SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON-FREDHOLM OPERATORS WITH THE DRIFT AND BI-LAPLACIAN

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Abstract: We study the solvability of some linear nonhomogeneous elliptic equations and establish that under certain technical conditions the convergence in L^2 of their right sides yields the existence and the convergence in H^4 of the solutions. The problems involve the fourth order differential operators with or without the Fredholm property, particularly the fourth derivative operator, on the whole real line or on a finite interval with periodic boundary conditions. We demonstrate that the transport term contained in these equations provides the regularization of the solutions.

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1. Introduction

Consider the equation

$$(-\Delta + V(x))u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant, and the function $V(x)$ is decaying to 0 at infinity. If $a \geq 0$, then the essential spectrum of the operator $A : E \to F$ corresponding to the left side of problem (1.1) contains the origin. Consequently, such operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work is devoted to the studies of the certain properties of the operators of this kind. Note that the elliptic problems containing the non-Fredholm operators were treated extensively in recent years (see [10], [11], [12], [13], [18], [19], [21], [22], [23], [24], [25], [26], [27], also [3]) along with their potential applications to the theory of reaction-diffusion equations (see [7], [8]). Fredholm structures, topological invariants and their applications were covered in

[9]. The works [14] and [17] deal with the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of the second order and of the operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in the second-order quasilinear elliptic systems of equations were discussed in [15]. Particularly, when $a = 0$, our operator A satisfies the Fredholm property in the certain properly chosen weighted spaces (see [1], [2], [3], [5], [6]). However, the case when a is nontrivial is significantly different and the approach developed in these articles cannot be applied.

One of the important questions about the equations with non-Fredholm operators is their solvability. We address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, so that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Because the operator A fails to satisfy the Fredholm property, the sequence u_n may not be convergent. We call a sequence u_n, so that $Au_n \to f$ in $L^2(\mathbb{R}^d)$ a solution in the sense of sequences of the equation $Au = f$ (see [18]). If such sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this problem. The solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of non-Fredholm operators this convergence may not hold or it can occur in some weaker sense. In such case, the solution in the sense of sequences may not imply the existence of the usual solution. In this article we will find the sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, we will determine the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent.

In the first part of the article we consider the equation with the transport term

$$\frac{d^4u}{dx^4} - b\frac{du}{dx} - au = f(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $a \geq 0$ and $b \in \mathbb{R}, \ b \neq 0$ are the constants and the right side is square integrable. The problem with the drift in the context of the Darcy's law describing the fluid motion in the porous medium was discussed in [24]. The transport term is crucial when studying the emergence and propagation of patterns arising in the theory of speciation (see [20]). Nonlinear propagation phenomena for the reaction-diffusion type equations containing the drift term was investigated in [4]. Existence of solutions for certain non-Fredholm integro-differential equations with the bi-Laplacian was considered in [26]. The article [11] deals with the solvability in the sense of sequences for some fourth order non-Fredholm operators. The equation analogous to (1.2) but without the drift term was studied in [27]. But in [27] it was assumed that the constant $a > 0$ since making a trivial there leads to the more singular situation. Clearly, the operator contained in the left side of (1.2)

$$L_{a,\ b} := \frac{d^4}{dx^4} - b\frac{d}{dx} - a : \quad H^4(\mathbb{R}) \to L^2(\mathbb{R}) \tag{1.3}$$

is non-selfadjoint. By means of the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ipx} dx, \quad p \in \mathbb{R} \tag{1.4}$$

it can be easily derived that the essential spectrum of the operator $L_{a,\ b}$ is given by

$$\lambda_{a,\ b}(p) := p^4 - a - ibp, \quad p \in \mathbb{R}.$$

Evidently, if $a > 0$ our operator $L_{a,\ b}$ is Fredholm, because the origin does not belong to its essential spectrum. But when a vanishes, the operator $L_{0,\ b}$ does not satisfy the Fredholm property since its essential spectrum contains the origin. Clearly, in the absence of the transport term we are dealing with the self-adjoint operator

$$\frac{d^4}{dx^4} - a : \quad H^4(\mathbb{R}) \to L^2(\mathbb{R}), \quad a > 0,$$

which is non-Fredholm (see [27]). Let us write down the corresponding sequence of approximate equations with $m \in \mathbb{N}$, namely

$$\frac{d^4 u_m}{dx^4} - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in \mathbb{R}, \tag{1.5}$$

where $a \geq 0$ and $b \in \mathbb{R},\ b \neq 0$ are the constants. The right sides of (1.5) tend to the right side of (1.2) in $L^2(\mathbb{R})$ as $m \to \infty$. We define the inner product of two functions

$$(f(x), g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\bar{g}(x) dx, \tag{1.6}$$

with a slight abuse of notations when these functions do not belong to $L^2(\mathbb{R})$. Indeed, if $f(x) \in L^1(\mathbb{R})$ and $g(x) \in L^\infty(\mathbb{R})$, then obviously the integral considered above is well defined, like for example in the case of the functions contained in the orthogonality relations (1.8) and (1.9) of Theorems 1.1 and 1.2 below. For our equation (1.2) on the finite interval $I := [0, 2\pi]$ with periodic boundary conditions (see (1.14)), we will use the inner product analogical to (1.6), replacing the real line with I. In the first part of the present article we will consider the space $H^4(\mathbb{R})$ equipped with the norm

$$\|u\|^2_{H^4(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \left\| \frac{d^4 u}{dx^4} \right\|^2_{L^2(\mathbb{R})}. \tag{1.7}$$

When dealing with the norm $H^4(I)$ later on, we will replace $\mathbb{R}$ with I in formula (1.7). Our first main result is as follows.

Theorem 1.1. *Let the constants* $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ *and* $f(x) \in L^2(\mathbb{R})$.

a) If $a > 0$, *then equation (1.2) admits a unique solution* $u(x) \in H^4(\mathbb{R})$.

b) If $a = 0$ *and* $xf(x) \in L^1(\mathbb{R})$, *then problem (1.2) possesses a unique solution* $u(x) \in H^4(\mathbb{R})$ *if and only if the orthogonality condition*

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{1.8}$$

is valid.

Evidently, the expression in the left side of (1.8) is well defined by virtue of the simple argument analogical to the proof of Fact 1 of [22]. Note that the argument of the case a) of the theorem above does not rely on the orthogonality conditions, as distinct from the analogous situation without a drift term described in [27]. Therefore, the introduction of the drift term provides the regularization for the solutions of our equations. Our next statement is about the solvability in the sense of sequences for our equation on whole the real line.

Theorem 1.2. *Let the constants* $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ *and* $m \in \mathbb{N},\ f_m(x) \in L^2(\mathbb{R})$, *so that* $f_m(x) \to f(x)$ *in* $L^2(\mathbb{R})$ *as* $m \to \infty$.

a) If $a > 0$, *then equations (1.2) and (1.5) have unique solutions* $u(x) \in H^4(\mathbb{R})$ *and* $u_m(x) \in H^4(\mathbb{R})$ *respectively, such that* $u_m(x) \to u(x)$ *in* $H^4(\mathbb{R})$ *as* $m \to \infty$.

b) If $a = 0$, *let* $xf_m(x) \in L^1(\mathbb{R})$, *so that* $xf_m(x) \to xf(x)$ *in* $L^1(\mathbb{R})$ *as* $m \to \infty$. *Furthermore,*

$$(f_m(x), 1)_{L^2(\mathbb{R})} = 0, \quad m \in \mathbb{N} \tag{1.9}$$

is valid. Then problems (1.2) and (1.5) admit unique solutions $u(x) \in H^4(\mathbb{R})$ *and* $u_m(x) \in H^4(\mathbb{R})$ *respectively, such that* $u_m(x) \to u(x)$ *in* $H^4(\mathbb{R})$ *as* $m \to \infty$.

The second part of our work is devoted to the studies of our equation on the finite interval with the periodic boundary conditions (see (1.14)), i.e. $I := [0,\ 2\pi]$, namely

$$\frac{d^4u}{dx^4} - b\frac{du}{dx} - au = f(x), \quad x \in I, \tag{1.10}$$

where $a \geq 0$ and $b \in \mathbb{R},\ b \neq 0$ are the constants and the right side of (1.10) is continuous and periodic. Clearly,

$$\|f\|_{L^1(I)} \leq 2\pi\|f\|_{C(I)} < \infty, \quad \|f\|_{L^2(I)} \leq \sqrt{2\pi}\|f\|_{C(I)} < \infty. \tag{1.11}$$

Thus, $f(x) \in L^2(I)$ as well. We use the Fourier transform

$$f_n := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)e^{-inx}dx, \quad n \in \mathbb{Z}, \tag{1.12}$$

so that

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{inx}}{\sqrt{2\pi}}.$$

Evidently, the non-selfadjoint operator contained in the left side of (1.10)

$$\mathcal{L}_{a,\, b} := \frac{d^4}{dx^4} - b\frac{d}{dx} - a : \quad H^4(I) \to L^2(I) \tag{1.13}$$

is Fredholm. By means of (1.12), it can be trivially checked that the spectrum of $\mathcal{L}_{a,\, b}$ is given by

$$\lambda_{a,\, b}(n) := n^4 - a - ibn, \quad n \in \mathbb{Z}$$

and the corresponding eigenfunctions are the Fourier harmonics $\dfrac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$. The eigenvalues of the operator $\mathcal{L}_{a,\, b}$ are simple, as distinct from the situation without the transport term, when the eigenvalues corresponding to $n \neq 0$ are two fold degenerate. The appropriate function space here $H^4(I)$ is given by

$$\{u(x) : I \to \mathbb{C} \mid u(x), u''''(x) \in L^2(I), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

$$u''(0) = u''(2\pi), \quad u'''(0) = u'''(2\pi)\}. \tag{1.14}$$

For the technical purposes, we use the following auxiliary constrained subspace

$$H_0^4(I) = \{u(x) \in H^4(I) \mid (u(x), 1)_{L^2(I)} = 0\}, \tag{1.15}$$

which is a Hilbert spaces as well (see e.g. Chapter 2.1 of [16]). Clearly, if $a > 0$, the kernel of the operator $\mathcal{L}_{a,\, b}$ is trivial. If $a = 0$, we consider

$$\mathcal{L}_{0,\, b} : \quad H_0^4(I) \to L^2(I).$$

Evidently, such operator has the trivial kernel as well. We write down the corresponding sequence of the approximate equations with $m \in \mathbb{N}$, namely

$$\frac{d^4 u_m}{dx^4} - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in I, \tag{1.16}$$

where $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ are the constants. The right sides of (1.16) are continuous, periodic and converge to the right side of (1.10) in $C(I)$ as $m \to \infty$. The goal of Theorems 1.3 and 1.4 below is to demonstrate the formal similarity of the results on the finite interval with periodic boundary conditions to the ones obtained for the whole real line case in Theorems 1.1 and 1.2 above.

Theorem 1.3. *Let the constants* $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ *and* $f(0) = f(2\pi),\ f(x) \in C(I)$.

a) If $a > 0$*, then equation (1.10) admits a unique solution* $u(x) \in H^4(I)$.

b) If $a = 0$, then problem (1.10) possesses a unique solution $u(x) \in H_0^4(I)$ if and only if the orthogonality condition

$$(f(x), 1)_{L^2(I)} = 0 \tag{1.17}$$

is valid.

The final main proposition of the article is devoted to the solvability in the sense of sequences for our equation on the finite interval I.

Theorem 1.4. *Let the constants $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ and $m \in \mathbb{N}$, so that $f_m(0) = f_m(2\pi)$. Furthermore, $f_m(x) \in C(I)$ and $f_m(x) \to f(x)$ in $C(I)$ as $m \to \infty$.*

a) If $a > 0$, then equations (1.10) and (1.16) have unique solutions $u(x) \in H^4(I)$ and $u_m(x) \in H^4(I)$ respectively, so that $u_m(x) \to u(x)$ in $H^4(I)$ as $m \to \infty$.

b) If $a = 0$, let

$$(f_m(x), 1)_{L^2(I)} = 0, \quad m \in \mathbb{N}. \tag{1.18}$$

Then problems (1.10) and (1.16) admit unique solutions $u(x) \in H_0^4(I)$ and $u_m(x) \in H_0^4(I)$ respectively, so that $u_m(x) \to u(x)$ in $H_0^4(I)$ as $m \to \infty$.

Note that in the cases a) of Theorems 1.3 and 1.4 above the argument does not rely on the orthogonality relations. When there are no transport terms in our problems, the case is more singular (see formulas (3.1) and (3.7) further down with $a = n_0^4,\ n_0 \in \mathbb{N}$).

2. The whole real line case

Proof of Theorem 1.1. Let us first demonstrate that it would be sufficient to solve our equation in $L^2(\mathbb{R})$. Indeed, if $u(x)$ is a square integrable solution of (1.2) on the whole real line, directly from this equation under the given conditions we derive that

$$\frac{d^4u}{dx^4} - b\frac{du}{dx} \in L^2(\mathbb{R})$$

as well. By virtue of the standard Fourier transform (1.4), we have $(p^4 - ibp)\widehat{u}(p) \in L^2(\mathbb{R})$, so that $\int_{-\infty}^{\infty} p^8 |\widehat{u}(p)|^2 dp < \infty$. Hence, $\frac{d^4u}{dx^4} \in L^2(\mathbb{R})$, such that $u(x) \in H^4(\mathbb{R})$ as well.

To establish the uniqueness of solutions of (1.2), we suppose that $u_1(x),\ u_2(x) \in H^4(\mathbb{R})$ satisfy (1.2). Then their difference $w(x) := u_1(x) - u_2(x) \in H^4(\mathbb{R})$ solves the homogeneous problem

$$\frac{d^4w}{dx^4} - b\frac{dw}{dx} - aw = 0.$$

Since the operator $L_{a,\,b}$ introduced in (1.3) does not possess any nontrivial zero modes in $H^4(\mathbb{R})$, the function $w(x)$ is trivial on the real line.
We apply the standard Fourier transform (1.4) to both sides of equation (1.2). This yields

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{p^4 - a - ibp}. \tag{2.1}$$

Thus,

$$\|u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{|\widehat{f}(p)|^2}{(p^4 - a)^2 + b^2 p^2} dp. \tag{2.2}$$

Let us first discuss the case a) of our theorem. Formula (2.2) gives us

$$\|u\|^2_{L^2(\mathbb{R})} \le \frac{1}{C} \|f\|^2_{L^2(\mathbb{R})} < \infty$$

as assumed. Here and further down C will designate a finite, positive constant.
Then we turn our attention to the situation when the parameter a vanishes. From (2.1), we easily express

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{p^4 - ibp} \chi_{\{|p| \le 1\}} + \frac{\widehat{f}(p)}{p^4 - ibp} \chi_{\{|p| > 1\}}. \tag{2.3}$$

Here and below χ_A will denote the characteristic function of a set $A \subseteq \mathbb{R}$. Evidently, the second term in the right side of (2.3) can be estimated from above in the absolute value by $\dfrac{|\widehat{f}(p)|}{|b|} \in L^2(\mathbb{R})$ because $f(x)$ is square integrable on the whole real line via our assumption. Let us write

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^p \frac{d\widehat{f}(s)}{ds} ds.$$

Hence, the first term in the right side of (2.3) can be expressed as

$$\frac{\widehat{f}(0)}{p^4 - ibp} \chi_{\{|p| \le 1\}} + \frac{\int_0^p \frac{d\widehat{f}(s)}{ds} ds}{p^4 - ibp} \chi_{\{|p| \le 1\}}. \tag{2.4}$$

By means of definition (1.4) of the standard Fourier transform, we easily obtain

$$\left| \frac{d\widehat{f}(p)}{dp} \right| \le \frac{1}{\sqrt{2\pi}} \|xf(x)\|_{L^1(\mathbb{R})}.$$

Thus, the second term in (2.4) can be bounded from above in the absolute value by

$$\frac{1}{\sqrt{2\pi}} \frac{\|xf(x)\|_{L^1(\mathbb{R})}}{|b|} \chi_{\{|p| \le 1\}} \in L^2(\mathbb{R}).$$

Clearly, the first term in (2.4) is contained in $L^2(\mathbb{R})$ if and only if $\widehat{f}(0)$ is trivial. This is equivalent to orthogonality relation (1.8). ∎

Let us proceed to establishing the solvability in the sense of sequences for our problem on whole the real line.

Proof of Theorem 1.2. First we suppose that equations (1.2) and (1.5) admit unique solutions $u(x) \in H^4(\mathbb{R})$ and $u_m(x) \in H^4(\mathbb{R}), \ m \in \mathbb{N}$ respectively, so that $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. This will imply that $u_m(x)$ also converges to $u(x)$ in $H^4(\mathbb{R})$ as $m \to \infty$. Clearly, from (1.2) and (1.5) we easily obtain that

$$\left\| \frac{d^4}{dx^4}(u_m - u) - b\frac{d(u_m - u)}{dx} \right\|_{L^2(\mathbb{R})} \le \|f_m - f\|_{L^2(\mathbb{R})} + a\|u_m - u\|_{L^2(\mathbb{R})}. \quad (2.5)$$

The right side of (2.5) tends to zero as $m \to \infty$ due to our assumptions. By means of the standard Fourier transform (1.4), we easily derive that

$$\int_{-\infty}^{\infty} p^8 |\widehat{u}_m(p) - \widehat{u}(p)|^2 dp \to 0, \quad m \to \infty.$$

Thus, $\dfrac{d^4 u_m}{dx^4} \to \dfrac{d^4 u}{dx^4}$ in $L^2(\mathbb{R})$ as $m \to \infty$. Therefore, $u_m(x) \to u(x)$ in $H^4(\mathbb{R})$ as $m \to \infty$ as well. We apply the standard Fourier transform (1.4) to both sides of (1.5). This gives us

$$\widehat{u}_m(p) = \frac{\widehat{f}_m(p)}{p^4 - a - ibp}, \quad m \in \mathbb{N}. \quad (2.6)$$

Let us first discuss the case a) of our theorem. By means of the part a) of Theorem 1.1, equations (1.2) and (1.5) admit unique solutions $u(x) \in H^4(\mathbb{R})$ and $u_m(x) \in H^4(\mathbb{R}), \ m \in \mathbb{N}$ respectively. By virtue of (2.6) along with (2.1), we obtain that

$$\|u_m - u\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} \frac{|\widehat{f}_m(p) - \widehat{f}(p)|^2}{(p^4 - a)^2 + b^2 p^2} dp.$$

Hence,

$$\|u_m - u\|_{L^2(\mathbb{R})} \le \frac{1}{C}\|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty$$

as assumed. Therefore, in the situation when $a > 0$ we have $u_m(x) \to u(x)$ in $H^4(\mathbb{R})$ as $m \to \infty$ by means of the argument above.

Let us conclude the proof of the theorem by treating the case when the parameter a is trivial. According to the result of the part a) of Lemma 3.3 of [21], under the stated assumptions

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \quad (2.7)$$

is valid. Then by virtue of the part b) of Theorem 1.1, problems (1.2) and (1.5) possess unique solutions $u(x) \in H^4(\mathbb{R})$ and $u_m(x) \in H^4(\mathbb{R}),\ m \in \mathbb{N}$ respectively if $a = 0$. Using (2.6) and (2.1), we derive

$$\widehat{u}_m(p) - \widehat{u}(p) = \frac{\widehat{f}_m(p) - \widehat{f}(p)}{p^4 - ibp}\chi_{\{|p|\le 1\}} + \frac{\widehat{f}_m(p) - \widehat{f}(p)}{p^4 - ibp}\chi_{\{|p|>1\}}. \tag{2.8}$$

Obviously, the second term in the right side of (2.8) can be bounded from above in the absolute value by $\dfrac{|\widehat{f}_m(p) - \widehat{f}(p)|}{|b|}$, so that

$$\left\| \frac{\widehat{f}_m(p) - \widehat{f}(p)}{p^4 - ibp}\chi_{\{|p|>1\}} \right\|_{L^2(\mathbb{R})} \le \frac{1}{|b|}\|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty$$

as assumed. Orthogonality conditions (2.7) and (1.9) give us

$$\widehat{f}(0) = 0, \quad \widehat{f}_m(0) = 0, \quad m \in \mathbb{N}.$$

Then we express

$$\widehat{f}(p) = \int_0^p \frac{d\widehat{f}(s)}{ds}ds, \quad \widehat{f}_m(p) = \int_0^p \frac{d\widehat{f}_m(s)}{ds}ds, \quad m \in \mathbb{N}, \tag{2.9}$$

so that it remains to estimate the norm of the term

$$\frac{\int_0^p [\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}]ds}{p^4 - ibp}\chi_{\{|p|\le 1\}}.$$

By means of the definition of the standard Fourier transform (1.4), we easily obtain that

$$\left|\frac{d\widehat{f}_m(p)}{dp} - \frac{d\widehat{f}(p)}{dp}\right| \le \frac{1}{\sqrt{2\pi}}\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})}.$$

Hence,

$$\left| \frac{\int_0^p [\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}]ds}{p^4 - ibp}\chi_{\{|p|\le 1\}} \right| \le \frac{\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|b|}\chi_{\{|p|\le 1\}},$$

such that

$$\left\| \frac{\int_0^p [\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}]ds}{p^4 - ibp}\chi_{\{|p|\le 1\}} \right\|_{L^2(\mathbb{R})} \le \frac{\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})}}{\sqrt{\pi}|b|} \to 0$$

as $m \to \infty$ due to the one of our assumptions. Therefore, $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. By virtue of the argument above we have that $u_m(x) \to u(x)$ in $H^4(\mathbb{R})$ as $m \to \infty$ in the situation b) of our theorem as well. ■

3. The equation on the finite interval

Proof of Theorem 1.3. Let us first establish that it would be sufficient to solve our problem in $L^2(I)$. Indeed, if $u(x)$ is a square integrable solution of (1.10), periodic on I along with its derivatives up to the third order inclusively, directly from our equation under the given conditions we derive that

$$\frac{d^4u}{dx^4} - b\frac{du}{dx} \in L^2(I).$$

By means of (1.12), we have that $(n^4 - ibn)u_n \in l^2$. Thus, $\sum_{n=-\infty}^{\infty} n^8 |u_n|^2 < \infty$, so that $\frac{d^4u}{dx^4} \in L^2(I)$. Thus, $u(x) \in H^4(I)$ as well.
To show the uniqueness of solutions of (1.10), we discuss the case of $a > 0$. When a vanishes, we are able to use the similar ideas in the constrained subspace $H_0^4(I)$. Let us suppose that $u_1(x),\ u_2(x) \in H^4(I)$ solve (1.10). Then their difference $w(x) := u_1(x) - u_2(x) \in H^4(I)$ satisfies the homogeneous equation

$$\frac{d^4w}{dx^4} - b\frac{dw}{dx} - aw = 0.$$

Since the operator $\mathcal{L}_{a,\, b}$ introduced in (1.13) does not have any nontrivial $H^4(I)$ zero modes, the function $w(x)$ vanishes identically in I.
We apply the Fourier transform (1.12) to both sides of problem (1.10). This yields

$$u_n = \frac{f_n}{n^4 - a - ibn}, \quad n \in \mathbb{Z}, \tag{3.1}$$

so that

$$\|u\|^2_{L^2(I)} = \sum_{n=-\infty}^{\infty} \frac{|f_n|^2}{(n^4 - a)^2 + b^2 n^2}. \tag{3.2}$$

First we deal with the case a) of our theorem. By virtue of (3.2), we have

$$\|u\|^2_{L^2(I)} \le \frac{1}{C} \|f\|^2_{L^2(I)} < \infty$$

as assumed (see (1.11)). By means of the argument above, $u(x) \in H^4(I)$ as well.
Let us conclude the proof of the theorem by treating the case when $a = 0$. From (3.1) we easily obtain that

$$u_n = \frac{f_n}{n^4 - ibn}, \quad n \in \mathbb{Z}. \tag{3.3}$$

Obviously, the right side of (3.3) is contained in l^2 if and only if

$$f_0 = 0, \tag{3.4}$$

so that

$$\|u\|^2_{L^2(I)} = \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{|f_n|^2}{n^8 + b^2n^2} \le \frac{1}{b^2}\|f\|^2_{L^2(I)} < \infty,$$

due to the one of our assumptions along with (1.11). The argument above implies that $u(x) \in H_0^4(I)$ as well. Evidently, (3.4) is equivalent to orthogonality condition (1.17). ■

Let us proceed to demonstrating the solvability in the sense of sequences for our problem on the interval I with periodic boundary conditions.

Proof of Theorem 1.4. Using the stated assumptions, we obtain that

$$|f(0) - f(2\pi)| \le |f(0) - f_m(0)| + |f_m(2\pi) - f(2\pi)| \le 2\|f_m - f\|_{C(I)} \to 0$$

as $m \to \infty$. Hence, $f(0) = f(2\pi)$. By means of (1.11) for $f_m(x),\ f(x)$ continuous on our interval I, we have $f_m(x),\ f(x) \in L^1(I) \cap L^2(I),\ m \in \mathbb{N}$. Formula (1.11) also gives us that

$$\|f_m(x) - f(x)\|_{L^1(I)} \le 2\pi\|f_m(x) - f(x)\|_{C(I)} \to 0, \quad m \to \infty. \tag{3.5}$$

Thus, $f_m(x) \to f(x)$ in $L^1(I)$ as $m \to \infty$. Analogously, (1.11) implies that

$$\|f_m(x) - f(x)\|_{L^2(I)} \le \sqrt{2\pi}\|f_m(x) - f(x)\|_{C(I)} \to 0, \quad m \to \infty. \tag{3.6}$$

Therefore, $f_m(x) \to f(x)$ in $L^2(I)$ as $m \to \infty$ as well. We apply the Fourier transform (1.12) to both sides of (1.16). This yields that

$$u_{m,n} = \frac{f_{m,n}}{n^4 - a - ibn}, \quad m \in \mathbb{N}, \quad n \in \mathbb{Z}. \tag{3.7}$$

First we discuss the situation a) of our theorem. By virtue of the part a) of Theorem 1.3, problems (1.10) and (1.16) have unique solutions $u(x) \in H^4(I)$ and $u_m(x) \in H^4(I),\ m \in \mathbb{N}$ respectively. By means of (3.1), (3.6) and (3.7), we have

$$\|u_m - u\|^2_{L^2(I)} = \sum_{n=-\infty}^{\infty} \frac{|f_{m,n} - f_n|^2}{(n^4 - a)^2 + b^2n^2} \le \frac{1}{C}\|f_m - f\|^2_{L^2(I)} \to 0, \quad m \to \infty.$$

Thus, $u_m(x) \to u(x)$ in $L^2(I)$ as $m \to \infty$. Let us show that $u_m(x)$ converges to $u(x)$ in $H^4(I)$ as $m \to \infty$. Indeed, by virtue of (1.10) and (1.16), we derive

$$\left\| \frac{d^4}{dx^4}(u_m - u) - b\frac{d(u_m - u)}{dx} \right\|_{L^2(I)} \le \|f_m - f\|_{L^2(I)} + a\|u_m - u\|_{L^2(I)}.$$

The right side of this inequality tends to zero as $m \to \infty$ due to (3.6). Using the Fourier transform (1.12), we arrive at

$$\sum_{n=-\infty}^{\infty} n^8 |u_{m,n} - u_n|^2 \to 0, \quad m \to \infty.$$

Hence, $\frac{d^4 u_m}{dx^4} \to \frac{d^4 u}{dx^4}$ in $L^2(I)$ as $m \to \infty$. Therefore, $u_m(x) \to u(x)$ in $H^4(I)$ as $m \to \infty$ as well.

Let us conclude our article with considering the case when the parameter a vanishes. By means of (1.18) along with (3.5), we obtain

$$|(f(x), 1)_{L^2(I)}| = |(f(x) - f_m(x), 1)_{L^2(I)}| \leq \|f_m - f\|_{L^1(I)} \to 0, \quad m \to \infty,$$

so that the limiting orthogonality condition

$$(f(x), 1)_{L^2(I)} = 0 \tag{3.8}$$

is valid. By virtue of the part b) of Theorem 1.3 above equations (1.10) and (1.16) admit unique solutions $u(x) \in H_0^4(I)$ and $u_m(x) \in H_0^4(I)$, $m \in \mathbb{N}$ respectively when a is trivial. Using formulas (3.1) and (3.7), we arrive at

$$u_{m,n} - u_n = \frac{f_{m,n} - f_n}{n^4 - ibn}, \quad m \in \mathbb{N}, \quad n \in \mathbb{Z}. \tag{3.9}$$

Orthogonality relations (3.8) and (1.18) give us that

$$f_0 = 0, \quad f_{m,0} = 0, \quad m \in \mathbb{N}.$$

Let us derive the upper bound for the norm as

$$\|u_m - u\|_{L^2(I)} = \sqrt{\sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{|f_{m,n} - f_n|^2}{n^8 + b^2 n^2}} \leq \frac{\|f_m - f\|_{L^2(I)}}{|b|} \to 0, \quad m \to \infty$$

via (3.6). Hence, $u_m(x) \to u(x)$ in $L^2(I)$ as $m \to \infty$. Therefore, $u_m(x) \to u(x)$ in $H_0^4(I)$ as $m \to \infty$ as well by means of the argument analogical to the one above in the proof of the situation a) of the theorem. ∎

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