On Newtonian Quantum Gravity for Stationary States of WIMPs

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Abstract.

We discuss the leading term in the semi-classical asymptotics of Newtonian quantum gravity for the Kepler problem. For dark matter, ice or dust particles in the gravitational field of a star or massive planet this explains how rapidly planets or ring systems can be formed by considering semi-classical equations for the asymptotics of the relevant Schrödinger wave function. We extend this treatment to isotropic harmonic oscillator potentials in two and three dimensions finding explicit solutions. Further in two dimensions, we find the explicit solutions for Newton's corresponding revolving orbits, explaining planetary perihelion advance in these terms. A general implicit approach to solving our equations is given using Cauchy characteristic curves giving necessary and sufficient conditions for existence and uniqueness of our solutions. Using this method we solve the two-dimensional Power Law Problem in our setting. Our methods give an insight to the behavior of semi-classical orbits in the neighborhood of classical orbits showing how complex constants of the motion emerge from the SO(4) symmetry group and what the particle densities have to be for circular spirals for different central potentials. Such constants have an important role in revolving orbits. Moreover, they and the aforementioned densities could give rise to observable effects in the early history of planetary ring systems. We believe the quantum spirals here associated with classical Keplerian elliptical orbits explain the evolution of galaxies, with a neutron star or black hole at their centre, from spiral to elliptical.

1. Introduction

In our previous papers, we have seen how in Schrödinger quantum mechanics, if the stationary state wave function $\psi \sim \exp\left(\frac{R+iS}{\epsilon^2}\right)$ as $\epsilon^2 = \hbar \sim 0$, for potential V, it is necessary that for real valued R and S,

$$\boldsymbol{\nabla} R.\boldsymbol{\nabla} S = 0, \qquad 2^{-1}(|\boldsymbol{\nabla} S|^2 - |\boldsymbol{\nabla} R|^2) + V = E \ \text{ and } \ S\rceil_{C_0} = S_0, \qquad (*)$$

where C_0 is the corresponding classical orbit, $\boldsymbol{\nabla} R\rceil_{C_0} = \mathbf{0}$ and S_0 is the corresponding Hamilton-Jacobi function, E being the energy.

These semi-classical equations and how to solve them will be the main concerns in the present paper. To the extent that for the Kepler-Coulomb problem they give the leading term in the semi-classical asymptotics of Newtonian quantum gravity, they are relevant in modelling the behaviour of ice, dust or dark matter particles in the early history of solar systems. In particular they show how quickly planets and ring systems can be formed. (Refs. [1],[5],[6],[7],[8],[14],[15],[16]). Further we argue that galactic evolution is in part governed by the first quantisation of the Kepler problem for elliptical orbits.

In this setting of a constrained Hamiltonian system, the orbit is given by $t \to \mathbf{X}_t^0, t \in \mathbb{R}_+$, being the time, $\mathbf{X}_t^0 \in \mathbb{R}^d$, d dimensions of space, where

$$\frac{d}{dt}\mathbf{X}_t^0 = (\boldsymbol{\nabla} S + \boldsymbol{\nabla} R)(\mathbf{X}_t^0).$$

Since

$$\frac{d}{dt}R(\mathbf{X}_t^0) = (\boldsymbol{\nabla} S + \boldsymbol{\nabla} R)(\mathbf{X}_t^0).\boldsymbol{\nabla} R(\mathbf{X}_t^0) = |\boldsymbol{\nabla} R|^2(\mathbf{X}_t^0) \geq 0,$$

R is monotonic increasing in time as the orbit is traversed, so $R(\mathbf{X}_t^0) \nearrow R_{\max}$, the maximum value of R. If, as you expect, this maximum value of R is attained on C_0, the classical orbit, then in the infinite time limit the orbit will converge to classical motion on C_0, the convergence coming from the Bohm potential $-|\boldsymbol{\nabla} R|^2$, since

$$2^{-1}(|\boldsymbol{\nabla} S|^2 + |\boldsymbol{\nabla} R|^2) + V - |\boldsymbol{\nabla} R|^2 = E.$$

A supreme example is provided by the atomic elliptic state, ψ, of Lena, Delande and Gay. ψ is localised on a classical Keplerian ellipse (with force centre at one focus) and minimises the associated SO(4) angular momentum uncertainty relations.(Refs. [5],[9],[11],[14],[16]). In this case, the classical Keplerian ellipse is C_0 and in the infinite time limit the corresponding orbit is described according to Kepler's laws for potential $V = -\frac{\mu}{r}$, μ gravitational mass at focus O, dimension d being 3. So in this semi-classical quantum mechanics of Newtonian gravity, if we take as our Schrödinger ensemble a cloud of identically prepared ice, dust particles or dark matter WIMPs, we have a simple spiral model for the formation of planets or ring systems. Moreover, we can see how long it takes to form a ring by estimating t, the infimum of s, such that

$$R_{\max} - R_0 = \int_0^s |\boldsymbol{\nabla} R|^2(\mathbf{X}_u^0)du.$$

In such a model it is plausible e.g. that gas giants such as Jupiter could be formed in a few million years as required in current planet building scenarios. A quick application is provided by the Levi-Civita transformation whereby $(x + iy) \to (x + iy)^2$, accompanied by the time-change from physical time t to a fictitious time s, give the elliptical orbit corresponding to motion on an ellipse (with force centre at the centre of the ellipse, not the focus) for potential $V = \omega^2 r^2/2$. Such a potential would be relevant if the ensemble of matter was inside a stable, homogeneous cloud of gravitating particles. The details of this are given in the next section (all part of the miracle of the inverse square law of force).(Ref. [11]).

Armed with these examples, the main theme of the paper is to try to extend the class of potentials for which our equations are soluble. Here we focus on the 2-dimensional situation, for $V_0 = -\frac{\mu}{r}$, and for $V = V_0 + \frac{C}{r^2}$, $r^2 = x^2 + y^2$. In the latter case we rediscover Newton's revolving orbits by considering certain complex constants of the motion. As an application we discuss the advance of the perihelion of Mercury in a general relativistic potential giving the agreement with what is observed. This is the content of section (3).

In section (4) we tackle equations directly using Cauchy's method of characteristics. As an illustrative example we solve the equations for the power law in 2-dimensions, $-\kappa r^p$, p being the power and a generalised LCKS transformation in two-dimensions. (Refs. [13],[24]). More importantly, we give a detailed set of results for the Keplerian motion in a neighbourhood of the classical orbit on C_0. Once again complex constants emerge quite naturally resonating with the methods for Newton's revolving orbits. Necessary and sufficient conditions are given for uniqueness and existence of solutions by converting our partial differential equations to ordinary differential equations.

For the sake of complteness and to help those readers not so familiar with celestial mechanics we include here a simple diagram (Figure 1) explaining why for the classical trajectory, $t \to \mathbf{X}_t^0 = \mathbf{r}(t)$, on the classical Keplerian ellipse with corresponding eccentric anomaly v,

$$\left(\frac{d^3}{dv^3} + \frac{d}{dv}\right)\mathbf{r} = 0,$$

where v satisfies the Kepler equation for physical time t,

$$v - e\sin v = \sqrt{\frac{\mu}{a^3}}t,$$

e being the orbital eccentricity, a the semi-major axis of the orbit and μ the gravitational mass at the force centre. Surprisingly this result requires no change in dependent coordinates as the Levi-Civita transformation demands and amounts to a mere time-change. Several authors on the internet refer to this and related results as deep mysteries of the Kepler problem. Our diagram and simple trigonometric identities dispel this mystery. Here t is the physical time measured from the pericentre, the point of nearest approach of the orbit to the force centre, $\mathbf{i} = \hat{\mathbf{r}}(0)$, $\mathbf{j} = \hat{\dot{\mathbf{r}}}(0)$ and if $\mathbf{r} = \overrightarrow{\mathrm{SP}}$,

$$\mathbf{r} = (a\cos v - ae)\mathbf{i} + \sqrt{1-e^2}a\sin v\mathbf{j}.$$

Setting $\mathbf{R} = \mathbf{r} + ae\mathbf{i}$ and $(') = \dfrac{d}{dv}$, we see that

$$\mathbf{R}'' = -\mathbf{R} \quad \text{and so} \quad \mathbf{r}''' + \mathbf{r}' = 0.$$

Since $\mathbf{R}(v) = \mathbf{R}(0)\cos v + \mathbf{R}'(0)\sin v$, we obtain

$$\mathbf{r}(v) = (\mathbf{r}(0) + ae\mathbf{i})\cos v + \mathbf{r}'(0)\sin v\mathbf{j} - ae\mathbf{i},$$

for $v \geq 0$. These results are important in understanding our Keplerian coordinates and important in their own right.

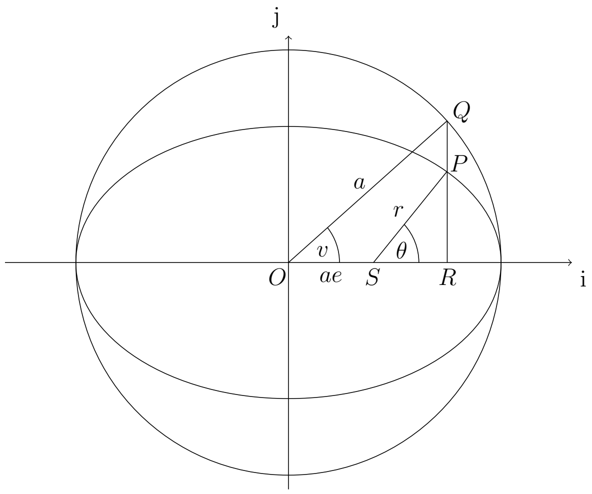

Figure 1.

$$r = a(1 - e\cos v), \qquad \tan\frac{v}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}, \qquad PR = \sqrt{1-e^2}QR.$$

2. Semi-classical Asymptotics of Newtonian Gravity and Celestial Mechanics

2.1. From Atomic to Astronomical Elliptic States

In this section we use some standard tools from celestial mechanics to help to reveal some of the secrets of impenetrable algebraic expressions coming from the SO(4) symmetry group of the Kepler problem. Our basic ingredient is the beautiful stationary state of Lena, Delande and Gay, $\psi_{n,e}(\mathbf{r})$, the atomic elliptic state,

$$\psi_{n,e}(\mathbf{r}) = C\exp\left(-\frac{n\mu r}{\lambda^2}\right)L_{n-1}(n\nu(\mathbf{r})),$$

$L_n, n = 0, 1, 2\ldots,$ being a Laguerre polynomial, $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$,

$$\nu(\mathbf{r}) = \frac{\mu}{\lambda^2}\left(r - \frac{x}{e} - \frac{iy\sqrt{1-e^2}}{e}\right).$$

Here our ellipse $\mathcal{E}_K$ has eccentricity e, $0 < e < 1$, with semimajor axis, $a = \dfrac{\lambda^2}{\mu}$, semilatus rectum, $\ell = a(1-e^2)$, in the plane $z = 0$, with semimajor axis parallel to the x-axis, with polar equation:

$$\frac{\ell}{r} = 1 + e\cos\theta, \qquad \ell = \frac{h^2}{\mu}, \qquad e = \sqrt{1 + \frac{2h^2E}{\mu^2}}, \qquad \text{the constants}$$

$E < 0$ being the energy, h the angular momentum about the z-axis and μ the gravitational mass at the focus of our ellipse $\mathcal{E}_K$. Most importantly, if H denotes the quantum mechanical Hamiltonian, $H = 2^{-1}|\mathbf{P}|^2 - \dfrac{\mu}{|\mathbf{Q}|}$, then

$$H\psi_{n,e} = E_n\psi_{n,e}, \qquad E_n = -\frac{\mu^2}{2\hbar^2 n^2}, \qquad \lambda = n\hbar, \qquad \hbar = \epsilon^2.$$

We define $\mathbf{Z}_{n,e}(\mathbf{r}) = \dfrac{\hbar}{i}\boldsymbol{\nabla}\ln\psi_{n,e} \to (\boldsymbol{\nabla}S - i\boldsymbol{\nabla}R)(\mathbf{r}) \stackrel{\text{def}}{=} \mathbf{Z}(\mathbf{r})$. We take the Bohr correspondence limit $n \nearrow \infty$, $\epsilon \searrow 0$, for fixed λ i.e. fixed energy $E < 0$. (Refs. [2],[4],[5],[16],[22]). In a previous paper we proved, in Cartesians, that

$$\mathbf{Z}(\mathbf{r}) = \frac{i\mu}{2\lambda}(1+\gamma)\frac{r}{|\mathbf{r}|} + \frac{\mu}{2\lambda e}\left(i,\ \sqrt{1-e^2}, 0\right),$$

where

$$\gamma(\mathbf{r}) = 1 - \lim_{n\nearrow\infty,\ \epsilon\searrow 0}\frac{L'_{n-1}(n\nu(\mathbf{r}))}{L_n(n\nu(\mathbf{r}))} = \sqrt{1 - \frac{4}{\nu(\mathbf{r})}}.$$

This gives the semi-classical state ψ_{sc},

$$\psi_{\text{sc}}(\mathbf{r}) = \nu(\mathbf{r})^{\frac{\lambda}{\hbar}}(1+\gamma(\mathbf{r}))^{\frac{2\lambda}{\hbar}}\exp\left(-\frac{\mu}{2\lambda\hbar}|\mathbf{r}| + \frac{\lambda}{2\hbar}(1-\gamma(\mathbf{r}))\nu(\mathbf{r})\right),$$

so to within an additive constant,

$$(R+iS)(\mathbf{r}) = -\frac{\mu}{\lambda}r + \frac{\lambda\nu(\mathbf{r})}{2}\left(1 - \sqrt{1 - \frac{4}{\nu(\mathbf{r})}}\right) - \lambda\ln\nu(\mathbf{r}) - 2\lambda\ln\left(1 - \sqrt{1 - \frac{4}{\nu(\mathbf{r})}}\right), \nu \text{ above.}$$

(Refs. [5],[6],[7],[8],[14],[15],[16]).

It is difficult to see any connection with the last equation and the Kepler ellipse detailed above. The key is the eccentric anomaly v for a point P on an ellipse.

Definition

For an ellipse $\mathcal{E}$, with major axis parallel to the x-axis, the eccentric anomaly, v, of a point P on $\mathcal{E}$ is the polar angle measured at the centre of $\mathcal{E}$ of the point Q on the circumcircle of $\mathcal{E}$, Q being the image under the parallel projection to the y-axis of P on $\mathcal{E}$ i.e. for cartesian coordinates (x, y) with origin at the right hand focus,

$$x = a(\cos v - e), \qquad y = a\sqrt{1-e^2}\sin v, \qquad r = \sqrt{x^2+y^2} = a(1 - e\cos v),$$

$$\tan\frac{v}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2},$$

(r, θ) polar coordinates of P on $\mathcal{E}$ measured from the focus.

2.2. Remarkable Formulae on $\mathcal{E}_K$

$$\nu = -\frac{(1-ee^{iv})^2}{ee^{iv}}, \qquad \gamma = \sqrt{1-\frac{4}{\nu}} = \frac{(1+ee^{iv})}{(1-ee^{iv})} = \alpha + i\beta;\ \alpha, \beta \in \mathbb{R},$$

where $1-\gamma = -\dfrac{2ee^{iv}}{(1-ee^{iv})}$ and $\dfrac{\nu}{2}(1-\gamma) = (1-ee^{iv})$.

This gives on our ellipse $\mathcal{E}_K$:

$$R + iS = -\frac{\mu}{\lambda}r + \lambda(1-ee^{iv}) - 2\lambda\ln 2 - \lambda\ln(-ee^{iv}).$$

So on $\mathcal{E}_K$, since $r = a(1-e\cos v)$ here,

$$R = -\frac{\mu}{\lambda}r + \lambda(1-e\cos v) - 2\lambda\ln 2 - \lambda\ln e,$$

e being the eccentricity of $\mathcal{E}_K$. So on $\mathcal{E}_K$ to within additive constants,

$$R = -2\lambda\ln 2 - \lambda\ln e, \qquad S = -\lambda(v + e\sin v),$$

v being the eccentric anomaly.

We leave as an exercise in vector algebra the proof of equations $(*)$.

Remark

In fact, if R achieves its global maximum on $\mathcal{E}_K$ and initial value of R is $R(\mathbf{x}_0)$, where $\mathbf{x}_0 = \mathbf{X}^0_{t=0}$, for all $\mathbf{x}_0 \in \mathbb{R}^3$, the passage time from $\mathbf{x}_0$ to $\mathcal{E}_K$, $\tau(\mathbf{x}_0, \mathcal{E}_K)$, is the minimum value of the time s such that

$$R_{\max} - R(\mathbf{x}_0) = \int_0^s |\boldsymbol{\nabla} R|^2(\mathbf{X}^0_u)du.$$

This result is spoiled by the singularity Σ of $(R+iS)$, $\Sigma = \{\nu \in \mathbb{C} : \mathcal{I}\nu = 0, 0 < \mathcal{R}\nu < 4\}$.

Theorem 2.1. $(R + iS)(\boldsymbol{r}) = -\dfrac{\mu}{\lambda}r + \lambda f(\nu)$, *where*

$$f(\nu) = \frac{\nu}{2}\left(1 - \sqrt{1 - \frac{4}{\nu}}\right) - \ln\nu - 2\ln\left(1 - \sqrt{1 - \frac{4}{\nu}}\right).$$

So in the complex plane f has a cut $\{\nu \in \mathbb{C} : \mathcal{I}\nu = 0, 0 < \mathcal{R}\nu < 4\} = \Sigma$, across which $\sqrt{\zeta}$, $\zeta = 1 - \dfrac{4}{\nu}$ flips sign. In fact

$$f(\nu) = \frac{2}{(1+\sqrt{\zeta})} - \ln\left(\frac{1-\sqrt{\zeta}}{1+\sqrt{\zeta}}\right) - \ln 4.$$

Setting $M(\Sigma) = \max_{\mathbf{r}\in\Sigma} R(\boldsymbol{r})$ the above remark is valid for $\boldsymbol{x}_0$, if $R(\boldsymbol{x}_0) > M(\Sigma)$.

Lemma 2.2.

$$M(\Sigma) = \lambda(2 - \ln 4).$$

Proof.

$$\Sigma = \{\nu = \nu_r + i\nu_i : \nu_i = 0, 0 < \nu_r < 4\}, \text{ so } y = 0.$$

On $y = 0$, $\nu(r, x) = \dfrac{\mu}{\lambda^2}\left(r - \dfrac{x}{e}\right)$ where $r = \sqrt{x^2 + z^2}$.

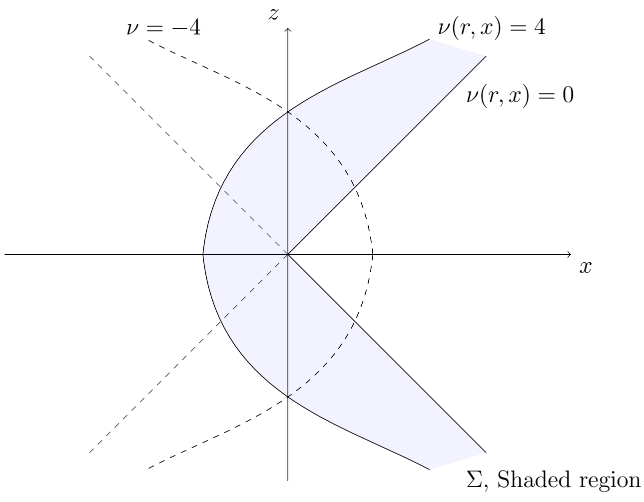

Figure 2.

$$
\begin{aligned}
R &= -\frac{\mu r}{\lambda} + \frac{\mu}{2\lambda}\left(r - \frac{x}{e}\right) - \lambda \ln 4 \\
R &= R_0 - \lambda \ln 4 \\
R_0 &= -\frac{\mu}{2\lambda}\left(r + \frac{x}{e}\right) = -\frac{\lambda}{2}\nu(r, -x)
\end{aligned}
$$

So $M(\Sigma) = \max_{\mathbf{r}\in\Sigma} R(\mathbf{r}) = 2\lambda - \lambda \ln 4 = \lambda(2 - \ln 4)$. $\square$

2.3. Keplerian Elliptic Coordinates in 2-Dimensions

The polar equation of an ellipse in the plane $z = 0$, with eccentricity $\tilde{e}$ and semilatus rectum $\tilde{\ell}$, with focus at origin O and major axis parallel to the x-axis reads

$$\frac{\tilde{\ell}}{r} = 1 + \tilde{e}\cos\theta,$$

which can be written as

$$\frac{\tilde{\ell}^2}{(1-\tilde{e}^2)} = x^2 + \frac{2\tilde{\ell}\tilde{e}x}{(1-\tilde{e}^2)} + \frac{y^2}{(1-\tilde{e}^2)}$$

i.e.

$$(x + \tilde{a}\tilde{e})^2 + \frac{y^2}{(1-\tilde{e}^2)} = \tilde{a}^2 = \frac{\tilde{\ell}^2}{(1-\tilde{e}^2)^2}.$$

So defining the coordinate v by

$$x + \tilde{a}\tilde{e} = \tilde{a}\cos v, \qquad y = \tilde{a}\sqrt{1-\tilde{e}^2}\sin v$$

and the coordinate u by $u = \tilde{e}$ gives if we set $\tilde{a} = \dfrac{2ae}{(e+u)}$,

$$x = \frac{2ae}{(e+u)}(\cos v - u), \qquad y = \frac{2ae}{(e+u)}\sqrt{1-u^2}\sin v.$$

A simple computation gives

$$r = \sqrt{x^2+y^2} = \frac{2ae}{(e+u)}(1 - u\cos v).$$

So v is the eccentric angle of a point P on the ellipse with equation above, with $\tilde{\ell} = \dfrac{2ae}{(e+u)}(1-u^2)$, $\tilde{e} = u$.

N.B. On Keplerian coordinates (u,v), we have assumed $r < 2a$, so the $u=$ constant contours are ellipses with eccentricity u, $0<u<1$, $v=$ eccentric anomaly on this ellipse. When $r>2a$ this changes because here $u<0$ and so the eccentricity of the ellipse in this region is $|u|$ and $v=\pi-$ eccentric anomaly.

We can now recast our equations of motion, $\dot{\mathbf{X}}_t^0 = (\boldsymbol{\nabla} R + \boldsymbol{\nabla} S)(\mathbf{X}_0^t), \mathbf{X}_0^t = (x,y)$, in terms of (u,v):

$$\dot{u} = b_u(u,v), \qquad \dot{v} = b_v(u,v).$$

We obtain for $c=\cos v$ and $s=\sin v$

$$b_u = \sqrt{\frac{\mu}{a^3}}\left(\frac{e+u}{2e}\right)^2 \frac{\sqrt{1-u^2}\left(\sqrt{1-u^2}(e+c-s)-\sqrt{1-e^2}(u+c-s)\right)}{(uc-1)((e+u)c+1+eu)}$$

$$b_v = \sqrt{\frac{\mu}{a^3}}\frac{1}{2e^2}\frac{\left((1+e^2+2ec)u^2+u(2e+(1+e^2)c+(1-e^2)s)-\sqrt{1-u^2}\sqrt{1-e^2}(e+c+s)u\right)}{(1-uc)(1+eu+(e+u)c)}$$

Observe that if we set $u=e$ in the equations for Keplerian coordinates we get the original Kepler ellipse for this problem. Observe that setting $u=e$ gives $b_u=0$ and

$$\dot{v} = \sqrt{\frac{\mu}{a^3}}\frac{1}{1-e\cos v}$$

i.e. Kepler's equation for the eccentric anomaly,

$$v - e\sin v = \sqrt{\frac{\mu}{a^3}}t.$$

When $z=0$, the coordinates u and v enable us to simplify the expressions in (x,y) for variables α and β, in 2-dimensions,

$$\alpha + i\beta = \sqrt{1-\frac{4}{\nu}}, \qquad \nu = \frac{\mu}{\lambda^2}\left(r-\frac{x}{e}-\frac{iy\sqrt{1-e^2}}{e}\right), \qquad r=\sqrt{x^2+y^2}$$

$$\alpha = \left(\frac{1}{2}\sqrt{\frac{\left(er-x-\frac{4\lambda^2 e}{\mu}\right)^2+(1-e^2)y^2}{(er-x)^2+(1-e^2)y^2}}+\frac{1}{2}\frac{\left(er-x-\frac{2\lambda^2 e}{\mu}\right)^2+(1-e^2)y^2-\frac{4\lambda^4 e^2}{\mu^2}}{(er-x)^2+(1-e^2)y^2}\right)^{\frac{1}{2}},$$

$$\beta = -\frac{2\lambda^2 e\sqrt{1-e^2}y}{\alpha\mu((er-x)^2+(1-e^2)y^2)}; \;\; \alpha = \frac{\sqrt{(1-u^2)(1-e^2)}}{(1+eu-(e+u)\cos v)}, \;\; \beta = \frac{(e+u)\sin v}{(1+eu-(e+u)\cos v)}.$$

Writing $\alpha = \sqrt{\frac{1}{2}(\sqrt{\alpha_1} + \alpha_2)}$, if we ask when α_1 is a perfect square, we discover α_1 is a perfect square on the ellipses $\mathcal{E}_u$ with polar equation

$$r = \frac{2ae}{(e+u)}(1 - u\cos v),$$

with focus at the origin O, for fixed u, the eccentricity of $\mathcal{E}_u$, $0 \le v \le 2\pi$, the eccentric anomaly.

3. Levi-Civita, Kustaanheimo-Stieffel (LCKS) Transformations

3.1. LCKS Transformations in Classical Mechanics

Consider the Kepler problem for $\zeta = \zeta(t) \in \mathbb{C}$, t being the physical time,

$$\frac{d^2\zeta}{dt^2} = -\mu|\zeta|^{-3}\zeta, \qquad t \ge 0.$$

Define $w \in \mathbb{C}$ by $\zeta = w^2$ and a new time variable s by $\dfrac{ds}{dt} = |\zeta(t)|^{-1}$. Then a simple calculation yields that the equation for energy conservation

$$2^{-1}\left|\frac{d\zeta}{dt}\right|^2 - \frac{\mu}{|\zeta|} = E, \qquad E < 0$$

a constant here, reduces to

$$2^{-1}\left|\frac{dw}{ds}\right|^2 - \frac{E}{4}|w|^2 = \frac{\mu}{4}$$

and working slightly harder for a constant ω

$$\frac{d^2w(s)}{ds^2} + \omega^2 w(s) = 0.$$

This is the original result due to Levi-Civita; energy conservation and equation of motion for an isotropic oscillator in 2-dimensions with energy $E' = \dfrac{\mu}{4}$ and circular frequency $\omega = \sqrt{-\dfrac{E}{4}}$. Here we investigate the transformation $(x+iy) \to (x+iy)^2$ or for $(x+iy) = re^{i\theta}$, $r \to r^2$, $\theta \to 2\theta$ and the time-change, $\dfrac{ds}{dt} = \dfrac{1}{r} = \sqrt{\dfrac{a}{\mu}}\dfrac{dv}{dt}$, v being the eccentric anomaly on $\mathcal{E}_K$, the ficticous time s, and physical time t (Ref. [24]).

Lemma 3.1. *Under the LCKS Transformation the Kepler ellipse with focus at the origin O*

$$\frac{\ell}{r} = 1 + e\cos\theta, \qquad 0 < e < 1, \qquad \ell > 0,$$

i.e. $r = a(1 - e\cos v)$, *is transformed to the ellipse with origin at the centre of the ellipse*

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1, \qquad p^2 = \frac{\ell}{(1+e)}, \qquad q^2 = \frac{\ell}{(1-e)}.$$

$$(x, y) = a(\cos v - e, \sqrt{1-e^2}\sin v) \to \left(\sqrt{a(1-e)}\cos\left(\frac{v}{2\sqrt{2}}\right), \sqrt{a(1+e)}\sin\left(\frac{v}{2\sqrt{2}}\right)\right).$$

N.B. The major axis for the transformed ellipse is parallel to the y-axis.

There is no LCKS transformation in 3-dimensions; there is one in 4-dimensions using quaternions, but here we work in 2-dimensions. Here $r = \sqrt{x^2 + y^2}$ and
$\nu = \frac{\mu}{\lambda^2}\left(r - \frac{x}{e} - \frac{iy\sqrt{1-e^2}}{e}\right)$, for our 2-dimensional Kepler-Coulomb problem.

3.2. Quantum Harmonic Oscillators

To continue we need to note that the elliptic harmonic oscillator state in 2-dimensions is

$$\psi_n^{\mathrm{HO}} = \exp\left(-\frac{n\omega r^2}{2\tilde{\lambda}}\right) H_n(\sqrt{n}\tilde{u}), \qquad n = 1, 2 \ldots,$$

where $\tilde{u} = \sqrt{\frac{\omega}{\tilde{\lambda}}\left((1-\alpha)\frac{x^2}{2} + (1+\alpha)\frac{y^2}{2} - i\beta xy\right)}$, $\alpha^2 - \beta^2 = 1$, for potential

$V(r) = \frac{1}{2}\omega^2 r^2$, H_n a Hermite polynomial (Ref. [11]).

For the quantum Hamiltonian

$$H = 2^{-1}\mathbf{P}^2 + V(\mathbf{Q}),$$

$$H\psi_n^{\mathrm{HO}} = \omega\hbar(n+1)\psi_n^{\mathrm{HO}}, \qquad \tilde{\lambda} = n\hbar = n\epsilon^2,$$

$\tilde{\lambda}$ being fixed as we take the limit $\epsilon \searrow 0$, $n \nearrow \infty$, so the limiting energy here is $\tilde{\lambda}\omega$. We need the result that for fixed $\tilde{\lambda}$,

$$\lim_{n\nearrow\infty} \frac{H_n'(\sqrt{n}u)}{\sqrt{n}H_n(\sqrt{n}u)} = u - \sqrt{u^2 - 2},$$

giving $\psi_n^{\mathrm{HO}} \cong \exp\left(\dfrac{\tilde{R}+i\tilde{S}}{\epsilon^2}\right)$ as $n \sim \infty$, where

$$(\tilde{R}+i\tilde{S})(x,y) = -\frac{1}{2}\omega r^2 + \frac{\tilde{\lambda}}{2}\tilde{u}^2\left(1-\sqrt{1-\frac{2}{\tilde{u}^2}}\right) + \tilde{\lambda}\ln\left(\tilde{u}+\tilde{u}\sqrt{1-\frac{2}{\tilde{u}^2}}\right).$$

Theorem 3.2. *Under the LCKS transformation:*

$$\nu(x,y) \to \frac{\mu}{\lambda^2}\left(x^2+y^2-\frac{1}{e}(x^2-y^2)-\frac{2ixy\sqrt{1-e^2}}{e}\right).$$

So defining ω *by* $\dfrac{\mu}{\lambda^2} = \dfrac{\omega}{\tilde{\lambda}}$, *if* $\dfrac{C\omega}{2} = \dfrac{\mu}{\lambda}$ *and* $C\tilde{\lambda} = 2\lambda$, *a simple calculation yields under LCKS*

$$(R+iS)(x,y) \to C(\tilde{R}+i\tilde{S})(x,y),$$

C *is a change of time scale here. Needless to say in 2-dimensions*

$$\boldsymbol{\nabla}\tilde{R}.\boldsymbol{\nabla}\tilde{S} = 0, \qquad 2^{-1}\left(|\boldsymbol{\nabla}\tilde{S}|^2 - |\boldsymbol{\nabla}\tilde{R}|^2\right) + V = \tilde{\lambda}\omega,$$

where $V = \dfrac{1}{2}\omega^2 r^2$.

The Levi-Civita-Kustaanheimo-Stieffel transformation in 2-dimensions gives a hint of the following result in 3-dimensions.

Theorem 3.3. *The isotropic harmonic oscillator elliptic state is, up to normalisation,*

$$\psi_n^{\mathrm{HO}} = exp\left(-\frac{\omega r^2}{2\tilde{\lambda}}\right)H_n(\sqrt{n}\tilde{u}),$$

where $r^2 = x^2+y^2+z^2$, $\tilde{u} = \sqrt{\dfrac{\omega}{\tilde{\lambda}}\left((1-\alpha)\dfrac{x^2}{2}+(1+\alpha)\dfrac{y^2}{2}-i\beta xy\right)}$, $\alpha = \dfrac{1}{e}$, $\beta = \dfrac{\sqrt{1-e^2}}{e}$, $\alpha^2-\beta^2 = 1$, e *being the eccentricity of the ellipse with equations*

$$z = 0, \qquad \frac{x^2}{p^2}+\frac{y^2}{q^2} = 1,$$

as above. H_n *being a Hermite polynomial.*

For quantum Hamiltonian $H = 2^{-1}\boldsymbol{P}^2 + 2^{-1}\omega^2\boldsymbol{Q}^2$,

$$H\psi_n^{\mathrm{HO}} = \omega\left(\tilde{\lambda} + \frac{3}{2}\hbar\right)\psi_n^{\mathrm{HO}}, \qquad \tilde{\lambda} = n\hbar = n\epsilon^2.$$

Proof. First $\psi_1(z) = \exp\left(-\frac{\omega z^2}{2\tilde{\lambda}}\right)$ is the ground state of the harmonic oscillator in 1-dimension and if

$$\psi_2(x,y) = \exp\left(-\frac{\omega(x^2+y^2)}{2\tilde{\lambda}}\right) H_n(\sqrt{n}\tilde{u}),$$

$$\boldsymbol{\nabla}\psi_1.\boldsymbol{\nabla}\psi_2 = 0, \ \text{ so } \ \Delta(\psi_1\psi_2) = \psi_1(\Delta\psi_2) + (\Delta\psi_1)\psi_2.$$

Further $\Delta\tilde{u} = 0$, so $\Delta H_n(\sqrt{n}\tilde{u}) = nH_n''(\sqrt{n}\tilde{u})|\boldsymbol{\nabla}\tilde{u}|^2$. Also,

$$\begin{aligned}\Delta\psi_2 &= \Delta\left(\exp\left(-\frac{\omega(x^2+y^2)}{2\tilde{\lambda}}\right) H_n(\sqrt{n}\tilde{u})\right) \\ &= \exp\left(-\tfrac{\omega(x^2+y^2)}{2\tilde{\lambda}}\right)\Delta H_n(\sqrt{n}\tilde{u}) + H_n(\sqrt{n}\tilde{u})\Delta\exp\left(-\tfrac{\omega(x^2+y^2)}{2\tilde{\lambda}}\right) \\ &\qquad +2\boldsymbol{\nabla}\left(\exp\left(-\tfrac{\omega(x^2+y^2)}{2\tilde{\lambda}}\right)\right).\boldsymbol{\nabla}H_n(\sqrt{n}\tilde{u}).\end{aligned}$$

A computation using the properties of Hermite polynomials yields the result

$$-\frac{\hbar^2}{2}\Delta\psi_n^{\mathrm{HO}} + \frac{\omega^2}{2}(x^2+y^2+z^2)\psi_n^{\mathrm{HO}} = \omega\left(\tilde{\lambda} + \frac{3}{2}\hbar\right)\psi_n^{\mathrm{HO}}, \qquad \tilde{\lambda} = n\hbar.$$

□

4. Newton's Revolving Orbits

Building on the results of the previous sections we now discuss the semi-classical system defined by a given central potential, V_0, with the addition of an inverse cube radial force. Specifically we shall consider the systems defined by $V_K = V_0 + \frac{(1-k^2)L^2}{2r^2}$, $k \in \mathbb{R}$, for $V_0 = -\frac{\mu}{r}$ and $V_0 = \frac{1}{2}\omega^2 r^2$, $r^2 = x^2 + y^2$. L is the angular momentum for the system defined by V_0. This system was well known to Newton (Ref. [3])and discussed in detail by Lynden-Bell (Ref. [12]). In essence the result derived for the classical motion is; if $r(\theta)$ describes the orbit under V_0 then $r\left(\frac{\theta}{k}\right)$ defines the orbit under V_K. The key to our semi-classical analysis will be the use of the Hamiltonian and a complex constant of the motion (which seems to have been given little attention in the literature) coupled with a simple transformation in the plane. We shall then observe that the semi-classical motion converges beautifully to the classical motion in the elliptical case as described above.

4.1. Constants of the Motion

Consider a particle of unit mass moving in the (X, Y) plane and governed by the Hamiltonian

$$H = 2^{-1}\mathbf{P}^2 + V_0(\mathbf{Q}).$$

Using polar coordinates (ρ, ϕ) and defining $\mathbf{P} = (P_X, P_Y) = (\dot{X}, \dot{Y})$ it can be shown that the systems above have constants of the motion, in time, defined respectively by:-

$$V_0 = -\frac{\mu}{\rho} : \qquad \frac{1}{2}(P_X^2 + P_Y^2) - \frac{\mu}{\rho} = E, \qquad E < 0,$$

$$\frac{\cos\left(\frac{\phi}{2}\right) P_X + \sin\left(\frac{\phi}{2}\right) P_Y - i\sqrt{-2E}\cos\left(\frac{\phi}{2}\right)}{\cos\left(\frac{\phi}{2}\right) P_Y - \sin\left(\frac{\phi}{2}\right) P_X - i\sqrt{-2E}\sin\left(\frac{\phi}{2}\right)} = -i\frac{\sqrt{1-e}}{\sqrt{1+e}}.$$

$E = -\dfrac{\mu^2}{2\lambda^2}$, is the energy and $0 < e < 1$ is the eccentricity.

$$V_0 = \frac{1}{2}\omega^2\rho^2 : \qquad \frac{1}{2}(P_X^2 + P_Y^2) + \frac{1}{2}\omega^2\rho^2 = E, \qquad E > 0,$$

$$\frac{P_X - i\omega X}{P_Y - i\omega Y} = -\frac{a}{b}i, \qquad a, b \in \mathbb{R}^+.$$

$E = \frac{1}{2}(a^2 + b^2)\omega^2$, is the energy for angular frequency ω.

The two systems are connected by the LCKS transformation and corresponding time-change in 2-dimensions as described in section 3.

We now move to the (x, y) plane and polar coordinates (r, θ) via the transformation $X = r\cos\left(\frac{\theta}{k}\right)$ and $Y = r\sin\left(\frac{\theta}{k}\right)$. If we identify $\mathbf{p} = (p_x, p_y) = (\dot{x}, \dot{y})$ it is not difficult to show that

$$P_X = \frac{1}{r}(xp_x + yp_y)\cos\left(\frac{\theta}{k}\right) - \frac{L}{r}\sin\left(\frac{\theta}{k}\right),$$

$$P_Y = \frac{1}{r}(xp_x + yp_y)\sin\left(\frac{\theta}{k}\right) + \frac{L}{r}\cos\left(\frac{\theta}{k}\right),$$

where L is the angular momentum of the underlying system defined by $V_0(\rho)$. We note that for $V_0 = -\frac{\mu}{\rho}$, $L = \lambda\sqrt{1-e^2}$ and for $V_0 = \frac{1}{2}\omega^2\rho^2$, $L = ab\omega$. Using this transformation, the above set of constants of the motion become

$$\frac{1}{2}(p_x^2+p_y^2)-\frac{\mu}{r}+\frac{(1-k^2)L^2}{2r^2}=E,$$

$$\frac{(xp_x+yp_y)\cos\left(\frac{\theta}{2k}\right)-L\sin\left(\frac{\theta}{2k}\right)-i\sqrt{-2E}r\cos\left(\frac{\theta}{2k}\right)}{(xp_x+yp_y)\sin\left(\frac{\theta}{2k}\right)+L\cos\left(\frac{\theta}{2k}\right)-i\sqrt{-2E}r\sin\left(\frac{\theta}{2k}\right)}=-i\frac{\sqrt{1-e}}{\sqrt{1+e}}$$

and

$$\frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}\omega^2r^2+\frac{(1-k^2)L^2}{2r^2}=E,$$

$$\frac{(xp_x+yp_y)\cos\left(\frac{\theta}{k}\right)-L\sin\left(\frac{\theta}{k}\right)-i\omega r^2\cos\left(\frac{\theta}{k}\right)}{(xp_x+yp_y)\sin\left(\frac{\theta}{k}\right)+L\cos\left(\frac{\theta}{k}\right)-i\omega r^2\sin\left(\frac{\theta}{k}\right)}=-\frac{a}{b}i,$$

respectively.

Observing that each pair of simultaneous equations can be solved for $(p_x,p_y)\in\mathbb{C}^2$ it is natural to define $\mathbf{Z}=(Z_x,Z_y)\in\mathbb{C}^2$ and replace $\mathbf{p}$ by $\mathbf{Z}$ in the above equations. Solving each pair of equations leads to the solution

$$Z_x=(\alpha+i\beta)\frac{x}{r^2}-\frac{y}{r^2}\sqrt{\beta^2-\alpha^2+2r^2(E-V_0(r))-(1-k^2)L^2-2\alpha\beta i},$$

$$Z_y=(\alpha+i\beta)\frac{y}{r^2}+\frac{x}{r^2}\sqrt{\beta^2-\alpha^2+2r^2(E-V_0(r))-(1-k^2)L^2-2\alpha\beta i},$$

where, for:-

$$V_0=-\frac{\mu}{r},\ \alpha=\frac{eL\sin\left(\frac{\theta}{k}\right)}{1+e\cos\left(\frac{\theta}{k}\right)},\ \beta=\frac{1}{\lambda}\left(\mu r-\frac{L^2}{1+e\cos\left(\frac{\theta}{k}\right)}\right),\ L=\lambda\sqrt{1-e^2},\ E=-\frac{\mu^2}{2\lambda^2},$$

and for

$$V_0=\frac{1}{2}\omega^2r^2,\qquad \alpha=\frac{(b^2-a^2)L\sin\left(\frac{2\theta}{k}\right)}{(a^2+b^2)+(b^2-a^2)\cos\left(\frac{2\theta}{k}\right)},\qquad \beta=\omega r^2-\frac{2abL}{(a^2+b^2)+(b^2-a^2)\cos\left(\frac{2\theta}{k}\right)},$$

$L=ab\omega$ and $E=\dfrac{1}{2}(a^2+b^2)\omega^2$.

Moreover, if we define $\mathbf{Z} = \boldsymbol{\nabla} S - i\boldsymbol{\nabla} R$, as in the previous sections we can extract $\boldsymbol{\nabla} S$ and $\boldsymbol{\nabla} R$. In each case, for the appropriate V_0, E, L, α and β,

$$\boldsymbol{\nabla} S = \left(\frac{1}{r^2}(\alpha x - \tilde{u}y), \frac{1}{r^2}(\alpha y + \tilde{u}x)\right),$$

$$\boldsymbol{\nabla} R = \left(-\frac{1}{r^2}(\beta x - \tilde{v}y), -\frac{1}{r^2}(\beta y + \tilde{v}x)\right),$$

where $\tilde{u} + i\tilde{v} = \sqrt{\beta^2 - \alpha^2 + 2r^2(E - V_0(r)) - (1-k^2)L^2 - 2\alpha\beta i}$. Explicitly

$$\tilde{u} = \left\{\frac{1}{2}(\beta^2 - \alpha^2 + 2r^2(E - V_0(r)) - (1-k^2)L^2) \right.$$
$$\left. + \frac{1}{2}\sqrt{(\beta^2 - \alpha^2 + 2r^2(E - V_0(r)) - (1-k^2)L^2)^2 + 4\alpha^2\beta^2}\right\}^{\frac{1}{2}}$$

and

$$\tilde{v} = -\frac{\alpha\beta}{\tilde{u}}$$

$$\tilde{v} = \left\{-\frac{1}{2}(\beta^2 - \alpha^2 + 2r^2(E - V_0(r)) - (1-k^2)L^2) \right.$$
$$\left. + \frac{1}{2}\sqrt{(\beta^2 - \alpha^2 + 2r^2(E - V_0(r)) - (1-k^2)L^2)^2 + 4\alpha^2\beta^2}\right\}^{\frac{1}{2}}$$

Needless to say, it is easy to show that, in each case,

$$2^{-1}(|\boldsymbol{\nabla} S|^2 - |\boldsymbol{\nabla} R|^2) + V = E, \qquad \boldsymbol{\nabla} R.\boldsymbol{\nabla} S = 0,$$

i.e. we have found solutions to the all important ‘semi- classical’ equations $(*)$.

4.2. Semi-classical and Classical Orbits

As defined previously the semi-classical orbit for these systems is governed by the dynamical equation

$$\frac{d}{dt}\mathbf{X}_t^0 = (\boldsymbol{\nabla} S + \boldsymbol{\nabla} R)(\mathbf{X}_t^0).$$

We have also seen that by virtue of its construction the classical orbit is attained in the infinite time limit as R reaches its maximum at which point $\boldsymbol{\nabla} R = \mathbf{0}$. Using the expression for $\boldsymbol{\nabla} R$ above this occurs when

$$\beta x - \tilde{v}y = 0 \text{ and } \beta y + \tilde{v}x = 0.$$

Assuming x and y are not simultaneously 0 i.e. the orbit does not reach the origin in finite time, these can be guaranteed if $\beta^2 + \tilde{v}^2 = 0 \implies \beta = 0$ and $\tilde{v} = 0$ simultaneously.

In the case of $V_0 = -\dfrac{\mu}{r}$,

$$\beta = 0 \implies \frac{\ell}{r} = 1 + e\cos\left(\frac{\theta}{k}\right),$$

where $\ell = \dfrac{L^2}{\mu} = \dfrac{\lambda^2(1-e^2)}{\mu}$.

In the case of $V_0 = \dfrac{1}{2}\omega^2 r^2$,

$$\beta = 0 \implies r^2 = \frac{2a^2b^2}{(a^2+b^2) + (b^2-a^2)\cos\left(\frac{2\theta}{k}\right)}.$$

Using the expression for $\tilde{v}$ we also see that $\tilde{v} = 0$ when $\beta = 0$ i.e. on the classical orbit. These results beautifully display the result of Newton: if $r(\theta)$ describes the orbit under V_0 then $r\left(\dfrac{\theta}{k}\right)$ defines the orbit under V_K.

4.3. The Significance of $\tilde{u}$ and α

Using the dynamical equation, we see that the polar coordinates (r, θ) satisfy particularly simple equations:

$$\frac{dr}{dt} = \frac{\alpha - \beta}{r}, \qquad \frac{d\theta}{dt} = \frac{\tilde{u} - \tilde{v}}{r^2}.$$

On the classical orbit $(\beta, \tilde{v}) = (0, 0)$. In addition, $\tilde{u} = kL$, so that

$$\frac{d\theta}{dt} = \frac{kL}{r^2} \implies r^2\frac{d\theta}{dt} = kL$$

the angular momentum. From this we deduce that $\dfrac{kL}{r}\dfrac{dr}{d\theta} = \alpha$, which is easy to verify in both cases described above.

The figures below show some simulations of semi-classical orbits which clearly display the convergence to the classical paths.

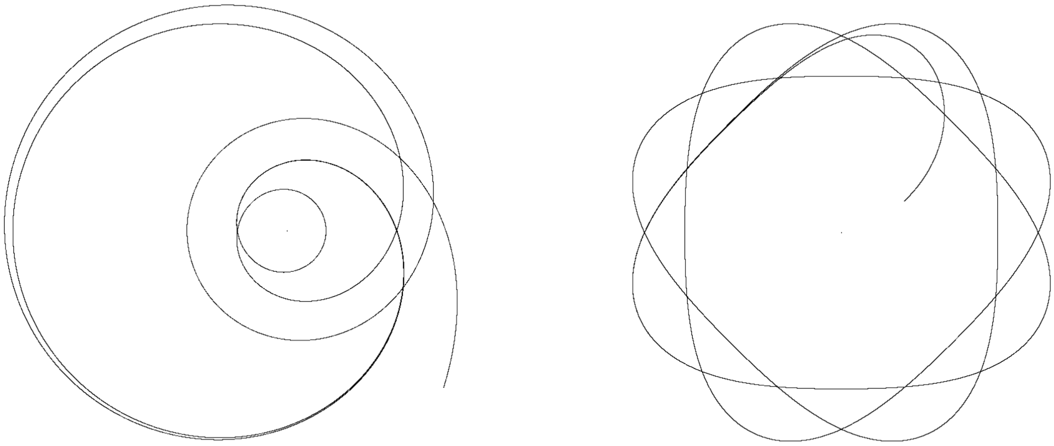

Figure 3.

$$V_0 = -\frac{\mu}{r} \ : \ k = 3 \qquad\qquad V_0 = \frac{1}{2}\omega^2 r^2 \ : \ k = \frac{3}{4}$$

We conclude this section with an application of revolving orbits which comes from general relativity. Here a simple correction to the Newtonian force law accounts for the perihelion precession of a planet, such as Mercury, orbiting a star. The simplest model gives the corrected potential $V(r) = -\frac{\mu}{r} - \frac{3\mu^2}{c^2 r^2}$, where c is the speed of light. We can identify this with a revolving orbit, with $k = \sqrt{1 + \frac{6\mu^2}{c^2 L^2}}$. It is then easy to deduce that the perihelion of the orbit precesses, approximately, by an angle of $\frac{6\pi\mu^2}{c^2 L^2}$ for each turn of the orbit.

In the setting of our semi-classical revolving orbits the convergence of an orbit of this type is illustrated in Figure 4 below.

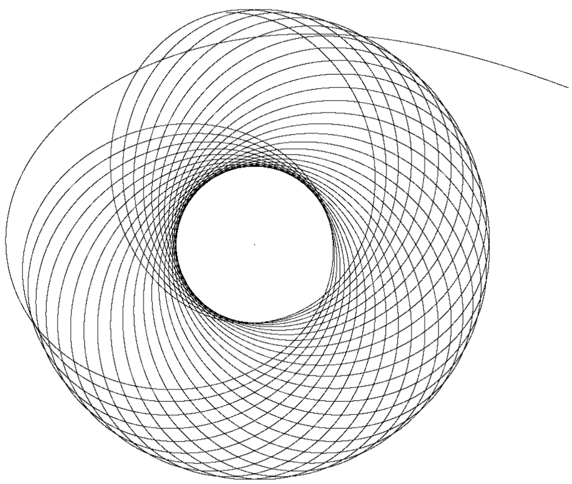

Figure 4.

5. Cauchy's Method of Characteristics and a Generalised LCKS Transformation in 2-dimensions

Here we focus on the orthogonality of the level curves of R and S,

$$\boldsymbol{\nabla} R.\boldsymbol{\nabla} S = 0,$$

assuming R is known exactly or approximately and that $S(x,y) = S_0(x,y)$, for $(x,y) \in \mathbb{C}_0$, curve of classical orbit, S_0 the Hamilton Jacobi function.

5.1. Method of Characteristics

Assume $\dfrac{\partial R}{\partial x} \neq 0$ then we need

$$\frac{\partial S}{\partial x} + \frac{\dfrac{\partial S}{\partial y}\dfrac{\partial R}{\partial y}}{\dfrac{\partial R}{\partial x}} = 0,$$

so if $y = y(x)$ and

$$\frac{dy}{dx} = \frac{\dfrac{\partial R}{\partial y}(x,y)}{\dfrac{\partial R}{\partial x}(x,y)}, \qquad \frac{d}{dx}S(x,y(x) = 0.$$

So for each point $(x_0, y_0) \in \mathbb{C}_0$ define $\dfrac{dy}{dx}(x; x_0, y_0)$ as above, with

$$\frac{dy}{dx}(x; x_0, y_0) = -\frac{\partial S_0}{\partial x}(x_0, y_0) \Bigg/ \frac{\partial S_0}{\partial y}(x_0, y_0)$$

and $y(x_0; x_0, y_0) = y_0,\ (x_0, y_0) \in \mathbb{C}_0$, then

$$S(x, y(x; x_0, y_0)) = S_0(x_0, y_0) \text{for each } (x_0, y_0) \in \mathbb{C}_0.$$

It follows that necessarily:

$$\frac{\partial S}{\partial x} + y' \frac{\partial S}{\partial y} = 0, \qquad \frac{\partial R}{\partial x} - \frac{1}{y'}\frac{\partial R}{\partial y} = 0,$$

so that

$$|\boldsymbol{\nabla} S|^2 = \left(\frac{\partial S}{\partial y}\right)^2 (1 + y'^2), \qquad |\boldsymbol{\nabla} R|^2 = \left(\frac{\partial R}{\partial y}\right)^2 \left(1 + \frac{1}{y'^2}\right).$$

Energy conservation then reduces to

$$(S_y)^2\left(1+\frac{R_y^2}{R_x^2}\right)-(R_y)^2\left(1+\frac{S_y^2}{R_x^2}\right)=2(E-V),$$

$R_x=\dfrac{\partial R}{\partial x}$ etc. In this case we have an implicit solution of Eqs($*$). Eliminating V we obtain:-

Lemma 5.1. *A necessary condition for semi-classical Eqs($*$) to have a solution is that*

$$\left(1+\frac{S_y^2}{R_x^2}\right)|\boldsymbol{\nabla}R|^2=\left(1+\frac{R_y^2}{S_x^2}\right)|\boldsymbol{\nabla}S|^2$$

and modulo some mild regularity assumptions on R and S, necessary and sufficient conditions are that additionally,

$$2(E-V)=|\boldsymbol{\nabla}R|^2\left(\frac{S_y^2}{R_x^2}-1\right).$$

Remarks

1. Sufficient regularity conditions can be deduced from Nagumo's results (Ref. [10]).

2. Above results reveal the structure of the level curves of R and S in the neighbourhood of C_0 on which R is known to have a maximum value and S_0 is known approximately. We return to this theme later.

3. It is easy to check the above conditions for all examples herein.

4. We need a more powerful result for explicit solutions.

5.2. Eplicit Solutions in 2-dimensions

Where the potential V is central, the classical angular momentum is conserved and the resulting motion can be assumed to be confined to the plane $z=0$. We work in $\mathcal{N}$, a 2-dimensional neighbourhood of an arc of C_0, the classical orbit, the neighbourhood being assumed to be simply connected. To be specific, $\mathcal{N}=\bigcup\limits_{\theta\in I,\delta(\theta)>0}N((r_0(\theta),\theta),\delta(\theta))$, in polars, $(r_0(\theta),\theta)$ being the centre and $\delta(\theta)>0$ being the radius of N, $(r_0(\theta),\theta)\in C_0$, and $\theta\in I$, a real interval. We assume that the quantum particle density of state, ψ, satisfies, $\rho(x,y) \sim C\exp\left(\frac{2R(x,y)}{\epsilon^2}\right)$, as $\epsilon^2 \sim 0$, the real valued R achieving its global maximum on C_0, where $\boldsymbol{\nabla} R = 0$ and $R = R_{\max}$.

To satisfy our equations $(*)$ we write $\boldsymbol{\nabla} S(x,y) = \lambda(x,y)\left(-\frac{\partial R}{\partial y}, \frac{\partial R}{\partial x}\right)$, where $\lambda(x,y) \sim O(|\boldsymbol{\nabla} R|^{-1})$ as $(x,y) \sim (x_0,y_0) \in C_0$. Assuming at all points of the arc of C_0 we have a continuously turning tangent $\mathbf{t}(C_0)$, with $|\mathbf{t}(C_0)| = 1$ it is necessary that

$$\boldsymbol{\nabla} S(x,y) \sim \lambda(x,y)|\boldsymbol{\nabla} R|(x,y)\mathbf{t}(C_0)(x_0,y_0)$$

as $(x,y) \sim (x_0,y_0)$. In any case

$$\frac{\partial S}{\partial x} = -\lambda\frac{\partial R}{\partial y}, \qquad \frac{\partial S}{\partial y} = \lambda\frac{\partial R}{\partial x}$$

and assuming C^2 behaviour

$$\frac{\partial^2 S}{\partial x\partial y} = \frac{\partial^2 S}{\partial y\partial x} = -\frac{\partial\lambda}{\partial y}\frac{\partial R}{\partial y} - \lambda\frac{\partial^2 R}{\partial y^2} = \frac{\partial\lambda}{\partial x}\frac{\partial R}{\partial x} + \lambda\frac{\partial^2 R}{\partial x^2},$$

i.e., λ has to satisfy the (R,V) equation:

$$\lambda\Delta R = -\boldsymbol{\nabla}\lambda.\boldsymbol{\nabla} R$$

and by energy conservation $\lambda = \pm\sqrt{\left(1 + \frac{2(E-V)}{|\boldsymbol{\nabla} R|^2}\right)}$.

Defining for $C(\mathbf{r}_0,\mathbf{r})$ a simple curve from $\mathbf{r}_0 \in C_0$ to $\mathbf{r} \in \mathcal{N}$, the line integral,

$$S(x,y) - S(x_0,y_0) = \int_{C(\mathbf{r}_0,\mathbf{r})}\left(-\lambda\frac{\partial R}{\partial y}, \lambda\frac{\partial R}{\partial x}\right).d\mathbf{r}$$

predicated on λ satisfying (R,V) equation gives a unique S defined in $\mathcal{N}$ with

$$\boldsymbol{\nabla} S = \pm\sqrt{|\boldsymbol{\nabla} R|^2 + 2(E-V)}(\widehat{\boldsymbol{\nabla} R^{\perp}}), \text{ off } C_0,$$

where $\boldsymbol{\nabla} R^{\perp} = \left(-\frac{\partial R}{\partial y}, \frac{\partial R}{\partial x}\right)$, $(\widehat{\boldsymbol{\nabla} R^{\perp}})$ being $\mathbf{t}(C_0)$ on the classical orbit.

Evidently as required $S\rceil_{C_0} = S_0$, the classical Hamilton Jacobi function.

We have proved modulo some mild regularity conditions:-

Theorem 5.2. *Assuming,* $\lambda = \pm\sqrt{\left(1 + \dfrac{2(E-V)}{|\boldsymbol{\nabla} R|^2}\right)}$*, satisfies the* (R, V) *equation*

$$\lambda \Delta R = -\boldsymbol{\nabla}\lambda.\boldsymbol{\nabla} R,$$

i.e. the flow with vector field $\lambda\boldsymbol{\nabla} R$ *is incompressible and has no sources or sinks,*

the solution to our equations ($*$*), in 2-dimensions, for a given* R*, is*

$$\boldsymbol{\nabla} S = \pm\sqrt{|\boldsymbol{\nabla} R|^2 + 2(E-V)}(\widehat{\boldsymbol{\nabla} R^{\perp}}).$$

In particular in a neighbourhood of C_0, the classical orbit, on the level surface,

$$R = R_0 = \Big\{(r,\theta) : r = r_0(\theta) + r_1(\theta),\ \frac{r_1^2}{2}\frac{\partial^2 R}{\partial r^2}(r_0, \theta_0) = R_0 - R_{\max}, \\ C : r = r_0(\theta), \theta \in (0, 2\pi),\ \text{say } as\ R \sim R_{\max}\Big\},$$

we obtain the leading term,

$$\boldsymbol{\nabla} S \sim \pm\sqrt{2(E-V)}\left(1 + \frac{|\boldsymbol{\nabla} R|^2}{4(E-V)}\right)\mathbf{t}(R = R_0),\ \text{as } R \sim R_{\max},$$

$\mathbf{t}(R = R_0)$ being the unit tangent to the level surface (contour), $R = R_0$, and tangent to C_0 if $R_0 = R_{\max}$.

This gives detailed information on the behaviour of the semi-classical limit in a narrow strip containing C_0 in terms or R, which we assume is known, at least as far as $\dfrac{\partial^2 R}{\partial r^2}(r_0, \theta)$, $\theta \in (0, 2\pi)$ and $|\boldsymbol{\nabla} R|^2$, in N.

Lemma 5.3. *Let* $R(x, y) = R_0$ *be a level surface of* R *in a neighbourhood,* $\mathcal{N}$*, of the classical orbit* C_0*, with polar equation,* $r = r_0$*,* $0 < \theta \leq 2\pi$*, then*

$$R(x, y) \sim R_{max} + \frac{(r - r_0(\theta))^2}{2}\frac{\partial^2 R}{\partial r^2}(r_0(\theta), \theta) + O\left((r - r_0(\theta))^3\right).$$

and the leading term for $|\boldsymbol{\nabla} R|^2$ *is given by,*

$$|\boldsymbol{\nabla} R|^2 \sim 2(R_0 - R_{max})\frac{\partial^2 R}{\partial r^2}(r_0, \theta)\left(1 + \frac{(r_0'(\theta))^2}{r^2}\right).$$

Proof. Modulo sufficient regularity,

$$\frac{\partial R}{\partial r} \sim (r - r_0(\theta))\frac{\partial^2 R}{\partial r^2}(r_0(\theta), \theta)$$

and

$$\frac{1}{r}\frac{\partial R}{\partial \theta} \sim -\frac{(r - r_0}{r}r_0'(\theta) + \frac{(r - r_0(\theta))^2}{2}r_0'(\theta)\frac{\partial^3 R}{\partial r^3}(r_0(\theta), \theta).$$

Since $R = R_{\max}$ on C_0, a simple computation gives the desired result. □

In 2-dimensions the formula for ∇S, apart from the obvious change of direction, merely includes a $|\nabla R|^2$ term, with $V \to V - \frac{|\nabla R|^2}{2}$ and

$$-\frac{|\nabla R|^2}{2} = \frac{(R_{\max} - R_0)}{r^2}\frac{\partial^2 R}{\partial r^2}(r_0, \theta)\left(r^2 + (r_0'(\theta))^2\right).$$

Since

$$\frac{d}{dt}\mathbf{X}_t^0 = (\nabla R + \nabla S)(\mathbf{X}_t^0),$$

we see that as $R_0 \sim R_{\max}$ on the level surface $R = R_0$,

$$h - h_0 \cong \frac{(R_{\max} - R_0)}{r}\frac{\partial^2 R}{\partial r^2}(r_0, \theta)\left(r^2 + (r_0'(\theta))^2\right) + \frac{\partial R}{\partial \theta}(r, \theta),$$

where $r_0'(\theta) = \frac{dr_0}{d\theta}$, showing how the shape of the classical orbit $r = r_0(\theta)$ affects the quantum correction to angular momentum, h_0 being the classical angular momentum. Needless to say for central potentials V, h_0 is a constant.

Corollary 5.3.1. *Modulo the (R, V) equation being satisfied and some mild regularity, in general, off the classical orbit C_0,*

$$r^2\dot{\theta} = \left(\frac{\partial R}{\partial \theta}(r, \theta) + r\sqrt{|\nabla R|^2 + 2(E - V)}\left(|\nabla R|^{-1}\frac{\partial R}{\partial r}\right)\right)\Bigg]_{R=R_0},$$

with above interpretation of the bracketed second term on C_0, where we reiterate we assume R is known. Similarly,

$$\dot{r} = \left(\frac{\partial R}{\partial r}(r, \theta) - \frac{\sqrt{|\nabla R|^2 + 2(E - V)}}{r}\left(|\nabla R|^{-1}\frac{\partial R}{\partial \theta}\right)\right)\Bigg]_{R=R_0}.$$

We conclude this section with the related result for circular orbits.

Theorem 5.4. *For the spiral circular orbit, C_0, in any central potential V, solving the (R, V) equation in polars (r, θ) yields:-*

$$R + iS = f(r) + ih\theta,$$

where, for $(') = \frac{d}{dr}$,

$$r^2 f'^2(r) = 4 \int_a^r \rho^2 \left(\frac{V'(\rho)}{2} + \frac{V(\rho) - E}{\rho} \right) d\rho,$$

with $\frac{V'(a)}{2} + \frac{V(a) - E}{a} = 0$, *$a$ being the radius of the circular orbit and E and h, the energy and angular momentum respectively. Moreover*

$$(f''(a))^2 = -\frac{d}{dr} \left(\frac{2(E - V(r))}{r} - V'(r) \right) \Bigg]_{r=a} = V''(a) + \frac{3h^2}{a^4},$$

so the particle density on C_0,

$$\rho \sim C_\epsilon exp \left(\frac{1}{\epsilon^2} \left(2R_{max} - \left(V''(a) + \frac{3h^2}{a^4} \right)^{\frac{1}{2}} (r - a)^2 \right) \right),$$

provided $V''(a) + \frac{3h^2}{a^4} > 0$, *i.e. the classical condition for a stable circular orbit. C_ϵ is determined by normalisation.*

Proof. Given $R + iS = f(r) + ih\theta$, the (R, V) equation implies

$$\frac{1}{r}(rf'(r))' + \frac{1}{2} \boldsymbol{\nabla} \ln \left(1 + \frac{2(E - V(r))}{f'^2(r)} \right) .f'(r)\hat{\mathbf{r}} = 0$$

i.e.

$$f'' f'^3 + f'^2 \left(\frac{f'^2 + 2(E - V)}{r} - V' \right) = 0$$

which can be integrated, with $f'(a) = 0$, to give

$$r^2 f'^2(r) = 4 \int_a^r u^2 \left(\frac{V'(u)}{2} + \frac{V(u) - E}{u} \right) du.$$

Since we need a repeated root at $r = a$ we require

$$\frac{V'(a)}{2} + \frac{V(a) - E}{a} = 0$$

as well as the classical condition $\left.\frac{d}{dr}\left(V + \frac{h^2}{2r^2}\right)\right]_{r=a} = 0$, where h is the angular momentum. Moreover, for $f'(r) \neq 0$,

$$f'' = -\frac{\left(\frac{f'^2 + 2(E - V)}{r} - V'\right)}{f'}.$$

An application of de l'Hopital's rule for $r \to a$ gives

$$(f''(a))^2 = -\left.\frac{d}{dr}\left(\frac{2(E - V(r))}{r} - V'(r)\right)\right]_{r=a} = V''(a) + \frac{3h^2}{a^4},$$

and the result for the particle density follows easily.

$\square$

An example for the above is provided by motion in the potential field

$$V(r) = -\frac{\mu}{r} + \frac{1}{2}\omega^2 r^2 + \frac{\omega\mu}{h} r$$

in 2-dimensions, for positive μ, ω and h. This represents a Coulomb (neutron star/black hole), harmonic oscillator and dark energy potential. The exact solution to our equations $(*)$, in this setting, is

$$R + iS = f(r) + iS = -\frac{1}{2}\omega r^2 - \frac{\mu}{h} r + h \ln r + ih\theta.$$

It is easy to check that $f'(r) = 0$ implies a circular orbit at

$$a = \frac{\sqrt{\mu^2 + 4\omega h^3} - \mu}{2\omega h}$$

and that $(f''(a))^2 = V''(a) + \frac{3h^2}{a^4} = \omega^2 + \frac{h^2 + 2\omega h a^2}{a^4} > 0$, giving a stable circular orbit.

A computer simulation for the corresponding semi-classical motion is shown in Figure 6 at the end of this paper.

5.3. Power Law Problem for Zero Energy in 2-dimensions

Quite recently the zero energy power law problem has attracted the attention of several researchers. An excellent survey is given in the paper by A.J. Makovski and K.J. Gorska (Ref. [13]). In our setting it reduces to solving the Eqs($*$) for $V(r) = -\kappa r^p$, most importantly,

$$2^{-1}(|\boldsymbol{\nabla} S|^2 - |\boldsymbol{\nabla} R|^2) - \kappa r^p = 0, \qquad \boldsymbol{\nabla} R.\boldsymbol{\nabla} S = 0, \qquad \boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right),$$

where $\boldsymbol{\nabla} = \boldsymbol{\nabla}_z$, $z = x + iy$, $r = \sqrt{(x^2 + y^2)} = |z|$. We linearise this problem by changing variables to $\boldsymbol{\nabla}_Z$, $Z = X + iY$, $Z = (z)^{(p+2)/2}$, then $f(z) = z^{(p+2)/2}$

$$|\boldsymbol{\nabla}_z f|^2 = |f'(z)|^2 |\boldsymbol{\nabla}_Z f|^2$$

giving

$$2^{-1}(|\boldsymbol{\nabla}_Z S|^2 - |\boldsymbol{\nabla}_Z R|^2) - \kappa |z|^p |f'(z)|^{-2} = 0$$

i.e.

$$2^{-1}(|\boldsymbol{\nabla}_Z S|^2 - |\boldsymbol{\nabla}_Z R|^2) - \kappa |z|^p \left(\frac{p+2}{2}\right)^{-2} |z|^{-p} = 0$$

i.e.

$$2^{-1}(|\boldsymbol{\nabla}_Z S|^2 - |\boldsymbol{\nabla}_Z R|^2) = \frac{4\kappa}{(p+2)^2}$$

Writing $R = \dfrac{\beta\tilde{\mu}}{\sqrt{\alpha^2 + \beta^2}}(\beta Y + \alpha X)$, $S = \dfrac{\alpha\tilde{\mu}}{\sqrt{\alpha^2 + \beta^2}}(-\beta X + \alpha Y)$, for constants $\alpha, \beta, \tilde{\mu}$, for cartesian X and Y,

$$\boldsymbol{\nabla} R = \frac{\beta\tilde{\mu}}{\sqrt{\alpha^2 + \beta^2}}(\alpha, \beta), \qquad \boldsymbol{\nabla} S = \frac{\alpha\tilde{\mu}}{\sqrt{\alpha^2 + \beta^2}}(-\beta, \alpha)$$

So, if $\tilde{\mu} = 2\sqrt{2\kappa}/(p+2)$, $\alpha^2 - \beta^2 = 1$,

$$\boldsymbol{\nabla} R.\boldsymbol{\nabla} S = 0, \qquad 2^{-1}(|\boldsymbol{\nabla} S|^2 - |\boldsymbol{\nabla} R|^2) = 2^{-1}\tilde{\mu}^2(\alpha^2 - \beta^2) = \frac{4\kappa}{(p+2)^2}$$

Hence, in the original variables $(x, y) = (r\cos\theta, r\sin\theta)$,

$$X = r^{(p+2)/2}\cos\left(\frac{(p+2)}{2}\theta\right), \qquad Y = r^{(p+2)/2}\sin\left(\frac{(p+2)}{2}\theta\right)$$

and

$$R = \frac{\beta\tilde{\mu}}{\sqrt{\alpha^2+\beta^2}}(\beta Y + \alpha X), \qquad S = \frac{\alpha\tilde{\mu}}{\sqrt{\alpha^2+\beta^2}}(-\beta X + \alpha Y),$$

solving Eqs($*$). The last example is the inspiration for the next section.

5.4. Holomorphic Change of Variables of Semi-Classical LCKS Transformation

Assume $\omega = \xi + i\eta = f(x+iy)$, $z = x+iy$, f being holomorphic and invertible for $z \in \mathbb{D} \subset \mathbb{C}; \xi, \eta, x, y \in \mathbb{R}$. We write:- $x = x(\xi,\eta), y = y(\xi,\eta); \xi = \xi(x,y), \eta = \eta(x,y)$,

and assume $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{\nabla}_z(R+S) = \begin{pmatrix} \partial_x(R+S) \\ \partial_y(R+S) \end{pmatrix}$.

Changing variables for any C^1 function $g(x,y)$, $(x,y) \to (\xi,\eta)$,

$$\begin{pmatrix} \dfrac{\partial g}{\partial x} \\ \dfrac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \dfrac{\partial g}{\partial \xi} \\ \dfrac{\partial g}{\partial \eta} \end{pmatrix}$$

i.e.

$$\boldsymbol{\nabla}_z g = \begin{pmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{pmatrix} \boldsymbol{\nabla}_w g.$$

From the Cauchy Riemann equations

$$\boldsymbol{\nabla}_z g.\boldsymbol{\nabla}_z h = |f'(z)|^2 \boldsymbol{\nabla}_w g.\boldsymbol{\nabla}_w h,$$

for any C^1 functions g, h. So for our functions R, S,

$$\boldsymbol{\nabla}_z R.\boldsymbol{\nabla}_z S = |f'(z)|^2 \boldsymbol{\nabla}_w R.\boldsymbol{\nabla}_w S$$

and

$$2^{-1}(|\boldsymbol{\nabla}_z S|^2 - |\boldsymbol{\nabla}_z R|^2) + (V-E) = 2^{-1}(|\boldsymbol{\nabla}_w S|^2 - |\boldsymbol{\nabla}_w R|^2)|f'(z)|^2 + (V-E).$$

So, assuming $f'(z) \neq 0$,

$$\boldsymbol{\nabla}_w R.\boldsymbol{\nabla}_w S = 0, \qquad 2^{-1}(|\boldsymbol{\nabla}_w S|^2 - |\boldsymbol{\nabla}_w R|^2) + |f'(z)|^{-2}(V - E) = 0.$$

Now, if $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \boldsymbol{\nabla}_z(R+S)$, $(\dot{}) = \dfrac{d}{dt}$, t being the time, since

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta}\\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta}\end{pmatrix}\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix}$$

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix} = \begin{pmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta}\\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta}\end{pmatrix}^{-1}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y}\\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y}\end{pmatrix}\begin{pmatrix}\dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x}\\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y}\end{pmatrix}\boldsymbol{\nabla}_w(R+S).$$

From the Cauchy-Riemann equations

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix} = |f'(z)|^2\begin{pmatrix}\dfrac{\partial}{\partial \xi}(R+S)\\ \dfrac{\partial}{\partial \eta}(R+S)\end{pmatrix},$$

so, if $ds = dt|f'(z(t))|^2$ is the time-change and $\begin{pmatrix}\xi(s)\\ \eta(s)\end{pmatrix}$ inherits the corresponding initial conditions as $\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$, $\dfrac{d}{ds} = (')$ we obtain

$$\begin{pmatrix}\xi'\\ \eta'\end{pmatrix} = \boldsymbol{\nabla}_w(R+S),$$

as expected. This follows from the obvious theorem for $\mathbb{D} \subset \mathbb{C}$, an open simply connected set.

Theorem 5.5. *If $f(z)$ is a holomorphic, invertible function of $z \in \mathbb{D}$ with $|f'(z)| \neq 0$ in $\mathbb{D}$, the transformation $\omega = f(z)$ transforms the Schrödinger equation in 2 dimensions,*

$$-2^{-1}\hbar^2\Delta_z\psi + (V-E)\psi = 0, \qquad z \in \mathbb{D},$$

to

$$-2^{-1}\hbar^2\Delta_w\psi + (V-E)|f'|^{-2}\psi = 0, \qquad w \in f(\mathbb{D}).$$

Proof. Cauchy-Riemann equations □

Corollary 5.5.1. *(Semi-classical LCKS tranformation)*

Modulo above assumptions on f, $w = f(z)$ transforms Eqs($*$) to:

$$2^{-1}(|\boldsymbol{\nabla}_w S|^2 - |\boldsymbol{\nabla}_w R|^2) + |f'(z)|^{-2}(V-E) = 0, \qquad \boldsymbol{\nabla}_w R.\boldsymbol{\nabla}_w S = 0$$

and if s is the above time-change, $\dfrac{d}{ds} = (')$,

$$\xi' = \frac{\partial}{\partial \xi}(R+S), \qquad \eta' = \frac{\partial}{\partial \eta}(R+S),$$

with appropriate initial conditions.

Further, if $R(x,y) = R_{\max}$ on C_0, and $|\boldsymbol{\nabla} R| = 0$ on C_0, transformed R inherits same properties guaranteeing convergence to corresponding classical orbit, $f(C_0)$, since

$$\frac{d}{ds}R(\xi,\eta) = \boldsymbol{\nabla}_w(R+S).\boldsymbol{\nabla}_w R = |\boldsymbol{\nabla} R|^2 \geq 0,$$

and, if the classical orbit is periodic, the time-change is valid since it will take infinite s time to reach $f(C_0)$

Remarks

1. (ξ, η) are orthogonal coordinates. We can regard new potential energy function as $W = |f'|^{-2}(V-E)$ and new energy is 0. So we can investigate the zero energy limit for new potential W.

2. It is now easy to check, $\omega = f(z)$, transforms constrained Hamiltonion system $H = \dfrac{p^2}{2} + V_{\text{eff}}(q)$, $(p - \boldsymbol{\nabla} R).\boldsymbol{\nabla} R = 0$, $V_{\text{eff}} = V - |\boldsymbol{\nabla} R|^2$, into the one corresponding with the above system.

6. Newtonian Quantum Gravity and Deviations from Keplerian Motion

The results in this section are heavily dependent on our work on constrained Hamiltonians underlying our asymptotic analysis in Refs. [14], [15] and [16]. These results are a simple consequence of taking the ultimate Bohr correspondence limit of the atomic elliptic state $\psi_{n,e}$, $\psi_{n,e} \sim \psi_{\text{sc}} = \exp\left(\frac{R+iS}{\epsilon^2}\right)$ as $\epsilon \sim 0$. They are predicated on the fact that R attains its global maximum on the Kepler ellipse $\mathcal{E}_K$, in the plane $z = 0$, with polar equation,

$$\frac{a\left(1-e^2\right)}{r}=1+e\cos\theta,$$

a being the semimajor axis of the elliptical orbit, parallel to the x-axis, e its eccentricity:

$$e=\sqrt{1+\frac{2L^2E}{\mu^2}},$$

where L is the orbital angular momentum, $E<0$ the total energy and μ is the gravitational mass at the force centre O. (see Ref. [6]). To simplify our results we need to assume the initial conditions:

$$R\left(\mathbf{X}^0_{t=0}\right)\geqslant M(\Sigma)=\lambda(2-\ln 4),\qquad \lambda=(\mu a)^{1/2}.$$

6.1. Convergence to the Plane $z=0$

Recall that

$$\frac{\dot z}{z}=\frac{-\mu(\alpha+\beta+1)}{2\lambda r},\qquad r=\sqrt{x^2+y^2+z^2}.$$

Since $\dot z=0$ when $z=0$, the orbiting particle takes an infinite time to reach the plane $z=0$, so we seek to understand the motion in this infinite time limit in which $u\to e$ and

$$\alpha\to\frac{(1-e^2)}{(1+e^2-2e\cos v)},\qquad \beta\to\frac{2e\sin v}{(1+e^2-2e\cos v)},$$

where $v\nearrow\infty$, v being the eccentric anomaly on ellipse $\mathcal{E}_K$, with

$$\frac{dv}{dt}=\sqrt{\frac{\mu}{a^3}}\frac{1}{(1-e\cos v)}$$

and

$$\tan^2\left(\frac{v}{2}\right)=\frac{(1-e)}{(1+e)}\tan^2\left(\frac{\theta}{2}\right),$$

where θ is the polar angle of orbiting particle. Hence, we obtain

$$\frac{dz}{dt}\to-\frac{1}{2}\frac{(2-e^2+2e\sin v)}{(1+e^2-2e\cos v)}\qquad\text{as}\qquad v\nearrow\infty,$$

where we assume $z(0)>0$, giving the rate of convergence in the form:

Theorem 6.1.

$$z(v) \to C\left(1+e^2-2e\cos v\right)^{-1/2} \exp\left(-\frac{1}{2}\frac{(2-e^2)}{(1-e^2)}\tan^{-1}\left(\frac{1+e}{1-e}\tan\left(\frac{v}{2}\right)\right)\right),$$

for some constant C, as $v \nearrow \infty$ for $0 < e < 1$.

6.2. Convergence to Kepler Ellipse in 3-Dimensions

The Keplerian coordinates (u, v) come into their own in this setting

$$x = \frac{2ae(\cos v - u)}{(e+u)}, \qquad y = \frac{2ae\sqrt{(1-u^2)}\sin v}{(e+u)}, \qquad v \in \mathbb{R}.$$

The Kepler ellipse $\mathcal{E}_K$ corresponds to $u = e$, the singularity Σ to $u = 1$ and $u = -e$ is the curve at infinity.

$$\alpha = \frac{\sqrt{(1-u^2)(1-e^2)}}{(1+eu-(e+u)\cos v)}, \qquad \beta = \frac{(e+u)\sin v}{(1+eu-(e+u)\sin v)},$$

$$b_x = \frac{\mu}{2\lambda e}\frac{\left(eu-(e-u)\cos v-(e+u)\sin v+\sqrt{(1-e^2)(1-u^2)}-1\right)}{(1-u\cos v)},$$

$$b_y = \frac{\mu}{2\lambda e}\frac{\left(\sqrt{1-u^2}(e\cos v-e\sin v+1)+\sqrt{1-e^2}(u\cos v+u\sin v-1)\right)}{(1-u\cos v)}$$

and the Jacobian

$$\Delta = \frac{\partial(x,y)}{\partial(u,v)} = \frac{4a^2e^2}{(e+u)^3\sqrt{1-u^2}}(u\cos v-1)((e+u)\cos v+1+eu).$$

In the infinite time limit we can take v to be the eccentric anomaly on the Kepler ellipse, $\mathcal{E}_K$, itself. With this proviso we consider the ordinary differential equation

$$\frac{du}{dv} = \frac{b_u(u,v)}{b_v(u,v)}, \qquad \frac{du}{dt} = b_u(u,v), \qquad \frac{dv}{dt} = b_v(u,v),$$

where in the limit

$$b_v = \sqrt{\frac{\mu}{a^3}}\frac{1}{(1-e\cos v)},$$

b_u as given in section 2.3. We then have the simple result:

Theorem 6.2. *For $v > v_0$, $(v - v_0)$ fixed as $v_0 \nearrow \infty$,*

$$\frac{u(v) - e}{u(v_0) - e} \to \left(\frac{\cos v_0 + \frac{1}{2}\left(e + e^{-1}\right)}{\cos v + \frac{1}{2}\left(e + e^{-1}\right)} \right)^{1/2} \exp\left(-(f(v) - f(v_0))\right)$$

where $f(v) = \dfrac{v}{2} + \tan^{-1}\left(\dfrac{1-e}{1+e}\tan\left(\frac{v}{2}\right)\right)$.

Proof. Since $b_u(e, v) = 0$,

$$\frac{d}{dv}\ln(u(v) - e) = \frac{\frac{du}{dv}}{(u-e)} = \frac{b_u(u,v)}{b_v(u,v)(u-e)} \to \frac{\frac{\partial b_u}{\partial u}(e,v)}{b_v(e,v)},$$

giving

$$\frac{d}{dv}\ln(u(v) - e) \to -\frac{1}{2}\frac{(1 + e\cos v - e\sin v)}{(\cos v + \frac{1}{2}(e + e^{-1}))},$$

from which the result follows. □

6.3. Transition to the Classical Era

The cognoscenti will regard the above as a 'half Nelson'. To them it will be important to repeat the above argument but include the Nelson noise coming from stochastic mechanics (Refs. [17],[18],[19],[20]). In this set up for orbiting particles of mass m_0, $\epsilon^2 = \frac{\hbar}{m_0}$ and we seek the limit as $m_0 \nearrow \infty$. Here we model this situation by assuming $\epsilon = \epsilon(u) \searrow 0$ as $u \to e$, as a result of mass accretion. In the last section $u(v)$, the solution of the above ordinary differential equation, corresponds to u^ϵ when $\epsilon = 0$ where late in the semi-classical era u^ϵ satisfies, as $\epsilon \sim 0$,

$$du^\epsilon = b_u(u^\epsilon, v)dt + \epsilon dN_u(t),$$

$$\frac{dv}{dt} = \sqrt{\frac{\mu}{a^3}}\frac{1}{(1 - e\cos v)}, \qquad N_u(t), \text{Nelson noise.}$$

So here we denote $u(v)$ as $u^0(v)$.

A simple calculation yields for $\mathbf{r}_v^\perp = \left(\dfrac{\partial y}{\partial v}, -\dfrac{\partial x}{\partial v}\right)$

$$dN_u(t) = \Delta^{-1}(\mathbf{X}_t^0)\mathbf{r}_v^{\perp}.d\mathbf{B}(t),$$

where $\mathbf{B}(t) = (B_x, B_y)$, is a BM($\mathbb{R}^2$) process.

A simple time-change to the elliptic anomaly v gives

$$dN_u(v) = \frac{(1 - e\cos v)^{1/2}}{a^2 e}\left(\frac{a^3}{\mu}\right)^{1/4}\frac{\left(a\sqrt{1-e^2}\cos v dB_x(v) + a\sin v dB_y(v)\right)}{(\cos v - e^{-1})(\cos v + \frac{1}{2}(e + e^{-1}))}.$$

So for W, a BM($\mathbb{R}$) process, correlated to B_x and B_y,

$$du^{\epsilon}(v) = \frac{b_u}{b_v}dv + \frac{\epsilon(u^{\epsilon})\sqrt{1-e^2}\sqrt{1+e\cos v)}}{(a\mu)^{1/4}(\cos v + \frac{1}{2}(e + e^{-1}))}dW(v).$$

Assuming $\epsilon(u = e) = 0$, letting $v \nearrow \infty$ gives

$$d\ln\left(\frac{u^{\epsilon} - e}{u^0 - e}\right) = \frac{\epsilon'(e)\sqrt{1-e^2}\sqrt{1-e\cos v}}{(a\mu)^{1/4}(\cos v + \frac{1}{2}(e + e^{-1}))}dW(v).$$

So, if u^{ϵ} and u^0 agree initially in this era,

$$(u^{\epsilon}(v)) = (u^0(v) - e)\exp\left(\frac{\epsilon'(e)\sqrt{1-e^2}}{(a\mu)^{1/4}}W(g(v))\right),$$

where

$$g(v) = \int_{v_0}^{v}\frac{(1 + e\cos v)}{(\cos v + \frac{1}{2}(e + e^{-1}))^2}dv.$$

For small eccentricities, $e \sim 0$, we have the log normal result:

$$\ln\left(\frac{u^{\epsilon} - e}{u^0 - e}\right) \sim \frac{2e\epsilon'(e)}{(a\mu)^{1/4}}W(v) \qquad \text{as} \qquad v \sim \infty,$$

where v is the eccentric anomaly on the Kepler ellipse $\mathcal{E}_{u=e}$.

More generally as $v \sim \infty$,

$$\ln\left(\frac{u^{\epsilon}(v) - e}{u^0(v) - e}\right) \sim \frac{2e\epsilon'(e)}{\sqrt{1-e^2}(a\mu)^{1/4}}W\left(\frac{2}{(1-e^2)}\tan^{-1}\left(\frac{1-e}{1+e}\tan\frac{v}{2}\right) + \frac{e\sin v}{(1 + 2e\cos v + e^2)}\right),$$

and

$$\left[\sqrt{\frac{\mu}{a^3}}(v^\epsilon(v)-v)\right]_\alpha^\beta \sim \int_\alpha^\beta \frac{\partial b_v}{\partial u}(e,v)(u^\epsilon(v)-e)(1-e\cos v)dv,$$

where v^ϵ is the eccentric anomaly on $\mathcal{E}_{u^\epsilon}$.

These are Nelsonian quantum corrections to Kepler's laws to first order of approximation in a neighbourhood of the classical orbit for the atomic elliptic state of Lena, Delande and Gay expressed in terms of Keplerian coordinates. We hope to investigate these more fully for $\epsilon = \epsilon(u) = \sqrt{\frac{\hbar}{m_0}}$, where $m_0 = m_0(u) \sim \infty$ as $u \to e$, m_0 so large that $\epsilon(u=e) = O(\epsilon_0^2)$, where $\epsilon_0 = \sqrt{\frac{\hbar}{m_{00}}}$, m_{00} being the initial mass of the orbiting particle before mass accretion starts.

6.4. Complex Constants of the Motion

For the stationary state, $\psi_{n,e}$, the Hamiltonian, $H = 2^{-1}\mathbf{P}^2 - \mu|\mathbf{Q}|^{-1}$, is a constant, $H = E$, $E < 0$ being the energy. Moreover, if $\tilde{\mathbf{A}} = (-2E)^{-1/2}\mathbf{A}$, $\mathbf{A}$ being the Hamilton–Lenz–Runge vector and $\mathbf{L}$ is the orbital angular momentum, $\mathbf{A}$ and $\mathbf{L}$ are quantum constants of the motion generating the dynamical symmetry group SO(4). Setting the Bohr correspondence limits equal to $\tilde{\mathbf{a}} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$, $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3)$, of $\mathbf{A}$ and $\mathbf{L}$ defined by the Bohr limits of cartesian coordinates,

$$\tilde{a}_i = \lim \psi_{n,e}^{-1}\tilde{A}_i\psi_{n,e}, \qquad \ell_i = \lim \psi_{n,e}^{-1}L_i\psi_{n,e}, \qquad i = 1,2,3,$$

where $\psi_{n,e} \neq 0$, for $\mathbf{Z} = \lim(-i\hbar\boldsymbol{\nabla}\ln\psi_{n,e})$, $\mathbf{Z} = -i\boldsymbol{\nabla}R + \boldsymbol{\nabla}S$, we obtain $\boldsymbol{\ell} = \mathbf{r}\wedge\mathbf{Z}(\mathbf{r})$, $\mathbf{a} = \mathbf{Z}\wedge(\mathbf{Z}\wedge\mathbf{r}) - \mu r^{-1}\mathbf{r}$, the semi-classical variables inheriting Pauli's identities for $\mathbf{L}$ and $\mathbf{A}$. So, defining $\tilde{\mathbf{a}} = (-2E)^{-1}\mathbf{a}$, we have the following identities:-

$$\begin{aligned} \ell_3 + \tilde{a}_1\sin\theta &= \lambda && (1)\\ \ell_1\cos\theta + i\ell_2 - \tilde{a}_3\sin\theta &= 0 && (2)\\ \tilde{a}_3\cos\theta + \ell_1\sin\theta &= 0 && (3)\\ \tilde{a}_1\cos\theta + i\tilde{a}_2 - \ell_3\sin\theta &= 0 && (4) \end{aligned}$$

where $\sin\theta = e$, the eccentricity. (Refs. [15],[16]). Here E has to be the classical energy. Combining equations for $\lambda = \mu(-2E)^{-1/2}$,

$$\begin{aligned} &(1)\times\sin\theta + (4)\times\cos\theta && \tilde{a}_1 + i\tilde{a}_2\cos\theta = \lambda\sin\theta && \text{(i)}\\ &(2)\times\cos\theta + (3)\times\sin\theta && \ell_1 + i\ell_2\cos\theta = 0 && \text{(ii)} \end{aligned}$$

So $\ell_1 = -i\ell_2 \cos\theta$ giving in (3)

$$\tilde{a}_3 - i\ell_2 \sin\theta = 0 \qquad \text{(iii)}$$
$$\ell_3 - i\tilde{a}_2 \sin\theta = \lambda\cos\theta \qquad \text{(iv)}$$

Writing $\boldsymbol{\ell} = \boldsymbol{\ell}^{\mathrm{r}} + i\boldsymbol{\ell}^{\mathrm{i}}$, $\tilde{\mathbf{a}} = \tilde{\mathbf{a}}^{\mathrm{r}} + i\tilde{\mathbf{a}}^{\mathrm{i}}$, r real part, i imaginary part, assuming non-zero denominators,

$$\frac{\ell_3^{\mathrm{i}}}{\tilde{a}_2^{\mathrm{r}}} = -\frac{\tilde{a}_3^{\mathrm{r}}}{\ell_2^{\mathrm{i}}} = \frac{\tilde{a}_3^{\mathrm{i}}}{\ell_2^{\mathrm{r}}} = e, \qquad \frac{\ell_1^{\mathrm{i}}}{\ell_2^{\mathrm{r}}} = -\frac{\ell_1^{\mathrm{r}}}{\ell_2^{\mathrm{i}}} = \frac{\tilde{a}_1^{\mathrm{i}}}{\tilde{a}_2^{\mathrm{r}}} = -\sqrt{1-e^2}.$$

This leaves the real parts of Equations (i) and (iv), giving, where $\psi_{\mathrm{sc}} \neq 0$,

$$\cos\theta = \frac{\lambda\ell_3^{\mathrm{r}} + \tilde{a}_1^{\mathrm{r}}\tilde{a}_2^{\mathrm{i}}}{(\ell_3^{\mathrm{r}})^2 + (\tilde{a}_1^{\mathrm{r}})^2}, \qquad \sin\theta = \frac{\lambda\tilde{a}_1^{\mathrm{r}} - \ell_3^{\mathrm{r}}\tilde{a}_2^{\mathrm{i}}}{(\ell_3^{\mathrm{r}})^2 + (\tilde{a}_1^{\mathrm{r}})^2}; \qquad \sin\theta = e.$$

(Ref. [21]).

6.5. 0/0 Limits for the Kepler Problem

The last two identities are the generalisations of Newton's results for planetary motion:

$$e = \sqrt{1 + \frac{2\Lambda E}{\mu}}, \qquad \Lambda = \frac{(\ell_3^{\mathrm{r}})^2}{\mu},$$

Λ being the semi-latus rectum of $\mathcal{E}_K$, the Keplerian ellipse. As $z \to 0$ in approaching the plane of $\mathcal{E}_K$, in a 3-dimensional neighbourhood of $\mathcal{E}_K$, with the exception of ℓ_3^{r} and $\tilde{a}_1^{\mathrm{r}}$, the above observables are zero in the limit or are small $O(|\boldsymbol{\nabla} R|)$ where $|\boldsymbol{\nabla} R| \to 0$ as we approach $\mathcal{E}_K$.

In fact $\boldsymbol{\ell}^{\mathrm{i}} = -(\mathbf{x} \wedge \boldsymbol{\nabla} R)$, $\boldsymbol{\ell}^{\mathrm{r}} = (\mathbf{x} \wedge \boldsymbol{\nabla} S)$, $\mathbf{a}^{\mathrm{i}} = -\boldsymbol{\nabla} S \wedge (\mathbf{x} \wedge \boldsymbol{\nabla} R) - \boldsymbol{\nabla} R \wedge (\mathbf{x} \wedge \boldsymbol{\nabla} S)$,

$\mathbf{a}^{\mathrm{r}} = -\boldsymbol{\nabla} S \wedge (\mathbf{x} \wedge \boldsymbol{\nabla} S) - \boldsymbol{\nabla} R \wedge (\mathbf{x} \wedge \boldsymbol{\nabla} R) - \frac{\mu}{|\mathbf{x}|}\mathbf{x}$ giving when $z = 0$, $\ell_1^{\mathrm{i}} = \ell_2^{\mathrm{i}} = \ell_1^{\mathrm{r}} = \ell_2^{\mathrm{r}} = 0$ and

$$\frac{\ell_3^{\mathrm{i}}}{\tilde{a}_2^{\mathrm{r}}} \to \frac{\left(y\dfrac{\partial R}{\partial x} - x\dfrac{\partial R}{\partial y}\right)\sqrt{-2E}}{r\left(\dfrac{\partial S}{\partial r}\dfrac{\partial R}{\partial y} + \dfrac{\partial S}{\partial y}\dfrac{\partial R}{\partial r}\right)},$$

$$\frac{\tilde{a}^{\mathrm{i}}_1}{\tilde{a}^{\mathrm{r}}_2} \to \frac{\left(\dfrac{\partial S}{\partial r}\dfrac{\partial R}{\partial r} + \dfrac{\partial R}{\partial r}\dfrac{\partial S}{\partial x}\right)}{\left(\dfrac{\partial S}{\partial r}\dfrac{\partial R}{\partial y} + \dfrac{\partial S}{\partial y}\dfrac{\partial R}{\partial r}\right)}.$$

Since the above are C^1 functions of u for $u \in (e, e + \delta e)$ by de l'Hopital

$$\frac{\ell^{\mathrm{i}}_3}{\tilde{a}^{\mathrm{r}}_2} \to \left.\frac{\dfrac{d}{du}\left(y\dfrac{\partial R}{\partial x} - x\dfrac{\partial R}{\partial y}\right)\sqrt{-2E}}{r\dfrac{d}{du}\left(\dfrac{\partial S}{\partial r}\dfrac{\partial R}{\partial y} + \dfrac{\partial S}{\partial y}\dfrac{\partial R}{\partial r}\right)}\right]_{u=0} \qquad \text{as } \ u \to e,$$

with a similar result for $\dfrac{\tilde{a}^{\mathrm{i}}_1}{\tilde{a}^{\mathrm{r}}_2}$. So for 2-dimensional motion in a neighbourhood of $\mathcal{E}_K$, differentiating with respect to u the identities $\tilde{a}^{\mathrm{i}}_1 + \sqrt{1-e^2}\tilde{a}^{\mathrm{r}}_2 = 0$, $\ell^{\mathrm{i}}_3 - e\tilde{a}^{\mathrm{r}}_2 = 0$,

$$\left.\frac{\tilde{a}^{\mathrm{i}\,\prime}_1}{\tilde{a}^{\mathrm{r}\,\prime}_2}\right]_{u=e} = -\sqrt{1-e^2}, \qquad \left.\frac{\ell^{\mathrm{i}\,\prime}_3}{\tilde{a}^{\mathrm{r}\,\prime}_2}\right]_{u=e} = e\sqrt{-2E}, \ \ \forall v,$$

i.e.

$$\frac{\tilde{a}^{\mathrm{i}}_1}{\tilde{a}^{\mathrm{r}}_2} = -\sqrt{1-e^2}, \qquad \frac{\ell^{\mathrm{i}}_3}{\tilde{a}^{\mathrm{r}}_2} = e\sqrt{-2E}, \ \ \forall v,$$

so the left hand sides are constant on $\mathcal{E}_K$ and

$$\sqrt{1-e^2} = \frac{\lambda\ell^{\mathrm{r}}_3 + \tilde{a}^{\mathrm{r}}_1\tilde{a}^{\mathrm{i}}_2}{(\ell^{\mathrm{r}}_3)^2 + (\tilde{a}^{\mathrm{r}}_1)^2}, \qquad e = \frac{\lambda\tilde{a}^{\mathrm{r}}_1 - \ell^{\mathrm{r}}_3\tilde{a}^{\mathrm{i}}_2}{(\ell^{\mathrm{r}}_3)^2 + (\tilde{a}^{\mathrm{r}}_1)^2}.$$

These results are inherited from SO(4) symmetry. Note we have proved the constancy of $\dfrac{\ell^{\mathrm{i}}_3}{\tilde{a}^{\mathrm{r}}_2}$ and $\dfrac{\tilde{a}^{\mathrm{i}}_1}{\tilde{a}^{\mathrm{r}}_2}$ on the Kepler ellipse $\mathcal{E}_K$, as well as the last expressions for e and $\sqrt{1-e^2}$, although the last two are just Newton's results. Similar results can be obtained for Isotropic Oscillators discussed herein.

7. Conclusion

We have seen in the Bohr correspondence limit the asymptotics of Newtonian gravity for dark matter particles in the atomic elliptic state, $\psi_{n,e}$, for the Kepler problem (and its image under the LCKS transformation - the isotropic harmonic oscillator elliptic state) naturally lead to orbits converging to their classical counterparts. This helps to explain how planets such as Jupiter could be formed in a few million years as well as planetary ring systems such as that of Saturn - something random collisions alone could not account for.

Further in this context we have seen how the Bohr correspondence limit of Pauli's equations for the generators of the dynamical symmetry group here, SO(4) and SU(3), lead to 8 identities involving the functions of space R and S, where $\psi \sim \exp\left(\frac{R+iS}{\epsilon^2}\right)$ as $\epsilon^2 = \frac{\hbar}{m} \to 0$, m being the particle mass. These identities embody complex constants of the motion, involving the real and imaginary parts of the semi-classical angular momentum and Hamilton Lenz Runge vectors. One of these gives an exact analogue in this semi-quantum setting of Newton's results for revolving orbits in Principia (Ref. [3]).

Of course some of these results reveal quantum corrections to Kepler's laws which could lead to experimental tests of the validity of our ideas. So here we have also given more details of the motion of semi-classical dark matter particles e.g. WIMPs in a neighbourhood of their corresponding classical Keplerian elliptical orbits. For WIMPs carrying some electrical charge in the presence of a magnetic field of a neutron star these results are still relevant. (See forthcoming paper Ref. [25]).

Finally we have investigated how Cauchy's method of characteristics enters into this setting in proving existence and uniqueness theorems for our underlying equations for the functions R and S for stationary states. Here we give just one example of how the quantum particle density limit in the form of R determines the corresponding function S. We show how for the zero energy power law this gives the correct limiting orbits in 2 dimensions. Needless to say such results are relevant for all central force problems in this context given angular momentum conservation means the resulting motion ultimately is planar. We plan to give more results of this in the future.

We conclude with the paradigm of semi-classical orbits arising in the Kepler problem - the semi-classical, circular spiral orbit, which we dub the quantum spiral, corresponding to classical circular orbits with radius a and centre O, with periodic time $\dfrac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}$, where μ is the gravitating point mass at O. One never encounters spirals in the classical Kepler problem but the first quantised Kepler problem abounds with them. Could this be a factor in the result that 72% $\to$ 80% of galaxies are spiral galaxies and less than 30% elliptical, spiral galaxies being younger than elliptical ones?

To simplify the presentation we choose unit of length to be a and unit of time to be $\dfrac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}$. With these assumptions, let $\mathbf{X}_t^0 = (x_t, y_t, z_t)$ and set $r_t = \sqrt{x_t^2 + y_t^2 + z_t^2}$, $z_t = r_t \cos(\theta_t)$ and $\phi_t = \tan^{-1}\left(\dfrac{y_t}{x_t}\right)$. R and S are given explicitly in Ref. [8] and it is proved that the semi-classical equation for r_t, for $t \geq 0$, is

$$\dot{r_t} = \frac{1}{r_t} - 1.$$

Given $r_0 > 1$, for $r_t > 1$, $\dot{r_t} < 0$, we have

$$r_t - 1 = T(t)\mathrm{e}^{-r_t},$$

where $T(t) = \mathrm{e}^{-t}\mathrm{e}^{r_0}(r_0 - 1) < 1$, if $t > r_0 + \ln(r_0 - 1)$. For sufficiently large t, we obtain from Lagrange's expansion

$$r_t = 1 + \sum_{n=1}^{\infty} \frac{(T(t))^n}{n!} D_a^{n-1}(\mathrm{e}^{-na})\rceil_{a=1}$$

i.e.

$$r_t = 1 + \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-nt}}{n!} \mathrm{e}^{-nr_0}(r_0 - 1)^n(-n)^{n-1}\mathrm{e}^{-n} \to 1 \qquad \text{as} \qquad t \nearrow \infty.$$

We also proved in the same reference that

$$z_t = r_t \cos(\theta_t) = \frac{r_t}{r_0} \exp\left(-\int_0^t \frac{ds}{r_s^2}\right) z_0 \to \frac{r_t}{r_0} z_0 \mathrm{e}^{-t} \to 0 \qquad \text{as} \qquad t \nearrow \infty$$

and, moreover,

$$r_t^2 \dot{\phi}_t \to 1 \qquad \text{as} \qquad t \nearrow \infty.$$

This is the simplest example of the spiral orbits converging to the Keplerian ellipses in the infinite time limit with eccentricity zero - see Figure 5 below. The same results obtain for any Keplerian ellipse with eccentricity e, $0 < e < 1$ in our semi-classical treatment.

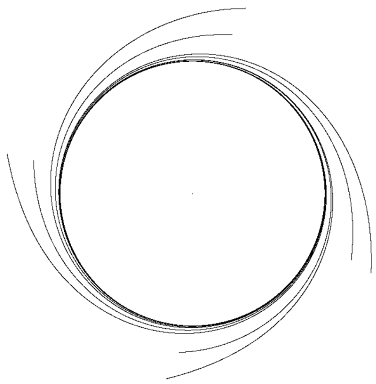

Figure 5. : $\psi_{n,n-1,n-1}(x, y, z) = (x + iy)^{n-1} \exp\left(-\frac{(x^2 + y^2 + z^2)^{\frac{1}{2}}}{\epsilon^2}\right).$

The above diagram gives the computer simulation of the transition of the structure of a spiral galaxy with 4 trailing arms to that of an elliptical galaxy with zero eccentricity. This is easily generalised and argues that there is an abundance of dark matter and other similar material in the observable universe behaving in a manner predicted by a first quantisation of the Kepler problem. Hence the present work on its asymptotics. We believe that the first place to look for these is the non-relativistic molecular gases in the arms of spiral galaxies.

So far we did not discuss in detail the requirements that the S term in our equations off the classical orbit be a gradient (c.f. (R, V) equation on page 21). A careful reading of Nelson's original ideas on stochastic mechanics tells us that when this fails there has to be some underlying motion of the aether invalidating the stochastic version of Newton's 2$^{\text{nd}}$ Law encapsulated in the Schrödinger formulation. In our formulation at the corresponding stage we can only conclude that there is no stationary state satisfying our requirements. Given the astrnomical length scales over which we are testing the asymptotics of this Schrödinger picture it is not surprising if this occurs here sometimes. What is truly amazing to our minds is that the asymptotics of the Schrödinger wavefunctions for states considered herein (especially the astronomical elliptic states of Lena, Delande and Gay) seem capable of explaining not only the rapid formation of ring systems for massive planets and solar systems themselves but also the evolution of some galaxies. These are the laboratories in which our ideas need to be tested, as our future work will show.

Finally we include a diagram resulting from the analysis at the end of section (5) of a Coulomb, harmonic oscillator and dark energy potential inspired by Whittaker (Ref. [26]). (See also Ref. [23]).

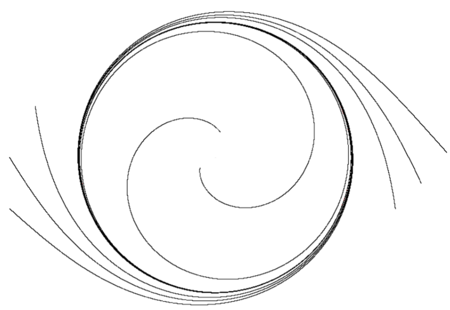

Figure 6.

Dedication and Thanks

This work is dedicated to Sir Michael Atiyah, a personal mentor and friend of one of us and an inspiration to us all, he is so sorely missed. It is also a pleasure to thank our teachers G.B.S, G.R.S., J.C.T. and D.W.

References

[1] Albeverio, S. Blanchard, P. and Høegh-Krohn, R. “A stochastic model for the orbits of planets and satellites: An interpretation of Titius-Bode law,” Exposition Math. 1, 365–373 (1983).

[2] Bohr, N., “Uber die anwendung der quantentheorie auf den atombau. I. Die grundpostulaten der quantentheorie,” Z. Phys. 13, 117–165 (1923).

[3] Chandrasekhar, S., Newton’s Principia fof the Common Reader (Clarendon Press, Oxford, 1995).

[4] Dirac, P.A.M., Lectures on Quantum Mechanics (Dover, New York, 1964).

[5] Durran, R., Neate, A., and Truman, A., “The divine clockwork: Bohr’s correspondence principle and Nelson’s stochastic mechanics for the atomic elliptic state”, J. Math. Phys. 49(3), 032102 (2008).

[6] Durran, R., Neate, A., Truman, A., and Wang, F.-Y., “On the divine clockwork: The spectral gap for the correspondence limit of the Nelson diffusion generator for the atomic elliptic state,” J. Math. Phys. 49(10), 102103 (2008).

[7] Durran, R., Neate, A., Truman, A., and Wang, F.-Y., “The correspondence limit of the atomic elliptic state,” Europhys. Lett. 84(3), 30003 (2008).

[8] Durran, R. and Truman, A., “Planetesimal diffusions,” in Stochastic Mechanics and Stochastic Processes (Swansea, 1986), Lecture Notes in Mathematics Vol. 1325 (Springer, Berlin, 1988), pp. 76–88.

[9] Gay, J. C., Delande, D., and Bommier, A., “Atomic quantum states with maximum localization on classical elliptical orbits”, Phys. Rev. A 39, 6587–6590 (1989).

[10] Hille, E., Lectures on Ordinary Differential Equations (Addison Wesley, Great Britain, 1969).

[11] Lena, C., Delande, D., and Gay, J. C., “Wave functions of atomic elliptic states”, Europhys. Lett. 15(7), 697–702 (1991).

[12] Lynden-Bell, D., “Hamilton’s Eccentricity Vector Generalised to Newton Wonders”, Observatory - Didcot. 126(1192), 176-182 (2006).

[13] Makowski, A.J. and Gorska, K.J. , “Fractional and integer angular momentum wave functions”, J. Phys. A 40, 11373-11383 (2007).

[14] Neate, A. and Truman, A., “A stochastic Burgers-Zeldovich model for the formation of planetary ring systems and the satellites of jupiter and saturn,” J. Math. Phys. 54(3), 033512 (2013).

[15] Neate, A. and Truman, A., “Semi-classical wave functions and semi-classical dynamics for the Kepler/Coulomb problem”, J. Phys. A 47(22), 225302 (2014).

[16] Neate, A. and Truman, A., “Semi-classical stochastic mechanics for Coulomb potential with applications to dark matter”, J. Math. Phys. 57(5), 052103 (2016).

[17] Nelson, E., Dynamical Theories of Brownian Motion (Princeton University Press, Princeton, NJ, 1967).

[18] Nelson, E., Quantum Fluctuations (Princeton University Press, Princeton, NJ, 1985).

[19] Nelson, E., “Field theory and the future of stochastic mechanics”, in Stochastic Processes in Classical and Quantum Systems (Proceedings, Ascona, Switzerland, 1985), Lecture Notes in Physics Vol. 262 (Springer, Berlin, 1986), pp. 438–469.

[20] Nelson, E., “Review of stochastic mechanics”, J. Phys.: Conf. Ser. 361(1), 012011 (2012).

[21] Pauli, W., “Über das wasserstoffspektrum vom standpunkt der neuen quantenmechanik”, Z. Phys. 36, 336–363 (1926).

[22] Rothe, H. J. and Rothe, K. D., Classical and Quantum Dynamics of Constrained Hamiltonian Systems, World Scientific Lecture Notes in Physics (World Scientific, 2010).

[23] Silbergleit, A. and Chernin, A., Kepler Problem in the Presence of Dark Energy, and the Cosmic Local Flow (Springer Briefs in Physics, 2019).

[24] Stiefel, E.L. and Scheifele, G., Linear and Regular Celestial Mechanics (Springer, New York, 1971).

[25] Truman, A., Neate, A. and Durran, R., “An Introduction to the Equatorial Orbitals of Toy Neutron Stars”, In preparation.

[26] Whittaker, E.T., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies (Cambridge University Press, 1937).

On completion of this work we received a copy of the following paper, containing a more complete set of references, which is also relevant to Nelson's stochastic mechanics:

Cresson, J., Nottale, L., and Lehner, T. "Stochastic modification of Newtonian dynamics and induced potential - Application to spiral galaxies and the dark potential" J. Math. Phys. 62, 072702 (2021).