, DPLV22, GK22, GU22, SV22, SVWZ22, GL23, BMV, DFM]), symmetry and classification results (see [BDVV21, BDVV23, BDVV23b, BDVV22]), etc.

In this paper, we contribute to the theory of mixed order operators as in (1.1) by providing new Maximum Principles for the associated parabolic equation.

The setting that we take into account also comprises that of an ecological niche, see [DV21], which requires the use of appropriate zero-flux conditions, as introduced in [DROV17, DPLV23].

More precisely, given a bounded domain $\Omega\subset\mathbb{R}^n$, a reflection condition for the fractional Laplacian has been put forth in [DROV17] as a natural extension of the Neumann boundary

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condition in the nonlocal setting, and reads as

$$\mathscr{N}_s u(x) = 0 \qquad \text{for all } x \in \mathbb{R}^n \setminus \Omega, \tag{1.2}$$

where

$$\mathscr{N}_s u(x) := \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy. \tag{1.3}$$

The rationale behind the condition in (1.2) is that it corresponds to a particle reflection of a Lévy process. Namely, a particle following a Lévy process leaves a given bounded domain almost surely in a finite time and the condition in (1.2) corresponds to an instantaneous reentering into the domain by following the same process distribution, see also [Von21] for a detailed probabilistic framework underpinning condition (1.3). As a matter of fact, this is the nonlocal counterpart of the classical feature of the Neumann condition in Gaussian diffusion, whose role corresponds to a "billiard" reflection at the boundary.

With this notation, a Neumann condition for mixed operators as in (1.1) was introduced in [DPLV23] and corresponds to these alternatives:

$$\begin{cases} \mathscr{N}_s u(x) = 0 \text{ for all } x \in \mathbb{R}^n \setminus \Omega, & \text{when } \alpha = 0, \\[2ex] \partial_\nu u(x) = 0 \text{ for all } x \in \partial\Omega, & \text{when } \beta = 0, \\[2ex] \begin{array}{l} \mathscr{N}_s u(x) = 0 \text{ for all } x \in \mathbb{R}^n \setminus \overline{\Omega} \\ \text{and } \partial_\nu u(x) = 0 \text{ for all } x \in \partial\Omega, \end{array} & \text{when } \alpha \neq 0 \text{ and } \beta \neq 0. \end{cases} \tag{1.4}$$

Here above ν denotes the outward unit normal to $\partial\Omega$ and ∂_ν the normal derivative of u.

In the rest of the paper, we will use the expression (α, β)-Neumann conditions as a short notation for (1.4).

Notice that the first two conditions in (1.4) correspond to the nonlocal Neumann condition of [DROV17] when only the Lévy-type anomalous diffusion takes place and to the classical Neumann condition when only the Gaussian diffusion is present.

Instead, the third condition in (1.4) corresponds to the case in which both classical and anomalous diffusions are present. In this case, the Neumann condition that arises involves both the classical and the nonlocal constraints. We stress that this prescription, maybe quite surprisingly, is not an "overdetermined" condition (indeed, overdetermined problems typically do not have solutions, apart from very special configurations): instead, the "double condition" given in (1.4) gives rise to well-posed problems and describes an ecological niche endowed with a zero-flux condition, see [DV21].

With this preliminary notation, the first result that we present is a new (weak) Maximum Principle for parabolic equations driven by operators of possibly mixed order and endowed with the corresponding zero-flux conditions of Neumann type:

Theorem 1.1. *Let Ω be a bounded domain of $\mathbb{R}^n$ with C^1 boundary, $T > 0$, $c \in \mathbb{R}$ and $f \in L^\infty(\Omega \times (0,T))$. Assume that u is a classical solution*[1] *of*

$$\begin{cases} \partial_t u(x,t) - \alpha\Delta u(x,t) + \beta(-\Delta)^s u(x,t) = (u(x,t) - c) f(x,t) & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ \text{with } (\alpha,\beta) - \text{Neumann conditions} & \text{in } (0,T). \end{cases} \tag{1.5}$$

[1]By classical solution we mean that, for all $t \in [0,T]$, $u(\cdot,t) \in L^\infty(\mathbb{R}^n) \cap C(\overline{\Omega}) \cap C^2(\Omega)$ and, for all $x \in \Omega$, $u(x,\cdot) \in C^1((0,T]) \cap C([0,T])$, with the (α,β)-Neumann conditions satisfied pointwise in $\mathbb{R}^n \setminus \Omega$ and $\partial\Omega$ respectively. This setting is taken for the sake of simplicity to allow us a specific focus on the maximum principle properties in this ideal scenario. We will devote a subsequent project to the detection of the optimal regularity theory for this type of equations (see [AFNRO23] for a regularity theory in the elliptic setting with nonlocal Neumann conditions, where the authors prove that solutions are C^α up to the boundary of the domain Ω).

Assume also that

- *if* $\beta = 0$, *then* Ω *satisfies the interior ball condition,*
- *if* $\alpha\beta \neq 0$, *then there exist* $\mu > 0$ *and* $\theta > 2s$ *such*[2] *that*

$$u \in L^\infty\big([0,T],\, C^\theta(\Omega_\mu)\big) \qquad \textit{with} \quad \Omega_\mu := \bigcup_{x\in\Omega} B_\mu(x). \tag{1.6}$$

Then,[3]

(*i*) *if* $u_0(x) \geqslant c$ *for any* $x \in \mathbb{R}^n$, *then* $u(x,t) \geqslant c$ *for any* $x \in \mathbb{R}^n$ *and* $t \in [0,T)$;
(*ii*) *if* $u_0(x) \leqslant c$ *for any* $x \in \mathbb{R}^n$, *then* $u(x,t) \leqslant c$ *for any* $x \in \mathbb{R}^n$ *and* $t \in [0,T)$.

We recall that an open set Ω is said to satisfy the interior ball condition if for any $x \in \partial\Omega$ there exists a ball $B \subset \Omega$ such that $x \in \partial B$.

As far as we know, the results presented are new even in the purely nonlocal case $\alpha = 0$ (the case $\beta = 0$ is classical, see e.g. [PW84]).

Theorem 1.1 will be pivotal in [DPLV] to analyze the survival strategies of a biological population that lives in an ecological niche and is subject to the so-called Allee effect.

The Allee effect (named after Warder Clyde Allee [All31, AB32, All38]) aims at detecting how survival rates are affected by the size of the population and what the correlation is between individual fitness and population density. In particular, it detects a critical density below which the population is severely endangered and at risk of extinction, see the [DPLV] and the references therein for a detailed mathematical analysis of the model.

In addition, we also prove the following Maximum Principle for operators of the type

$$L[u] := \alpha\Delta u - \beta(-\Delta)^s u - \partial_t u.$$

Theorem 1.2. *Let* $t_1 > 0$ *and* $M \in \mathbb{R}$. *Suppose that* $L[u] \geqslant 0$ *in* $\Omega \times (0,t_1]$.

If $\beta = 0$, *assume in addition that* Ω *is connected.*

If $u \leqslant M$ *in* $\mathbb{R}^n \times (0,t_1]$ *and* $u(x_1,t_1) = M$ *for some* $x_1 \in \Omega$, *then* $u(x,t) = M$ *for every* $(x,t) \in \Omega \times (0,t_1]$.

Notice in particular that the assumption that $u \leqslant M$ in $\mathbb{R}^n \times (0,t_1]$ in the statement of Theorem 1.2 can be relaxed to $u \leqslant M$ in $\Omega \times (0,t_1]$ if one also assumes the (α,β)-Neumann conditions.

The rest of this paper is devoted to the proofs of the main results. Namely, in Section 2 we establish the Maximum Principle stated in Theorem 1.1 and in Section 3 we provide the proof of Theorem 1.2, together with some auxiliary results of independent interest.

2. Proof of Theorem 1.1

In this section we provide the proof of the weak Maximum Principle in Theorem 1.1.

Proof of Theorem 1.1. We start with the proof of (i). For this, suppose that $u(x,0) \geqslant c$ for any $x \in \mathbb{R}^n$.

Since $f \in L^\infty(\Omega \times (0,T))$, there exists $M > 0$ such that $|f(x,t)| \leqslant M$ for any $x \in \mathbb{R}^n$ and $t \in [0,T)$. We set

$$k := \lfloor 4MT \rfloor + 1 \in \mathbb{N} \tag{2.1}$$

and we define the set

$$I_i := \left(\frac{iT}{k}, \frac{(i+1)T}{k}\right], \qquad \text{for all } \ i \in \{0,\dots,k-1\}.$$

[2]As customary, if $\theta > 1$, the notation $u \in C^\theta(\Omega_\mu)$ means $u \in C^{k,\vartheta}(\Omega_\mu)$ with $k \in \mathbb{N}$, $\vartheta \in (0,1]$ and $\theta = k + \vartheta$.

[3]We point out that, here and throughout the paper, we use the notation "for all $x \in \mathbb{R}^n$" even when $\beta = 0$. As a matter fact, in this case it suffices to write "for all $x \in \Omega$" in the statement of Theorem 1.1, but we use this more compact notation for the sake of readability.

We first observe that, for all $i \in \{0,\dots,k-1\}$,

$$\text{if } u\left(x,\tfrac{iT}{k}\right) \geqslant c \text{ for any } x\in\mathbb{R}^n, \text{ then } u(x,t)\geqslant c \text{ for any } x\in\mathbb{R}^n \text{ and any } t\in I_i. \tag{2.2}$$

Indeed, let $i \in \{0,\dots,k-1\}$ and suppose that $u\left(x,\frac{iT}{k}\right) \geqslant c$ for any $x\in\mathbb{R}^n$. Assume by contradiction that $u(\overline{x}_\star,\overline{t})<c$ for some $\overline{x}_\star\in\mathbb{R}^n$ and $\overline{t}\in I_i$.

Now we observe that we can pick the point $\overline{x}_\star$ such that $\overline{x}_\star\in\overline{\Omega}$. More explicitly, we claim that

$$\text{there exists } \overline{x}\in\overline{\Omega} \text{ such that } u(\overline{x},\overline{t})<c. \tag{2.3}$$

To check this, we first observe that if $\beta=0$ then the claim in (2.3) is obvious (recall the footnote 3 on page 3).

If instead $\beta\neq0$, we suppose by contradiction that $u(y,\overline{t})\geqslant c$ for each $y\in\overline{\Omega}$. Then, we use the nonlocal Neumann condition to see that, for all $x\in\mathbb{R}^n\setminus\overline{\Omega}$ and $t\in(0,T)$,

$$0=\int_\Omega \frac{u(x,t)-u(y,t)}{|x-y|^{n+2s}}\,dy.$$

In particular, taking $x:=\overline{x}_\star$ and $t:=\overline{t}$, we find that

$$0=\int_\Omega \frac{u(\overline{x}_\star,\overline{t})-u(y,\overline{t})}{|\overline{x}_\star-y|^{n+2s}}\,dy.$$

Since, for all $y\in\Omega$, we have that $u(\overline{x}_\star,\overline{t})-u(y,\overline{t})\leqslant u(\overline{x}_\star,\overline{t})-c<0$, the above equation leads to a contradiction, thus proving (2.3).

Now, by means of (2.3), since u is continuous up to the boundary of Ω, without loss of generality we can suppose that

$$\overline{x}\in\Omega.$$

Now we define

$$\begin{aligned} \varepsilon &:= \min\left\{\frac{T}{k},\frac{1}{2M}-\frac{T}{k},\frac{k}{4T}(c-u(\overline{x},\overline{t}))\right\}\\ \text{and}\quad v(x,t) &:= u(x,t)+\varepsilon\left(t-\frac{iT}{k}\right)+\varepsilon^2. \end{aligned}$$

Notice that $\varepsilon>0$ (recall our choice of k in (2.1)), and therefore, in light of the assumption in (2.2), for all $x\in\mathbb{R}^n$,

$$v\left(x,\frac{iT}{k}\right)=u\left(x,\frac{iT}{k}\right)+\varepsilon\left(\frac{iT}{k}-\frac{iT}{k}\right)+\varepsilon^2\geqslant c+\varepsilon^2>c.$$

Moreover, since $\varepsilon\leqslant\frac{k}{4T}(c-u(\overline{x},\overline{t}))$, and also $\varepsilon\leqslant\frac{T}{k}$, we have that

$$v(\overline{x},\overline{t})=u(\overline{x},\overline{t})+\varepsilon\left(\overline{t}-\frac{iT}{k}\right)+\varepsilon^2\leqslant c+\varepsilon\left(-\frac{4T}{k}+\overline{t}-\frac{iT}{k}+\varepsilon\right)\leqslant c-\frac{2T}{k}\varepsilon<c.$$

Thus, by the continuity of u (and hence of v), we have that there exist $x^\star\in\overline{\Omega}$ and $t^\star\in\left(\frac{iT}{k},\overline{t}\right)$ such that $v(x,t)>c$ for any $x\in\Omega$ and $t\in\left[\frac{iT}{k},t^\star\right)$ and $v(x^\star,t^\star)=c$.

We treat separately the cases $x^\star\in\Omega$ and $x^\star\in\partial\Omega$.

First, suppose that $x^\star\in\Omega$. Therefore, for any $x\in\Omega$, we have that

$$v(x,t^\star)\geqslant c=v(x^\star,t^\star). \tag{2.4}$$

Notice that, by continuity, this holds true for all $x\in\overline{\Omega}$. In addition, if $\beta\neq0$, for all $x\in\mathbb{R}^n\setminus\overline{\Omega}$,

$$\begin{aligned} 0&=\int_\Omega \frac{u(x,t^\star)-u(y,t^\star)}{|x-y|^{n+2s}}\,dy=\int_\Omega \frac{v(x,t^\star)-v(y,t^\star)}{|x-y|^{n+2s}}\,dy\\ &\leqslant\int_\Omega \frac{v(x,t^\star)-c}{|x-y|^{n+2s}}\,dy=\big(v(x,t^\star)-c\big)\int_\Omega\frac{dy}{|x-y|^{n+2s}}, \end{aligned}$$

which implies that $v(x,t^\star)\geqslant c$. Accordingly, the inequality in (2.4) is satisfied for all $x\in\mathbb{R}^n$.

As a consequence,

$$\nabla v(x^\star,t^\star)=0 \qquad \text{and} \qquad \Delta v(x^\star,t^\star)\geqslant 0. \tag{2.5}$$

Furthermore,

$$(-\Delta)^s v(x^\star,t^\star)=\int_{\mathbb{R}^n}\frac{v(x^\star,t^\star)-v(y,t^\star)}{|x^\star-y|^{n+2s}}\,dy\leqslant 0. \tag{2.6}$$

Moreover, we have that $v(x^\star,t)\geqslant c=v(x^\star,t^\star)$ for any $t\in\left[\frac{iT}{k},t^\star\right)$, and therefore

$$\partial_t v(x^\star,t^\star)\leqslant 0. \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), we find that

$$\Big((\partial_t-\alpha\Delta+\beta(-\Delta)^s)v\Big)(z^\star)\leqslant 0. \tag{2.8}$$

Furthermore, a direct computation, using also the equation in (1.5), leads to

$$\begin{aligned}\Big((\partial_t-\alpha\Delta+\beta(-\Delta)^s)v\Big)(x^\star,t^\star)&=\Big((\partial_t-\alpha\Delta+\beta(-\Delta)^s)u\Big)(x^\star,t^\star)+\varepsilon\\&=(u(x^\star,t^\star)-c)f(x^\star,t^\star)+\varepsilon\\&=(v(x^\star,t^\star)-c)f(x^\star,t^\star)-f(x^\star,t^\star)\left[\varepsilon\left(t^\star-\frac{iT}{k}\right)+\varepsilon^2\right]+\varepsilon\\&=-f(x^\star,t^\star)\left[\varepsilon\left(t^\star-\frac{iT}{k}\right)+\varepsilon^2\right]+\varepsilon\\&\geqslant -M\varepsilon\left(\frac{T}{k}+\varepsilon\right)+\varepsilon\\&\geqslant\frac{\varepsilon}{2},\end{aligned}$$

in light of the definition of ε.

This is in contradiction with (2.8), and thus the claim in (2.2) is proved in this case.

Now we deal with the case $x^\star\in\partial\Omega$. We point out that if there exists another point $x^\sharp\in\Omega$ such that $v(x^\sharp,t^\star)=c$, then we repeat the argument above with $x^\sharp$ in place of $x^\star$ and we reach the desired contradiction. Therefore, we can suppose, without loss of generality, that $v(x,t^\star)>c$ for all $x\in\Omega$.

Consequently, if $\alpha=0$, from the nonlocal Neumann condition we conclude that

$$0=\mathscr{N}_s u(x^\star,t^\star)=\mathscr{N}_s v(x^\star,t^\star)=\int_\Omega\frac{v(x^\star,t^\star)-v(y,t^\star)}{|x^\star-y|^{n+2s}}\,dy=\int_\Omega\frac{c-v(y,t^\star)}{|x^\star-y|^{n+2s}}\,dy<0.$$

This gives the desired contradiction and establishes the claim in (2.2) in this case.

Now, if instead $\beta=0$, we define the set

$$E:=\left\{(x,t)\in\Omega\times\left(\frac{iT}{k},\frac{(i+1)T}{k}\right)\;:\;|x^\star-x|^2+|t^\star-t|^2<r^2\right\}$$

and observe that, thanks to the continuity of v, we can take r small enough such that $|v(x,t)-c|<\varepsilon^2$ for every $(x,t)\in E$.

As a consequence, for every $(x,t)\in E$,

$$|u(x,t)-c|\leqslant|v(x,t)-c|+\varepsilon\left(t-\frac{iT}{k}\right)+\varepsilon^2<\varepsilon\left(t-\frac{iT}{k}\right)+2\varepsilon^2,$$

and therefore

$$\begin{aligned}(\partial_t-\alpha\Delta)v(x,t)&=(\partial_t-\alpha\Delta)u(x,t)+\varepsilon\\&=(u(x,t)-c)f(x,t)+\varepsilon\\&\geqslant -M|u(x,t)-c|+\varepsilon\\&\geqslant -M\left[\varepsilon\left(t-\frac{iT}{k}\right)+2\varepsilon^2\right]+\varepsilon\\&\geqslant -M\varepsilon\left(\frac{T}{k}+2\varepsilon\right)+\varepsilon\\&\geqslant \frac{MT}{k},\end{aligned}$$

thanks to the definition of ε. Namely, v satisfies $(\partial_t-\alpha\Delta)v>0$ in E.

Now, we define $E_{t^\star}:=\{(x,y)\in E\,:\,t\leqslant t^\star\}$ and observe that v achieves its minimum in $(x^\star,t^\star)\in\partial E$.

Moreover, since Ω satisfies the interior ball condition, we can construct a ball $B\subset\mathbb{R}^{n+1}$ such that $\overline{B}\cap\partial E=\{(x^\star,t^\star)\}$ and $B\cap\{t\leqslant t^\star\}\subset E_{t^\star}$, and also $v(x,t)>c$ in $E_{t^\star}$.

Thus, all the assumptions of Theorem 6 in Chapter 3 of [PW84] are satisfied, and therefore we conclude that

$$\frac{\partial v}{\partial\nu}(x^\star,t^\star)<0.$$

On the other hand, since $x^\star\in\partial\Omega$, from the boundary condition in (1.5) we have that

$$\frac{\partial v}{\partial\nu}(x^\star,t^\star)=\frac{\partial u}{\partial\nu}(x^\star,t^\star)=0.$$

Hence, we have reached the desired contradiction, which completes the proof of (2.2) when $\beta=0$.

Now, we focus on the case $\alpha\beta\neq0$. In this case, up to a rigid motion, we can suppose that $x^\star$ is the origin and that the outward unit normal of Ω at $x^\star$ equals $-e_n$. We also consider $\rho\in(0,\mu)$ small enough such that $\Omega\cap B_\rho$ is the supergraph of a C^1 function and $\Omega\setminus B_\rho\neq\varnothing$.

We observe that

$$\frac{\partial v}{\partial\nu}(x^\star,t^\star)=\frac{\partial u}{\partial\nu}(x^\star,t^\star)=0. \tag{2.9}$$

Also, recalling the assumption in (1.6), we have that $v^\star:=v(\cdot,t^\star)\in C^\theta(\Omega_\mu)$, and therefore, using (2.9) when $\theta>1$, for all $\varepsilon\in(0,\rho)$,

$$v^\star(-\varepsilon e_n)-c=v^\star(-\varepsilon e_n)-v^\star(0)\leqslant C\varepsilon^\theta,$$

for some $C>0$.

As a result, changing variable $z:=y/\varepsilon$,

$$\begin{aligned}&\int_{\Omega\cap B_\rho}\frac{v^\star(-\varepsilon e_n)-v^\star(y)}{|\varepsilon e_n+y|^{n+2s}}\,dy\leqslant\int_{\Omega\cap B_\rho}\frac{v^\star(-\varepsilon e_n)-c}{|\varepsilon e_n+y|^{n+2s}}\,dy\leqslant C\varepsilon^\theta\int_{\Omega\cap B_\rho}\frac{dy}{|\varepsilon e_n+y|^{n+2s}}\\&\qquad\leqslant C\varepsilon^{\theta-2s}\int_{(\Omega\cap B_\rho)/\varepsilon}\frac{dz}{|e_n+z|^{n+2s}},\end{aligned}$$

which is infinitesimal as ε goes to zero.

From this and the nonlocal Neumann condition, using the Lebesgue Dominated Convergence Theorem, it follows that

$$\begin{aligned}0&=\lim_{\varepsilon\searrow0}\int_\Omega\frac{v^\star(-\varepsilon e_n)-v^\star(y)}{|\varepsilon e_n+y|^{n+2s}}\,dy\\&\leqslant\lim_{\varepsilon\searrow0}\int_{\Omega\setminus B_\rho}\frac{v^\star(-\varepsilon e_n)-v^\star(y)}{|\varepsilon e_n+y|^{n+2s}}\,dy\end{aligned}$$

$$= \int_{\Omega\setminus B_\rho} \frac{v^\star(0) - v^\star(y)}{|y|^{n+2s}}\,dy$$
$$= \int_{\Omega\setminus B_\rho} \frac{c - v^\star(y)}{|y|^{n+2s}}\,dy,$$

which is strictly negative. Thus, the desired contradiction is established, and the proof of (2.2) is thereby completed.

We remark that (2.2) and the fact that $u(x,0) \geqslant c$ for any $x \in \mathbb{R}^n$ entail that

$$u(x,t) \geqslant c \quad \text{for any } x \in \mathbb{R}^n \text{ and for any } t \in I_i, \text{ for all } i \in \{0,\dots,k-1\},$$

which in turn gives that

$$u(x,t) \geqslant c \quad \text{for any } x \in \mathbb{R}^n \text{ and for any } t \in [0,T).$$

This concludes the proof of (i).

Now, in order to prove the statement in (ii), one can set $\widetilde{u}(x,t) := -u(x,t)$, $\widetilde{u}_0(x) := -u_0(x)$ and $\widetilde{c} := -c$ and notice that $\widetilde{u}$ is a solution of

$$\begin{cases} \partial_t \widetilde{u}(x,t) - \alpha\Delta\widetilde{u}(x,t) + \beta(-\Delta)^s\widetilde{u}(x,t) = (\widetilde{u}(x,t) - \tilde{c})f(x,t) & \text{in } \Omega\times(0,T),\\ \widetilde{u}(x,0) = \widetilde{u}_0(x) & \text{in } \Omega,\\ \text{with } (\alpha,\beta) - \text{Neumann conditions} & \text{in } (0,T). \end{cases}$$

Then, one can apply the statement in (i) to $\widetilde{u}$ and $\widetilde{c}$ to obtain the desired claim in (ii). □

3. Proof of Theorem 1.2

In this section we will provide the proof of Theorem 1.2. To reach this goal, we will also establish some intermediate results of independent interest.

We recall that the operator $L[u]$ is defined as

$$L[u] := \alpha\Delta u - \beta(-\Delta)^s u - \partial_t u.$$

We first provide the following auxiliary statement, which falls into the category of weak Maximum Principles.

Theorem 3.1. *Let $D \subset \Omega \times (0,T)$. Suppose that*

$$L[u] > 0 \text{ in } D. \tag{3.1}$$

If $\beta \neq 0$, assume also that

$$\sup_{(x,t)\in\mathbb{R}^n\times(0,T)} u(x,t) = \sup_{(x,t)\in D} u(x,t). \tag{3.2}$$

Then, u cannot achieve its maximum in the interior of D.

Proof. We assume by contradiction that u achieves its maximum at a point $(\bar{x},\bar{t})$ in the interior of $D \subset \Omega\times(0,T)$. As a consequence,

$$\partial_t u(\bar{x},\bar{t}) = 0, \qquad \nabla u(\bar{x},\bar{t}) = 0 \qquad \text{and} \qquad \Delta u(\bar{x},\bar{t}) \leqslant 0.$$

Moreover, if $\beta \neq 0$, in light of the additional assumption in (3.2), we have that $u(\bar{x},\bar{t}) \geqslant u(x,t)$ for every $x \in \mathbb{R}^n$ and $t \in (0,T)$, and so

$$(-\Delta)^s u(\bar{x},\bar{t}) = \int_{\mathbb{R}^n} \frac{u(\bar{x},\bar{t}) - u(y,\bar{t})}{|x-y|^{n+2s}}\,dy \geqslant 0.$$

Gathering these pieces of information, we have that $L[u] \leqslant 0$ in D, which is in contradiction with (3.1). The proof of Theorem 3.1 is thereby complete. □

We now deal with the strong Maximum Principle in Theorem 1.2. For this, we first make the following observation:

Lemma 3.2. *Let $s \in (0,1]$, $r>0$ and $\gamma>1$. We define $\mathbb{R}^n \ni x \mapsto v_\gamma(x) := e^{-\gamma|x|^2}$.*

Then, there exists $\gamma_0>1$, only depending on n, s, and r, such that, for all $\gamma \geqslant \gamma_0$ and $x \in \mathbb{R}^n \setminus B_r$,

$$(-\Delta)^s v_\gamma(x) < 0.$$

Proof. When $s=1$, the claim is a direct calculation, so we focus on the case $s \in (0,1)$ and we observe that

$$\begin{aligned}\Phi(x) \;&:=\; e^{\gamma|x|^2}\int_{\mathbb{R}^n} \frac{e^{-\gamma|x+y|^2}+e^{-\gamma|x-y|^2}-2e^{-\gamma|x|^2}}{|y|^{n+2s}}\,dy\\ &= \int_{\mathbb{R}^n} \frac{e^{-\gamma(2x\cdot y+|y|^2)}+e^{\gamma(2x\cdot y-|y|^2)}-2}{|y|^{n+2s}}\,dy.\end{aligned}$$

Now we set $\varepsilon := \frac{1}{\sqrt{\gamma}} \in (0,1)$ and we see that

$$\int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{e^{-\gamma(2x\cdot y+|y|^2)}+e^{\gamma(2x\cdot y-|y|^2)}-2}{|y|^{n+2s}}\,dy \geqslant -2\int_{\mathbb{R}^n\setminus B_\varepsilon}\frac{dy}{|y|^{n+2s}} = -\frac{C}{\varepsilon^{2s}},$$

for some positive constant C depending only on n and s, and accordingly

$$\Phi(x) \geqslant \int_{B_\varepsilon} \frac{e^{-\gamma(2x\cdot y+|y|^2)}+e^{\gamma(2x\cdot y-|y|^2)}-2}{|y|^{n+2s}}\,dy - \frac{C}{\varepsilon^{2s}}. \tag{3.3}$$

We also define $\hat{x} := \frac{x}{|x|}$ and we remark that, when $y \in B_\varepsilon$, $x \in \mathbb{R}^n \setminus B_r$ and $\hat{x}\cdot y \geqslant \varepsilon/2$, we have that

$$\begin{aligned}&e^{-\gamma(2x\cdot y+|y|^2)}+e^{\gamma(2x\cdot y-|y|^2)}-2 \geqslant e^{\gamma(2x\cdot y-|y|^2)}-2 \geqslant e^{\gamma(\varepsilon|x|-|y|^2)}-2\\ &\qquad\geqslant e^{\gamma(\varepsilon r-\varepsilon^2)}-2 \geqslant e^{\gamma\varepsilon r/2}-2 = e^{\frac{\sqrt{\gamma}r}{2}}-2 \geqslant \frac{e^{\frac{\sqrt{\gamma}r}{2}}}{2},\end{aligned}$$

as long as ε is small enough (i.e., as long as γ is large enough).

From this, we infer that

$$\begin{aligned}&\int_{B_\varepsilon\cap\{\hat{x}\cdot y\geqslant\varepsilon/2\}} \frac{e^{-\gamma(2x\cdot y+|y|^2)}+e^{\gamma(2x\cdot y-|y|^2)}-2}{|y|^{n+2s}}\,dy \geqslant \frac{e^{\frac{\sqrt{\gamma}r}{2}}}{2}\int_{B_\varepsilon\cap\{\hat{x}\cdot y\geqslant\varepsilon/2\}}\frac{dy}{|y|^{n+2s}}\\ &\qquad= \frac{e^{\frac{\sqrt{\gamma}r}{2}}}{2\varepsilon^{2s}}\int_{B_1\cap\{\hat{x}\cdot Y\geqslant 1/2\}}\frac{dY}{|Y|^{n+2s}} = \frac{c_0\, e^{\frac{\sqrt{\gamma}r}{2}}}{\varepsilon^{2s}},\end{aligned}$$

for some $c_0>0$.

Also,

$$\begin{aligned}&\int_{B_\varepsilon\cap\{\hat{x}\cdot y\leqslant-\varepsilon/2\}} \frac{e^{-\gamma(2x\cdot y+|y|^2)}+e^{\gamma(2x\cdot y-|y|^2)}-2}{|y|^{n+2s}}\,dy\\ &\qquad= \int_{B_\varepsilon\cap\{\hat{x}\cdot z\geqslant\varepsilon/2\}} \frac{e^{-\gamma(-2x\cdot z+|z|^2)}+e^{\gamma(-2x\cdot z-|z|^2)}-2}{|z|^{n+2s}}\,dz\\ &\qquad= \int_{B_\varepsilon\cap\{\hat{x}\cdot z\geqslant\varepsilon/2\}} \frac{e^{-\gamma(2x\cdot z+|z|^2)}+e^{\gamma(2x\cdot z-|z|^2)}-2}{|z|^{n+2s}}\,dz\\ &\qquad\geqslant \frac{c_0\, e^{\frac{\sqrt{\gamma}r}{2}}}{\varepsilon^{2s}}.\end{aligned}$$

These observations lead to

$$\int_{B_\varepsilon\cap\{|\hat{x}\cdot y|\geqslant\varepsilon/2\}} \frac{e^{-\gamma(2x\cdot y+|y|^2)}+e^{\gamma(2x\cdot y-|y|^2)}-2}{|y|^{n+2s}}\,dy \geqslant \frac{2c_0\, e^{\frac{\sqrt{\gamma}r}{2}}}{\varepsilon^{2s}}.$$

This and (3.3) give that

$$\Phi(x) \geqslant \int_{B_\varepsilon\cap\{|\hat{x}\cdot y|<\varepsilon/2\}} \frac{e^{-\gamma(2x\cdot y+|y|^2)}+e^{\gamma(2x\cdot y-|y|^2)}-2}{|y|^{n+2s}}\,dy+\frac{2c_0\,e^{\frac{\sqrt{\gamma}r}{2}}}{\varepsilon^{2s}}-\frac{C}{\varepsilon^{2s}}. \tag{3.4}$$

We also observe that, for all $y \in B_\varepsilon$,

$$\begin{aligned} & e^{-\gamma(2x\cdot y+|y|^2)}+e^{\gamma(2x\cdot y-|y|^2)}-2 \\ =\; & \Big(e^{-\gamma(2x\cdot y+|y|^2)}+e^{\gamma(2x\cdot y-|y|^2)}-2e^{-\gamma|y|^2}\Big)+2\Big(e^{-\gamma|y|^2}-1\Big) \\ =\; & 2e^{-\gamma|y|^2}\Big(\cosh(2\gamma x\cdot y)-1\Big)+2\Big(e^{-\gamma|y|^2}-1\Big) \\ \geqslant\; & 0-2\gamma|y|^2. \end{aligned}$$

We insert this information into (3.4) and we find that

$$\begin{aligned} \Phi(x) &\geqslant -2\gamma\int_{B_\varepsilon}\frac{dy}{|y|^{n+2s-2}}+\frac{2c_0\,e^{\frac{\sqrt{\gamma}r}{2}}}{\varepsilon^{2s}}-\frac{C}{\varepsilon^{2s}} \\ &= \frac{2c_0\,e^{\frac{\sqrt{\gamma}r}{2}}}{\varepsilon^{2s}}-\frac{C_\star(\gamma+1)}{\varepsilon^{2s}}, \end{aligned}$$

for some constant $C_\star > 0$.

Hence, if γ is large enough, we deduce that

$$\Phi(x) \geqslant \frac{c_0\,e^{\frac{\sqrt{\gamma}r}{2}}}{\varepsilon^{2s}} > 0,$$

which gives the desired result. □

We use the barrier constructed in Lemma 3.2 to establish the following intermediate result:

Lemma 3.3. *Let u satisfy*

$$L[u] \geqslant 0 \text{ in } \Omega\times(0,T),$$

for some $T > 0$.

Let B be an open ball centered at $(\overline{x},\overline{t})\in\Omega\times(0,T)$ with radius R and such that $\overline{B}\subset\Omega\times(0,T)$.

Suppose that $u \leqslant M$ in $\mathbb{R}^n\times(0,T)$, that $u < M$ in B and that $u(x_M,t_M) = M$ at some point $(x_M,t_M)\in\partial B$.

Then, we have that $(x_M,t_M) = (\overline{x},\overline{t}\pm R)$.

Proof. We may assume without loss of generality that (x_M,t_M) is the only point on the boundary of B where u achieves its maximum in B, otherwise we can replace B with a smaller ball B' such that $B'\subset B$ and $\partial B'\cap\partial B = \{(x_M,t_M)\}$. Accordingly, we can suppose that

$$u(x,t) < M \quad \text{in } \overline{B}\setminus\{(x_M,t_M)\}. \tag{3.5}$$

We notice that it suffices to prove that

$$x_M = \overline{x}, \tag{3.6}$$

since this would imply that $t_M = \overline{t}\pm R$, thus giving the desired result.

Hence, from now on, we focus on the proof of (3.6). For this, we argue by contradiction and we suppose that $x_M \neq \overline{x}$. Thus, we can construct a ball $\widetilde{B}\subset\Omega\times(0,T)$ centered at (x_M,t_M) and with radius $\widetilde{R}$ such that

$$\widetilde{R} < |x_M-\overline{x}|. \tag{3.7}$$

We can write $\partial\widetilde{B} = C'\cup C''$, where $C' := \partial\widetilde{B}\cap\overline{B}$ and $C'' := \partial\widetilde{B}\setminus C'$. In light of (3.5), we have that $u < M$ in C', and therefore there exists $\eta > 0$ such that

$$u \leqslant M-\eta \quad \text{in } C'. \tag{3.8}$$

Moreover, since $u \leqslant M$ in $\Omega \times (0,T)$, we have that

$$u \leqslant M \quad \text{in } C''. \tag{3.9}$$

Now, given $\gamma > 0$ (to be chosen sufficiently large in what follows), we define the auxiliary function

$$v(x,t) := e^{-\gamma(|x-\overline{x}|^2+(t-\overline{t})^2)} - e^{-\gamma R^2}.$$

We note that $v > 0$ in B, $v = 0$ on ∂B and $v < 0$ in $(\mathbb{R}^n \times (0,T)) \setminus B$.

Moreover, we compute

$$L[v](x,t) = 2\gamma e^{-\gamma(|x-\overline{x}|^2+(t-\overline{t})^2)}\Big(2\gamma\alpha|x-\overline{x}|^2 - \alpha n + (t-\overline{t})\Big) - \beta(-\Delta)^s v(x,t).$$

If x belongs to the closure of $\widetilde{B}$, then, recalling also (3.7), we see that $|x-\overline{x}| \geqslant |x_M-\overline{x}| - \widetilde{R} > 0$. As a consequence, we can take γ sufficiently large, possibly in dependence of n, α, $\overline{x}$, x_M, R and $\widetilde{R}$, such that $L[v] \geqslant -\beta(-\Delta)^s v$ in the closure of $\widetilde{B}$.

Also, we notice that, if x belongs to the closure of $\widetilde{B}$,

$$|x-\overline{x}|^2 + (t-\overline{t})^2 \geqslant \big(|x_M-\overline{x}| - \widetilde{R}\big)^2 + (t-\overline{t})^2 \geqslant \big(|x_M-\overline{x}| - \widetilde{R}\big)^2.$$

Therefore, we are in the position of applying Lemma 3.2 (here with $r := |x_M-\overline{x}| - \widetilde{R}$) and obtain that $(-\Delta)^s v < 0$ in the closure of $\widetilde{B}$, possibly increasing γ, that now may depend also on s.

From these observations, we conclude that $L[v] > 0$ in the closure of $\widetilde{B}$, provided that γ is chosen sufficiently large (which is now fixed once and for all).

Now, for all $\varepsilon > 0$, we define the function $w(x,t) := u(x,t) + \varepsilon v(x,t)$ and we observe that

$$L[w] = L[u] + \varepsilon L[v] > 0 \quad \text{in } \widetilde{B}. \tag{3.10}$$

Moreover, in light of (3.8), we have that, in C',

$$w \leqslant M - \eta + \varepsilon\big(1 - e^{-\gamma R^2}\big),$$

and therefore we can take ε small enough, possibly in dependence of η, γ and R, such that $w < M$ in C'.

Furthermore, from (3.9) we deduce that, in $C'' \subset (\Omega \times (0,T)) \setminus B$,

$$w \leqslant M + \varepsilon v < M.$$

Gathering these pieces of information, we conclude that $w < M$ on $\partial\widetilde{B} = C' \cup C''$.

On the other hand, since $v = 0$ on ∂B, we see that

$$w(x_M,t_M) = u(x_M,t_M) + \varepsilon v(x_M,t_M) = u(x_M,t_M) = M.$$

This implies that the maximum of w in $\widetilde{B}$ is achieved at an interior point.

Now, in light of (3.10), our aim is to exploit Theorem 3.1 in $\widetilde{B}$, and thus reach a contradiction. To employ this strategy, we need to check that if $\beta \neq 0$ then

$$w(x,t) \leqslant M \text{ in } (\mathbb{R}^n \times (0,T)) \setminus \widetilde{B}. \tag{3.11}$$

To this end, we use (3.5) to see that $u < M$ in $\overline{B} \setminus \widetilde{B}$, and therefore there exists $\eta' > 0$ such that

$$u \leqslant M - \eta' \text{ in } \overline{B} \setminus \widetilde{B}.$$

As a result,

$$w \leqslant M - \eta' + \varepsilon\big(1 - e^{-\gamma R^2}\big) \text{ in } \overline{B} \setminus \widetilde{B}.$$

Thus, we can take ε small enough, possibly depending on η', γ and R, such that $w < M$ in $\overline{B} \setminus \widetilde{B}$.

Moreover, since $v(x,t) < 0$ in $\big(\mathbb{R}^n \times (0,T)\big) \setminus B$, we have that

$$w \leqslant M + \varepsilon v < M \text{ in } (\mathbb{R}^n \times (0,T)) \setminus B,$$

hence (3.11) holds true.

Accordingly, we reach the desired contradiction in light of Theorem 3.1. The claim in (3.6) is thereby established, as desired. □

We now show that if there is a point $(x_0, t_0) \in \Omega \times (0, T)$ such that $u(x_0, t_0) = M$, then u is identically equal to M along every segment in $\Omega \times \{t_0\}$ containing (x_0, t_0). The rigorous statement goes as follows:

Lemma 3.4. *Let u satisfy*

$$L[u] \geqslant 0 \text{ in } \Omega \times (0, T),$$

for some $T > 0$.

Suppose that $u \leqslant M$ in $\Omega \times (0, T)$ and that $u < M$ at some point $(x_0, t_0) \in \Omega \times (0, T)$.

If $\ell \subset \Omega \times \{t_0\}$ is any line segment which contains (x_0, t_0), then $u < M$ on ℓ.

Proof. We argue by contradiction and suppose that there exists $x_1 \in \Omega$ with $(x_1, t_0) \in \ell$ such that $u(x_1, t_0) = M$.

We define $x_\tau := \tau x_0 + (1 - \tau)x_1$ for all $\tau \in [0, 1]$. Up to changing the choice of x_1, we can assume that

$$u(x_\tau, t_0) < M \quad \text{for all } \tau \in (0, 1]. \tag{3.12}$$

Let d_0 be defined as

$$d_0 := \min \left\{ 1, \min_{\tau \in [0,1]} \left\{ \operatorname{dist}\Big((x_\tau, t_0), \partial\big(\Omega \times (0, T)\big) \Big) \right\} \right\}$$

$$\text{with} \qquad \operatorname{dist}\Big((x_\tau, t_0), \partial\big(\Omega \times (0, T)\big) \Big) := \min_{z \in \partial(\Omega \times (0,T))} |(x_\tau, t_0) - z|.$$

Moreover, for all $\tau \in (0, d_0)$, we define $d(x_\tau)$ to be the distance of (x_τ, t_0) from the nearest point in $\Omega \times (0, T)$ where $u = M$ (notice that this minimization procedure is made over a non-empty set, since $u(x_1, t_0) = M$). With this definition, since $u(x_1, t_0) = M$, we have that

$$d(x_\tau) \leqslant |x_\tau - x_1| \quad \text{for all } \tau \in (0, d_0). \tag{3.13}$$

The aim is now to exploit Lemma 3.3. For this, given $\tau \in (0, d_0)$, we take B to be the ball centered at (x_τ, t_0) with radius $d(x_\tau)$ and we observe that $u < M$ in B and there exists a point $(x_M, t_M) \in \partial B$ such that $u(x_M, t_M) = M$. Thus, we are in the position of applying Lemma 3.3, obtaining that either $(x_M, t_M) = (x_\tau, t_0 + d(x_\tau))$ or $(x_M, t_M) = (x_\tau, t_0 - d(x_\tau))$. This gives that either $u(x_\tau, t_0 + d(x_\tau)) = M$ or $u(x_\tau, t_0 - d(x_\tau)) = M$.

Now, if $\delta > 0$ is sufficiently small, we see that

$$\Big| (x_{\tau+\delta}, t_0) - (x_\tau, t_0 \pm d(x_\tau)) \Big| = \sqrt{d(x_\tau)^2 + \delta^2 |x_0 - x_1|^2},$$

and so

$$d(x_{\tau+\delta}) \leqslant \sqrt{d(x_\tau)^2 + \delta^2 |x_0 - x_1|^2} < d(x_\tau) + \frac{\delta^2 |x_0 - x_1|^2}{2d(x_\tau)}. \tag{3.14}$$

Moreover, since $(x_\tau, t_0 \pm d(x_\tau))$ are the nearest points to (x_τ, t_0) where $u = M$, we have that

$$B_{\sqrt{d(x_\tau)^2 - \delta^2 |x_0 - x_1|^2}}(x_{\tau+\delta}) \subset B_{x_\tau}(x_\tau),$$

and so

$$d(x_{\tau+\delta}) \geqslant \sqrt{d(x_\tau)^2 - \delta^2 |x_0 - x_1|^2}. \tag{3.15}$$

Now, suppose that $d(x_\tau) > 0$ and let $0 < \delta |x_0 - x_1| < d(x_\tau)$. We divide the interval $(\tau, \tau+\delta)$ into N subintervals of equal lenght and use first (3.14) (with τ replaced by $\tau + j\delta/N$ and δ replaced by δ/N) and then (3.15) (with δ replaced by δ/N) to find that, for every $j = 0, 1, \ldots, N - 1$,

$$d\left(x_{\tau + \frac{j+1}{N}\delta} \right) - d\left(x_{\tau + \frac{j}{N}\delta} \right) \leqslant \frac{\delta^2 |x_0 - x_1|^2}{2N^2 d\left(x_{\tau + \frac{j}{N}\delta} \right)} \leqslant \frac{\delta^2 |x_0 - x_1|^2}{2N^2 \sqrt{d(x_\tau)^2 - \delta^2 |x_0 - x_1|^2}}.$$

Consequently,

$$d(x_{\tau+\delta})-d(x_\tau)=\sum_{j=0}^{N-1}\left[d\left(x_{\tau+\frac{j+1}{N}\delta}\right)-d\left(x_{\tau+\frac{j}{N}\delta}\right)\right]\leqslant\frac{\delta^2|x_0-x_1|^2}{2N\sqrt{d(x_\tau)^2-\delta^2|x_0-x_1|^2}}.$$

Thus, taking the limit as $N\to+\infty$, we conclude that $d(x_{\tau+\delta})\leqslant d(x_\tau)$, for any $\delta>0$ sufficiently small. This says that $d(x_\tau)$ is nonincreasing in τ.

On the other hand, from (3.13) we have, for all $\tau\in(0,d_0)$,

$$d(x_\tau)\leqslant|x_\tau-x_1|=\tau|x_0-x_1|,$$

and therefore

$$\lim_{\tau\to0}d(x_\tau)=0.$$

This and the fact that $d(x_\tau)$ is nonincreasing imply that $d(x_\tau)=0$ for all $\tau\in(0,d_0)$, that is $u(x_\tau,t_0)=M$ for all $\tau\in(0,d_0)$.

This is in contradiction with (3.12), and so the proof of Lemma 3.4 is completed. □

We can also provide the following result:

Lemma 3.5. *Let $(x_1,t_1)\in\Omega\times(0,T)$ and $R>0$ such that $\overline{B_R(x_1,t_1)}\subset\Omega\times(0,T)$. Let also*

$$B_{t_1}:=\big\{(x,t)\in B_R(x_1,t_1)\,:\,t\leqslant t_1\big\}.$$

Let u satisfy

$$L[u]\geqslant0\ \textit{in}\ B_{t_1}.$$

Suppose that

$$u(x,t)<M\ \textit{for every}\ (x,t)\in B_{t_1}\ \textit{such that}\ t<t_1. \tag{3.16}$$

If $\beta\neq0$, suppose in addition that

$$u(x,t)\leqslant M\ \textit{for every}\ x\in\mathbb{R}^n\ \textit{and}\ t\in(0,t_1]. \tag{3.17}$$

Then, we have that $u(x_1,t_1)<M$.

Proof. Suppose by contradiction that $u(x_1,t_1)=M$.

For $\gamma>0$, we define the function

$$v(x,t):=e^{-(|x-x_1|^2+\gamma(t-t_1))}-1,$$

and compute

$$L[v](x,t)=e^{-\gamma(t-t_1)}\left[e^{-|x-x_1|^2}\big(4\alpha|x-x_1|^2-2\alpha n+\gamma\big)-(-\Delta)^se^{-|x-x_1|^2}\right].$$

Notice that $|(-\Delta)^se^{-|x-x_1|^2}|\leqslant C$, for some $C>0$ depending only on n and s, and therefore

$$L[v](x,t)\geqslant e^{-\gamma(t-t_1)}\left[e^{-|x-x_1|^2}\big(4\alpha|x-x_1|^2-2\alpha n+\gamma\big)-C\right].$$

Thus, we can choose γ sufficiently large (that is now fixed once and for all), possibly in dependence of n, s, α and R, such that $L[v](x,t)>0$ in $B_R(x_1,t_1)$.

Now, the paraboloid

$$|x-x_1|^2+\gamma(t-t_1)=0 \tag{3.18}$$

is tangent to the hyperplane $t=t_1$ at (x_1,t_1) and lies below the hyperplane $t=t_1$. We denote by D the intersection between B_{t_1} and the paraboloid in (3.18), that is

$$D:=\big\{(x,t)\in B_{t_1}\,:\,|x-x_1|^2+\gamma(t-t_1)\leqslant0\big\},$$

and write $\partial D=C'\cup C''$, where

$$C':=\big\{(x,t)\in\partial B_{t_1}\,:\,|x-x_1|^2+\gamma(t-t_1)\leqslant0\big\}$$

and $C'':=\partial D\setminus C'$.

By the assumption in (3.16), we have that $u < M$ in C', and so there exists $\eta > 0$ such that

$$u \leqslant M - \eta \quad \text{in } C'.$$

We now define the function $w(x,t) := u(x,t) + \varepsilon v(x,t)$, for all $\varepsilon > 0$. Noticing also that $v = 0$ in C'', taking ε small enough, we have the following:

(*i*) $L[w] = L[u] + \varepsilon L[v] > 0$ in D,
(*ii*) $w \leqslant M - \eta + \varepsilon(e^{\gamma R} - 1) < M$ in C',
(*iii*) $w = u < M$ in $C'' \setminus \{(x_1, t_1)\}$.

Now, in the case $\beta \neq 0$, we claim that

$$w \leqslant M \text{ in } \big(\mathbb{R}^n \times (0, t_1)\big) \setminus D. \tag{3.19}$$

To check this, we notice that if $(x,t) \in ((\mathbb{R}^n \times (0,t_1)) \setminus D) \cap \{|x - x_1|^2 + \gamma(t - t_1) \leqslant 0\}$ then $v(x,t) \leqslant e^{\gamma t_1} - 1$.

Moreover, from (3.16), there exists $\eta' > 0$ such that $u \leqslant M - \eta'$, and so

$$w \leqslant M - \eta' + \varepsilon(e^{\gamma t_1} - 1),$$

thus, up to taking ε smaller, we have

$$w \leqslant M \text{ in } ((\mathbb{R}^n \times (0,t_1)) \setminus D) \cap \{|x - x_1|^2 + \gamma(t - t_1) \leqslant 0\}. \tag{3.20}$$

When $(x,t) \in ((\mathbb{R}^n \times (0,t_1)) \setminus D) \cap \{|x - x_1|^2 + \gamma(t - t_1) > 0\}$, we have that $v < 0$, and so from (3.17)

$$w \leqslant M + \varepsilon v < M \text{ in } ((\mathbb{R}^n \times (0,t_1)) \setminus D) \cap \{|x - x_1|^2 + \gamma(t - t_1) > 0\}.$$

From this and (3.20), we obtain (3.19).

In light of (3.19) and condition (*i*), we are in the position of exploiting Theorem 3.1, deducing that w attains its maximum on the boundary of D.

Therefore, in view of (*ii*) and (*iii*), we conclude that the maximum of w is achieved at the point (x_1, t_1).

Consequently, we have that $\partial_t w(x_1, t_1) \geqslant 0$. Thus, since $\partial_t v(x_1, t_1) = -\gamma < 0$, we get

$$\partial_t u(x_1, t_1) = \partial_t w(x_1, t_1) - \varepsilon \partial_t v(x_1, t_1) \geqslant \varepsilon\gamma > 0.$$

On the other hand, since (x_1, t_1) is a local maximum point for u on $t = t_1$, we see that $\Delta u(x_1, t_1) \leqslant 0$. Also, in light of (3.17),

$$\begin{aligned}
(-\Delta)^s u(x_1, t_1) &= \frac{1}{2}\int_{\mathbb{R}^n} \frac{2u(x_1,t_1) - u(x_1 + y, t_1) - u(x_1 - y, t_1)}{|y|^{n+2s}}\, dy \\
&= \frac{1}{2}\int_{\mathbb{R}^n} \frac{2M - u(x_1 + y, t_1) - u(x_1 - y, t_1)}{|y|^{n+2s}}\, dy \\
&\geqslant 0.
\end{aligned}$$

Gathering these pieces of information, we obtain that $L[u](x_1, t_1) \leqslant 0$, which provides the desired contradiction. □

With this preliminary work, we are in the position of completing the proof of Theorem 1.2.

Proof of Theorem 1.2. Let Ω' be the connected component of Ω that contains x_1. In this way, we have that $(x_1, t_1) \in \Omega' \times (0, t_1]$. Now, we first focus on the proof of the claim in Theorem 1.2 in Ω', that is, we establish that

$$u(x,t) = M \text{ for all } (x,t) \in \Omega' \times (0, t_1]. \tag{3.21}$$

Notice that if we prove that

$$u(x,t) = M \text{ for all } (x,t) \in \Omega' \times (0, t_1), \tag{3.22}$$

then, by continuity, we would have that (3.21) holds true. Therefore, we focus on the proof of (3.22).

To this aim, we suppose, by contradiction, that there exist $(x_0,t_0) \in \Omega' \times (0,t_1)$ such that $u(x_0,t_0) < M$. Since Ω' is connected, there exists a path made of line segments connecting (x_0,t_0) and (x_1,t_0), and thus from Lemma 3.4 we have that

$$u(x_1,t_0) < M. \tag{3.23}$$

Now, let $\tau \in (t_0,t_1]$ be such that $u(x_1,t) < M$ for all $t \in [t_0,\tau)$ and

$$u(x_1,\tau) = M. \tag{3.24}$$

Notice that such a τ exists in light of the assumption $u(x_1,t_1) = M$.

Now, in lieu of (3.23), from Lemma 3.4, used here with $T := t_1$ and ℓ a segment in $\Omega' \times \{t_0\}$ containing (x_1,t_0), we infer that $u < M$ on ℓ. By continuity, there exists $R > 0$ such that $u(x,t) < M$ for all $|x - x_1| < R$ and $t \in [t_0,\tau)$.

Finally, Lemma 3.5 implies that $u(x_1,\tau) < M$, which is in contradiction with (3.24). This establishes the desired claim in (3.22).

Now, if $\beta = 0$, we have that Ω is connected, and therefore the claim in Theorem 1.2 follows from (3.21).

If instead $\beta \neq 0$, in order to complete the proof of Theorem 1.2, it remains to show that $u(x,t) = M$ for all $(x,t) \in (\Omega \setminus \Omega') \times (0,t_1]$. By continuity, this boils down to checking that $u(x,t) = M$ for all $(x,t) \in (\Omega \setminus \Omega') \times (0,t_1)$.

To this end, let Ω'' be a connected component of Ω such that $\Omega' \cap \Omega'' = \varnothing$ and assume by contradiction that there exist $(\bar{x},\bar{t}) \in \Omega'' \times (0,t_1)$ such that

$$u(\bar{x},\bar{t}) < M. \tag{3.25}$$

On the other hand, from (3.22) we know that u is constant in $\Omega' \times (0,t_1)$, hence if $(\widetilde{x},\bar{t}) \in \Omega' \times (0,t_1)$ we have that $\partial_t u(\widetilde{x},\bar{t}) = 0$ and $\Delta u(\widetilde{x},\bar{t}) = 0$. Accordingly,

$$0 \leqslant L[u](\widetilde{x},\bar{t}) = \alpha\Delta u(\widetilde{x},\bar{t}) - \beta(-\Delta)^s u(\widetilde{x},\bar{t}) - \partial_t u(\widetilde{x},\bar{t}) = -\beta(-\Delta)^s u(\widetilde{x},\bar{t}).$$

As a consequence, we have that $(-\Delta)^s u(\widetilde{x},\bar{t}) \leqslant 0$, namely

$$0 \geqslant \int_{\mathbb{R}^n} \frac{u(\tilde{x},\bar{t}) - u(y,\bar{t})}{|\tilde{x} - y|^{n+2s}}\,dy = \int_{\mathbb{R}^n} \frac{M - u(y,\bar{t})}{|\tilde{x} - y|^{n+2s}}\,dy \geqslant 0.$$

This implies that $u(y,\bar{t}) = M$ for every $y \in \mathbb{R}^n$, in contradiction with (3.25).

This completes the proof of Theorem 1.2. □

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