ON SOME REGULARITY PROPERTIES OF MIXED LOCAL AND NONLOCAL ELLIPTIC EQUATIONS

XIFENG SU, ENRICO VALDINOCI, YUANHONG WEI, AND JIWEN ZHANG

Abstract. This article is concerned with "up to $C^{2,\alpha}$-regularity results" about a mixed local-nonlocal nonlinear elliptic equation which is driven by the superposition of Laplacian and fractional Laplacian operators.

First of all, an estimate on the L^∞ norm of weak solutions is established for more general cases than the ones present in the literature, including here critical nonlinearities.

We then prove the interior $C^{1,\alpha}$-regularity and the $C^{1,\alpha}$-regularity up to the boundary of weak solutions, which extends previous results by the authors [X. Su, E. Valdinoci, Y. Wei and J. Zhang, Math. Z. (2022)], where the nonlinearities considered were of subcritical type.

In addition, we establish the interior $C^{2,\alpha}$-regularity of solutions for all $s \in (0,1)$ and the $C^{2,\alpha}$-regularity up to the boundary for all $s \in (0, \frac{1}{2})$, with sharp regularity exponents.

For further perusal, we also include a strong maximum principle and some properties about the principal eigenvalue.

Contents

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1. Introduction

The present paper is concerned with some regularity results about the following mixed local and nonlocal elliptic equation

$$\begin{cases} -\Delta u + (-\Delta)^s u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where $s \in (0,1)$ and Ω is a bounded open set in $\mathbb{R}^n$.

As customary, the fractional Laplacian $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = c_{n,s}\, \text{P.}V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy,$$

in which $c_{n,s} > 0$ is a suitable normalization constant, whose explicit value does not play a role here, and P.$V.$ means that the integral is taken in the Cauchy principal value sense.

The superposition operator

$$\mathcal{L} = -\Delta + (-\Delta)^s,$$

is of nonlocal type, due to the presence of the fractional Laplacian, hence equation (1) is a nonlocal elliptic equation in a bounded domain Ω with homogeneous Dirichlet exterior data.

We also recall that the fractional Laplacian is the generator of a Lévy flight, while the Laplacian is the generator of a Brownian motion. Based on this, equation (1) can naturally model the combined effect of Brownian motions and Lévy flights. As observed in [DV21], these operators describe a biological species whose individuals diffuse either by a random walk or by a jump process. The analysis of different types of mixed operators motivated by biological questions has also been carried out in [MPV13, PV18].

Elliptic equations driven by the above mixed operator have recently received a great attention from different points of view, including viscosity solution theory [JK05, BI08, BJK10, BCCI12], existence and non-existence theory [AC21, BDVV22a, SVWZ22a], eigenvalue problems [DPFR19, DPLV22], shape optimization results [BDVV23], symmetry properties of the solutions [BVDV21], etc.

In contrast with the cases in which the pure classical Laplacian and the pure fractional Laplacian are separately taken into account, this kind of problems combines the classical setting and the features typical of nonlocal operators in a framework that is not scale-invariant. Hence, different scales will lead to different features of the mixed operator from the case of single classical Laplacian or single fractional Laplacian, thereby making either one of them prevail, or both somewhat coexist in the end.

The purpose of this article concentrates on establishing up to $C^{2,\alpha}$-regularity theory for mixed local-nonlocal elliptic equations, both in the interior and up to the boundary.

Our results are divided according to the following aspects:.

- First of all, we extend the regularity result of [SVWZ22b] to some critical problems. More precisely, we prove the boundedness in L^∞ of any weak solution of nonlinear problem (1) with the critical Sobolev embedding exponent
$$2^* = \begin{cases} \frac{2n}{n-2} & \text{if } n > 2, \\ \infty & \text{if } n \leqslant 2. \end{cases}$$
- We then deduce a $C^{1,\alpha}$-regularity result up to the boundary. This will actually be obtained from the $W^{2,p}$-regularity of weak solutions of problem (1). Moreover, we discuss the interior $C^{1,\alpha}$-regularity for all $\alpha \in (0,1)$.
- Next we show the interior $C^{2,\alpha}$-regularity for all $s \in (0,1)$. We develop a bespoke argument to get over a difficulty caused by the nonlinear terms. Our argument combines a truncation technique with the above $C^{1,\alpha}$ estimates. We notice when g depends only on x, the interior $C^{2,\alpha}$-regularity of problem (1) is studied in [BDVV22b]. Our result is thereby a nonlinear version of [BDVV22b]. However, even in the case of linear equations, our method is different from [BDVV22b] since the working space that we choose here is different.
- Finally, we show the $C^{2,\alpha}$-regularity up to the boundary for (1) when $s \in (0, \frac{1}{2})$ and discuss the case when $s \in (\frac{1}{2}, 1)$. A restriction on the fractional exponent for this type of regularity is somewhat unavoidable, as shown by one-dimensional examples in which one picks a solution of (1) in $\Omega := (0,1)$ which behaves, near 0^+, as $ax + O(x^2)$, for some constant a.

  Also, the nonlinear term brings some difficulties for the bootstrap argument, in view of technical regularity issues related to the composition of functions with low and high smoothness.
- As a byproduct of our main results, we also provide in the appendix some auxiliary results in a formulation which is "ready for use" in forthcoming papers, such as a strong maximum principle and a characterization of the principal eigenvalue.

Now we start presenting our main results. First off, in the strategy that we proposed above, the L^∞-regularity theory goes as follows.

Theorem 1.1 (L^∞-regularity)**.** *Let u be a weak solution*[1] *of* (1). *Assume that there exist $c > 0$ and $q \in [2, 2^*]$ such that*

$$|g(x,t)| \leqslant c(1 + |t|^{q-1}) \quad \text{for a.e. } x \in \Omega, t \in \mathbb{R}. \tag{1.2}$$

Then, $u \in L^\infty(\Omega)$.

[1] The precise definition of weak solution will be recalled in Definition 2.1.

More precisely, there exists a constant $C_0 = C_0(c, n, q, \Omega) > 0$ *independent of u, such that*

$$\|u\|_{L^\infty(\Omega)} \leqslant C_0 \left(1 + \int_\Omega |u|^{2^*\beta_1}\, dx\right)^{\frac{1}{2^*(\beta_1 - 1)}},$$

where $\beta_1 := \frac{2^*+1}{2}$.

Remark 1.2. In [SVWZ22b], we established the L^∞ boundedness of weak solution, provided that the exponent of the nonlinear term is controlled by

$$2_s^* = \begin{cases} \frac{2n}{n-2s} & \text{if } n > 2s, \\ \infty & \text{if } n \leqslant 2s. \end{cases}$$

Here 2_s^* is the critical Sobolev embedding exponent for the space $H^s(\mathbb{R}^n)$ and characterizes the critical case for the operator $(-\Delta)^s$.

Obviously, $2^* \geqslant 2_s^*$ (and the inequality is strict when $n > 2s$). This suggests that, in several cases of interest, the critical exponent is actually 2^* rather than 2_s^*, due to the interpolation inequality, and correspondingly Theorem 1.1 can be seen as a sharpening of [SVWZ22b, Theorem 1.1].

Based on the L^∞-regularity theory, one can employ the $W^{2,p}$ theory of weak solutions to deduce a global $C^{1,\alpha}$-regularity as follows.

Theorem 1.3 ($C^{1,\alpha}$-regularity up to the boundary)**.** *Let u be a weak solution of* (1)*. Assume that g satisfies* (1.1) *and* $\partial\Omega$ *is of class* $C^{1,1}$.

Then, $u \in C^{1,\alpha}(\overline{\Omega})$ *for any* $\alpha \in (0, \min\{1, 2 - 2s\})$.

More precisely, for s and α *as above, it holds that*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant C\left(1 + \|u\|^{2^*}_{L^\infty(\Omega)}\right),$$

where the positive constant C depends only on n, c, α, Ω, s.

We remark that Theorems 1.1 and 1.3 can be seen as a critical exponent version of the L^∞-regularity and of the $C^{1,\alpha}$-regularity in [SVWZ22b], respectively.

In light of Theorem 1.3, one obtains an interior $C^{2,\alpha}$-regularity result for the problem (1) provided that $\partial\Omega$ is $C^{1,1}$: we present this result in Theorem 1.4 below for the sake of completeness, however this regularity assumption on the domain can be removed, as showcased in the forthcoming Theorem 1.6.

Theorem 1.4 (Interior $C^{2,\alpha}$-regularity)**.** *Let u be a weak solution of* (1)*. Assume that* $\partial\Omega$ *is of class* $C^{1,1}$ *and that* $g \in C^\alpha_{\mathrm{loc}}(\Omega \times \mathbb{R})$ *satisfies* (1.1) *for any given* $\alpha \in (0, 1)$*. Then* $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$.

More specifically, if W is an open domain with $\overline{W} \subset \Omega$*, and denote*

$$\rho := \mathit{dist}(W, \partial\Omega), \quad \text{and} \quad W_{\frac{\rho}{2}} := \left\{x \in \Omega : \mathit{dist}(x, W) < \frac{\rho}{2}\right\}.$$

Then,

$$\|u\|_{C^{2,\alpha}(\overline{W})} \leqslant C\left(\|g\|_{L^\infty(\overline{W}_{\rho/2} \times \mathbb{R})} + \|g\|_{C^\alpha(\overline{W}_{\rho/2} \times \mathbb{R})} \|u\|_{C^1(\overline{\Omega})} + \|u\|_{C^{1,1-s}(\overline{\Omega})}\right)$$

for some constant $C > 0$ *depending only on* n, s, Ω, α, W.

The above results utilize the external condition in (1) and the regularity of the domain. We now present an alternative interior regularity theory for weak solutions of the equation without specifying any external condition and without any regularity assumption on the boundary of the domain. Namely, by combining a suitable truncation method and a covering argument, we obtain the interior $C^{1,\alpha}$-regularity and the interior $C^{2,\alpha}$-regularity, as formulated in the following two results:

Theorem 1.5 (Interior $C^{1,\alpha}$-regularity). *Let $g(x,t) \in L^\infty_{\rm loc}(\Omega \times \mathbb{R})$, $u \in X^1$ be a bounded weak solution of*

$$-\Delta u + (-\Delta)^s u = g(x,u) \qquad \text{in } \Omega, \tag{1.3}$$

where X^1 will be defined in (2).

Assume that V is an open domain with $\overline{V} \subset \Omega$.

Then, $u \in C^{1,\alpha}_{\rm loc}(\Omega)$ for any $\alpha \in (0,1)$.

More specifically,

$$\|u\|_{C^{1,\alpha}(\overline{V})} \leqslant C\left(\|g\|_{L^\infty(V_{\frac{3\rho}{4}}\times\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)$$

where the constant $C > 0$ depends on n, s, α, V.

Theorem 1.6 (Interior $C^{2,\alpha}$-regularity). *Let $g(x,t) \in C^\alpha_{\rm loc}(\Omega\times\mathbb{R})$ for any $\alpha \in (0,1)$, $u \in X^1$ be a bounded weak solution of*

$$-\Delta u + (-\Delta)^s u = g(x,u) \qquad \text{in } \Omega. \tag{1.4}$$

Assume that V is an open domain with $\overline{V} \subset \Omega$.

Then, $u \in C^{2,\alpha}(\overline{V})$.

Furthermore,

$$\|u\|_{C^{2,\alpha}(\overline{V})} \leqslant C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{C^\alpha(\overline{V}_{\frac{7\rho}{8}}\times\mathbb{R}^n)}\right)\left(1 + \|g\|_{C^\alpha(\overline{V}_{\frac{7\rho}{8}}\times\mathbb{R}^n)}\right),$$

where the constant $C > 0$ depends on n, s, α, V.

Now we also consider the global $C^{2,\alpha}$-regularity up to the boundary:

Theorem 1.7 ($C^{2,\alpha}$-regularity up to the boundary). *Suppose that $\partial\Omega$ is of class $C^{2,\alpha}$. Let $s \in (0, \frac{1}{2})$ and $\alpha \in (0,1)$ be such that*

$$\alpha + 2s \leqslant 1.$$

If u is a weak solution of (1) *and $g \in C^\alpha(\overline{\Omega}, \mathbb{R})$ satisfies* (1.1)*, then $u \in C^{2,\alpha}(\overline{\Omega})$.*

Moreover, we have that

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leqslant C\left(\|g\|_{C^\alpha(\overline{\Omega}\times\mathbb{R})}\|u\|_{C^1(\overline{\Omega})} + \|g\|_{L^\infty(\Omega\times\mathbb{R})} + \|u\|_{L^\infty(\Omega)}\right)$$

for some positive constant C depending only on n, s, α, Ω.

Remark 1.8. In [BDVV22b, Theorem 1.6], a related $C^{2,\alpha}$-regularity result is proved for linear equation. This result relies on the smoothness of the linear term and is not directly applicable to the nonlinear equation (1).

In our context, for the interior $C^{2,\alpha}$-regularity, we will deal with several technical issues coming from the nonlinearity $g(x, u(x))$ to guarantee that one can carry out the iterative process. For the $C^{2,\alpha}$-regularity up to the boundary, we also treat the mixed operator with $s \in (0, \frac{1}{2})$ as a perturbation of the classical Laplacian and apply the classical results of local problems to obtain the main result.

The rest of this paper is organized as follows. In Section 2, we first introduce several preliminaries and define the working space. Then, the L^∞-boundedness of weak solutions is established in Section 3, while Section 4 is devoted to the $C^{1,\alpha}$-regularity up to the boundary and the interior $C^{1,\alpha}$-regularity.

Next, the interior $C^{2,\alpha}$-regularity and the $C^{2,\alpha}$-regularity up to the boundary are shown in Section 5.

In the appendix we collect some simple consequences of the main results, such as a strong maximum principle and a variational characterization for the principal eigenvalue. These results are not stated in their maximal generality, but rather in a form which is ready to use in nonlinear analysis problems (see [SVWZ22a]).

2. Preliminaries and additional results

Let $s \in (0, 1)$ be given and $\Omega \subset \mathbb{R}^n$ be an open bounded set. We introduce our working space, which is slightly different from e.g. [BDVV22b, SVWZ22b].

Let us first define the semi-norm

$$[u]_s := \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dxdy \right)^{\frac{1}{2}}.$$

In terms of the mixed operator $\mathcal{L}$, a "natural" function space to consider is the following space

$$X^1 := \left\{ \begin{array}{l} u : \mathbb{R}^n \to \mathbb{R} \text{ is Lebesgue measurable:} \\ u|_\Omega \in H^1(\Omega), \ [u]_s \text{ is finite} \end{array} \right\} \tag{2.1}$$

equipped with the norm

$$\left(\|u\|^2_{H^1(\Omega)} + [u]_s^2 \right)^{\frac{1}{2}}.$$

Moreover, in order to study the solvability of Dirichlet problem (1), an appropriate space to employ is the following space[2] of functions

$$X^1_0 := \left\{ \begin{array}{l} u : \mathbb{R}^n \to \mathbb{R} \text{ is Lebesgue measurable:} \\ u|_\Omega \in H^1_0(\Omega), \ [u]_s \text{ is finite}, \ u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \end{array} \right\}$$

equipped with the norm

$$\left(\|u\|^2_{H^1_0(\Omega)} + [u]_s^2 \right)^{\frac{1}{2}}.$$

[2]We prefer to use the function space $H^1_0(\Omega)$ rather than $H^1(\Omega)$, or $H^1(\mathbb{R}^n)$ with zero data outside Ω, to avoid trace theorems which may impose additional regularity assumptions on Ω. Notice indeed that no regularity on Ω is assumed in Theorem 1.1. See however the forthcoming Lemma 2.1 addressing the equivalence of the norms involved in each of these choices.

Due to the Poincaré inequality, the above norm in X_0^1 is also equivalent to

$$\|u\|_{X_0^1} := \left(\|\nabla u\|_2^2 + [u]_s^2\right)^{\frac{1}{2}}.$$

Obviously, X_0^1 is a Hilbert space equipped with the inner product

$$\langle u, v\rangle_{X_0^1} = \int_\Omega \nabla u \cdot \nabla v \, dx + \iint_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dxdy, \quad \forall\, u, v \in X_0^1.$$

The following is the definition of weak solutions of (1) in X_0^1:

Definition 2.1 (Weak solution)**.** *We say that $u \in X_0^1$ is a weak solution of the mixed elliptic equation* (1) *, if u satisfies*

$$\int_\Omega \langle \nabla u, \nabla \varphi\rangle + \iint_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, dxdy = \int_\Omega g(x, u(x))\varphi(x) \, dx, \quad \textit{for any } \varphi \in X_0^1.$$

We investigate some key facts of X_0^1 in the following lemmata on equivalence of norms, embedding and density results.

Lemma 2.1. *In the space X_0^1, we have the following equivalence of norms:*

$$\|\cdot\|_{X_0^1} \sim \|\cdot\|_{H_0^1(\Omega)}.$$

Proof. Let $u \in H_0^1(\Omega)$ and $u = 0$ a.e. in $\mathbb{R}^n \backslash \Omega$. Using the change of variable $z = y - x$, one has

$$\begin{aligned} &\int_\Omega \int_{\mathbb{R}^n \backslash \Omega} \frac{|u(x)|^2}{|x - y|^{n+2s}} \, dydx \\ &= \int_\Omega \int_{\{x+z \notin \Omega\} \cap \{|z| \leqslant 1\}} \frac{|u(x)|^2}{|z|^{n+2s}} \, dzdx + \int_\Omega \int_{\{x+z \notin \Omega\} \cap \{|z| > 1\}} \frac{|u(x)|^2}{|z|^{n+2s}} \, dzdx \\ &=: A_1 + A_2. \end{aligned}$$

We observe that when $x + z$ lies in the complement of Ω we have

$$|u(x)|^2 = |u(x + z) - u(x)|^2 = \left| \int_0^1 \nabla u(x + tz) \cdot z \, dt \right|^2 \leq \int_0^1 |\nabla u(x + tz)|^2 |z|^2 \, dt$$

and therefore, making use of the change of variable $y := x + tz$ and the fact that u vanishes outside Ω, we find that

$$A_1 \leqslant C_1(n, s)\|u\|_{H_0^1(\Omega)}^2. \tag{2.2}$$

Moreover,

$$A_2 \leqslant C_2(n, s)\|u\|_{L^2(\Omega)}^2. \tag{2.3}$$

Also, it follows from [DNPV12] that

$$\iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dxdy \leqslant C_3(n, s)\|u\|_{H_0^1(\Omega)}^2. \tag{2.4}$$

Combining (2), (2) and (2), we deduce that

$$[u]_s^2 \leqslant C(n,s)\|u\|^2_{H_0^1(\Omega)},$$

for some suitable constant $C(n,s)$.

As a consequence of this, we obtain the equivalence between both norms $\|\cdot\|_{X_0^1}$ and $\|\cdot\|_{H_0^1}$. □

Lemma 2.2. *The embedding $X_0^1 \hookrightarrow L^{2^*}(\Omega)$ is continuous.*

Proof. Let $u \in X_0^1$, invoking the Sobolev inequality, and we have

$$\|u\|_{L^{2^*}(\Omega)} \leqslant C\|u\|_{H_0^1(\Omega)} \leqslant C\|u\|_{X_0^1}$$

for some suitable positive constant C. □

Lemma 2.3. *Let $\mathscr{C}_0^\infty := \left\{u \in C(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n\backslash\Omega,\ u|_\Omega \in C_0^\infty(\Omega)\right\}$ and denote by $\overline{\mathscr{C}_0^\infty}$ the closure of $\mathscr{C}_0^\infty$ with respect to X_0^1-norm. Then, we have that $\overline{\mathscr{C}_0^\infty} = X_0^1$.*

Proof. It is not difficult to see that $\overline{\mathscr{C}_0^\infty} \subset X_0^1$. Besides, according to Lemma 2.1 and the fact $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, one can deduce that $X_0^1 \subset \overline{\mathscr{C}_0^\infty}$. □

3. L^∞-regularity

In this section, we show that the weak solutions $u \in X_0^1$ are bounded in $L^\infty(\Omega)$. To prove this result, we will use the Moser iteration method.

To begin with, we construct an auxiliary function φ and provide several fundamental properties of φ, which are useful for the iterative procedure.

Lemma 3.1. *Given $\beta > 1$ and $T > 0$, we define*

$$\varphi(t) = \begin{cases} -\beta T^{\beta-1}(t+T) + T^\beta, & \text{if } t \leqslant -T \\ |t|^\beta, & \text{if } -T < t < T, \\ \beta T^{\beta-1}(t-T) + T^\beta, & \text{if } t \geqslant T. \end{cases}$$

Let $u \in X_0^1$, then $\varphi(u) \in X_0^1$.

One can refer to [SVWZ22b] for the details of the proof. Note that the difference in the present paper is that we deal with 2^* rather than 2_s^*.

Lemma 3.2. *Under the assumptions of Theorem 1.1, let β_1 is such that $2\beta_1 - 1 = 2^*$. Then, $u \in L^{2^*\beta_1}(\Omega)$.*

Proof. Thanks to the convexity of φ, one may deduce that

$$(-\Delta)^s\varphi(u) \leqslant \varphi'(u)(-\Delta)^s u \tag{3.1}$$

in the sense of distributions.

Moreover, we recall that

$$\|u\|^2_{L^{2_s^*}(\Omega)} \leqslant S \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dxdy,$$

where S is a constant depending only on n and s.

Hence, by (3), we have that

$$\begin{aligned}\|\varphi(u)\|^2_{L^{2^*_s}(\Omega)} &\leqslant S \iint_{\mathbb{R}^{2n}} \frac{|\varphi(u(x)) - \varphi(u(y))|^2}{|x-y|^{n+2s}}\,dxdy \\ &= 2S\int_{\mathbb{R}^n} \varphi(u)(-\Delta)^s\varphi(u)\,dx \leqslant 2S\int_{\mathbb{R}^n}\varphi(u)\,\varphi'(u)\,(-\Delta)^s u\,dx.\end{aligned} \tag{3.2}$$

We multiply the first equation of (1) by $\varphi(u)\varphi'(u)$ and integrate over $\mathbb{R}^n$. In this way, we obtain that

$$-\int_{\mathbb{R}^n}(\varphi(u)\varphi'(u))\Delta u\,dx + \int_{\mathbb{R}^n}\varphi(u)\,\varphi'(u)\,(-\Delta)^s u\,dx = \int_{\mathbb{R}^n}\varphi(u)\varphi'(u)g(x,u)\,dx. \tag{3.3}$$

The first term on the left side of (3) can be rewritten as

$$\begin{aligned}&-\int_\Omega(\varphi(u)\varphi'(u))\Delta u\,dx = \int_\Omega \nabla u\cdot\nabla(\varphi(u)\varphi'(u))\,dx \\ &= \int_\Omega|\nabla u|^2|\varphi'(u)|^2\,dx + \int_\Omega|\nabla u|^2\varphi(u)\,\varphi''(u)\,dx.\end{aligned} \tag{3.4}$$

Equations (3), (3) and (3) lead to

$$\int_\Omega|\nabla u|^2|\varphi'(u)|^2\,dx \leqslant \int_{\mathbb{R}^n}\varphi(u)\,\varphi'(u)\,g(x,u)\,dx.$$

Hence, recalling (1.1) and utilizing the Sobolev inequality, we obtain

$$\begin{aligned}\|\varphi(u)\|^2_{L^{2^*}(\Omega)} &\leqslant C\|\nabla(\varphi(u))\|^2_{L^2(\Omega)} = C\int_\Omega|\varphi'(u)\nabla u|^2\,dx \\ &\leqslant C\int_{\mathbb{R}^n}\varphi(u)\varphi'(u)g(x,u)\,dx \leqslant C\int_{\mathbb{R}^n}\varphi(u)|\varphi'(u)|(1+|u|^{2^*-1})\,dx \\ &= C\Big(\int_{\mathbb{R}^n}\varphi(u)|\varphi'(u)|\,dx + \int_{\mathbb{R}^n}\varphi(u)|\varphi'(u)||u|^{2^*-1}\,dx\Big),\end{aligned}$$

where the constant C depends only on n, Ω and c.

Employing the estimates

$$\varphi(u)\leqslant|u|^\beta, \qquad |\varphi'(u)|\leqslant\beta|u|^{\beta-1} \qquad \text{and} \qquad |u\varphi'(u)|\leqslant\beta\varphi(u),$$

we see that

$$\Big(\int_\Omega(\varphi(u))^{2^*}\,dx\Big)^{2/2^*} \leqslant C\beta\Big(\int_\Omega|u|^{2\beta-1}\,dx + \int_\Omega(\varphi(u))^2|u|^{2^*-2}\,dx\Big). \tag{3.5}$$

Notice that C is a positive constant that does not depend on β and that the last integral on the right-hand-side of the inequality (3) is well defined for every $T > 0$.

Furthermore, for any given $R > 0$, we apply the Hölder inequality and obtain that

$$\begin{aligned}&\int_\Omega(\varphi(u))^2|u|^{2^*-2}\,dx = \int_{\{|u|\leqslant R\}}(\varphi(u))^2|u|^{2^*-2}\,dx + \int_{\{|u|>R\}}(\varphi(u))^2|u|^{2^*-2}\,dx \\ &\leqslant \int_{\{|u|\leqslant R\}}\frac{(\varphi(u))^2}{|u|}R^{2^*-1}\,dx + \Big(\int_\Omega(\varphi(u))^{2^*}\,dx\Big)^{2/2^*}\Big(\int_{\{|u|>R\}}|u|^{2^*}\,dx\Big)^{\frac{2^*-2}{2^*}}.\end{aligned} \tag{3.6}$$

By the Monotone Convergence Theorem, one can choose $R > 0$ large enough such that

$$\left(\int_{\{|u|>R\}} |u|^{2^*}\, dx \right)^{\frac{2^*-2}{2^*}} \leqslant \frac{1}{2C\beta_1},$$

where C is the constant in (3).

Therefore, one can reabsorb the last term in (3) into the left hand side of (3) to get

$$\left(\int_\Omega (\varphi(u))^{2^*}\, dx \right)^{2/2^*} \leqslant 2C\beta_1 \Big(\int_\Omega |u|^{2^*}\, dx + R^{2^*-1} \int_\Omega \frac{(\varphi(u))^2}{|u|}\, dx \Big).$$

Now, using that $\varphi(u) \leqslant |u|^{\beta_1}$, we conclude that the terms in the right hand side of the above inequality is bounded independently of T.

Sending $T \to +\infty$, we obtain that

$$\left(\int_\Omega |u|^{2^*\beta_1}\, dx \right)^{2/2^*} \leqslant 2C\beta_1 \Big(\int_\Omega |u|^{2^*}\, dx + R^{2^*-1} \int_\Omega |u|^{2^*}\, dx \Big) < +\infty, \tag{3.7}$$

as desired. □

Now we complete the proof of the L^∞-regularity result:

Proof of Theorem 1.1. For $m \geqslant 1$, we define β_{m+1} such that

$$2\beta_{m+1} + 2^* - 2 = 2^*\beta_m.$$

As in [SVWZ22b], we infer that

$$\left(1 + \int_\Omega |u|^{2^*\beta_{m+1}}\, dx\right)^{\frac{1}{2^*(\beta_{m+1}-1)}} \leqslant (C\beta_{m+1})^{\frac{1}{2(\beta_{m+1}-1)}} \left(1 + \int_\Omega |u|^{2^*\beta_m}\, dx\right)^{\frac{1}{2^*(\beta_m-1)}}.$$

We now define $C_{m+1} := C\beta_{m+1}$ and

$$A_m := \left(1 + \int_\Omega |u|^{2^*\beta_m}\, dx\right)^{\frac{1}{2^*(\beta_m-1)}}.$$

In particular, note that $A_1 = \left(1 + \int_\Omega |u|^{2^*\beta_1}\, dx\right)^{\frac{1}{2^*(\beta_1-1)}}$ is bounded by (3).

It is not difficult to verify that there exists a constant $C_0 > 0$ independent of m, such that

$$A_{m+1} \leqslant \prod_{k=2}^{m+1} C_k^{\frac{1}{2(\beta_k-1)}} A_1 \leqslant C_0 A_1. \tag{3.8}$$

We stress that once (3) is established, then, by the Hölder inequality, we conclude that $u \in L^p(\Omega)$, for every $p \in [1, +\infty)$. Furthermore, a limit argument implies that

$$\|u\|_{L^\infty(\Omega)} \leqslant C_0 A_1 < +\infty,$$

where C_0 depends only on n, q, Ω, c, which finishes the proof of Theorem 1.1. □

4. $C^{1,\alpha}$-REGULARITY

In this section, we focus on the $C^{1,\alpha}$-regularity up to the boundary and on the interior $C^{1,\alpha}$-regularity for operator $\mathcal{L}$.

4.1. **$C^{1,\alpha}$-regularity up to the boundary.** The $C^{1,\alpha}$-regularity up to the boundary result for $\mathcal{L}$ is now an immediate corollary of the following $W^{2,p}$-regularity theorem.

Theorem 4.1. *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$ and $s \in (0,1)$. If $f \in L^p(\Omega)$ where p satisfies*

$$\begin{cases} 1 < p < \infty, & \text{if } s \in (0, \frac{1}{2}], \\ n < p < \frac{n}{2s-1}, & \text{if } s \in (\frac{1}{2}, 1). \end{cases} \tag{4.1}$$

Then, the problem

$$-\Delta u + (-\Delta)^s u = f, \qquad \text{in } \Omega$$

has a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Furthermore,

$$\|u\|_{W^{2,p}} \leqslant C\Big(\|u\|_{L^p} + \|f\|_{L^p}\Big), \tag{4.2}$$

where the positive constant C depends only on Ω, n, s, p.

Theorem 4.1 can be obtained by following an argument given in [SVWZ22b]. For the reader's convenience we sketch this argument here, by pointing out the following steps:

Step 1. Obtain L^p-estimates on the operator $(-\Delta)^s$;

Step 2. Apply the fixed point theorem to deduce that the problem

$$-\Delta u + \lambda u = f - (-\Delta)^s u, \qquad \text{in } \Omega$$

has a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for $\lambda > 0$ large enough;

Step 3. Employ the maximum principle in [BDVV22b, Theorem 1.2] and the bootstrap method to conclude.

With this, we can now complete the proof of the $C^{1,\alpha}$-regularity theory by proceeding as follows.

Proof of Theorem 1.3. Let $u \in X_0^1$ be the weak solution of (1). Theorem 1.1 gives that the map $x \mapsto g(x, u(x))$ belongs to $L^\infty(\Omega)$.

Moreover, due to Theorem 4.1, we deduce that there exists a unique solution $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for every p satisfying (4.1). By combining this with the Sobolev inequality, we obtain that

(1) if $s \in (0, \frac{1}{2}]$, then for any $\alpha \in (0,1)$, $v \in C^{1,\alpha}(\overline{\Omega})$;
(2) if $s \in (\frac{1}{2}, 1)$, then for any $\alpha \in (0, 2-2s)$, $v \in C^{1,\alpha}(\overline{\Omega})$.

Finally, the Lax-Milgram Theorem yields that $u = v$. In particular, according to (4.1), (1.1) and the Sobolev inequality, we obtain that

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant C\left(\|u\|_{L^\infty(\Omega)} + \|g(\cdot, u(\cdot))\|_{L^\infty(\Omega)}\right) \leqslant C\left(\|u\|_{L^\infty(\Omega)} + c(1 + \|u\|_{L^\infty(\Omega)}^{2^*-1})\right),$$

where the positive constant C depends only on n, α, Ω, s. □

4.2. **Interior $C^{1,\alpha}$-regularity.** The interior $C^{1,\alpha}$-regularity for the mixed operator $\mathcal{L}$ will be derived by a mollifier technique and a cutoff argument. We split the proof of Theorem 1.5 into three steps.
Step 1. Regularize the solution u by the standard mollifier to get some properties about u_ε. See subsection 4.2.1.
Step 2. Use the cut off argument for u_ε and employing [FRRO22, Proposition 2.18] to get the interior $C^{1,\alpha}$-norm of u_ε in B_R for any $R > 0$ small enough. See subsection 4.2.2.
Step 3. Apply Proposition 4.3 and Arzelà-Ascoli Theorem to get the inclusion. See subsection 4.2.3.

We now introduce some notations.

(a) Let V be an open set with $V \Subset \Omega$, and denote

$$\rho := \operatorname{dist}(V, \partial\Omega), \quad \text{and} \quad V_\delta := \{x \in \Omega : \operatorname{dist}(x, V) < \delta\}. \tag{4.1}$$

(b) For any given $x_0 \in V_{\frac{\rho}{4}}$ where $V_{\frac{\rho}{4}}$ is given in (4.2), denote

$$0 < R < \min\left(\frac{1}{2}, \frac{\rho}{10}\right), \quad B_R(x_0) := \{x \in \Omega : |x - x_0| < R\}.$$

(c) Denote the interior norm as follows:

$$[u]_{\alpha;B_R(x_0)} = \sup_{x,y\in B_R(x_0)} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \qquad 0 < \alpha < 1;$$

$$|u|'_{k;B_R(x_0)} = \sum_{j=0}^{k} R^j \|D^j u\|_{L^\infty(B_R(x_0))};$$

$$|u|'_{k,\alpha;B_R(x_0)} = |u|'_{k;B_R(x_0)} + R^{k+\alpha}[D^k u]_{\alpha;B_R(x_0)}.$$

4.2.1. *A Mollifier technique.* Let $u \in X^1$ solve the equation (1.5) and η_ε is the standard mollifier. For every $x \in \mathbb{R}^n$, we denote

$$u_\varepsilon(x) := (\eta_\varepsilon * u)(x) = \int_\Omega \eta_\varepsilon(x - y)u(y)\,dy = \int_{|y|\leqslant\varepsilon} \eta_\varepsilon(y)u(x - y)\,dy \quad \forall 0 < \varepsilon < R.$$

Recall that if u solves the following equation in the weak sense

$$-\Delta u + (-\Delta)^s u = g(x, u) \qquad \text{in } \Omega,$$

then, for every $\varepsilon \in (0, R)$ small enough,

$$-\Delta(\eta_\varepsilon * u) + (-\Delta)^s(\eta_\varepsilon * u) = \eta_\varepsilon * g(x, u) \qquad \text{in } V_{\frac{3\rho}{4}}. \tag{4.2}$$

We may assume that $g_\varepsilon := \eta_\varepsilon * g(x, u)$. Then, u_ε and g_ε satisfy the following equation

$$-\Delta u_\varepsilon + (-\Delta)^s u_\varepsilon = g_\varepsilon \qquad \text{in } V_{\frac{3\rho}{4}}.$$

Moreover, from the following two Propositions 4.1 and 4.2, we can see that

$$u_\varepsilon \in C^2(\overline{V}_{\frac{3\rho}{4}}) \cap L^\infty(\mathbb{R}^n) \quad \text{and} \quad g_\varepsilon \in L^\infty(B_R(x_0)) \qquad \forall 0 < \varepsilon < R.$$

Proposition 4.1. *Assume that u is a bounded solution of* (1.5)*, then $u_\varepsilon \in L^\infty(\mathbb{R}^n)$. In particular,*

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leqslant \|u\|_{L^\infty(\mathbb{R}^n)} \qquad \forall 0 < \varepsilon < R. \tag{4.3}$$

Proof. $\forall x \in \mathbb{R}^n$, one has

$$|u_\varepsilon(x)| \leqslant \int_\Omega \eta_\varepsilon(y)|u(x_1 - y)|\, dy \leqslant \|u\|_{L^\infty(\mathbb{R}^n)}.$$

Consequently, one can deduce that $u_\varepsilon \in L^\infty(\mathbb{R}^n)$ and (4.1). □

Proposition 4.2. *Let $g \in L^\infty_{\text{loc}}(\Omega \times \mathbb{R}^n)$, for any $\varepsilon \in (0, R)$, then*

$$\|g_\varepsilon(\cdot, u(\cdot))\|_{L^\infty(B_R(x_0))} \leqslant \|g\|_{L^\infty(\overline{B}_{2R}(x_0)\times\mathbb{R}^n)}. \tag{4.4}$$

Proof. For every $x \in B_R(x_0)$, one has

$$|g_\varepsilon(x, u(x))| = \int_{|y|\leqslant\varepsilon} |\eta_\varepsilon(y)g(x - y, u(x - y))|\, dy \leqslant \|g\|_{L^\infty(\overline{B}_{2R}(x_0)\times\mathbb{R}^n)}.$$ □

4.2.2. *A priori $C^{1,\alpha}$-estimate.* We now use the cut off argument for u_ε to estimate the $C^{1,\alpha}$-norm of u_ε.

Consider a cut off function $\phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ satisfying

$$\begin{cases} \phi \equiv 1 \text{ on } B_{\frac{3}{2}}(0) \\ \text{supp}(\phi) \subset B_2(0); \\ 0 \leqslant \phi \leqslant 1 \text{ on } \mathbb{R}^n. \end{cases} \tag{4.5}$$

Denote

$$\phi^R(x) := \phi\left(\frac{x - x_0}{R}\right). \tag{4.6}$$

Thus, one has that

$$B_{4R}(x_0) \subset V_{\frac{3\rho}{4}} \subset \Omega, \quad \text{supp}(\phi^R) \subset B_{2R}(x_0).$$

Lemma 4.2. *We assume that $g(\cdot, u(\cdot)) \in L^\infty(\Omega)$. Suppose that $u \in C^2(\overline{V}_{\frac{3\rho}{4}}) \cap L^\infty(\mathbb{R}^n)$ solves the following equation*

$$-\Delta u + (-\Delta)^s u = g \qquad \text{in } V_{\frac{3\rho}{4}}.$$

Then, there exists $\psi \in L^\infty(B_R(x_0), \mathbb{R})$ such that $v := \phi^R u$ satisfies

$$-\Delta v + (-\Delta)^s v = \psi \qquad \text{in } V_{\frac{3\rho}{4}}. \tag{4.7}$$

In addition,

$$R^2\|\psi\|_{L^\infty(B_R(x_0))} \leqslant C_{\rho,n,s}\left(\|g(\cdot, u(\cdot))\|_{L^\infty(B_R(x_0))} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) \tag{4.8}$$

for some positive constant $C_{n,s,\rho}$.

Proof. From the definition of the function v, we know that

$$v \in C_0^2(B_{2R}(x_0)) \subset C_0^2(\mathbb{R}^n).$$

Furthermore, to check (4.2), one can calculate that

$$\psi = g(x,u) + \Delta(u(1-\phi^R)) - (-\Delta)^s(u(1-\phi^R)) \quad \text{ in } V_{\frac{3\rho}{4}}.$$

We also notice that $\phi^R = 1$ in $B_{3R/2}(x_0)$ and that

$$\psi = g(x,u) - (-\Delta)^s(u(1-\phi^R)) \quad \text{ in } B_{3R/2}(x_0).$$

We now obtain suitable L^∞- estimates on the function $(-\Delta)^s(u(1-\phi^R))$. Observing that for $x \in B_R(x_0)$, we have that

$$(-\Delta)^s(u(1-\phi^R))(x) = \int_{\mathbb{R}^n\setminus B_{3R/2}(x_0)} \frac{-u(1-\phi^R)(y)}{|x-y|^{n+2s}}\,dy.$$

From this and the fact that

(4.9)
$$|x-y| \geqslant |y-x_0-R| \geqslant |y-x_0|-R \geqslant \frac{1}{5}(R+|y-x_0|) \quad \forall y \in \mathbb{R}^n\setminus B_{3R/2}(x_0),\, x \in B_R(x_0),$$

one obtains that

$$\begin{aligned}\|(-\Delta)^s(u(1-\phi^R))\|_{L^\infty(B_R(x_0))} &\leqslant 5^{n+2s} \int_{\mathbb{R}^n\setminus B_{3R/2}(x_0)} \frac{|u(y)|}{(R+|y-x_0|)^{n+2s}}\,dy \\ &\leqslant C_{n,s}\|u\|_{L^\infty(\mathbb{R}^n)} R^{-2s}.\end{aligned}$$

Since $g(\cdot,u(\cdot)) \in L^\infty(B_R(x_0))$, we have

$$R^2\|\psi\|_{L^\infty(B_R(x_0))} \leqslant C_{\rho,n,s}\left(\|g(\cdot,u(\cdot))\|_{L^\infty(B_R(x_0))} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)$$

for some positive constant $C_{n,s,\rho}$. □

According to (4.2.1), we know that for every $\varepsilon \in (0,R)$ small enough, $u_\varepsilon \in C^2(\overline{V}_{\frac{3\rho}{4}})$ solves the following equation

$$-\Delta u_\varepsilon + (-\Delta)^s u_\varepsilon = g_\varepsilon \qquad \text{in } V_{\frac{3\rho}{4}},$$

where g_ε satisfies the estimate (4.2).

Consider the cut off function ϕ^R given by (4.2.2), let $v_\varepsilon := \phi^R u_\varepsilon$, then applying Lemma 4.2, one has there exists $\psi_\varepsilon \in L^\infty(B_R(x_0),\mathbb{R})$ such that v_ε satisfies

$$-\Delta v_\varepsilon + (-\Delta)^s v_\varepsilon = \psi_\varepsilon \qquad \text{in } V_{\frac{3\rho}{4}},$$

where

$$\psi_\varepsilon = g_\varepsilon(x,u) + \Delta(u_\varepsilon(1-\phi^R)) - (-\Delta)^s(u_\varepsilon(1-\phi^R)) \quad \text{ in } V_{\frac{3\rho}{4}}.$$

Also, $\phi^R = 1$ in $B_{3R/2}(x_0)$ and

$$\psi_\varepsilon = g_\varepsilon(x,u) - (-\Delta)^s(u_\varepsilon(1-\phi^R)) \quad \text{ in } B_{3R/2}(x_0).$$

In particular, employing (4.2), one finds that

$$R^2\|\psi_\varepsilon\|_{L^\infty(B_R(x_0))} \leqslant C_{\rho,n,s}\left(\|g_\varepsilon(\cdot,u(\cdot))\|_{L^\infty(B_R(x_0))} + \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\right),$$

where $C_{\rho,n,s}$ is a positive constant.

We notice that, for $v_\varepsilon \in C_0^\infty(B_{2R}(x_0))$, one has

$$
\begin{aligned}
R^2\|(-\Delta)^s v_\varepsilon\|_{L^\infty(B_R(x_0))} &\leqslant C_{n,s,\rho}\left(R^{1+\beta}[Dv_\varepsilon]_{0,\beta;B_{2R}(x_0)} + \|v_\varepsilon\|_{L^\infty(B_{2R}(x_0))}\right)\\
&= C_{n,s,\rho}\Big(\|\phi^R u_\varepsilon\|_{L^\infty(B_{2R}(x_0))} + R^{1+\beta}[\phi^R Du_\varepsilon + u_\varepsilon D\phi^R]_{0,\beta;B_{2R}(x_0)}\Big)\\
&\leqslant C_{n,s,\rho}|u_\varepsilon|'_{1,\beta;B_{2R}(x_0)},
\end{aligned}
\tag{4.10}
$$

for every $\beta \in (\max\{0, 2s-1\}, 1)$.

As a result, for any given $\alpha \in (\max\{0, 2s-1\}, 1)$, taking $\beta = \frac{\max\{0,2s-1\}+\alpha}{2}$, invoking [FRRO22, Proposition 2.18], and utilizing (4.2.2) and interpolation inequality, one has for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$
\begin{aligned}
|v_\varepsilon|'_{1,\alpha;B_{R/2}(x_0)} &\leqslant C_{n,\alpha}\left(\|v_\varepsilon\|_{L^\infty(B_R(x_0))} + R^2\left(\|\psi_\varepsilon\|_{L^\infty(B_R(x_0))} + \|(-\Delta)^s v_\varepsilon\|_{L^\infty(B_R(x_0))}\right)\right)\\
&\leqslant C_{n,\alpha,\rho,\alpha}\Big(\|v_\varepsilon\|_{L^\infty(B_R(x_0))} + R^2\|\psi_\varepsilon\|_{L^\infty(B_R(x_0))} + |u_\varepsilon|'_{1,\beta;B_{2R}(x_0)}\Big)\\
&\leqslant C_{n,\alpha,\rho,\alpha}\Big(\|v_\varepsilon\|_{L^\infty(B_R(x_0))} + R^2\|\psi_\varepsilon\|_{L^\infty(B_R(x_0))}\\
&\qquad + \delta|u_\varepsilon|'_{1,\alpha;B_{2R}(x_0)} + C_\delta\|u_\varepsilon\|_{L^\infty(B_{2R}(x_0))}\Big).
\end{aligned}
$$

Therefore, for every $x_0 \in V_{\frac{\rho}{4}}$, applying Propositions 4.1 and 4.2, we know that $\forall\delta > 0$, there exists C_δ such that

$$
\begin{aligned}
&\|u_\varepsilon\|'_{1,\alpha;B_{R/2}(x_0)} = |v_\varepsilon|'_{1,\alpha;B_{R/2}(x_0)}\\
&\leqslant C_{n,s,\alpha,\rho}\left(R^2\|\psi_\varepsilon\|_{L^\infty(B_R(x_0))} + \delta|u_\varepsilon|'_{1,\alpha;B_{2R}(x_0)} + C_\delta\|u_\varepsilon\|_{L^\infty(B_{2R}(x_0))}\right)\\
&\leqslant C_{n,s,\alpha,\rho}\Big(\|g\|_{L^\infty(V_{3\rho/4}\times\mathbb{R}^n)} + C_\delta\|u\|_{L^\infty(\mathbb{R}^n)} + \delta|u_\varepsilon|'_{1,\alpha;B_{2R}(x_0)}\Big),
\end{aligned}
\tag{4.11}
$$

for every $R \in \left(0, \frac{\rho}{10}\right)$, $\varepsilon \in (0, R)$, and for some constant $C_{n,s,\alpha,\rho} > 0$.

4.2.3. *Interior $C^{1,\alpha}$-regularity.* Based on (4.2.2), we can obtain the following Proposition 4.3 by adapting [BFV14, Lemma 8], which can guarantee that the $C^{1,\alpha}$-norm of u_ε is uniformly bounded in ε in some ball.

Proposition 4.3. *Let $y \in \mathbb{R}^n, d > 0$, and $u \in C^{1,\alpha}(B_d(y))$. Suppose that for any $\delta > 0$, there exists $\Lambda_\delta > 0$ such that, for any $x \in B_d(y)$ and any $r \in (0, d - |x-y|]$, we have*

$$|u|'_{1,\alpha;B_{r/8}(x)} \leqslant \Lambda_\delta + \delta|u|'_{1,\alpha;B_{r/2}(x)}.$$

Then, there exist constants $\delta_0, C > 0$, depending only on n, α, d, such that

$$\|u\|_{C^{1,\alpha}(B_{d/8}(y))} \leqslant C\Lambda_{\delta_0}.$$

Proof. We notice that, since $r < d$, one has

$$|u|'_{1,\alpha;B_{r/8}(x)} \leqslant |u|'_{1,\alpha;B_d(y)},$$

which implies that

$$M := \sup_{x\in B_d(y), r\in(0,d-|x-y|]} |u|'_{1,\alpha;B_{r/8}(x)} < +\infty.$$

We now use a covering argument: pick $\lambda \in (0, \frac{1}{2}]$, to be chosen later, and fixed any $x \in B_d(y)$ and $r \in (0, d - |x - y|]$, we cover $B_{r/8}(x)$ with finitely many balls $\{B_{\lambda r/8}(x_k)\}_{k=1}^N$ with $x_k \in B_{r/8}(x)$, for some N depending only on λ, n.

Furthermore, we observe that $B_{r/2}(x_k) \subset B_d(y)$. Indeed,

$$|x_k - y| + \frac{r}{2} \leqslant |x_k - x| + |x - y| + \frac{r}{2} < r + |x - y| < d.$$

Consequently, since $\lambda \leqslant \frac{1}{2}$, we use (5.2) (with $x = x_k$ and r scaled to λr) to get

$$|u|'_{1,\alpha;B_{\lambda r/8}(x_k)} \leqslant \Lambda_\delta + \delta|u|'_{1,\alpha;B_{\lambda r/2}(x_k)}. \tag{4.12}$$

Then, using [BFV14, Lemma 7] with $\rho := r/8$ and $\lambda = 1/8$, recalling the definition of M, and employing (4.2.3), one has

$$\begin{aligned} |u|'_{1,\alpha;B_{r/8}(x)} &\leqslant C\sum_{k=1}^N |u|'_{1,\alpha;B_{\lambda r/8}(x_k)} \leqslant C_\lambda N\Lambda_\delta + C_\lambda\delta\sum_{k=1}^N |u|'_{1,\alpha;B_{\lambda r/2}(x_k)} \\ &\leqslant CN\Lambda_\delta + C\delta\sum_{k=1}^N |u|'_{1,\alpha;B_{r/16}(x_k)} \leqslant CN\Lambda_\delta + CN\delta M. \end{aligned}$$

It follows from the definition of M that

$$M \leqslant CN\Lambda_\delta + CN\delta M,$$

by choosing $\delta_0 = 1/(CN)$, one has

$$M \leqslant 2CN\Lambda_{\delta_0}.$$

Thus we have proved that

$$|u|'_{1,\alpha;B_{r/8}(x)} \leqslant 2CN\Lambda_{\delta_0}. \qquad \forall x \in B_d(y), r \in (0, d - |x - y|].$$

Setting $x = y, r = d$, we have

$$|u|'_{1,\alpha;B_{d/8}(y)} \leqslant 2N\Lambda_{\delta_0},$$

which implies the desired result that

$$\|u\|_{C^{1,\alpha}(\overline{B}_{d/8}(y))} \leqslant 2\max\left\{d^{-1-\alpha}, 1\right\} CN\Lambda_{\delta_0}. \qquad \square$$

Proof of Theorem 1.5. First of all, one can choose $d = \rho/5$ in Proposition 4.3, so that

$$B_d(y) \subset V_{\rho/4} \quad \text{and} \quad d/4 < \frac{\rho}{10}, \quad \forall y \in V.$$

As a consequence, applying Proposition 4.3, utilizing (4.2.2), we know that

$$\|u_\varepsilon\|_{C^{1,\alpha}(\overline{B}_{\rho/40}(y))} \leqslant C\left(\|g\|_{L^\infty(V_{\frac{3\rho}{4}}\times\mathbb{R}^n)} + C_{\delta_0}\|u\|_{L^\infty(\mathbb{R}^n)}\right),$$

for every $y \in V$, where the constant $C > 0$ depends on n, s, α, ρ.

Applying the Arzelà-Ascoli Theorem, we obtain that $u \in C^{1,\alpha}(\overline{B}_{\rho/40}(y))$. More specifically, one has that

$$\|u\|_{C^{1,\alpha}(\overline{B}_{\rho/40}(y))} \leqslant C\left(\|g\|_{L^\infty(V_{\frac{3\rho}{4}}\times\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right),$$

for every $y \in V$ and $\alpha \in (\max\{0, 2s-1\}, 1)$, and for some constant $C > 0$ depending on n, s, α, ρ. □

Using the cover argument again, there exist finitely many balls $\left\{B_{\rho/40}(y_k)\right\}_{k=1}^N$ with $y_k \in V$, for some N depending on n, ρ, V, such that

$$V \subset \bigcup_{k=1}^N B_{\rho/40}(y_k).$$

Thus, we can obtain

$$\|u\|_{C^{1,\alpha}(\overline{V})} \leqslant C\left(\|g\|_{L^\infty(V_{\frac{3\rho}{4}}\times\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right),$$

for every $\alpha \in (0, 1)$, where the constant $C > 0$ depends on n, s, α, ρ.

5. $C^{2,\alpha}$-regularity

In this section, we derive the interior $C^{2,\alpha}$-regularity and the $C^{2,\alpha}$-regularity up to the boundary for the mixed operator $\mathcal{L}$.

5.1. Interior $C^{2,\alpha}$-regularity with smooth boundary.

The aim of this section is to establish an interior $C^{2,\alpha}$-regularity theory for the problem (1). To this end, we should overcome the additional difficulty caused by the nonlinearity $g(x, u(x))$. More precisely, since the nonlinearity g depends on unknown function u, both the Hölder continuity of nonlinearity g and the regularity of u impact the regularity of $g(x, u(x))$ as a function of x.

From the proof of Theorem 1.3 for the $C^{1,\alpha}$-regularity obtained in Section 4, one cannot choose the Hölder exponent α arbitrarily in $(0, 1)$. This restriction could at a first glance prevent us from deriving the interior $C^{2,\alpha}$-regularity for any $\alpha \in (0, 1)$. Therefore, to overcome this difficulty, we make a detour round such restriction and carry out the proof of Theorem 1.4 via the following two steps:

(a) Establish a “weak version” of Theorem 1.4 in subsection 5.1.1;
(b) Employ an interpolation inequality and a truncation argument to obtain Theorem 1.4 in subsection 5.1.2.

5.1.1. *“Weak version” of the regularity result*.

Now we provide a “weak version” of Theorem 1.4, in which the regularity exponent obtained in the case $s \in (\frac{1}{2}, 1)$ is suboptimal (we will then utilize this result as a “pivot” to obtain Theorem 1.4 in its full generality):

Theorem 5.1. *Let us assume, in addition to the hypothesis of Theorem 1.3, that $g \in C^\gamma_{\mathrm{loc}}(\Omega \times \mathbb{R})$ with $\gamma \in (0, 1)$. Then, we have $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha < \min\{\gamma, \min\{1, 2-2s\}\}$.*

We remark that Theorem 5.1 above provides an interior C^{2,α_0}-estimate of the solution u for some $\alpha_0 \in (0, 2-2s)$, which allows us to invoke an interpolation inequality (see e.g. (5.1.2)) to gain the interior $C^{1,\alpha}$-estimate of u for each $\alpha \in (0,1)$. Moreover, by applying the same arguments as in the proof of Theorem 5.1 (such as a truncation argument and a mollifier technique), one can finally complete the proof of Theorem 1.4 in Subsection 5.1.2.

Let us first give the following C^α-estimates of a truncation function.

Lemma 5.2. *Under the assumptions of Theorem 1.3, let V be an open set with $V \Subset \Omega$, and denote*

$$\rho := \mathit{dist}(V, \partial\Omega), \qquad \mathit{and} \qquad V_\delta := \{x \in \Omega : \ \mathit{dist}(x, V) < \delta\}.$$

Consider a cut off function $\phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ satisfying

$$\begin{cases} \phi \equiv 1 \ \mathit{on}\ V_{\frac{\rho}{4}}, \\ \mathit{supp}(\phi) \subset V_{\frac{\rho}{2}} \subset \Omega, \\ 0 \leqslant \phi \leqslant 1 \ \mathit{on}\ \mathbb{R}^n. \end{cases}$$

Then, there exists $\psi \in C^\alpha(\overline{V}_{\frac{7\rho}{8}}, \mathbb{R})$ such that $v := \phi u$ is a solution of

$$\begin{cases} -\Delta v + (-\Delta)^s v = \psi & \mathit{in}\ V_{\frac{7\rho}{8}}, \\ v = 0 & \mathit{on}\ \mathbb{R}^n \setminus V_{\frac{7\rho}{8}}. \end{cases} \tag{5.1}$$

In particular,

$$\|\psi\|_{C^\alpha(\overline{V}_{\frac{7\rho}{8}})} \leqslant C\left(1 + \|g\|_{C^\gamma(\overline{V}_{\frac{7\rho}{8}} \times \mathbb{R})}\right)\left(1 + \|u\|_{C^{1,\alpha}(\overline{\Omega})}\right), \quad \forall \alpha \in (0, \min\{\gamma, \min\{1, 2-2s\}\}),$$

where C is a positive constant depending only on n, s, Ω, V.

Proof. From Theorem 1.3 and the definition of the function v, we know that

$$v \in C_0^{1,\alpha}(\overline{V}_{\frac{\rho}{2}}) \subset C_0^{1,\alpha}(\mathbb{R}^n).$$

Furthermore, to check (5.2), one can calculate that

$$\begin{aligned} \psi &= \phi g(x,u) + u(-\Delta)\phi + u(-\Delta)^s\phi + 2\nabla u \nabla \phi \\ &\quad + \int_{\mathbb{R}^n} \frac{(u(x) - u(x+z))(\phi(x) - \phi(x+z))}{|z|^{n+2s}}\, dz. \end{aligned} \tag{5.2}$$

Indeed,

$$-\Delta v = \phi(-\Delta)u + u(-\Delta)\phi + 2\nabla u \nabla \phi,$$

and

$$(-\Delta)^s v = \phi(-\Delta)^s u + u(-\Delta)^s \phi + \int_{\mathbb{R}^n} \frac{(u(x) - u(x+z))(\phi(x) - \phi(x+z))}{|z|^{n+2s}}\, dz.$$

Recalling the fact that

$$-\Delta u + (-\Delta)^s u = g(x,u) \qquad \text{in } \Omega,$$

one infers that

$$-\Delta v + (-\Delta)^s v = \phi g(x,u) + u(-\Delta)\phi + u(-\Delta)^s\phi + 2\nabla u\nabla\phi + \int_{\mathbb{R}^n} \frac{(u(x)-u(x+z))(\phi(x)-\phi(x+z))}{|z|^{n+2s}}\,dz,$$

which is (5.1.1), as desired.

Step 1. C^α**- estimates on the function** $\phi g(x,u)$**.** For each $i = \{1,\dots,n\}$, since $g \in C^\gamma_{\mathrm{loc}}(\Omega\times\mathbb{R})$ and $u \in C^{1,\alpha}(\overline{\Omega})$, for $x_1, x_2 \in V_{\frac{7\rho}{8}}$ with $x_1 \neq x_2$, we have that

$$\begin{aligned}\frac{|g(x_1,u(x_1)) - g(x_2,u(x_2))|}{|x_1-x_2|^\alpha} &\leqslant \frac{\|g\|_{C^\gamma(\overline{V}_{7\rho/8}\times\mathbb{R})}|(x_1-x_2, u(x_1)-u(x_2))|^\gamma}{|x_1-x_2|^\alpha}\\ &\leqslant \frac{\|g\|_{C^\gamma(\overline{V}_{7\rho/8}\times\mathbb{R})}\,(|x_1-x_2| + |u(x_1)-u(x_2)|)^\gamma}{|x_1-x_2|^\alpha}\\ &\leqslant \frac{\|g\|_{C^\gamma(\overline{V}_{7\rho/8}\times\mathbb{R})}\left(|x_1-x_2| + \|u\|_{C^1(\overline{\Omega})}|x_1-x_2|\right)^\gamma}{|x_1-x_2|^\alpha}\\ &\leqslant (\mathrm{diam}(\Omega))^{\gamma-\alpha}\|g\|_{C^\gamma(\overline{V}_{7\rho/8}\times\mathbb{R})}\left(1+\|u\|_{C^1(\overline{\Omega})}\right).\end{aligned}$$

Consequently,

$$[g(\cdot,u(\cdot))]_{C^\alpha(\overline{V}_{7\rho/8})} \leqslant (\mathrm{diam}(\Omega))^{\gamma-\alpha}\|g\|_{C^\gamma(\overline{V}_{7\rho/8}\times\mathbb{R})}\left(1+\|u\|_{C^1(\overline{\Omega})}\right).$$

Thus, for all $\alpha \in (0,\gamma]$, we have

$$\begin{aligned}\|\phi g(\cdot,u(\cdot))\|_{C^\alpha(\overline{V}_{7\rho/8})} &\leqslant [\phi]_{C^\alpha(\overline{V}_{7\rho/8})}\|g(\cdot,u(\cdot))\|_{L^\infty(V_{7\rho/8})} + [g(\cdot,u(\cdot))]_{C^\alpha(\overline{V}_{7\rho/8})}\|\phi\|_{L^\infty(V_{7\rho/8})}\\ &\quad + \|\phi g(\cdot,u(\cdot))\|_{L^\infty(V_{7\rho/8})}\\ &\leqslant C_1\|g\|_{C^\gamma(\overline{V}_{7\rho/8}\times\mathbb{R})}\left(1+\|u\|_{C^1(\overline{\Omega})}\right)\end{aligned}$$

for some suitable constant $C_1 > 0$.

Step 2. C^α**- estimates on the function** $u(-\Delta)\phi + u(-\Delta)^s\phi + 2\nabla u\nabla\phi$**.** For every $x \in V_{\frac{7\rho}{8}}$, we have that

$$\left|u(-\Delta)\phi + u(-\Delta)^s\phi + 2\nabla u\nabla\phi\right| \leqslant C_2\|u\|_{C^1(\overline{\Omega})},$$

and

$$\begin{aligned}&[u(-\Delta)\phi + u(-\Delta)^s\phi + 2\nabla u\nabla\phi]_{C^\alpha(\overline{V}_{7\rho/8})}\\ \leqslant&[u]_{C^\alpha(\overline{V}_{7\rho/8})}\|(-\Delta)\phi + (-\Delta)^s\phi\|_{L^\infty(V_{7\rho/8})} + [(-\Delta)\phi + (-\Delta)^s\phi]_{C^\alpha(\overline{V}_{7\rho/8})}\|u\|_{L^\infty(V_{7\rho/8})}\\ &\quad + 2\left([\nabla u]_{C^\alpha(\overline{V}_{7\rho/8})}\|\nabla\phi\|_{L^\infty(V_{7\rho/8})} + [\nabla\phi]_{C^\alpha(\overline{V}_{7\rho/8})}\|\nabla u\|_{L^\infty(V_{7\rho/8})}\right)\\ \leqslant&C_3\|u\|_{C^{1,\alpha}(\overline{\Omega})},\end{aligned}$$

for some suitable constant $C_2, C_3 > 0$.

Step 3. C^α**- estimates on the function** $h(x) := \int_{\mathbb{R}^n}\frac{(u(x)-u(x+z))(\phi(x)-\phi(x+z))}{|z|^{n+2s}}\,dz$**.** Let us first estimate the L^∞-norm of h. To this end, for every $x \in V_{\frac{7\rho}{8}}$, since $u = 0$ on

$\mathbb{R}^n \backslash \Omega$, one has that

$$\begin{aligned}|h(x)| &= \left| \int_{\mathbb{R}^n} \frac{(u(x) - u(x+z))(\phi(x) - \phi(x+z))}{|z|^{n+2s}} \, dz \right| \\ &\leqslant \int_{\{|z|<1\}} \frac{[u]_{C^1(\overline{\Omega})}[\phi]_{C^1(\overline{\Omega})}}{|z|^{n+2s-2}} \, dz + \int_{\{|z|\geqslant 1\}} \frac{4\|u\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)}}{|z|^{n+2s}} \, dz \leqslant C_4 \|u\|_{C^1(\overline{\Omega})}.\end{aligned}$$

Now we estimate the C^α-norm of h. For this, given $x_1, x_2 \in V_{\frac{7\rho}{8}}$ with $x_1 \neq x_2$, let $r := |x_1 - x_2|$. We distinguish two cases.

Case 1. If $r \geqslant 1$, then

$$\frac{|h(x_1) - h(x_2)|}{|x_1 - x_2|^\alpha} \leqslant 2\|h\|_{L^\infty(V_{\frac{7\rho}{8}})} \leqslant 2C_4 \|u\|_{C^1(\overline{\Omega})}. \tag{5.3}$$

Case 2. If $r < 1$, then

$$\begin{aligned}\frac{|h(x_1) - h(x_2)|}{|x_1 - x_2|^\alpha} &= \left| \int_{\mathbb{R}^n} \frac{(u(x_1) - u(x_1+z) - u(x_2) + u(x_2+z))(\phi(x_2) - \phi(x_2+z))}{|z|^{n+2s} r^\alpha} \, dz \right| \\ &\quad + \left| \int_{\mathbb{R}^n} \frac{(u(x_1) - u(x_1+z))(\phi(x_1) - \phi(x_1+z) - \phi(x_2) + \phi(x_2+z))}{|z|^{n+2s} r^\alpha} \, dz \right| \\ &:= I_1 + I_2.\end{aligned}$$

For I_1, one sees that

$$\begin{aligned}&\left| \int_{|z|<1} \frac{(u(x_1) - u(x_1+z) - u(x_2) + u(x_2+z))(\phi(x_2) - \phi(x_2+z))}{|z|^{n+2s} r^\alpha} \, dz \right| \\ &\quad\leqslant \int_{|z|<1} \frac{|(u(x_1) - u(x_1+z) - u(x_2) + u(x_2+z))| \|\nabla\phi\|_{L^\infty(\Omega)}}{|z|^{n+2s-1} r^\alpha} \, dz \\ &\quad\leqslant \|\nabla\phi\|_{L^\infty(\Omega)} \int_{|z|<r} \frac{|(u(x_1) - u(x_1+z) - u(x_2) + u(x_2+z))|}{|z|^{n+2s-1+\alpha}} \, dz \\ &\qquad + \|\nabla\phi\|_{L^\infty(\Omega)} \int_{r<|z|<1} \frac{2[u]_{C^1(\overline{\Omega})} r^\alpha |z|^{1-\alpha}}{|z|^{n+2s-1} r^\alpha} \, dz \\ &\quad\leqslant 2\|\nabla\phi\|_{L^\infty(\Omega)} \int_{|z|<1} \frac{2[u]_{C^1(\overline{\Omega})}}{|z|^{n+2s-2+\alpha}} \, dz \leqslant C_5 [u]_{C^1(\overline{\Omega})}, \quad \forall \alpha \in (0, \min\{1, 2-2s\}),\end{aligned} \tag{5.4}$$

and

(5.5)

$$\begin{aligned}
&\left|\int_{|z|\geqslant 1} \frac{(u(x_1)-u(x_1+z)-u(x_2)+u(x_2+z))(\phi(x_2)-\phi(x_2+z))}{|z|^{n+2s}r^{\alpha}}\,dz\right| \\
&\qquad \leqslant 2\|\phi\|_{L^\infty(\Omega)}\int_{|z|\geqslant 1}\frac{|(u(x_1)-u(x_1+z)-u(x_2)+u(x_2+z))|}{|z|^{n+2s}r^{\alpha}}\,dz \\
&\qquad \leqslant 2\|\phi\|_{L^\infty(\Omega)}\int_{|z|\geqslant 1}\frac{2[u]_{C^\alpha(\overline{\Omega})}r^{\alpha}}{|z|^{n+2s}r^{\alpha}}\,dz \\
&\qquad \leqslant 2\|\phi\|_{L^\infty(\Omega)}\int_{|z|\geqslant 1}\frac{2[u]_{C^\alpha(\overline{\Omega})}}{|z|^{n+2s}}\,dz \leqslant C_6[u]_{C^\alpha(\overline{\Omega})}, \quad \forall\alpha\in(0,\min\{1,2-2s\}).
\end{aligned}$$

Combining (5.1.1) with (5.1.1), we deduce that

$$I_1 \leqslant \left(C_5[u]_{C^1(\overline{\Omega})}+C_6[u]_{C^\alpha(\overline{\Omega})}\right), \qquad \forall\alpha\in(0,\min\{1,2-2s\}), \tag{5.6}$$

for some suitable constants $C_5, C_6 > 0$.

For I_2, using similar arguments to the estimate of I_1, one has

(5.7)

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \frac{(u(x_1)-u(x_1+z))(\phi(x_1)-\phi(x_1+z)-\phi(x_2)+\phi(x_2+z))}{|z|^{n+2s}r^{\alpha}}\,dz\right| \\
&\leqslant \int_{|z|<1}\frac{|(\phi(x_1)-\phi(x_1+z)-\phi(x_2)+\phi(x_2+z))|\|\nabla u\|_{L^\infty(\Omega)}}{|z|^{n+2s-1}r^{\alpha}}\,dz \\
&\leqslant \|\nabla u\|_{L^\infty(\Omega)}\int_{|z|<r}\frac{|\phi(x_1)-\phi(x_1+z)-\phi(x_2)+\phi(x_2+z)|}{|z|^{n+2s-1+\alpha}}\,dz \\
&\qquad + \|\nabla u\|_{L^\infty(\Omega)}\int_{r<|z|<1}\frac{2[\phi]_{C^1(\overline{\Omega})}r^{\alpha}|z|^{1-\alpha}}{|z|^{n+2s-1}r^{\alpha}}\,dz \\
&\leqslant 2\|\nabla u\|_{L^\infty(\Omega)}\int_{|z|<1}\frac{2[\phi]_{C^1(\overline{\Omega})}}{|z|^{n+2s-2+\alpha}}\,dz \leqslant C_7[u]_{C^1(\overline{\Omega})}, \quad \forall\alpha\in(0,\min\{1,2-2s\}),
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{|z|\geqslant 1} \frac{(u(x_1)-u(x_1+z))(\phi(x_1)-\phi(x_1+z)-\phi(x_2)+\phi(x_2+z))}{|z|^{n+2s}r^{\alpha}}\,dz\right| \\
&\qquad \leqslant 2\|u\|_{L^\infty(\Omega)}\int_{|z|\geqslant 1}\frac{|\phi(x_1)-\phi(x_1+z)-\phi(x_2)+\phi(x_2+z)|}{|z|^{n+2s}r^{\alpha}}\,dz \\
&\qquad \leqslant 2\|u\|_{L^\infty(\Omega)}\int_{|z|\geqslant 1}\frac{2[\phi]_{C^\alpha(\overline{\Omega})}r^{\alpha}}{|z|^{n+2s}r^{\alpha}}\,dz \\
&\qquad \leqslant 2\|u\|_{L^\infty(\Omega)}\int_{|z|\geqslant 1}\frac{2[\phi]_{C^\alpha(\overline{\Omega})}}{|z|^{n+2s}}\,dz \leqslant C_8\|u\|_{L^\infty(\Omega)}, \quad \forall\alpha\in(0,\min\{1,2-2s\}).
\end{aligned} \tag{5.8}$$

From (5.1.1) and (5.1.1) one concludes that

$$I_2 \leqslant C_7[u]_{C^1(\overline{\Omega})}+C_8\|u\|_{L^\infty(\Omega)}, \qquad \forall\alpha\in(0,\min\{1,2-2s\}), \tag{5.9}$$

for some suitable constants $C_7, C_8 > 0$.

Combining (5.1.1) and (5.1.1), and recalling (5.1.1), we conclude that

$$\frac{|h(x_1) - h(x_2)|}{|x_1 - x_2|^\alpha} \leqslant C_8 \|u\|_{C^{1,\alpha}(\overline{\Omega})}, \qquad \forall \alpha \in (0, \min\{1, 2-2s\}).$$

As a consequence, we derive that

$$\|h\|_{C^\alpha(\overline{V}_{7\rho/8})} \leqslant C\|u\|_{C^{1,\alpha}(\overline{\Omega})}, \qquad \forall \alpha \in (0, \min\{1, 2-2s\}).$$

In light of **Step 1**–**Step 3**, we obtain that $\psi \in C^\alpha(\overline{V}_{\frac{7\rho}{8}})$ and

$$\|\psi\|_{C^\alpha(\overline{V}_{7\rho/8})} \leqslant C\left(\|g\|_{L^\infty(\overline{V}_{7\rho/8}\times\mathbb{R})} + \|g\|_{C^\gamma(\overline{V}_{7\rho/8}\times\mathbb{R})}\|u\|_{C^1(\overline{\Omega})} + \|u\|_{C^{1,\alpha}(\overline{\Omega})}\right), \tag{5.10}$$

for any $\alpha \in (0, \min\{\gamma, \min\{1, 2-2s\}\})$ and for some suitable constant $C > 0$ depending only on n, s, Ω, V. □

We next use a mollifier to regularize v and obtain the $C^{2,\alpha}$-estimates on the convolution between the mollifier and v, then we will pass to the limit. This will allow us to complete the proof of Theorem 5.1. The details are as follows.

Proof of Theorem 5.1. We notice that the function $v := \phi u$ satisfies the following equation

$$\begin{cases} -\Delta v + (-\Delta)^s v = \psi & \text{in } V_{\frac{7\rho}{8}}, \\ v = 0 & \text{on } \mathbb{R}^n \backslash V_{\frac{7\rho}{8}}, \end{cases} \tag{5.11}$$

where $\psi \in C^\alpha(\overline{V}_{\frac{7\rho}{8}})$ and $v \in C_0^{1,\alpha}(\overline{V_{\frac{\rho}{2}}})$.

Recall that if v solves (5.1.1) in the weak sense and η_ε is a standard mollifier then

$$-\Delta(\eta_\varepsilon * v) + (-\Delta)^s(\eta_\varepsilon * v) = \eta_\varepsilon * \psi \qquad \text{in } V_{\frac{3\rho}{4}},$$

for $0 < \varepsilon < \frac{\rho}{10}$ small enough, and $\operatorname{supp}(\eta_\varepsilon * v) \subset V_{\frac{3\rho}{4}}$.

Indeed, $v = 0$ outside $V_{\frac{\rho}{2}}$, and

$$(\eta_\varepsilon * v)(x) = \int_{|z|<1} \eta(z) v(x - \varepsilon z)\, dz = 0, \quad \forall x \in \mathbb{R}^n \backslash V_{\frac{3\rho}{4}}.$$

Hence, we can regularize v, obtain the desired estimates, then pass to the limit. In this way, we may assume that $v_\varepsilon := \eta_\varepsilon * v \in C_0^\infty(V_{\frac{3\rho}{4}})$ and $\psi_\varepsilon := \eta_\varepsilon * \psi \in C^\infty(\overline{V}_{\frac{3\rho}{4}})$, then v_ε and ψ_ε satisfy the following equation

$$\begin{cases} -\Delta v_\varepsilon + (-\Delta)^s v_\varepsilon = \psi_\varepsilon & \text{in } V_{\frac{3\rho}{4}}, \\ v_\varepsilon = 0 & \text{on } \mathbb{R}^n \backslash V_{\frac{3\rho}{4}}. \end{cases} \tag{5.12}$$

According to the global estimate for the Dirichlet problem (5.1.1) and the fact that, for $v_\varepsilon \in C_0^\infty(V_{\frac{3\rho}{4}})$, there exists $C_{1/2} > 0$ such that

$$\|(-\Delta)^s v_\varepsilon\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}})} \leqslant \frac{1}{2}\|v_\varepsilon\|_{C^{2,\alpha}(\overline{V}_{\frac{3\rho}{4}})} + C_{1/2}\|v_\varepsilon\|_{L^\infty(V_{\frac{3\rho}{4}})},$$

we have

$$\begin{aligned}\|v_\varepsilon\|_{C^{2,\alpha}(\overline{V}_{\frac{3\rho}{4}})} &\leqslant C_{n,\alpha,V_{\frac{3\rho}{4}}}\left(\|v_\varepsilon\|_{L^\infty(V_{\frac{3\rho}{4}})}+\|\psi_\varepsilon\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}})}+\|(-\Delta)^s v_\varepsilon\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}})}\right)\\ &\leqslant C(n,\alpha,V_{\frac{3\rho}{4}})\left(\|v_\varepsilon\|_{L^\infty(V_{\frac{3\rho}{4}})}+\|\psi_\varepsilon\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}})}\right),\end{aligned}\tag{5.13}$$

where in the last inequality we have used the fact that if $w\in C_0^\infty(\mathbb{R}^n)$, then for all $\tau>0$ there exists $C_\tau>0$ such that

$$\|(-\Delta)^s w\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}})}\leqslant \tau\|w\|_{C^{2,\alpha}(\overline{V}_{\frac{3\rho}{4}})}+C_\tau\|w\|_{L^\infty(V_{\frac{3\rho}{4}})}.$$

Furthermore, exploiting [SVWZ22b, Lemma 4.8], there exists $\omega\in C^\infty(\overline{V}_{\frac{3\rho}{4}})\cap W^{1,\infty}(\mathbb{R}^n)$ such that for every $s\in(0,1)$

$$-\Delta\omega+(-\Delta)^s\omega\geqslant 1,\quad \text{in } V_{\frac{3\rho}{4}},\quad \omega>0 \text{ in } \overline{V}_{\frac{3\rho}{4}},\quad 0\leqslant\omega\leqslant C_0 \text{ in } \mathbb{R}^n,$$

for some constant C_0 depending only on $n, s, V_{\frac{3\rho}{4}}$.

As a result, we find that

$$-\Delta\left(\omega-\frac{v_\varepsilon}{\|\psi_\varepsilon\|_{L^\infty(O)}}\right)+(-\Delta)^s\left(\omega-\frac{v_\varepsilon}{\|\psi_\varepsilon\|_{L^\infty(O)}}\right)\geqslant 0\qquad \text{in } V_{\frac{3\rho}{4}}.$$

Applying the maximum principle (see [BDVV22b, Theorem 1.2]), we obtain

$$\|v_\varepsilon\|_{L^\infty(V_{\frac{3\rho}{4}})}\leqslant C_0\|\psi_\varepsilon\|_{L^\infty(V_{\frac{3\rho}{4}})}\qquad \text{in } V_{\frac{3\rho}{4}}.$$

As a consequence of this and (5.1.1), we have

$$\|v_\varepsilon\|_{C^{2,\alpha}(\overline{V}_{\frac{3\rho}{4}})}\leqslant C_{n,s,\alpha,V_{\frac{3\rho}{4}}}\left(\|\psi_\varepsilon\|_{L^\infty(V_{\frac{3\rho}{4}})}+\|\psi_\varepsilon\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}})}\right),\quad \forall 0<\varepsilon<\frac{\rho}{10}.$$

Therefore,

$$\|\eta_\varepsilon * v\|_{C^{2,\alpha}(\overline{V}_{\frac{3\rho}{4}})}\leqslant C_{n,s,\alpha,V_{\frac{3\rho}{4}}}\|\eta_\varepsilon*\psi\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}})}\leqslant C\|\psi\|_{C^\alpha(\overline{V}_{\frac{7\rho}{8}})},$$

for some suitable constant $C>0$ independent of v,ε.

As a result, utilizing the Arzelà-Ascoli Theorem, one finds that $v\in C^{2,\alpha}(\overline{V}_{\frac{3\rho}{4}})$, and consequently that $u\in C^{2,\alpha}(\overline{V}_{\frac{\rho}{4}})$.

Moreover, it follows from (5.1.1) that

$$\|u\|_{C^{2,\alpha}(\overline{V}_{\frac{\rho}{4}})}\leqslant C\|\psi\|_{C^\alpha(\overline{V}_{\frac{7\rho}{8}})}\leqslant C\left(\|g\|_{L^\infty(\overline{V}_{7\rho/8}\times\mathbb{R})}+\|g\|_{C^\gamma(\overline{V}_{7\rho/8}\times\mathbb{R})}\|u\|_{C^1(\overline{\Omega})}+\|u\|_{C^{1,\alpha}(\overline{\Omega})}\right),\tag{5.14}$$

for any $\alpha\in(0,\min\{\gamma,\min\{1,2-2s\}\})$ and for some positive constant C depending only on $n, s, \Omega, V, \alpha, \gamma$. □

5.1.2. ***Proof of the interior $C^{2,\alpha}$-regularity***. In light of the assumptions of Theorem 1.4, let us denote

$$\beta := \min\left\{1-s, \frac{\alpha}{2}\right\}. \tag{5.15}$$

Without loss of generality, we suppose that V is a $C^{2,\beta}$ domain in Ω with $\overline{V} \subset \Omega$. Exploiting the interpolation inequality (see e.g. [GT01, Lemma 6.35]), for each $\alpha \in (0,1)$, one can deduce that

$$\|u\|_{C^{1,\alpha}(\overline{V}_{\frac{\rho}{4}})} \leqslant \|u\|_{C^{2,\beta}(\overline{V}_{\frac{\rho}{4}})} + C_{V,s,\beta}\|u\|_{L^\infty(V_{\frac{\rho}{4}})},$$

for some suitable constant $C_{V,s,\beta}$.

By means of Theorem 5.1, combining (5.1.2) with (5.1.1), one has

$$\begin{aligned}\|u\|_{C^{1,\alpha}(\overline{V}_{\frac{\rho}{4}})} &\leqslant C\left(\|g\|_{L^\infty(\overline{V}_{7\rho/8}\times\mathbb{R})} + \|g\|_{C^\alpha(\overline{V}_{7\rho/8}\times\mathbb{R})}\|u\|_{C^1(\overline{\Omega})} + \|u\|_{C^{1,\beta}(\overline{\Omega})}\right) + C_{V,s,\beta}\|u\|_{L^\infty(V_{\frac{\rho}{4}})}\\ &\leqslant C\left(\|g\|_{L^\infty(\overline{V}_{7\rho/8}\times\mathbb{R})} + \|g\|_{C^\alpha(\overline{V}_{7\rho/8}\times\mathbb{R})}\|u\|_{C^1(\overline{\Omega})} + \|u\|_{C^{1,1-s}(\overline{\Omega})}\right)\end{aligned} \tag{5.16}$$

for some positive constant C depending only on n, s, Ω, V, α.

Based on the $C^{1,\alpha}_{\text{loc}}$-estimate of the solution u, using a truncation argument on the solution u on the set V and adapting the techniques of Theorem 5.1, we will now complete the proof of Theorem 1.4. The details go as follows.

We first provide the C^α-estimate of a truncation function.

Lemma 5.3. *Under the assumptions of Theorem 1.3, let W be an open set with $W \Subset V \Subset \Omega$, and denote*

$$\delta := \ dist(W, \partial V),$$

Select a cut off function $\Phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ satisfying

$$\begin{cases}\Phi \equiv 1 \ on \ W,\\ supp(\Phi) \subset W_{\frac{\delta}{2}} \subset V,\\ 0 \leqslant \Phi \leqslant 1 \ on \ \mathbb{R}^n.\end{cases}$$

Then, there exists $\Psi \in C^\alpha(\overline{\Omega}, \mathbb{R})$ such that $w := \Phi u$ is a solution of

$$\begin{cases}-\Delta w + (-\Delta)^s w = \Psi & in \ V,\\ w = 0 & on \ \mathbb{R}^n\setminus V.\end{cases} \tag{5.17}$$

In particular,

$$\|\Psi\|_{C^\alpha(\overline{V})} \leqslant C\left(1 + \|g\|_{C^\alpha(\overline{V}_{\frac{7\rho}{8}}\times\mathbb{R})}\right)^2\left(1 + \|u\|_{C^{1,\alpha}(\overline{\Omega})}\right) \quad \alpha \in (0,1),$$

and C is a positive constant depending only on $n, s, V, \Phi, \Omega, \alpha$.

Proof. From (5.1.2) and the definition of the function w, we know that

$$\begin{aligned}\|w\|_{C^{1,\alpha}(\overline{W}_{\frac{\delta}{2}})} &= \|\Phi u\|_{C^{1,\alpha}(\overline{W}_{\frac{\delta}{2}})} \leqslant \|\Phi u\|_{C^{1,\alpha}(\overline{V})} \leqslant C_{W,V}\|u\|_{C^{1,\alpha}(\overline{V})}\\ &\leqslant C_{W,V,n,s,\Omega,\alpha}\left(1+\|g\|_{C^{\alpha}(\overline{V}_{\frac{7\rho}{8}}\times\mathbb{R})}\right)\left(1+\|u\|_{C^{1,1-s}(\overline{\Omega})}\right),\end{aligned}$$

where $C_{W,V,n,s,\Omega,\alpha}$ is a suitable positive constant.

Moreover, observing that $\operatorname{supp}(w) \subset W_{\frac{\delta}{2}}$, one has that

$$w \in C_0^{1,\alpha}(\overline{W}_{\frac{\delta}{2}}) \subset C_0^{1,\alpha}(\mathbb{R}^n).$$

Furthermore, to check (5.3), one can calculate that

$$\begin{aligned}\Psi &= \Phi g(x,u) + u(-\Delta)\Phi + u(-\Delta)^s\Phi + 2\nabla u\nabla\Phi\\ &\quad + \int_{\mathbb{R}^n}\frac{(u(x)-u(x+z))(\Phi(x)-\Phi(x+z))}{|z|^{n+2s}}\,dz.\end{aligned} \tag{5.18}$$

Indeed,

$$-\Delta w = \Phi(-\Delta)u + u(-\Delta)\Phi + 2\nabla u\nabla\Phi,$$

and

$$(-\Delta)^s w = \Phi(-\Delta)^s u + u(-\Delta)^s\Phi + \int_{\mathbb{R}^n}\frac{(u(x)-u(x+z))(\Phi(x)-\Phi(x+z))}{|z|^{n+2s}}\,dz.$$

Recalling the fact that

$$-\Delta u + (-\Delta)^s u = g(x,u) \qquad \text{in } \Omega,$$

we see that

$$\begin{aligned}-\Delta w + (-\Delta)^s w &= \Phi g(x,u) + u(-\Delta)\Phi + u(-\Delta)^s\Phi + 2\nabla u\nabla\Phi\\ &\quad + \int_{\mathbb{R}^n}\frac{(u(x)-u(x+z))(\Phi(x)-\Phi(x+z))}{|z|^{n+2s}}\,dz,\end{aligned}$$

which is (5.1.2), as desired.

Here below, in analogy with the proof of Lemma 5.2, we state **Step 1** in a straightforward way. Instead, **Step 2** and **Step 3** need suitable modifications (specifically, we will make use of (5.1.2) to show (5.1.2) below).

Step 1. C^{α}- estimate on the function $\Phi g(x,u)$, that is

$$\|\Phi g(\cdot,u(\cdot))\|_{C^{\alpha}(\overline{V})} \leqslant C_1\|g\|_{C^{\alpha}(\overline{V}\times\mathbb{R})}\left(1+\|u\|_{C^1(\overline{\Omega})}\right)$$

for some suitable constant $C_1 > 0$.

Step 2. C^{α}- estimate on the function $u(-\Delta)\Phi + u(-\Delta)^s\Phi + 2\nabla u\nabla\Phi$. For every $x \in V$,

$$\left|u(-\Delta)\Phi + u(-\Delta)^s\Phi + 2\nabla u\nabla\Phi\right| \leqslant C_2\|u\|_{C^1(\overline{\Omega})},$$

and

$$
\begin{aligned}
&[u(-\Delta)\Phi + u(-\Delta)^s\Phi + 2\nabla u\nabla\Phi]_{C^\alpha(\overline{V})}\\
\leqslant&[u]_{C^\alpha(\overline{V})}\|(-\Delta)\Phi + (-\Delta)^s\Phi\|_{L^\infty(V)} + [(-\Delta)\Phi + (-\Delta)^s\Phi]_{C^\alpha(\overline{V})}\|u\|_{L^\infty(V)}\\
&\quad + 2\left([\nabla u]_{C^\alpha(\overline{V})}\|\nabla\Phi\|_{L^\infty(V)} + [\nabla\Phi]_{C^\alpha(\overline{V})}\|\nabla u\|_{L^\infty(V)}\right)\\
\leqslant&C_3\left(\|u\|_{C^1(\overline{\Omega})} + \|u\|_{C^{1,\alpha}(\overline{V})}\right),
\end{aligned}
\tag{5.19}
$$

for some suitable constant $C_2, C_3 > 0$.

Step 3. C^α**- estimate on the function** $H(x) := \int_{\mathbb{R}^n} \frac{(u(x)-u(x+z))(\Phi(x)-\Phi(x+z))}{|z|^{n+2s}}\,dz$. Let us first estimate the L^∞-norm of H.

For every $x \in V$, since $u = 0$ on $\mathbb{R}^n\backslash\Omega$, one has that

$$
|h(x)| = \left|\int_{\mathbb{R}^n} \frac{(u(x)-u(x+z))(\Phi(x)-\Phi(x+z))}{|z|^{n+2s}}\,dz\right| \leqslant C_4\|u\|_{C^1(\overline{\Omega})}.
$$

Now we estimate the C^α-norm of H. For this, given $x_1, x_2 \in V$ with $x_1 \neq x_2$, let $r := |x_1 - x_2|$. We distinguish two cases.

Case 1. If $r \geqslant \frac{\rho}{10}$, then

$$
\frac{|h(x_1)-h(x_2)|}{|x_1-x_2|^\alpha} \leqslant 20\|h\|_{L^\infty(V)} \leqslant 20C_4\rho^{-\alpha}\|u\|_{C^1(\overline{\Omega})}.
\tag{5.20}
$$

Case 2. If $r < \frac{\rho}{10}$, then

$$
\begin{aligned}
\frac{|h(x_1)-h(x_2)|}{|x_1-x_2|^\alpha} =& \left|\int_{\mathbb{R}^n} \frac{(u(x_1)-u(x_1+z)-u(x_2)+u(x_2+z))(\Phi(x_2)-\Phi(x_2+z))}{|z|^{n+2s}r^\alpha}\,dz\right|\\
&+\left|\int_{\mathbb{R}^n} \frac{(u(x_1)-u(x_1+z))(\Phi(x_1)-\Phi(x_1+z)-\Phi(x_2)+\Phi(x_2+z))}{|z|^{n+2s}r^\alpha}\,dz\right|\\
:=& I_1 + I_2.
\end{aligned}
$$

For I_1, if $|z| < \frac{\rho}{10}$, then $x_i \pm z \in V_{\frac{\rho}{4}}$, one sees that

$$
\begin{aligned}
&\left|\int_{|z|<\frac{\rho}{10}} \frac{(u(x_1)-u(x_1+z)-u(x_2)+u(x_2+z))(\Phi(x_2)-\Phi(x_2+z))}{|z|^{n+2s}r^\alpha}\,dz\right|\\
\leqslant& \int_{|z|<\frac{\rho}{10}} \frac{|(u(x_1)-u(x_1+z)-u(x_2)+u(x_2+z))|\|\nabla\Phi\|_{L^\infty(\Omega)}}{|z|^{n+2s-1}r^\alpha}\,dz\\
\leqslant&\|\nabla\Phi\|_{L^\infty(\Omega)}\int_{|z|<\frac{\rho}{10}} \frac{\int_0^1 |Du(t(x_1+z)+(1-t)x_1) - Du(t(x_2+z)+(1-t)x_2)|\,dt}{|z|^{n+2s-2}r^\alpha}\,dz\\
\leqslant&\|\nabla\Phi\|_{L^\infty(\Omega)}\int_{|z|<\frac{\rho}{10}} \frac{2[u]_{C^{1,\alpha}(\overline{V}_{\rho/4})}r^\alpha}{|z|^{n+2s-2}r^\alpha}\,dz \leqslant C_5\rho^{2-2s}[u]_{C^{1,\alpha}(\overline{V}_{\rho/4})},
\end{aligned}
\tag{5.21}
$$

and

$$
\begin{aligned}
&\left| \int_{|z|\geqslant \frac{\rho}{10}} \frac{(u(x_1)-u(x_1+z)-u(x_2)+u(x_2+z))(\Phi(x_2)-\Phi(x_2+z))}{|z|^{n+2s}r^{\alpha}}\,dz \right| \\
&\leqslant 2\|\Phi\|_{L^\infty(\Omega)} \int_{|z|\geqslant \frac{\rho}{10}} \frac{|(u(x_1)-u(x_1+z)-u(x_2)+u(x_2+z))|}{|z|^{n+2s}r^{\alpha}}\,dz \\
&\leqslant 2\|\Phi\|_{L^\infty(\Omega)} \int_{|z|\geqslant \frac{\rho}{10}} \frac{2[u]_{C^\alpha(\overline{\Omega})}r^{\alpha}}{|z|^{n+2s}r^{\alpha}}\,dz \leqslant C_6\rho^{-2s}[u]_{C^\alpha(\overline{\Omega})}.
\end{aligned} \tag{5.22}
$$

Combining (5.1.2) with (5.1.2), we deduce that

$$
I_1 \leqslant \left(C_5\rho^{2-2s}[u]_{C^{1,\alpha}(\overline{V}_{\rho/4})} + C_6\rho^{-2s}[u]_{C^\alpha(\overline{\Omega})} \right) \tag{5.23}
$$

for some suitable constants $C_5, C_6 > 0$.

For I_2, using similar arguments to the estimate of I_1, one has

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^n} \frac{(u(x_1)-u(x_1+z))(\Phi(x_1)-\Phi(x_1+z)-\Phi(x_2)+\Phi(x_2+z))}{|z|^{n+2s}r^{\alpha}}\,dz \right| \\
\leqslant &\int_{|z|<1} \frac{|(\Phi(x_1)-\Phi(x_1+z)-\Phi(x_2)+\Phi(x_2+z))|\|\nabla u\|_{L^\infty(\Omega)}}{|z|^{n+2s-1}r^{\alpha}}\,dz \\
\leqslant &2\|\nabla u\|_{L^\infty(\Omega)} \int_{|z|<1} \frac{2[\Phi]_{C^{1,\alpha}(\mathbb{R}^n)}r^{\alpha}|z|^1}{|z|^{n+2s-1}r^{\alpha}}\,dz \leqslant C_7[u]_{C^1(\overline{\Omega})},
\end{aligned} \tag{5.24}
$$

and

$$
\begin{aligned}
&\left| \int_{|z|\geqslant 1} \frac{(u(x_1)-u(x_1+z))(\Phi(x_1)-\Phi(x_1+z)-\Phi(x_2)+\phi(x_2+z))}{|z|^{n+2s}r^{\alpha}}\,dz \right| \\
\leqslant &2\|u\|_{L^\infty(\Omega)} \int_{|z|\geqslant 1} \frac{2[\Phi]_{C^\alpha(\overline{\Omega})}r^{\alpha}}{|z|^{n+2s}r^{\alpha}}\,dz \leqslant C_8\|u\|_{L^\infty(\Omega)}.
\end{aligned} \tag{5.25}
$$

From (5.1.2) and (5.1.2) one concludes that

$$
I_2 \leqslant C_7[u]_{C^1(\overline{\Omega})} + C_8\|u\|_{L^\infty(\Omega)}, \tag{5.26}
$$

for some suitable constants $C_7, C_8 > 0$.

Combining (5.1.2) and (5.1.2), and recalling (5.1.2), we conclude that

$$
\frac{|h(x_1)-h(x_2)|}{|x_1-x_2|^{\alpha}} \leqslant C_8\left(\|u\|_{C^1(\overline{\Omega})} + [u]_{C^{1,\alpha}(\overline{V}_{\rho/4})} \right).
$$

As a consequence, we derive that

$$
\|h\|_{C^\alpha(\overline{V})} \leqslant C\left(\|u\|_{C^1(\overline{\Omega})} + [u]_{C^{1,\alpha}(\overline{V}_{\rho/4})} \right).
$$

In light of **Step 1–Step 3**, and combining (5.1.1) with (5.1.2), we obtain that $\Psi \in C^\alpha(\overline{V})$ and

(5.27)

$$\begin{aligned}
\|\Psi\|_{C^\alpha(\overline{V})} &\leqslant C\left(\|g\|_{L^\infty(\overline{V}\times\mathbb{R})} + \|g\|_{C^\alpha(\overline{V}\times\mathbb{R})}\|u\|_{C^1(\overline{\Omega})} + \|u\|_{C^1(\overline{\Omega})} + [u]_{C^{1,\alpha}(\overline{V}_{\rho/4})}\right)\\
&\leqslant C\left(\|g\|_{L^\infty(\overline{V}\times\mathbb{R})} + \|g\|_{C^\alpha(\overline{V}\times\mathbb{R})}\|u\|_{C^1(\overline{\Omega})} + \|u\|_{C^1(\overline{\Omega})} + \|u\|_{C^{2,\beta}(\overline{V}_{\frac{\rho}{4}})} + \|u\|_{L^\infty(V_{\frac{\rho}{4}})}\right)\\
&\leqslant C\left(\|g\|_{L^\infty(\overline{V}_{7\rho/8}\times\mathbb{R})} + \|g\|_{C^\alpha(\overline{V}_{7\rho/8}\times\mathbb{R})}\|u\|_{C^1(\overline{\Omega})} + \|u\|_{C^{1,1-s}(\overline{\Omega})}\right), \quad \forall \alpha \in (0,1),
\end{aligned}$$

for some constant $C > 0$ depending only on $n, s, \Omega, V, W, \alpha$. □

The gist of this argument consists of employing a mollifier to regularize w, establishing the $C^{2,\alpha}$-estimates on the mollification, and then passing to the limit.

Proof of Theorem 1.4. We observe that the function $w := \Phi u$ satisfies the following equation

$$\begin{cases} -\Delta w + (-\Delta)^s w = \Psi & \text{in } V,\\ w = 0 & \text{on } \mathbb{R}^n \setminus V, \end{cases}$$

where $\Psi \in C^\alpha(\overline{V})$ and $w \in C_0^{1,\alpha}(\overline{W}_{\frac{\delta}{2}})$.

Hence, by the mollifier argument, we can consider the functions

$$w_\varepsilon := \eta_\varepsilon * w, \quad \Psi_\varepsilon := \eta_\varepsilon * \Psi, \qquad \forall 0 < \varepsilon < \frac{\delta}{10}.$$

In this way, we have that $w_\varepsilon \in C_0^\infty(W_{\frac{3\delta}{4}})$, that$\Psi_\varepsilon \in C^\infty(\overline{W}_{\frac{3\delta}{4}})$ and that

$$\begin{cases} -\Delta w_\varepsilon + (-\Delta)^s w_\varepsilon = \Psi_\varepsilon & \text{in } W_{\frac{3\delta}{4}},\\ w_\varepsilon = 0 & \text{on } \mathbb{R}^n \setminus W_{\frac{3\delta}{4}}. \end{cases}$$

We readily demonstrate the $C^{2,\alpha}$-estimate of w_ε, referring to (5.1.1), finding that

$$\begin{aligned}
\|w_\varepsilon\|_{C^{2,\alpha}(\overline{W}_{\frac{3\delta}{4}})} &\leqslant C_{n,\alpha,W_{\frac{3\delta}{4}}}\left(\|w_\varepsilon\|_{L^\infty(W_{\frac{3\delta}{4}})} + \|\Psi_\varepsilon\|_{C^\alpha(\overline{W}_{\frac{3\delta}{4}})} + \|(-\Delta)^s w_\varepsilon\|_{C^\alpha(\overline{W}_{\frac{3\delta}{4}})}\right)\\
&\leqslant C(n,\alpha,W_{\frac{3\delta}{4}})\left(\|w_\varepsilon\|_{L^\infty(W_{\frac{3\delta}{4}})} + \|\Psi_\varepsilon\|_{C^\alpha(\overline{W}_{\frac{3\delta}{4}})}\right).
\end{aligned}$$

As a consequence of this, and combining [SVWZ22b, Lemma 4.8] with the maximum principle (see [BDVV22b, Theorem 1.2]), we conclude that

$$\|w_\varepsilon\|_{C^{2,\alpha}(\overline{W}_{\frac{3\delta}{4}})} \leqslant C_{n,s,\alpha,W_{\frac{3\delta}{4}}}\left(\|\Psi_\varepsilon\|_{L^\infty(W_{\frac{3\delta}{4}})} + \|\Psi_\varepsilon\|_{C^\alpha(\overline{W}_{\frac{3\delta}{4}})}\right), \quad \forall 0 < \varepsilon < \frac{\delta}{10}.$$

Therefore,

$$\|\eta_\varepsilon * w\|_{C^{2,\alpha}(\overline{W}_{\frac{3\delta}{4}})} \leqslant C_{n,s,\alpha,W_{\frac{3\delta}{4}}}\|\eta_\varepsilon * \Psi\|_{C^\alpha(\overline{W}_{\frac{3\delta}{4}})} \leqslant C\|\Psi\|_{C^\alpha(\overline{V})}, \tag{5.28}$$

for some constant $C > 0$ independent of w, ε.

As a result, applying the Arzelà-Ascoli Theorem and exploiting the definition of w, we obtain that $u \in C^{2,\alpha}(\overline{W})$. Moreover, it follows from (5.1.2) and (5.1.2) that

$$\|u\|_{C^{2,\alpha}(\overline{W})} \leqslant C\|\Psi\|_{C^{\alpha}(\overline{V})} \leqslant C\left(\|g\|_{L^{\infty}(\overline{V}_{7\rho/8}\times\mathbb{R})} + \|g\|_{C^{\alpha}(\overline{V}_{7\rho/8}\times\mathbb{R})}\|u\|_{C^{1}(\overline{\Omega})} + \|u\|_{C^{1,1-s}(\overline{\Omega})}\right)$$

for some constant $C > 0$ depending only on $n, s, \Omega, V, \alpha, W$. □

5.2. **General interior $C^{2,\alpha}$-regularity.** Now we discuss how to remove the $C^{1,1}$ smoothness assumption of the domain. For this, we apply a suitable truncation argument again of the solution u with an interior $C^{1,\alpha}$-regularity argument to have a version of $C^{2,\alpha}$ regularity result stated in Theorem 1.6.

We split the proof of Theorem 1.6 into two steps:

(a) Employ a mollifier technique and a cut off argument to get the interior $C^{2,\alpha}$-norm of convolution u_ε in subsection 5.2.1.

(b) Apply Proposition 5.2 and Arzelà-Ascoli Theorem to get the inclusion in subsection 5.2.2.

5.2.1. *A mollifier technique and a truncation argument.* Let $u \in X^1$ solve the equation (1.6), then the convolutions u_ε and g_ε satisfy the following equation

$$-\Delta u_\varepsilon + (-\Delta)^s u_\varepsilon = g_\varepsilon \qquad \text{in } V_{\frac{3\rho}{4}}. \tag{5.1}$$

Moreover, from Proposition 4.1 and the following Proposition 5.1, we can see that

$$u_\varepsilon \in C^{2,\alpha}(\overline{V}_{\frac{3\rho}{4}}) \cap L^\infty(\mathbb{R}^n) \quad \text{and} \quad g_\varepsilon \in C^\alpha(\overline{B}_R(x_0)) \qquad \forall 0 < \varepsilon < R.$$

Proposition 5.1. *Let $g \in C^\alpha_{\text{loc}}(\Omega \times \mathbb{R}^n)$ for any $\alpha \in (0,1)$ and $\varepsilon \in (0,R)$, If $u \in C^1_{\text{loc}}(\Omega)$, then*

$$\|g_\varepsilon(\cdot, u(\cdot))\|_{C^\alpha(\overline{B}_R(x_0))} \leqslant \|g\|_{C^\alpha(\overline{B}_{2R}(x_0)\times\mathbb{R}^n)} \left(1 + \|Du\|_{L^\infty(B_{2R}(x_0))}\right). \tag{5.2}$$

Proof. For every $x \in B_R(x_0)$, one has

$$|g_\varepsilon(x)| = \left|\int_{|y|\leqslant\varepsilon} \eta_\varepsilon(y) g(x-y, u(x-y))\, dy\right| \leqslant \|g(\cdot, u(\cdot))\|_{L^\infty(B_{2R}(x_0))},$$

and, for all $x_1, x_2 \in B_R(x_0)$ with $x_1 \neq x_2$, one has that

$$\|g_\varepsilon(\cdot, u(\cdot))\|_{C^\alpha(\overline{B}_R(x_0))} \leqslant \|g\|_{C^\alpha(\overline{B}_{2R}(x_0)\times\mathbb{R}^n)} \left(1 + \|Du\|_{L^\infty(B_{2R}(x_0))}\right).$$ □

We now use the cut off argument again for u_ε to get the $C^{2,\alpha}$-norm of u_ε.

Lemma 5.4. *For any $\alpha \in (0,1)$, we assume that $g(\cdot, u(\cdot)) \in C^\alpha_{\text{loc}}(\Omega)$. Suppose that $u \in C^{2,\alpha}(\overline{V}_{\frac{3\rho}{4}}) \cap L^\infty(\mathbb{R}^n)$ solves the following equation*

$$-\Delta u + (-\Delta)^s u = g \qquad \text{in } V_{\frac{3\rho}{4}}.$$

Then, there exists $\psi \in C^\alpha(B_R(x_0), \mathbb{R})$ such that $v := \phi^R u$ satisfies

$$-\Delta v + (-\Delta)^s v = \psi \qquad \text{in } V_{\frac{3\rho}{4}},$$

where $\phi^R(x)$ *is defined in* (4.2.2). *In particular,*

$$R^2|\psi|'_{0,\alpha;B_R(x_0)} \leqslant C_{\rho,n,s}\left(R^2|g(\cdot,u(\cdot))|'_{0,\alpha;B_R(x_0)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) \tag{5.3}$$

for some positive constant $C_{n,s,\rho}$.

Proof. From the definition of the function v, we know that

$$v \in C_0^{2,\alpha}(B_{2R}(x_0)) \subset C_0^{2,\alpha}(\mathbb{R}^n).$$

We also notice that $\phi^R = 1$ in $B_{3R/2}(x_0)$ and that

$$\psi = g(x,u) - (-\Delta)^s(u(1-\phi^R)) \quad \text{ in } B_{3R/2}(x_0).$$

Step 1. C^α**- estimates on the function** $g(x,u)$**.** Since $g(\cdot,u(\cdot)) \in C^\alpha(\overline{B}_R(x_0))$, we have

$$R^2|g(\cdot,u(\cdot))|'_{0,\alpha;B_R(x_0)} = R^2\|g(\cdot,u(\cdot))\|_{L^\infty(B_R(x_0))} + R^{2+\alpha}[g(\cdot,u(\cdot))]_{0,\alpha;B_R(x_0)}.$$

Step 2. C^α**- estimates on the function** $(-\Delta)^s(u(1-\phi^R))$**.**

Observing that for $x \in B_R(x_0)$, we have

$$(-\Delta)^s(u(1-\phi^R))(x) = \int_{\mathbb{R}^n\setminus B_{3R/2}(x_0)} \frac{-u(1-\phi^R)(y)}{|x-y|^{n+2s}}\,dy.$$

From the above expression and the inequality (4.2.2), one obtains that

$$\begin{aligned}\|(-\Delta)^s(u(1-\phi^R))\|_{L^\infty(B_R(x_0))} \leqslant& 5^{n+2s}\int_{\mathbb{R}^n\setminus B_{3R/2}(x_0)} \frac{|u(y)|}{(R+|y-x_0|)^{n+2s}}\,dy\\ \leqslant& C_{n,s}\|u\|_{L^\infty(\mathbb{R}^n)}R^{-2s}.\end{aligned} \tag{5.4}$$

For every $x_1, x_2 \in B_R(x_0)$ with $x_1 \neq x_2$,

$$\begin{aligned}&|(-\Delta)^s(u(1-\phi^R))(x_1) - (-\Delta)^s(u(1-\phi^R))(x_2)|\\ \leqslant& \int_{\mathbb{R}^n\setminus B_{3R/2}(x_0)} |-u(1-\phi^R)(y)|\left|\frac{1}{|x_1-y|^{n+2s}} - \frac{1}{|x_2-y|^{n+2s}}\right|dy\\ \leqslant& \int_{|x_1-x_2|>\frac{|x_1-y|}{2}\cap\mathbb{R}^n\setminus B_{3R/2}(x_0)} |u(y)|\left|\frac{1}{|x_1-y|^{n+2s}} - \frac{1}{|x_2-y|^{n+2s}}\right|dy\\ &+\int_{|x_1-x_2|\leqslant\frac{|x_1-y|}{2}\cap\mathbb{R}^n\setminus B_{3R/2}(x_0)} |u(y)|\left|\frac{1}{|x_1-y|^{n+2s}} - \frac{1}{|x_2-y|^{n+2s}}\right|dy\\ :=& I_1 + I_2.\end{aligned}$$

For I_1, we see that

$$\begin{aligned}I_1 &\leqslant 2\int_{|x_1-x_2|>\frac{|x_1-y|}{2}\cap\mathbb{R}^n\setminus B_{3R/2}(x_0)} \frac{|u(y)||x_1-x_2|^\alpha}{|x_1-y|^{n+2s+\alpha}}\,dy\\ &\leqslant 2\int_{\mathbb{R}^n\setminus B_{3R/2}(x_0)} \frac{|u(y)||x_1-x_2|^\alpha}{|x_1-y|^{n+2s+\alpha}}\,dy\\ &\leqslant C_{n,s}|x_1-x_2|^\alpha\int_{\mathbb{R}^n} \frac{|u(y)|}{(R+|y-x_0|)^{n+2s+\alpha}}\,dy\\ &\leqslant C_{n,s}\|u\|_{L^\infty(\mathbb{R}^n)}|x_1-x_2|^\alpha R^{-2s-\alpha}.\end{aligned} \tag{5.5}$$

We also observe that, for every $t \in [0,1]$,

$$|tx_1 + (1-t)x_2 - y| \geqslant |x_1 - y| - |x_1 - x_2| \geqslant \frac{|x_1 - y|}{2}, \quad \text{if} \quad |x_1 - x_2| \leqslant \frac{|x_1 - y|}{2},$$

so that,

$$\begin{aligned}
I_2 &\leqslant \int_{|x_1-x_2|\leqslant\frac{|x_1-y|}{2}\cap\mathbb{R}^n\setminus B_{3R/2}(x_0)} |u(y)| \left| \int_0^1 \frac{|x_1 - x_2|}{|tx_1 + (1-t)x_2 - y|^{n+2s+1}}\, dt \right| dy \\
&\leqslant C_{n,s} \int_{\mathbb{R}^n\setminus B_{3R/2}(x_0)} |u(y)| \left| \int_0^1 \frac{|x_1 - x_2|^{\alpha}|x_1 - y|^{1-\alpha}}{|x_1 - y|^{n+2s+1}}\, dt \right| dy \\
&\leqslant C_{n,s} \int_{\mathbb{R}^n\setminus B_{3R/2}(x_0)} |u(y)| \frac{|x_1 - x_2|^{\alpha}}{|x_1 - y|^{n+2s+\alpha}}\, dy \qquad (5.6)\\
&\leqslant C_{n,s}|x_1 - x_2|^{\alpha} \int_{\mathbb{R}^n} \frac{|u(y)|}{(R + |y - x_0|)^{n+2s+\alpha}}\, dy \\
&\leqslant C_{n,s}\|u\|_{L^\infty(\mathbb{R}^n)}|x_1 - x_2|^{\alpha}R^{-2s-\alpha}.
\end{aligned}$$

Combining (5.2.1) and (5.2.1), one has

$$[(-\Delta)^s(u(1-\phi^R))]_{0,\alpha;B_R(x_0)} \leqslant C_{n,s}\|u\|_{L^\infty(\mathbb{R}^n)}R^{-2s-\alpha}.$$

As a consequence of this and (5.2.1), we have that

$$\begin{aligned}
&R^2|(-\Delta)^s(u(1-\phi^R))|'_{0,\alpha;B_R(x_0)} \\
=&R^2\|(-\Delta)^s(u(1-\phi^R))\|_{L^\infty(B_R(x_0))} + R^{2+\alpha}[(-\Delta)^s(u(1-\phi^R))]_{\alpha;B_R(x_0)} \\
\leqslant&C_{n,s}\|u\|_{L^\infty(\mathbb{R}^n)}R^{2-2s}.
\end{aligned}$$

In light of **Step 1** and **Step 2**, we obtain

$$\begin{aligned}
R^2|\psi|'_{0,\alpha;B_R(x_0)} &= R^2\|\psi\|_{L^\infty(B_R(x_0))} + R^{2+\alpha}[\psi]_{\alpha;B_R(x_0)} \\
&\leqslant C_{\rho,n,s}\left(R^2|g(\cdot,u(\cdot))|'_{0,\alpha;B_R(x_0)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)
\end{aligned}$$

for some positive constant $C_{n,s,\rho}$. □

According to (5.2.1), we know that for every $0 < \varepsilon < R$ small enough, $u_\varepsilon \in C^{2,\alpha}(\overline{V}_{\frac{3\rho}{4}})$ solves the following equation

$$-\Delta u_\varepsilon + (-\Delta)^s u_\varepsilon = g_\varepsilon \qquad \text{in } V_{\frac{3\rho}{4}},$$

where g_ε satisfies the estimate (5.1).

Consider the cut off function ϕ^R given by (4.2.2), let $v_\varepsilon := \phi^R u_\varepsilon$, then applying Lemma 5.4, one has there exists $\psi_\varepsilon \in C^\alpha(B_R(x_0),\mathbb{R})$ such that v_ε satisfies

$$-\Delta v_\varepsilon + (-\Delta)^s v_\varepsilon = \psi_\varepsilon \qquad \text{in } V_{\frac{3\rho}{4}},$$

where

$$\psi_\varepsilon = g_\varepsilon(x,u) - (-\Delta)^s(u_\varepsilon(1-\phi^R)) \quad \text{in } B_{3R/2}(x_0).$$

In particular, employing (5.4), one finds that

$$R^2|\psi_\varepsilon|'_{0,\alpha;B_R(x_0)} \leqslant C_{\rho,n,s}\left(R^2|g_\varepsilon(\cdot,u(\cdot))|'_{0,\alpha;B_R(x_0)} + \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\right),$$

where $C_{\rho,n,s}$ is a positive constant.

Observing that for $v_\varepsilon \in C_0^\infty(B_{2R}(x_0))$, for all $\delta > 0$, there exists $C_\delta > 0$ such that

$$
\begin{aligned}
R^2|(-\Delta)^s v_\varepsilon|'_{0,\alpha;B_R(x_0)} &= R^2\|(-\Delta)^s v_\varepsilon\|_{L^\infty(B_R(x_0))} + R^{2+\alpha}[(-\Delta)^s v_\varepsilon]_{\alpha;B_R(x_0)} \\
&\leqslant C_{n,s,\rho}\Big(R^2\|D^2 v_\varepsilon\|_{L^\infty(B_{2R}(x_0))} + \|v_\varepsilon\|_{L^\infty(B_{2R}(x_0))} \\
&\qquad + R^{2+\alpha_0}[D^2 v_\varepsilon]_{\alpha_0;B_{2R}(x_0)} + R\|Dv_\varepsilon\|_{L^\infty(B_{2R}(x_0))}\Big) \\
&= C_{n,s,\rho}\Big(R^2\|u_\varepsilon D^2\phi^R + 2Du_\varepsilon D\phi^R + \phi^R D^2 u_\varepsilon\|_{L^\infty(B_{2R}(x_0))} \\
&\qquad + \|\phi^R u_\varepsilon\|_{L^\infty(B_{2R}(x_0))} + R\|\phi^R Du_\varepsilon + u_\varepsilon D\phi^R\|_{L^\infty(B_{2R}(x_0))} \\
&\qquad + R^{2+\alpha_0}[u_\varepsilon D^2\phi^R + 2Du_\varepsilon D\phi^R + \phi^R D^2 u_\varepsilon]_{\alpha_0;B_{2R}(x_0)}\Big) \\
&\leqslant C_{n,s,\rho}|u_\varepsilon|'_{2,\alpha_0;B_{2R}(x_0)} \leqslant \delta|u_\varepsilon|'_{2,\alpha;B_{2R}(x_0)} + C_\delta\|u_\varepsilon\|_{L^\infty(B_{2R}(x_0))},
\end{aligned} \tag{5.7}
$$

where

$$
\alpha_0 = \begin{cases} 0 & \alpha < 2-2s \\ \alpha-(1-s) & \alpha \geqslant 2-2s. \end{cases}
$$

Thus, combining [GT01, Theorem 4.6] and (5.2.1), one has for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$
\begin{aligned}
|v_\varepsilon|'_{2,\alpha;B_{R/2}(x_0)} &\leqslant C_{n,\alpha}\left(\|v_\varepsilon\|_{L^\infty(B_R(x_0))} + R^2\left(|\psi_\varepsilon|'_{0,\alpha;B_R(x_0)} + |(-\Delta)^s v_\varepsilon|'_{0,\alpha;B_R(x_0)}\right)\right) \\
&\leqslant C_{n,\alpha,\rho,\alpha}\Big(\|v_\varepsilon\|_{L^\infty(B_R(x_0))} + R^2|\psi_\varepsilon|'_{0,\alpha;B_R(x_0)} \\
&\qquad + \delta|u_\varepsilon|'_{2,\alpha;B_{2R}(x_0)} + C_\delta\|u_\varepsilon\|_{L^\infty(B_{2R}(x_0))}\Big).
\end{aligned}
$$

Therefore, for every $x_0 \in V_{\frac{\rho}{4}}$, recalling the definition of v_ε and Proposition 4.1, and applying Proposition 5.1, we know that $\forall\delta > 0$, there exists C_δ such that

$$
\begin{aligned}
&|u_\varepsilon|'_{2,\alpha;B_{R/2}(x_0)} = |v_\varepsilon|'_{2,\alpha;B_{R/2}(x_0)} \\
&\leqslant C_{n,s,\alpha,\rho}\left(R^2|\psi_\varepsilon|'_{0,\alpha;B_R(x_0)} + \delta|u_\varepsilon|'_{2,\alpha;B_{2R}(x_0)} + C_\delta\|u\|_{L^\infty(B_{2R}(x_0))}\right) \\
&\leqslant C\Big(\|g\|_{C^\alpha(\overline{B}_{2R}(x_0)\times\mathbb{R}^n)}\left(1 + \|Du\|_{L^\infty(B_{2R}(x_0))}\right) + \|u\|_{L^\infty(\mathbb{R}^n)} \\
&\qquad + \delta|u_\varepsilon|'_{2,\alpha;B_{2R}(x_0)} + C_\delta\|u\|_{L^\infty(B_{2R}(x_0))}\Big) \\
&\leqslant C\left(\|g\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}}\times\mathbb{R}^n)}\left(1 + \|Du\|_{L^\infty(V_{\frac{3\rho}{4}})}\right) + C_\delta\|u\|_{L^\infty(\mathbb{R}^n)} + \delta|u_\varepsilon|'_{2,\alpha;B_{2R}(x_0)}\right),
\end{aligned} \tag{5.8}
$$

for every $R \in \left(0, \frac{\rho}{10}\right)$ and $\varepsilon \in (0, R)$, where the constant $C > 0$ depends only on n, s, α, ρ.

5.2.2. *Interior $C^{2,\alpha}$-regularity.* In light of the estimate (5.2.1), we can obtain the following Proposition 5.2 by adapting Proposition 4.3, which can guarantee the $C^{2,\alpha}$-norm of u_ε is uniformly bounded in ε in some ball.

Proposition 5.2. *Let $y \in \mathbb{R}^n, d > 0$, and $u \in C^{2,\alpha}(B_d(y))$. Suppose that for any $\delta > 0$, there exists $\Lambda_\delta > 0$ such that, for any $x \in B_d(y)$ and any $r \in (0, d - |x - y|]$, we have*

$$|u|'_{2,\alpha;B_{r/8}(x)} \leqslant \Lambda_\delta + \delta |u|'_{2,\alpha;B_{r/2}(x)}. \tag{5.9}$$

Then, there exist constants $\delta_0, C > 0$, depending only on n, α, d, such that

$$\|u\|_{C^{2,\alpha}(B_{d/8}(y))} \leqslant C\Lambda_{\delta_0}.$$

Proof of Theorem 1.6. First of all, one can choose $d = \rho/5$ in Proposition 5.2, so that

$$B_d(y) \subset V_{\rho/4} \quad \text{and} \quad d/4 < \frac{\rho}{10}, \quad \forall y \in V.$$

As a consequence, applying Proposition 5.2, and utilizing (5.2.1), we know that

$$\|u_\varepsilon\|_{C^{2,\alpha}(\overline{B}_{\rho/40}(y))} \leqslant C\left(\|g\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}} \times \mathbb{R}^n)}\left(1 + \|Du\|_{L^\infty(V_{\frac{3\rho}{4}})}\right) + C_{\delta_0}\|u\|_{L^\infty(\mathbb{R}^n)}\right)$$

for every $y \in V$, where the constant $C > 0$ depends on n, s, α, ρ.

Applying the Arzelà-Ascoli Theorem, we obtain that $u \in C^{2,\alpha}(\overline{B}_{\rho/40}(y))$. In particular, one has

$$\|u\|_{C^{2,\alpha}(\overline{B}_{\rho/40}(y))} \leqslant C\left(\|g\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}} \times \mathbb{R}^n)}\left(1 + \|Du\|_{L^\infty(V_{\frac{3\rho}{4}})}\right) + \|u\|_{L^\infty(\mathbb{R}^n)}\right),$$

for every $y \in V$, and for some constant $C > 0$ depending on n, s, α, ρ.

Using the cover argument again, and employing Theorem 1.5, we can obtain the desired result that

$$\begin{aligned}
\|u\|_{C^{2,\alpha}(\overline{V})} &\leqslant C\left(\|g\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}} \times \mathbb{R}^n)}\left(1 + \|Du\|_{L^\infty(V_{\frac{3\rho}{4}}))}\right) + \|u\|_{L^\infty(\mathbb{R}^n)}\right) \\
&\leqslant C\left(\|g\|_{C^\alpha(\overline{V}_{\frac{3\rho}{4}} \times \mathbb{R}^n)}\left(1 + \|g\|_{L^\infty(V_{\frac{7\rho}{8}} \times \mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right) + \|u\|_{L^\infty(\mathbb{R}^n)}\right) \\
&\leqslant C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{C^\alpha(\overline{V}_{\frac{7\rho}{8}} \times \mathbb{R}^n)}\right)\left(1 + \|g\|_{C^\alpha(\overline{V}_{\frac{7\rho}{8}} \times \mathbb{R}^n)}\right)
\end{aligned}$$

where the constant $C > 0$ depends on n, s, α, ρ. □

5.3. $C^{2,\alpha}$**-regularity up to the boundary.** The main ideas used here come from [BDVV, Theorem B.1]. We split the proof of Theorem 1.7 into the following two subsections:

Section 5.3.1: Estimate $\|(-\Delta)^s u\|_{C^\alpha(\overline{\Omega})}$.
Section 5.3.2: Apply the method of continuity to complete the proof of Theorem 1.7.

5.3.1. *Hölder estimate of the fractional Laplacian term.* Let $\alpha \in (0,1)$, we consider the function space $\mathcal{B}_\alpha(\Omega)$ defined as follows:

$$\mathcal{B}_\alpha(\Omega) := \left\{u \in C(\mathbb{R}^n) : u \equiv 0 \text{ in } \mathbb{R}^n \backslash \Omega \text{ and } u|_\Omega \in C^{2,\alpha}(\overline{\Omega})\right\}.$$

Lemma 5.5. *Let Ω be a $C^{2,\alpha}$ domain in $\mathbb{R}^n$, $u \in \mathcal{B}_\alpha(\Omega)$, $s \in (0, \frac{1}{2})$ and $\alpha + 2s < 1$.*
Then, $(-\Delta)^s u \in C^\alpha(\overline{\Omega})$ and

$$\|(-\Delta)^s u\|_{C^\alpha(\overline{\Omega})} \leqslant C(\|u\|_{C^{\alpha+2s}(\overline{\Omega})} + \|u\|_{C^2(\overline{\Omega})}),$$

where C depends only on n, s, α, Ω.

We split the proof of Lemma 5.5 into two steps:
Step 1. Estimate $[(-\Delta)^s u]_{C^\alpha(\overline{\Omega})}$ for every $s \in (0,1)$ (see Lemma 5.6).
Step 2. Estimate the L^∞-norm of $(-\Delta)^s u$ for every $s \in (0,1)$ (see Lemma 5.7).

Lemma 5.6. *Suppose that Ω is a $C^{2,\alpha}$ domain in $\mathbb{R}^n$, $u \in \mathcal{B}_\alpha(\Omega)$, $s \in (0, \frac{1}{2})$ and $\alpha + 2s \leqslant 1$.*
Then,

$$[(-\Delta)^s u]_{C^\alpha(\mathbb{R}^n)} \leqslant C[u]_{C^{\alpha+2s}(\overline{\Omega})},$$

where C depends only on n, s, α.

Proof. Let us estimate the difference $(-\Delta)^s u(x_1) - (-\Delta)^s u(x_2)$ for $x_1, x_2 \in \mathbb{R}^n$. For this, we note that

$$|(-\Delta)^s u(x_1) - (-\Delta)^s u(x_2)| \leqslant A_1 + A_2,$$

where

$$A_1 = \left| \int_{B(r)} \frac{u(x_1) - u(x_1+z) - u(x_2) + u(x_2+z)}{|z|^{n+2s}}\, dz \right|,$$
$$A_2 = \left| \int_{\mathbb{R}^n \setminus B(r)} \frac{u(x_1) - u(x_1+z) - u(x_2) + u(x_2+z)}{|z|^{n+2s}}\, dz \right|.$$

Case 1. Let $\beta := \alpha + 2s \leqslant 1$. For A_1, we use that $|u(x_i) - u(x_i+z)| \leqslant [u]_{C^\beta(\overline{\Omega})}|z|^\beta$ for $i = 1, 2$. Therefore,

$$A_1 \leqslant \int_{B(r)} \frac{2[u]_{C^\beta(\overline{\Omega})}|z|^\beta}{|z|^{n+2s}}\, dz \leqslant C(n,s,\alpha)[u]_{C^\beta(\overline{\Omega})} r^\alpha.$$

For A_2, we use that $|u(x_1) - u(x_2)| \leqslant [u]_{C^\beta(\overline{\Omega})}|x_1 - x_2|^\beta$ for $i = 1, 2$ As a consequence,

$$A_2 \leqslant \int_{\mathbb{R}^n \setminus B(r)} \frac{2[u]_{C^\beta(\overline{\Omega})}|x_1 - x_2|^\beta}{|z|^{n+2s}}\, dz \leqslant C(n,s)[u]_{C^\beta(\overline{\Omega})} r^{-2s}|x_1 - x_2|^\beta.$$

Taking $r = |x_1 - x_2|$ and adding A_1 and A_2, we obtain

$$[(-\Delta)^s u]_{C^\alpha(\mathbb{R}^n)} \leqslant C_{n,s,\alpha}[u]_{C^\beta(\overline{\Omega})},$$

where $C_{n,s,\alpha} > 0$ is a suitable constant. □

Lemma 5.7. *Let $\Omega \subset \mathbb{R}^n$ be a $C^{2,\alpha}$ domain and $u \in \mathcal{B}_\alpha(\Omega)$. Then, for every $s \in (0,1)$ we have that $(-\Delta)^s u \in L^\infty(\mathbb{R}^n)$ and*

$$\|(-\Delta)^s u\|_{L^\infty(\mathbb{R}^n)} \leqslant C\|u\|_{C^2(\overline{\Omega})},$$

where $C > 0$ depends only on n, s, Ω.

Proof. We choose $x_0 \in \Omega$ such that
$$d_0 =: \operatorname{dist}(x_0, \partial\Omega) = \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega).$$
Then, we have that
$$\begin{aligned}|(-\Delta)^s u(x_0)| &\leqslant \left|\int_{|z|<d_0} \frac{u(x_0+z)-u(x_0)}{|z|^{n+2s}}\,dx\right| + \left|\int_{|z|\geqslant d_0} \frac{u(x_0+z)-u(x_0)}{|z|^{n+2s}}\,dx\right| \\ &\leqslant \int_{|z|<d_0} \frac{\int_0^1 |\nabla u(x_0+tz)|\,dt}{|z|^{n+2s-1}}\,dx + 2\|u\|_{L^\infty(\Omega)} \int_{|z|\geqslant d_0} \frac{1}{|z|^{n+2s}}\,dx \qquad (5.1) \\ &\leqslant c_{n,s}(d_0^{1-2s} + d_0^{-2s})\|u\|_{C^1(\overline{\Omega})},\end{aligned}$$
where $c_{n,s} > 0$ is another constant independent of u.

Combining Lemma 5.6 with (5.3.1), for every $x \in \overline{\Omega}$, one has
$$\begin{aligned}|(-\Delta)^s u(x)| &\leqslant [(-\Delta)^s u]_{C^\alpha(\mathbb{R}^n)}|x-x_0|^\alpha + |(-\Delta)^s u(x_0)| \\ &\leqslant C_{n,s,\alpha} \operatorname{diam}(\Omega)^\alpha \|u\|_{C^{\alpha+2s}(\overline{\Omega})} + |(-\Delta)^s u(x_0)| \\ &\leqslant C_{n,s,\alpha}\|u\|_{C^{\alpha+2s}(\overline{\Omega})} + c_{n,s}(d_0^{1-2s} + d_0^{-2s})\|u\|_{C^1(\overline{\Omega})}.\end{aligned}$$
As a consequence of this and Lemma 5.6, we finally obtain
$$\|(-\Delta)^s u\|_{C^\alpha(\overline{\Omega})} = \|(-\Delta)^s u\|_{L^\infty(\overline{\Omega})} + [(-\Delta)^s u]_{C^\alpha(\overline{\Omega})} \leqslant C_{n,s,\Omega,\alpha}(\|u\|_{C^{\alpha+2s}(\overline{\Omega})} + \|u\|_{C^1(\overline{\Omega})}),$$
for some suitable $C_{n,s,\Omega,\alpha} > 0$. □

5.3.2. *Proof of the $C^{2,\alpha}$-regularity up to the boundary.* First of all, we establish the following facts:

Lemma 5.8. (1) *$\mathcal{L}u \in C^\alpha(\overline{\Omega})$ for every $u \in \mathcal{B}_\alpha(\Omega)$;*
(2) *there exists a constant $C = C(n, s, \alpha) > 0$ such that*
$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leqslant C\left(\|\mathcal{L}u\|_{C^\alpha(\overline{\Omega})} + \sup_{x\in\Omega}|u|\right) \qquad \forall u \in \mathcal{B}_\alpha(\Omega). \tag{5.2}$$

Proof. As regards (1), it follows directly from Lemma 5.5 and the fact that $\Delta u \in C^\alpha(\overline{\Omega})$ if $u \in \mathcal{B}_\alpha(\Omega)$. We now turn to prove (5.8).

To this end, we observe that $\partial\Omega$ is of class $C^{2,\alpha}$, which allows us to apply [GT01, Theorem 6.14]: there exists a constant $C = C(n, \alpha) > 0$ such that
$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leqslant C\left(\|\Delta u\|_{C^\alpha(\overline{\Omega})} + \sup_{x\in\Omega}|u|\right), \tag{5.3}$$
for every $u \in C^{2,\alpha}(\overline{\Omega})$ satisfying $u \equiv 0$ in $\mathbb{R}^n\setminus\Omega$.

Thus, combining Lemma 5.5 and (5.3.2), for every $u \in \mathcal{B}_\alpha(\Omega)$ one has
$$\begin{aligned}\|u\|_{C^{2,\alpha}(\overline{\Omega})} &\leqslant C\left(\|\Delta u\|_{C^\alpha(\overline{\Omega})} + \sup_{x\in\Omega}|u|\right) = C\left(\|\mathcal{L}u - (-\Delta)^s u\|_{C^\alpha(\overline{\Omega})} + \sup_{x\in\Omega}|u|\right) \\ &\leqslant C\left(\|\mathcal{L}u\|_{C^\alpha(\overline{\Omega})} + \|(-\Delta)^s u\|_{C^\alpha(\overline{\Omega})} + \sup_{x\in\Omega}|u|\right) \qquad (5.4) \\ &\leqslant \bar{C}\left(\|\mathcal{L}u\|_{C^\alpha(\overline{\Omega})} + \|u\|_{C^{\alpha+2s}(\overline{\Omega})} + \|u\|_{C^1(\overline{\Omega})} + \sup_{x\in\Omega}|u|\right).\end{aligned}$$

Owing again to the regularity of $\partial\Omega$, we can utilize the global interpolation inequality (see e.g. [GT01, Chapter 6]): there exists a constant $\theta = \theta(\bar{C}) > 0$ such that

$$\|u\|_{C^{\alpha+2s}(\overline{\Omega})} + \|u\|_{C^1(\overline{\Omega})} \leqslant \frac{1}{2\overline{C}}\|u\|_{C^{2,\alpha}(\overline{\Omega})} + \theta \sup_{x\in\Omega}|u|.$$

As a consequence of this and (5.3.2), we obtain the estimate in (5.8), as desired. □

We now use the method of continuity (See. e.g. [GT01, Theorem 5.2]) to complete the proof.

We first notice that $\mathcal{B}_\alpha(\Omega)$ is endowed with a structure of Banach space by the norm

$$\|u\|_{\mathcal{B}_\alpha(\Omega)} := \|u\|_{C^{2,\alpha}(\overline{\Omega})} \qquad \forall u \in \mathcal{B}_\alpha(\Omega).$$

Furthermore, for every $t \in [0,1]$, we set

$$\mathcal{L}_t = (1-t)(-\Delta) + t\mathcal{L} = -\Delta + t(-\Delta)^s.$$

Utilizing (1)-(2) in Lemma 5.8, one concludes that $\mathcal{L}_t$ maps $\mathcal{B}_\alpha(\Omega)$ to $C^\alpha(\overline{\Omega})$ and

$$\|u\|_{\mathcal{B}_\alpha(\Omega)} := \|u\|_{C^{2,\alpha}(\overline{\Omega})} \leqslant C\left(\|\mathcal{L}_t u\|_{C^\alpha(\overline{\Omega})} + \sup_{x\in\Omega}|u|\right) \qquad \forall u \in \mathcal{B}_\alpha(\Omega), \tag{5.5}$$

where $C > 0$ is a suitable constant independent of u and t.

Furthermore, using [BDVV22b, Theorem 4.7], it is not difficult to see that

$$\sup_{x\in\Omega}|u| = \sup_{x\in\mathbb{R}^n}|u| \leqslant \rho\|\mathcal{L}_t u\|_{L^\infty(\Omega)} \leqslant \rho\|\mathcal{L}_t u\|_{C^\alpha(\overline{\Omega})}, \tag{5.6}$$

where ρ is an another constant independent of u and t.

Due to (5.3.2) and (5.3.2), we can apply the method of continuity in the following settings:

(i) $\mathcal{B}_\alpha(\Omega)$ and $C^\alpha(\Omega)$ are Banach space;

(ii) $\mathcal{L}_0$ and $\mathcal{L}_1$ are linear and bounded from $\mathcal{B}_\alpha(\Omega)$ to $C^\alpha(\overline{\Omega})$;

(iii) There exists a constant $\hat{C} > 0$ such that

$$\|u\|_{\mathcal{B}_\alpha(\Omega)} \leqslant \hat{C}\|\mathcal{L}_t u\|_{C^\alpha(\overline{\Omega})} \qquad \forall u \in \mathcal{B}_\alpha(\Omega) \text{ and } t \in [0,1].$$

Since $\mathcal{L}_0 = -\Delta$ is surjective, one can deduce that $\mathcal{L}_1 = \mathcal{L}$ is also surjective: for every $f \in C^\alpha(\overline{\Omega})$ there exists unique $u \in \mathcal{B}_\alpha(\Omega)$ such that

$$\mathcal{L}u = f \qquad \text{a.e. in } \Omega.$$

We observe that $u \in C^{2,\alpha}(\overline{\Omega})$ and $u \equiv 0$ in $\mathbb{R}^n\backslash\Omega$.

Proof of Theorem 1.7. Let $u \in X_0^1$ be the weak solution of (1). Theorem 1.3 implies that the map $x \mapsto g(x, u(x))$ lies in $C^\alpha(\overline{\Omega})$.

Moreover, there exists unique solution $v \in \mathcal{B}_\alpha(\Omega) \subset X_0^1$. In addition, the Lax-Milgram Theorem yields that $u = v$, and in particular, from (5.8), we deduce that

$$\begin{aligned}
\|u\|_{C^{2,\alpha}(\overline{\Omega})} &\leqslant C\left(\|\mathcal{L}u\|_{C^\alpha(\overline{\Omega})} + \sup_{x\in\Omega}|u|\right) \leqslant C\left(\|g(\cdot,u(\cdot))\|_{C^\alpha(\overline{\Omega})} + \sup_{x\in\Omega}|u|\right)\\
&\leqslant C\left([g]_{C^\alpha(\overline{\Omega}\times\mathbb{R})}\left(1 + \|Du\|_{L^\infty(\Omega)}\right) + \|g\|_{L^\infty(\overline{\Omega}\times\mathbb{R}^n)} + \|u\|_{L^\infty(\Omega)}\right)\\
&\leqslant C\left(\|g\|_{C^\alpha(\overline{\Omega}\times\mathbb{R})}\left(1 + \|u\|_{C^1(\overline{\Omega})}\right) + \|u\|_{L^\infty(\Omega)}\right),
\end{aligned}$$

for some constant C depending only on n, s, α, Ω. Hence, the proof of Theorem 1.7 is completed. $\square$

Appendix A. The strong maximum principle

We collect here some auxiliary results which follow as a simple byproduct of our main theorems. These results are not stated in their full generality, but rather in a way which makes them directly utilizable in nonlinear analysis problems (in particular, we will use them in the present form in the article [SVWZ22a]).

First, we present a strong maximum principle (see [BDVV22b] for a weak maximum principle).

Theorem A.1 (Strong maximum principle)**.** *Let Ω be a bounded $C^{1,1}$ domain, u be a weak solution of* (1). *Assume that $g \in C^{\alpha}_{\text{loc}}(\Omega \times \mathbb{R})$ satisfies* (1.1) *for any given $\alpha \in (0, 1)$ and $g \geqslant 0$.*

Then, $u \in C^{2,\alpha}_{\text{loc}}(\Omega) \cap C(\mathbb{R}^n)$ for any $\alpha \in (0, 1)$ and furthermore

$$u > 0 \quad \text{in } \Omega.$$

Proof. We divide the proof into the following two steps.

Step 1. We first show that $u \geqslant 0$ in Ω. Since u is a nontrivial weak solution of (1), utilizing Theorems 1.1 and 1.4, one deduces that

$$u \in C^{2,\alpha}_{\text{loc}}(\Omega) \cap C(\mathbb{R}^n).$$

Moreover, due to $g(x, u) \geqslant 0$,

$$-\Delta u + (-\Delta)^s u \geqslant 0, \text{ a.e. in } \Omega \text{ and } u \equiv 0 \text{ in } \mathbb{R}^n \backslash \Omega.$$

Suppose by contradiction that there exists a point $\eta \in \Omega$ such that $u(\eta) < 0$. Since $u \in C(\mathbb{R}^n)$ and $\overline{\Omega}$ is compact, one can find $x_0 \in \overline{\Omega}$ such that

$$u(x_0) = \min_{\overline{\Omega}} u(x) < 0.$$

We notice that $x_0 \in \Omega$ because u is continuous and $u = 0$ in $\mathbb{R} \backslash \Omega$.

Thus, $\Delta u(x_0) \geqslant 0$ and so

$$\begin{aligned} 0 \leqslant \big(-\Delta + (-\Delta)^s\big)u(x_0) &= -\Delta u(x_0) + \int_{\mathbb{R}^n} \frac{u(x_0) - u(y)}{|x_0 - y|^{n+2s}}\, dy \\ &\leqslant \int_{\mathbb{R}^n} \frac{u(x_0) - u(y)}{|x_0 - y|^{n+2s}}\, dy \leqslant 0. \end{aligned} \tag{A.7}$$

Here, the last inequality holds due to the fact that $u(x_0) \leqslant u(x)$ for all $x \in \mathbb{R}^n$. Hence, we obtain that $u \equiv 0$ throughout $\mathbb{R}^n$, which is a contradiction.

Step 2. We now show that $u > 0$ in Ω. We argue by contradiction.

If this were not true, there would exist a point $\xi \in \Omega$ such that $u(\xi) = 0$, which is an interior minimum point for u in Ω. Replacing x_0 with ξ in (A), due to Step 1, one has $u \equiv 0$ throughout $\mathbb{R}^n$, which is a contradiction. $\square$

Appendix B. The principal eigenvalue problem

Now we give a characterization of the principal eigenvalue and eigenfunction.

Theorem B.1 (Variational properties for the principal eigenvalue). *Consider the eigenvalue problem*

$$\begin{cases} -\Delta u + (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{B.8}$$

Then,

(a) *We have*

$$\lambda_1 = \min \left\{ \|u\|^2_{X_0^1} \ \mid \ u \in X_0^1, \|u\|_{L^2(\Omega)} = 1 \right\} > 0.$$

(b) λ_1 *is simple.*
(c) *The first eigenfunction* φ_1 *is of* $C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ *class.*
(d) $\varphi_1 \geqslant 0$.

Furthermore, if Ω *is a* $C^{1,1}$ *domain, then*

(e) $\varphi_1 \in C^{2,\alpha}_{\mathrm{loc}}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.
(f) $\varphi_1 > 0$ *in* Ω *and* $\varphi_1 = 0$ *in* $\mathbb{R}^n \backslash \Omega$.

Proof. As for **(a)** and **(d)**, we set

$$I(u) := \|u\|^2_{X_0^1}, \qquad \mathcal{S} = \left\{ u \in X_0^1, \|u\|_{L^2(\Omega)} = 1 \right\}.$$

Let $\{u_j\} \subset \mathcal{S}$ be a minimizing sequence for I, that is

$$\lim_{j\to\infty} I(u_j) = \inf_{\mathcal{S}} \ \|u\|^2_{X_0^1} \geqslant 0.$$

Then $\{u_j\}$ is bounded in X_0^1. Up to a subsequence, still defined by $\{u_j\}$, there exists $u^* \in X_0^1$ such that

$$u_j \rightharpoonup u^* \text{ in } X_0^1;$$
$$u_j \to u^* \text{ in } L^2(\Omega).$$

Since $\|u_j\|_{L^2(\Omega)} = 1$, we know that $\|u^*\|_{L^2(\Omega)} = 1$ and $u^* \in \mathcal{S}$. According to the weak lower semi-continuity of the norm, one has that

$$\lim_{j\to\infty} I(u_j) \geqslant I(u^*) \geqslant \inf_{\mathcal{S}} \ \|u\|^2_{X_0^1},$$

which implies that $I(u^*) = \inf_{\mathcal{S}} \ \|u\|^2_{X_0^1}$.

We claim that for any $v \in X_0^1$,

$$\langle u^*, v \rangle_{X_0^1} = I(u^*) \int_\Omega u^* v \, dx. \tag{B.9}$$

In fact, let $t \in (-1,1)$, $u_t = \frac{u^*+tv}{\|u^*+tv\|_2}$, then $u_t \in \mathcal{S}$ and one has

$$\begin{aligned} I(u_t) &= \frac{\|u^*\|^2_{X_0^1} + 2t\langle u^*, v\rangle_{X_0^1} + t^2\|v\|^2_{X_0^1}}{1 + 2t\int_\Omega u^* v\,dx + t^2|v|_2^2} \\ &= \frac{\|u^*\|^2_{X_0^1} + 2t\langle u^*, v\rangle_{X_0^1} + t^2\|v\|^2_{X_0^1}}{1 - 4t^2(\int_\Omega u^* v\,dx)^2 + 2t^2|v|_2^2 + t^4|v|_2^4}\left(1 - 2t\int_\Omega u^* v\,dx + t^2|v|_2^2\right) \\ &\leqslant \frac{\|u^*\|^2_{X_0^1} + 2t\langle u^*, v\rangle_{X_0^1} + t^2\|v\|^2_{X_0^1}}{(1 - t^2|v|_2^2)^2}\left(1 - 2t\int_\Omega u^* v\,dx + t^2|v|_2^2\right) \\ &= \frac{1}{(1 - t^2|v|_2^2)^2}\left(I(u^*) + 2t\big(\langle u^*, v\rangle_{X_0^1} - I(u^*)\int_\Omega u^* v\,dx\big) + P(t^2)\right). \end{aligned}$$

If there existed $v \in X_0^1$ such that

$$\langle u^*, v\rangle_{X_0^1} - I(u^*)\int_\Omega u^* v\,dx = l > 0,$$

then for $t < 0$ small enough,

$$I(u_t) \leqslant \frac{1}{(1 - t^2|v|_2^2)^2}\Big(I(u^*) + 2tl + P(t^2)\Big) < I(u^*),$$

which would contradict the definition of u^*. We can use the same argument to verify that the case of $l < 0$ cannot hold as well, thus deducing that (B) is valid.

As a consequence of (B), we infer that λ_1 is attained at some $\varphi_1 \in X_0^1$, with $\|\varphi_1\|_{L^2(\Omega)} = 1$ and $I(\varphi_1) = \lambda_1$.

In order to prove that $\varphi_1 \geqslant 0$ a.e. in Ω, we first need to show that if φ_1 is an eigenfunction related to λ_1, then both φ_1 and $|\varphi_1|$ realize the minimum in the definition of λ_1.

Indeed, let $x_1 \in \{x \in \mathbb{R}^n, \varphi_1 > 0\}$ and $x_2 \in \{x \in \mathbb{R}^n, \varphi_1 < 0\}$. Then,

$$||\varphi_1(x_1)| - |\varphi_1(x_2)|| = |\varphi_1(x_1) + \varphi_1(x_2)| < \varphi_1(x_1) - \varphi_1(x_2) = |\varphi_1(x_1) - \varphi_1(x_2)|. \tag{B.10}$$

If both the sets $\{x \in \mathbb{R}^n, \varphi_1 > 0\}$ and $\{x \in \mathbb{R}^n, \varphi_1 < 0\}$ have positive measure, exploiting (B), we get $I(|\varphi_1|) < I(\varphi_1) = \lambda_1$, whence, since $|\varphi_1| \in \mathcal{S}$, we have that $I(|\varphi_1|) = I(\varphi_1) = \lambda_1$, which implies that $\varphi_1 \geqslant 0$ or $\varphi_1 \leqslant 0$ a.e. in Ω. By replacing φ_1 by $|\varphi_1|$, we may conclude that $\varphi_1 \geqslant 0$.

Next, we turn to prove **(b)**. Suppose by contradiction that there exists another eigenfunction $\varphi_2 \in X_0^1$, $\varphi_1 \neq \varphi_2$, corresponding to λ_1. Setting $\bar{\varphi}_2 = \frac{\varphi_2}{\|\varphi_2\|_{L^2(\Omega)}}$ and $u_1 = \varphi_1 - \bar{\varphi}_2$.

We claim that $u_1 = 0$ a.e. in Ω. In fact, it is easy to see that u_1 is also the eigenfunction related to λ_1, so we deduce $u_1 \leqslant 0$ or $u_1 \geqslant 0$ a.e. in Ω, that is, $\varphi_1 \leqslant \bar{\varphi}_2$ or $\varphi_1 \geqslant \bar{\varphi}_2$ a.e. in Ω, which implies that ${\varphi_1}^2 \leqslant \bar{\varphi}_2^2$ or $\varphi_1^2 \geqslant \bar{\varphi}_2^2$. However, $\|\varphi_1\|_{L^2(\Omega)} = 1 = \|\bar{\varphi}_2\|_{L^2(\Omega)}$, we get $\varphi_1^2 = \bar{\varphi}_2^2$, hence $u_1 = 0$ a.e. in Ω. Therefore, if $\varphi \in X_0$ is an eigenfunction related to λ_1, then $\varphi = |\varphi|_2\varphi_1$.

Finally, as for **(c)** **(e)** and **(f)**, since φ_1 is a weak solution of (B.1) and satisfies the assumption (1.1), applying Theorems 1.1 and 1.3, one has $\varphi_1 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Moreover, it follows from Theorem 1.4 that $\varphi_1 \in C^{2,\alpha}_{\mathrm{loc}}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$. As a consequence of this and **(a)**, employing the strong maximum principle, we obtain that $\varphi_1 > 0$. $\square$

References

[AC21] Nicola Abatangelo and Matteo Cozzi. An elliptic boundary value problem with fractional nonlinearity. *SIAM J. Math. Anal.*, 53(3):3577–3601, 2021.

[BCCI12] Guy Barles, Emmanuel Chasseigne, Adina Ciomaga, and Cyril Imbert. Lipschitz regularity of solutions for mixed integro-differential equations. *J. Differential Equations*, 252(11):6012–6060, 2012.

[BDVV] Stefano Biagi, Serena Dipierro, Enrico Valdinoci, and Eugenio Vecchi. A Faber-Krahn inequality for mixed local and nonlocal operators. *J. Anal. Math.*

[BDVV22a] Stefano Biagi, Serena Dipierro, Enrico Valdinoci, and Eugenio Vecchi. A Brezis-Nirenberg type result for mixed local and nonlocal operators. 2022. Preprint.

[BDVV22b] Stefano Biagi, Serena Dipierro, Enrico Valdinoci, and Eugenio Vecchi. Mixed local and nonlocal elliptic operators: regularity and maximum principles. *Comm. Partial Differential Equations*, 47(3):585–629, 2022.

[BDVV23] Stefano Biagi, Serena Dipierro, Enrico Valdinoci, and Eugenio Vecchi. A Hong-Krahn-Szegö inequality for mixed local and nonlocal operators. *Math. Eng.*, 5(1):Paper No. 014, 25, 2023.

[BFV14] Begoña Barrios, Alessio Figalli, and Enrico Valdinoci. Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 13(3):609–639, 2014.

[BI08] Guy Barles and Cyril Imbert. Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 25(3):567–585, 2008.

[BJK10] Imran H. Biswas, Espen R. Jakobsen, and Kenneth H. Karlsen. Viscosity solutions for a system of integro-PDEs and connections to optimal switching and control of jump-diffusion processes. *Appl. Math. Optim.*, 62(1):47–80, 2010.

[BVDV21] Stefano Biagi, Eugenio Vecchi, Serena Dipierro, and Enrico Valdinoci. Semilinear elliptic equations involving mixed local and nonlocal operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 151(5):1611–1641, 2021.

[DNPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[DPFR19] Leandro M. Del Pezzo, Raúl Ferreira, and Julio D. Rossi. Eigenvalues for a combination between local and nonlocal p-Laplacians. *Fract. Calc. Appl. Anal.*, 22(5):1414–1436, 2019.

[DPLV22] Serena Dipierro, Edoardo Proietti Lippi, and Enrico Valdinoci. Linear theory for a mixed operator with Neumann conditions. *Asymptot. Anal.*, 128(4):571–594, 2022.

[DV21] Serena Dipierro and Enrico Valdinoci. Description of an ecological niche for a mixed local/nonlocal dispersal: an evolution equation and a new Neumann condition arising from the superposition of Brownian and Lévy processes. *Phys. A*, 575:Paper No. 126052, 20, 2021.

[FRRO22] Xavier Fernández-Real and Xavier Ros-Oton. *Regularity theory for elliptic PDE*, volume 28 of *Zurich Lectures in Advanced Mathematics*. EMS Press, Berlin, [2022] ©2022.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[JK05] Espen R. Jakobsen and Kenneth H. Karlsen. Continuous dependence estimates for viscosity solutions of integro-PDEs. *J. Differential Equations*, 212(2):278–318, 2005.

[MPV13] Eugenio Montefusco, Benedetta Pellacci, and Gianmaria Verzini. Fractional diffusion with Neumann boundary conditions: the logistic equation. *Discrete Contin. Dyn. Syst. Ser. B*, 18(8):2175–2202, 2013.

[PV18] Benedetta Pellacci and Gianmaria Verzini. Best dispersal strategies in spatially heterogeneous environments: optimization of the principal eigenvalue for indefinite fractional Neumann problems. *J. Math. Biol.*, 76(6):1357–1386, 2018.

[SVWZ22a] Xifeng Su, Enrico Valdinoci, Yuanhong Wei, and Jiwen Zhang. Multiple solutions for mixed local and nonlocal elliptic equations arising from the Lévy type processes. 2022. Preprint.

[SVWZ22b] Xifeng Su, Enrico Valdinoci, Yuanhong Wei, and Jiwen Zhang. Regularity results for solutions of mixed local and nonlocal elliptic equations. *Math. Z.*, 302(3):1855–1878, 2022.

School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems (Ministry of Education), Beijing Normal University, No. 19, XinJieKouWai St., HaiDian District, Beijing 100875, P. R. China

Email address: `xfsu@bnu.edu.cn, billy3492@gmail.com`

Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, WA 6009 Crawley, Australia

Email address: `enrico.valdinoci@uwa.edu.au`

School of Mathematics, Jilin University, No. 2699, Qianjin St., Changchun 130012, P. R. China

Email address: `weiyuanhong@jlu.edu.cn`

School of Mathematical Sciences, Beijing Normal University, No. 19, XinJieKouWai St., HaiDian District, Beijing 100875, P. R. China

Email address: `jwzhang628@mail.bnu.edu.cn`