Does the Weyl invariant based proposal provide an accurate description of gravitational entropy?

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(Dated: July 12, 2022)

The evolution of the universe is described by the second law of thermodynamics, which states that the entropy at a point farther in time would have a greater entropy than that of a point in time earlier. The usual description of entropy is by thermodynamic information, where the universe at a time very close to the initial singularity is homogeneous and therefore of a lower entropy state. As time progresses, structure formations increase, which imply an increasing entropy. A geometric description of this was proposed by Penrose, who looked for an account of gravitational clumping in terms of the Weyl curvature, in the Weyl curvature hypothesis. This hypothesised that the increasing entropy of the universe can be described as the increasing Weyl curvature, starting from zero at the initial singularity, and eventually tidal distortions increase due to clumping of matter due to gravitation, and structure formations account for the increasing anisotropies. In this paper, we look at gravitational entropy briefly, looking at the conditions such a proposal should satisfy, and some examples of where this proposal reduces to the familiar entropy and where this proposal requires further investigation.

I. INTRODUCTION

The evolution of the universe has always puzzled physicists, and one of the most interesting puzzles in cosmology and in describing the evolution of the universe is how the entropy of the universe is modelled. Following Penrose's hypothesis [13] on using the Weyl tensor to quantify some sort of "gravitational entropy", a large volume of literature has been contributed to this question. The dominance of the Weyl tensor over the Riemann tensor in particular, is an interesting point used to describe the universe's evolution. Since the Weyl tensor vanishes only for conformally flat spacetimes such as the Friedmann-Robertson spacetime, and following that the early universe must have had a lower order of entropy than at the present, the question then is, *if the universe's entropy always increases, can some form of the Weyl curvature be used to describe the universe?*
The notion of gravitational entropy is fundamental to the WCH, since describing entropy as a result of the Weyl tensor seems rather unnatural. The idea of this can be described via black hole thermodynamics by considering that the information lost "into" a black hole. For a description of this topic, refer to section 2. Since entropy encoded into the black hole must follow the second law of thermodynamics, the information in the black hole must somehow be a quantification of the black hole itself, as was shown by Bekenstein in his thought experiment. However, since the most elementary case of Schwarzschild black holes in themselves are vacuum solutions, a *gravitational* description of entropy seems appropriate, at least in terms of a "good" function. Seemingly, this function must remain non-zero in vacuum cases in order to describe entropy, and must also be a curvature contribution. Therefore the geometric nature of the function must be similar to that of the Riemann tensor, obeying the symmetries

$$C_{abcd} = C_{badc}$$
$$C_{abcd} = -C_{bacd}$$
$$C_{abcd} + C_{acdb} + C_{adbc} = 0$$

Therefore, the Weyl tensor seems to be an appropriate choice given that it is not necessarily zero and that it describes curvature contributions. This is important especially considering that in the case of black holes, the model is a vacuum solution but with non-zero entropy.
Recall that the Einstein field equations are of the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = kT_{\mu\nu} \tag{1}$$

The Friedmann-Robertson-Walker metric is a solution to the field equations (1), described by a scale factor $a(t)$ and a sectional curvature k:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{2}$$

It is known that this solution is conformally flat, i.e. the metric describing the FRW spacetime follows that there exists some function λ such that $g \sim \lambda^2\eta$, where η is the Minkowskian metric. Since the Weyl tensor vanishes only in the case of conformally flat spacetime and the early universe was closer to the FRW model due to lower entropy, the Weyl tensor can be considered to describe the state of the universe in the form of gravitational entropy, since naturally structure formations increase in future states of the universe [5]. Due to this, the entropy of the universe increases, and the Weyl tensor becomes non-zero due to the state of the universe. Penrose hypothesised that since the Weyl curvature would increase as a result of asymmetries and structure formations, the monotonicity of the Weyl tensor can be considered as the

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gravitational entropy described earlier. While a mathematical proof of this is yet to be found, it is interesting to note that there have been many papers accounting for black hole thermodynamics and gravitational entropy (e.g. [1, 2, 4, 6, 8]).

A. The Weyl tensor

The Weyl tensor is somewhat similar to the Riemann tensor in two ways. Firstly, it is a measure of tidal forces, and secondly, it obeys the same symmetries as the Riemann tensor. In $n-$dimensions, the Weyl tensor can be written as

$$C_{abcd} = R_{abcd} - (\frac{1}{n-2}(g_{ac}R_{db} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}) + \frac{1}{(n-1)(n-2)}(g_{ad}R_{bc} - g_{ac}R_{bd})R) \quad (3)$$

As said previously, the Weyl tensor is only vanishing in the case of conformally flat spacetimes. Further, the trace of the Weyl tensor is zero, and $C^a_{bac} = 0$. Due to this, the construction of an invariant using the Weyl tensor C_{abcd} can be done similar to forming the Kretschmann scalar of the Riemann tensor by "squaring", i.e.

$$W = C_{abcd}C^{abcd}$$

In order to capture the strictly increasing nature of the Weyl tensor, we can say that the entropy of the universe is governed by the second law of thermodynamics, which says that [7, 17, 18]

$$\frac{\Delta S}{\Delta t} \geq 0 \quad (4)$$

At times very close to $t = 0$, the universe is a perfect FRW spacetime and has very low entropy, and this can be attributed in some sense to the vanishing nature of the Weyl tensor for FRW spacetime. At times $t > 0$, the universe experiences inflation and structure formations, and due to the clumping of matter the gravitational entropy increases, and therefore the Weyl curvature increases. While the original hypothesis did not point to cosmology directly, it regarded to the reproduction of the entropy formula for black holes in terms of gravitational entropy. Further, the dominance of the Weyl tensor over the Riemann tensor can be seen in the case stated previously, where the vacuum solution nature of the Schwarzschild solution can be pointed out. This can be done by looking at the trace form of the energy-momentum tensor in (1) by rewriting the field equations as

$$R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \quad (5)$$

Clearly, when $T_{\mu\nu} = 0$, the expression reduces to $R_{\mu\nu} = 0$. The landscape of WCH has this as a fundamental point, since it is not necessary that the gravitational entropy be zero in vacuum cases, nor be non-zero in cases where $T_{\mu\nu}$ is not zero, particularly for FRW spacetimes. The description then is that the Weyl tensor is zero at the initial singularity at $t = 0$, and monotonically increases with time, representing a gravitational analog of thermodynamic entropy.
But what are the conditions on the gravitational entropy model chosen? It is clear that there exist specific constraints on the gravitational description of entropy, and as we will see the mathematical formulation of this must satisfy these conditions. Since the entropy of the universe can never decrease, the first condition is that the gravitational entropy must never decrease and must be strictly increasing, accounting for the structure formations in the universe. Secondly, the preset condition on the Weyl tensor being non-vanishing must hold true, i.e. the only cases where the entropy is zero is when the spacetime is conformally flat. Thirdly, it must reduce to the familiar entropy formula in the case of black holes. The last point is of particular interest, since the problem in itself concerns more or less the gravitational entropy more than a cosmological description at this stage. Therefore, in the case of non-charged black holes, the interpretation of the entropy formula is the point of interest with respect to gravitational entropy. This can be attributed to the fact that black hole formation results in larger entropy states than that of ordinary forms of matter such as stars. After the gravitational collapse of a star, the resulting black hole[27] obtains the familiar Hawking-Bekenstein relation, which must be the general form for the formulation of gravitational entropy.

B. Thermodynamics

A mathematical formulation can then be attempted by first considering this important point, by basing the notion of gravitational entropy on the condition that it must be of the familiar Hawking-Bekenstein form, which is to say that $S \sim A$. We define the gravitational analogs of thermodynamic components as

$$T_g dS_g = dU_g + p_g dV \quad (6)$$

Where T_g is the temperature, U_g is the internal energy, p_g is the pressure and S_g is the entropy of the gravitational field. While S_g is the point of interest here, no components of the above equation actually contribute to the field equations as the T, p or S contributions from matter fields on (M, g). The contribution from S_g is of particular interest here, and we wish to see what the above equation reduces to in the case of Schwarzschild black holes in an attempt to relate to the Hawking-Bekenstein relation. On choosing specific coordinates (here the Gullstrand-Painlev coordinates [5]) and adopting the tetrad formalism, we see that the gravitational entropy term reduces to $S_g \sim A$, which we use as a reverse-chronological condition on the gravitational entropy. While previously we had made it necessary for the black hole to remain uncharged, a description of charged black holes can also be found – however, this would face problems with the case of extremal Reissner-Nordstrom black holes as we will discuss soon.

C. Remarks

The picture of the formulation of gravitational entropy can be broken into two parts – the black holes sector, where we are concerned with the interpretation of gravitational entropy in terms of black hole thermodynamics, and the cosmological sector, where we are interested in knowing how gravitational entropy can be used to describe the way the early universe behaved, referring to the anisotropies present in those times. The *CET proposal* [5] refers to the use of the Bel-Robinson tensor in general relativity. The definition of the gravitational entropy in the CET proposal is more or less the general definition of gravitational entropy as stated previously, accounting for the conditions that S_g satisfies. In the next section, we will overview black hole thermodynamics and pave way for the gravitational entropy proposal, where we will describe the gravitational entropy in terms of a surface integral using a scalar field. We then consider a suitable scalar and use the divergence theorem to define the entropy density. To solve the integral we consider a spatial metric and define the spatial volume element, which would allow us to find the entropy. However, note that there is an important point to take into consideration – the integral must not contain the singularity to prevent a divergence. In order to ensure this, we can consider a small sphere of infinitesimal radius around the singularity and remove this sphere. In the CET proposal, this is not so – the removal of a spherical element in order to prevent the divergence of the integral is not a necessary operation, and is replaced by the usual Newmann-Penrose formalism construction of the gravitational entropy, where we consider the Cartan invariants DW and Ψ_2, where W is the Weyl tensor and Ψ_2 is the Newmann-Penrose scalar. Further, a variation of the parameters of the black hole, such as the entropy of the black hole would require a variation of the horizon, and therefore the mass of the black hole.

II. GRAVITATIONAL ENTROPY

The thermodynamics underlying black holes is very interesting for a number of reasons. Firstly, black holes in themselves pose a problem with information and the second law of thermodynamics, since matter falling into the universe contributes to the entropy of the black hole, which evaporates steadily due to Hawking radiation. Secondly, the description of entropy of black holes must be in some way directly related to the area of the event horizon, as proposed by Bekenstein. The problem with the first point is that when matter falls into a black hole, it seems to "disappear" from the viewpoint of an observer in the universe. Therefore, an appropriate description of black holes and information falling into the black hole must be provided [7].

A. Schwarazschild black hole

Following what may be considered one of the most important debates in general relativity, Bekenstein [17] proposed a thought experiment where a box of gas with a mass m and a temperature T_{box} was lowered into a black hole. Assuming that the box has a length l close to the thermal wavelength of the box $\hbar/T$, and since the lowering of the box of gas into the black hole would decrease the entropy of the universe for an observer, this would be of the order $-\frac{ml}{\hbar}$. The increase in the horizon area would be of the order Gml, and therefore the relation between the area of the horizon and the entropy of the black hole would be of the form

$$S_{BH} \sim \frac{\Delta A}{\hbar G} \tag{7}$$

This is the form that we wish the gravitational entropy S_g to reduce to in cases of black holes. The second law of thermodynamics takes the form

$$S_{BH} = S_T + S_{HB} \tag{8}$$

Where S_{BH} is the entropy of the black hole and S_{HB} is the Hawking-Bekenstein entropy of the black hole.
For our gravitational entropy proposal we wish the gravitational entropy to be S_{HB}, and therefore we have

$$S_g = S_{HB} \tag{9}$$

We will now describe the formulation for the gravitational entropy.
We will start by defining S_g as the surface integral [1]

$$S_g = k_s \int_\sigma \Psi \mathbf{e}_r \cdot d\sigma \tag{10}$$

Where σ is the surface of the horizon. The scalar field Ψ is built on the Weyl invariant seen previously, and $\Psi \mathbf{e}_r$ denotes that the vector field is radial. It is trivial to use the divergence theorem to transform this into a volume integral:

$$S_g = k_s \int_V \nabla \cdot \Psi \mathbf{e}_r \, dV \tag{11}$$

The entropy density s_d can be defined in terms of the divergence of the field – however, we must take the absolute value of this in order to ensure that the value is always non-negative. Then, we have

$$s_d = k_s |\nabla \cdot \Psi \mathbf{e}_r| \tag{12}$$

We will consider the example of a Schwarzschild-like solution and attempt a gravitational entropy description of the solution using the above formalism. Consider the scalar Ψ to be of the form

$$\Psi = \sqrt{\frac{C_{abcd}C^{abcd}}{R_{abcd}R^{abcd}}} \tag{13}$$

Where we considered the Weyl invariant and the Kretschmann invariant to define the scalar Ψ. The simplest case of $\Psi = C_{abcd}C^{abcd}$ is an unviable option, since the entropy determined by this scalar does not follow the usual area-entropy relation. Recall that the Schwarzschild solution is described by a metric of the form

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{14}$$

Here the exponential terms are of the form $e^{2\alpha(r)} = \left(e^{(2\beta(r)}\right)^{-1} = \left(1 - \frac{2GM}{c^2 r}\right)$. In this case the Kretschmann scalar and Weyl invariant are both equal, and so in this case the gravitational entropy is maximum. Then, considering some small sphere of radius ϵ around the singularity and remove this in order to ensure the integral does not diverge. Then, given that the scalar Ψ reduces to unity due to the equivalence of the Weyl invariant and the Kretschmann scalar, the integral can be found trivially from (10) by first identifying that we are working in a three-dimensional setting. Then, a spatial metric h_{ij} by removing $i0$ components from the metric $g_{\mu\nu}$:

$$h_{ij} = g_{ij} - \frac{g_{i0}g_{j0}}{g_{00}}$$

Under the Schwarzschild setting (14) this metric would be the diagonal $\mathrm{diag}(e^{2\beta(r)}, r^2, r^2\sin^2\theta)$. We use this to define the $d\sigma$ surface element:

$$d\sigma = \sqrt{\frac{h}{h_{11}}}d\theta d\phi \mathbf{e}_r$$

Then, we remove a spherical element of radius ϵ:

$$S_g = k_s \int (R_{sch}^2 - \epsilon^2)\sin\theta d\theta d\phi$$

Which reduces to

$$S_g = 4\pi k_s (R_{sch}^2 - \epsilon^2)$$

If we let the sphere of radius ϵ be zero, we will be left with:

$$S_g = 4\pi k_s R_{Sch}^2 \tag{15}$$

Where $4\pi R_{Sch}^2$ is the area of the event horizon, and therefore $S \sim A$ via our gravitational entropy formalism. Since S_g must be the Hawking-Bekenstein formula (9), and therefore the constant k_s can be found to be of the form

$$k_s = \frac{k_B c^3}{4G\hbar} \tag{16}$$

We can then find the entropy density s_d from (12), which would allow us to ensure that the entropy density becomes asymptotically null as the value of r increases, where the solution becomes asymptotically Minkowskian. In fact, this is not a surprise, since the CET proposal showed that describing this entropy leads to the Hawking-Bekenstein formula. This description was made by considering the Schwarzschild solution in the appropriate coordinates. The gravitational forms of entropy density ρ_g and the temperature would be [5]

$$\rho_g = \frac{2m}{8\pi r^3} \tag{17}$$

$$T_g = \frac{m}{2\pi r^2 \sqrt{|1 - \frac{2m}{r}|}} \tag{18}$$

Further, the entropy density would be given in terms of ρ_g and T_g, allowing us to calculate the entropy density and therefore the entropy, which would reduce to the familiar relation (7). Note that there is a constant that we set to 1, which appears beforehand in the expression for ρ_g and T_g which we substituted as unity. This was found out by requiring that the entropy was the Hawking-Bekenstein entropy, which would require the constant to be unity.
So far we have looked at gravitational entropy in the case of the Schwarzschild solution. We will now turn our attention to the special case of de Sitter spacetime, where the gravitational entropy proposal is challenged.

B. de Sitter spacetime and modified gravity

The de Sitter solutions to the field equations are defined by a cosmological constant Λ, which affects the geometry of the universe – if Λ is positive, we refer it to the *de Sitter solution*, whereas if Λ is negative, we refer it as the *anti-de Sitter solution*. The geometry of the universe is based on the constant sectional curvature $k = +1$, 0 or -1, and the last two cases require a negative Λ. The de Sitter spacetime is defined by setting $M = 0$ for a Schwarzschild de Sitter spacetime, and is therefore an elementary case of this spacetime. The SdS spacetime has a Weyl invariant independent of Λ, while the Kretschmann scalar is a function of Λ. Adopting the previously seen formalism, we consider the W/S form, which would give us the value of Ψ. Then, we again identify a sphere of radius ϵ and remove this from the integral:

$$S_g = k_s \int (\Psi(R_{SdS})R_{SdS}^2 - \Psi(\epsilon)\epsilon^2)\sin\theta d\theta d\phi$$

Where R_{SdS} is the black hole horizon radius. This would again reduce to the individual $\Psi(R_{SdS})$ and $\Psi(\epsilon)$ terms, which would leave us with

$$S_g = 4\pi k_s (\Psi(R_{SdS})R_{SdS}^2 - \Psi(\epsilon)\epsilon^2)$$

This would reduce to (15) when $\Lambda = 0$, which is the usual Schwarzschild solution. The dS case is when $M = 0$, and therefore the Weyl invariant is zero, since it is only a function of M. In this solution, the radius $R_{dS} = \sqrt{\frac{3}{\Lambda}}$ is a cosmological horizon, and Gibbons and Hawking [18] showed that this has an entropy similar to that of black hole entropy, only with area of the black hole being defined in terms of R_{dS} rather than an event horizon radius. It is trivial to see that the Weyl curvature of a dS spacetime is zero, and therefore the gravitational entropy this suggested would be zero, which is a contradiction to the well-known area-entropy relation for the dS spacetime as said above. Therefore, the Weyl curvature-based entropy proposal proves to be inconsistent with the dS spacetime. Corrections using different curvature invariants proves to be ineffective, since any factor would also vanish due to the Weyl tensor being zero in the dS spacetime.

In fact, this is also the case of *extreme* Reissner-Nordstrom black holes as shown by [6], which showed that as the general case of a Reissner-Nordstrom black hole tends to the extremal case, the entropy of the black hole becomes zero, which also is a problem for the gravitational entropy proposal. Therefore, alternatives must be considered, at least for the dS spacetime[28].

The gravitational entropy proposal also has problems with the modification of the Hawking-Bekenstein relation in the case of modified theories of gravity. For instance, in $f(R)$ gravity, the Hawking-Bekenstein relation is a function of $f(R)$:

$$S_{f(R)} = \frac{f'(R)A}{4G\hbar} \tag{19}$$

Where the prime denotes the derivative of the function f(R) w.r.t the spatial r coordinate. (19) clearly imposes a further restriction on the gravitational entropy proposal – since $f'(R)$ affects the entropy, there is a form of "background theory effect", where different $f(R)$ theories would produce a non-general form of the Hawking-Bekenstein entropy. Since (7) no longer holds true without any other parameters, the gravitational entropy proposal needs to take into account of this extension too. Clearly, the determined Hawking-Bekenstein entropy in $f(R)$ gravity affects the entropy and is not necessarily of the form of (7) with a denominator 4, since $f(R) = R$ is only a general case of general reltivity as an $f(R)$ theory. Therefore, under different background theories (i.e. theories that have different forms of $f(R)$, such as the Nojiri-Odintsov $f(R)$ theory, which has the function $f(R) = R + \alpha R^m - \beta R^{-n}$ – clearly, the resulting $f'(R)$ does not yield the usual Hawking-Bekenstein entropy), the entropy of the black hole changes, only following the area-entropy proportionality. This is also the case in other modified theories of gravity such as Lovelock gravity, where the general $D = 5$ case is that of Gauss-Bonnett gravity, described by

$$S = \frac{1}{k}\int \left(R+\alpha[R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\beta\gamma}R^{\mu\nu\beta\gamma}]\right) d^5x\sqrt{-g} \tag{20}$$

Where $d^5x\sqrt{-g}$ is simply the five-dimensional volume element, $k = 16\pi G$ and α is a constant parameter. In this case, the black hole entropy again receives a modification, which must also be held accountable by the gravitational entropy proposal.

C. Wormhole gravitational entropy

The initial hypothesis was that the Weyl invariant $W = C_{abcd}C^{abcd}$ functions as a measure of entropy, starting from zero at the initial singularity when the universe has low entropy, accounting for the homogeneity of the universe and increases strictly. This was shown to be inconsistent in the case of isotropic singularities, in which case the $W-$form of entropy description is contradicted [3]. In order to correct this, a modified form of the scalar by also considering the square of the Ricci tensor $S = R_{\mu\nu}R^{\mu\nu}$:

$$\frac{W}{S} = \frac{C_{abcd}C^{abcd}}{R_{ab}R^{ab}}$$

Under this, the expression (10) may be modified so that the scalar Ψ is no longer solely described by W but by S too. In [1], it was shown that while choosing between curvature invariants for describing gravitational entropy, the above form of Ψ is not totally satisfactory. This form of entropy was also shown to have problems by [25], and therefore a different description of entropy might work. The next attempt would be to define a curvature invariant using the square of the Riemann tensor, which would be the Kretschmann scalar form we have seen previously. Describing entropy models using the Newmann-Penrose formalism has been investigated in several cases, such as [2, 8, 22]. These investigations show that the gravitational entropy proposal does retain the reduction to the Hawking-Bekenstein relation condition, and therefore the proposal is quite strong.

In fact, the approach mentioned in section II A can be applied to wormhole solutions to investigate the entropy density variation of such solutions. Wormholes are topological solutions that join two points from the same or different universe [26]. For instance, considering the Ellis wormhole metric

$$ds^2 = -dt^2 + dr^2 + (r^2 + r_0^2)d\Omega^2 \tag{21}$$

Where r_0 is the "throat" radius of the wormhole. Adopting the scalar W/S, the entropy density is given by

$$s_d = \frac{4k_s}{3}\left|\frac{r}{(r^2 + r_0^2)}\right| \tag{22}$$

This approach allows us to understand the specifics of wormhole solutions such as the gravitational entropy density of the solution, and the comparison between the CET approach and the usual approach using [1] has been concluded in [24]. The motivation behind investigating

wormholes that are traversable is the existence of exotic matter, i.e. matter violating the energy conditions and how the gravitational entropy description works in such a background. The variation of the gravitational entropy in the model is also an important point, where different wormhole solutions have a different variation of gravitational entropy – any formalism must describe this variation consistently. The calculation of gravitational entropy via the Rudjord approach and the CET approach both yield the gravitational entropy variation, but at different regions in the wormhole solution function differently. For a more local description of the solution, the CET proposal yields a consistent result of the gravitational entropy, but the calculations of temperature yield inconsistent results under different wormhole models.
We will finally discuss the CET proposal and some examples of black hole solutions via this proposal.

III. THE CET PROPOSAL

The Newmann-Penrose formalism [20] is adopted throughout this section, and we consider the Clifton, Ellis and Tavakol's adoption of the Bel-Robinson tensor to describe the gravitational entropy. We define the Bel-Robinson tensor in the usual format as

$$T_{\alpha\beta\gamma\delta} = \frac{1}{4}(C_{\lambda\alpha\beta\rho}C^{\lambda\rho}_{\gamma\delta} + C^{*}_{\lambda\alpha\beta\rho}C^{*\lambda\rho}_{\gamma\delta}) \tag{23}$$

From this, we construct a symmetric trace-free tensor t_{ab} that is defined as the square-root of the Bel-Robinson tensor. We define the complex null tetrad required as

$$\begin{aligned} l^a &= \tfrac{1}{\sqrt{2}}(x^a - iy^a) \\ n^a &= \tfrac{1}{\sqrt{2}}(v^a - z^a) \\ m^a &= \tfrac{1}{\sqrt{2}}(v^a + z^a) \end{aligned}$$

Where x^a, y^a and z^a are spacelike unit vectors and v^a is the matter 4−velocity. CET showed that by following a series of calculations, the local gravitational entropy is

$$S_g = \int_V \frac{\rho_g}{T_g} dV \tag{24}$$

The CET approach has been considered by many sources of literature, and has been applied to black holes, cosmologies and wormhole solutions. For instance, considering the AdS Reissner-Nordstrom solution, where the parameters specified are M, Q, Λ, with Q being the charge, the Newmann-Penrose scalar Ψ_2 and DC_{abcd} are required, where $D \equiv n^{\mu}\nabla_{\mu}$. The metric can be written in terms of the tetrad as

$$ds^2 = -2l_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)} \tag{25}$$

Where the bar denotes complex conjugation. We choose the null coframe, defined by $l_a l^a = n_a n^a = m_a m^a = \bar{m}_a \bar{m}^a = 0$. The approach using the subtraction of a spherical element of radius ϵ around the singularity in the previously seen approach can be discarded by this approach, where we can directly calculating the gravitational entropy, which would reduce to the familiar Hawking-Bekenstein entropy formula, and this approach has been found useful in several literatures. The changed forms of the scalar Ψ can be incorporated into the approach specified above. Using this, we can switch from the original Weyl invariant form C_{abcd} to the factors including $R_{\mu\nu}R^{\mu\nu}$ and the Kretschmann scalar.
This proposal can be applied to the FLRW case, where the metric is as seen in (2). Adopting a null coframe [8] (with k=1),

$$\begin{aligned} l_a &= \tfrac{1}{\sqrt{2}}\left(dt - \tfrac{a(t)dr}{1-r^2}\right) \\ n_a = l_a &= \tfrac{1}{\sqrt{2}}\left(dt + \tfrac{a(t)dr}{1-r^2}\right) \\ m_a &= \tfrac{1}{\sqrt{2}}(ra(t)(d\theta + i\sin\theta d\phi)) \end{aligned}$$

From this, we can find the gravitational entropy by choosing the limits of integration to the horizon radius. This reduces to a form of the Hawking-Bekenstein entropy, specified by the horizon radius, called the Gibbons-Hawking entropy [18].

CONCLUSION

The gravitational entropy proposal has two fascinating elements – firstly in terms of black holes, where the reduction of the gravitational entropy to the entropy formula is found, and secondly in terms of approaches towards describing cosmological evolution. The current understanding of gravitational entropy in cosmology was fuelled by Penrose's hypothesis that the Weyl curvature tensor can be used to describe a form of gravitational entropy, and this in turn was to describe the evolution of the universe as the monotonically increasing nature of the Weyl curvature due to gravitational clumping of matter in later stages of the universe, which described the structure formations and therefore the anisotropies formed, which were due to the increasing entropy of the universe. While the cosmological aspects of WCH are not the only aspects of gravitational entropy that are of interest, the cosmological implications of gravitational entropy descriptions are huge – particularly, it would allow us to understand the formalism of gravitational entropy in the large picture of the evolution of the universe under different conditions. Other aspects of gravitational entropy that are of interest are different black holes models, corrections to the usual gravitational entropy in modified gravity and in wormhole solutions. In further works, the relation of WCH to the conformal cyclic cosmologies (CCC) hypothesis by Penrose will be investigated, where different conformally related metrics corresponding to two "aeons" will be used to understand the WCH, and the notion of gravitational entropy in other forms of wormholes such as charged wormholes will be investigated. In this paper, we looked at some of the key aspects of the gravitational entropy landscape, and we looked at two approaches towards mathematically defining gravitational entropy, one by Rudjord and Gron, and the Newmann-Penrose formalism approach by Clifton, Ellis and Tavakol. We looked at some of the requirements of the formalism suggested by [1], and looked at places where these two proposals can be used to mathematically define a consistent description of gravitational entropy. We applied the approach suggested by [1] in the case of a Schwarzschild black hole, and applied this to a SdS case. We then discussed points where this approach requires modifications, such as the case of dS spacetimes, extremal Reissner-Nordstrom black holes and $f(R)$ gravity, and briefly looked at the idea of gravitational entropy approaches in the case of wormhole solutions. We finally looked at the CET proposal, which applies to general relativistic cases and has been adopted by several literatures.

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[27] Again, provided this is not a charged solution.

[28] As it was previously mentioned by [1], the extremal Reissner-Nordstrom black hole case is more or less impossible, and therefore this case may be considered of somewhat less priority to that of the dS spacetime.