SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON-FREDHOLM OPERATORS RELATED TO THE DOUBLE SCALE ANOMALOUS DIFFUSION IN HIGHER DIMENSIONS

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Abstract: We study solvability of certain linear nonhomogeneous equations involving the sum of the two distinct fractional powers of a Schrödinger operator in higher dimensions and establish that under reasonable technical assumptions the convergence in $L^2(\mathbb{R}^d)$ of the right sides implies the existence and the convergence in $L^2(\mathbb{R}^d)$ of the solutions. The problems contain the operators without the Fredholm property and we use the methods of the spectral and scattering theory for the Schrödinger type operators similarly to our preceding work [21].
AMS Subject Classification: 35J15, 35R11
Key words: solvability conditions, non-Fredholm operators, function spaces

1. Introduction

Consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ converges to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$, which corresponds to the left side of equation (1.1) contains the origin. Consequently, this operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work deals with the studies of certain properties of the operators of this kind. Let us recall that the elliptic equations involving the non-Fredholm operators were treated extensively in recent years (see [10], [18], [19], [20], [22], [23], [24], [25], [26], [27], [28], also [4]) along with their potential applications to the theory of reaction-diffusion problems (see [8], [9]). Non-Fredholm operators are also very important when studying the wave systems with an infinite number of the localized traveling waves (see [1]). In particular, when $a = 0$ the operator A satisfies the Fredholm property in certain properly chosen weighted spaces (see [2], [3], [5], [6], [4]). However, the case of $a \neq 0$ is considerably different and the method developed in these works cannot be applied.

One of the important issues concerning the equations with non-Fredholm operators is their solvability. We address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Designate by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n,\ n \in \mathbb{N}.$$

Because the operator A does not satisfy the Fredholm property, the sequence u_n may not be convergent. We call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of problem $Au = f$ (see [18]). If this sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this problem. The solution in the sense of sequences is equivalent in this case to the usual solution. However, in the case of the non-Fredholm operators, this convergence may not hold or it can occur in some weaker sense. In this case, the solution in the sense of sequences may not imply the existence of the usual solution. In the present work we will find the sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent. Solvability in the sense of sequences for the non-Fredholm Schrödinger type operators raised to a fractional power minus a nonnegative constant was discussed in [21]. The present work is our modest attempt to generalize these results. In the first part of the article we consider the equation

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}^{s_1}u+$$
$$+\{-\Delta_x + V(x) - \Delta_y + U(y)\}^{s_2}u = f(x, y), \quad x, y \in \mathbb{R}^3 \tag{1.2}$$

with the powers $0 < s_1 < s_2 < 1$. The operator in the left side of problem (1.2)

$$H_{U,\,V} := \{-\Delta_x + V(x) - \Delta_y + U(y)\}^{s_1} + \{-\Delta_x + V(x) - \Delta_y + U(y)\}^{s_2} \tag{1.3}$$

is defined by virtue of the spectral calculus. Here and below the Laplacians Δ_x and Δ_y are acting on the x and y variables respectively. The sum of the two Schrödinger type operators involved in both terms in the right side of (1.3) has the physical meaning of the cumulative hamiltonian of the two non interacting three dimensional quantum particles in external potentials. The fractional powers of second order differential operators are actively used, for example in the studies of the anomalous diffusion problems (see e.g. [29], [30], [31] and the references therein). The probabilistic realization of the anomalous diffusion was discussed in [15]. The equations involving the sum of the disctinct fractional powers of a differential operator similarly to (1.2) above are relevant to the studies of the double scale anomalous diffusion. The form boundedness criterion for the relativistic Schrödinger operator was established in [14]. The article [13] is devoted to proving the imbedding theorems and the studies of the spectrum of a certain pseudodifferential operator.

The scalar potential functions involved in operator $H_{U,\,V}$ are assumed to be shallow and short-range, satisfying the assumptions analogous to the ones of [22] and [23].

Assumption 1.1. *The potential functions* $V(x), U(y) : \mathbb{R}^3 \to \mathbb{R}$ *satisfy the bounds*

$$|V(x)| \leq \frac{C}{1+|x|^{3.5+\varepsilon}}, \quad |U(y)| \leq \frac{C}{1+|y|^{3.5+\varepsilon}}$$

with a certain $\varepsilon > 0$ *and* $x, y \in \mathbb{R}^3$ *a.e. so that*

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|V\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1, \tag{1.4}$$

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|U\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{9}}\|U\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}^{\frac{8}{9}} < 1 \tag{1.5}$$

and

$$\sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi, \quad \sqrt{c_{HLS}}\|U\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Here C denotes a finite positive constant and c_{HLS} given on p.98 of [12] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\right| \leq c_{HLS}\|f_1\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^2, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

The norm of a function $f_1 \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, $d \in \mathbb{N}$ is designated as $\|f_1\|_{L^p(\mathbb{R}^d)}$. By means of Lemma 2.3 of [23], under Assumption 1.1 above on the scalar potentials, the operator

$$-\Delta_x + V(x) - \Delta_y + U(y)$$

on $L^2(\mathbb{R}^6)$ is self-adjoint and is unitarily equivalent to $-\Delta_x - \Delta_y$ via the product of the wave operators (see [11], [17])

$$\Omega_V^{\pm} := s - \lim_{t\to\mp\infty} e^{it(-\Delta_x+V(x))}e^{it\Delta_x}, \quad \Omega_U^{\pm} := s - \lim_{t\to\mp\infty} e^{it(-\Delta_y+U(y))}e^{it\Delta_y},$$

with the limits here understood in the strong L^2 sense (see e.g. [16] p.34, [7] p.90). Therefore, operator (1.3) has only the essential spectrum

$$\sigma_{ess}(H_{U,\,V}) = [0, +\infty)$$

and no nontrivial $L^2(\mathbb{R}^6)$ eigenfunctions. Hence, operator (1.3) does not satisfy the Fredholm property. The functions of the continuos spectrum of the first differential operator involved in (1.3) are the solutions the Schrödinger equation

$$[-\Delta_x + V(x)]\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3,$$

in the integral form the Lippmann-Schwinger equation

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{1.6}$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_{k_1}(x))_{L^2(\mathbb{R}^3)} = \delta(k-k_1),\ k, k_1 \in \mathbb{R}^3.$$

hold. The integral operator involved in (1.6)

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi(x) \in L^\infty(\mathbb{R}^3).$$

We consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ and its norm $\|Q\|_\infty < 1$ under our Assumption 1.1 via Lemma 2.1 of [23]. In fact, this norm is bounded above by the k-independent quantity $I(V)$, which is the left side of bound (1.4). Analogously, for the second differential operator involved in (1.3) the functions of its continuous spectrum solve

$$[-\Delta_y + U(y)]\eta_q(y) = q^2\eta_q(y), \quad q \in \mathbb{R}^3,$$

in the integral formulation

$$\eta_q(y) = \frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta_q)(z)dz, \tag{1.7}$$

such that the the orthogonality conditions

$$(\eta_q(y), \eta_{q_1}(y))_{L^2(\mathbb{R}^3)} = \delta(q-q_1),\ q, q_1 \in \mathbb{R}^3$$

are valid. The integral operator involved in (1.7) is

$$(P\eta)(y) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta)(z)dz, \quad \eta(y) \in L^\infty(\mathbb{R}^3).$$

For $P : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ its norm $\|P\|_\infty < 1$ under Assumption 1.1 by means of Lemma 2.1 of [23]. As above, such norm can be estimated from above by the q-independent quantity $I(U)$, which is the left side of inequality (1.5). By virtue of the spectral theorem, we have

$$H_{U,\,V}\varphi_k(x)\eta_q(y) = [(k^2+q^2)^{s_1} + (k^2+q^2)^{s_2}]\varphi_k(x)\eta_q(y).$$

Let us denote by the double tilde sign the generalized Fourier transform with the product of these functions of the continuous spectrum

$$\tilde{\tilde{f}}(k,q) := (f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}, \quad k, q \in \mathbb{R}^3. \tag{1.8}$$

(1.8) is a unitary transform on $L^2(\mathbb{R}^6)$. Our first main result is as follows.

Theorem 1.2. *Let Assumption 1.1 hold and $f(x,y) \in L^1(\mathbb{R}^6) \cap L^2(\mathbb{R}^6)$. Then equation (1.2) admits a unique solution $u(x,y) \in L^2(\mathbb{R}^6)$.*

We turn our attention to the issue of the solvability in the sense of sequences for our problem. The corresponding sequence of approximate equations with $n \in \mathbb{N}$ is given by

$$\{-\Delta_x + V(x) - \Delta_y + U(y)\}^{s_1} u_n + \\ +\{-\Delta_x + V(x) - \Delta_y + U(y)\}^{s_2} u_n = f_n(x,y), \quad x, y \in \mathbb{R}^3 \tag{1.9}$$

where $0 < s_1 < s_2 < 1$ and the right sides tend to the right side of (1.2) in $L^2(\mathbb{R}^6)$ as $n \to \infty$.

Theorem 1.3. *Let Assumption 1.1 hold, $n \in \mathbb{N}$ and $f_n(x,y) \in L^1(\mathbb{R}^6) \cap L^2(\mathbb{R}^6)$, so that $f_n(x,y) \to f(x,y)$ in $L^1(\mathbb{R}^6)$ as $n \to \infty$ and $f_n(x,y) \to f(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$. Then equations (1.2) and (1.9) possess unique solutions $u(x,y) \in L^2(\mathbb{R}^6)$ and $u_n(x,y) \in L^2(\mathbb{R}^6)$ respectively, so that $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n \to \infty$.*

The second part of the article is devoted to the studies of the problem

$$\{-\Delta_x - \Delta_y + U(y)\}^{s_1} u + \\ +\{-\Delta_x - \Delta_y + U(y)\}^{s_2} u = \phi(x,y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^3 \tag{1.10}$$

where $d \in \mathbb{N}$ and the powers $0 < s_1 < s_2 < 1$. The scalar potential function involved in (1.10) is shallow and short-range under our Assumption 1.1. The operator

$$L_U := \{-\Delta_x - \Delta_y + U(y)\}^{s_1} + \{-\Delta_x - \Delta_y + U(y)\}^{s_2} \tag{1.11}$$

here is defined by means of the spectral calculus. The sum of the free negative Laplacian and the Schrödinger type operator involved in both terms in the right side of (1.11) has the physical meaning of the cumulative hamiltonian of a free d dimensional particle and a three dimensional particle in an external potential. The particles do not interact. As above, the operator

$$-\Delta_x - \Delta_y + U(y)$$

on $L^2(\mathbb{R}^{d+3})$ is self-adjoint and is unitarily equivalent to $-\Delta_x - \Delta_y$. Hence, operator (1.11) has only the essential spectrum

$$\sigma_{ess}(L_U) = [0, \infty),$$

and no nontrvial $L^2(\mathbb{R}^{d+3})$ eigenfunctions. Thus, operator (1.11) is non Fredholm. By means of the spectral theorem

$$L_U \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}} \eta_q(y) = [(k^2+q^2)^{s_1} + (k^2+q^2)^{s_2}] \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}} \eta_q(y).$$

Let us consider another useful generalized Fourier transform with the standard Fourier harmonics and the perturbed plane waves, namely

$$\tilde{\tilde{\phi}}(k,q) := \left(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{d}{2}}}\eta_q(y)\right)_{L^2(\mathbb{R}^{d+3})}, \quad k \in \mathbb{R}^d, \quad q \in \mathbb{R}^3. \tag{1.12}$$

(1.12) is a unitary transform on $L^2(\mathbb{R}^{d+3})$. We have the following statement.

Theorem 1.4. *Let the potential function $U(y)$ satisfy Assumption 1.1 and $\phi(x,y) \in L^1(\mathbb{R}^{d+3}) \cap L^2(\mathbb{R}^{d+3}),\ d \in \mathbb{N}$. Then equation (1.10) has a unique solution $u(x,y) \in L^2(\mathbb{R}^{d+3})$.*

The final result of the article deals with the issue of the solvability in the sense of sequences for our problem (1.10). The corresponding sequence of approximate equations with $n \in \mathbb{N},\ x \in \mathbb{R}^d, \quad d \in \mathbb{N}, \quad y \in \mathbb{R}^3,\ 0 < s_1 < s_2 < 1$ is given by

$$\{-\Delta_x - \Delta_y + U(y)\}^{s_1} u_n +$$
$$+\{-\Delta_x - \Delta_y + U(y)\}^{s_2} u_n = \phi_n(x,y). \tag{1.13}$$

The right sides of (1.13) tend to the right side of (1.10) in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.

Theorem 1.5. *Let the potential function $U(y)$ satisfy Assumption 1.1, $n \in \mathbb{N}$ and $\phi_n(x,y) \in L^1(\mathbb{R}^{d+3}) \cap L^2(\mathbb{R}^{d+3})$ $d \in \mathbb{N}$, so that $\phi_n(x,y) \to \phi(x,y)$ in $L^1(\mathbb{R}^{d+3})$ as $n \to \infty$ and $\phi_n(x,y) \to \phi(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$. Then equations (1.10) and (1.13) admit unique solutions $u(x,y) \in L^2(\mathbb{R}^{d+3})$ and $u_n(x,y) \in L^2(\mathbb{R}^{d+3})$ respectively, so that $u_n(x,y) \to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n \to \infty$.*

Note that for the statements of the theorems above we do not require any orthogonality conditions for the right sides of our equations. We proceed to the proofs of our results.

2. Solvability in the sense of sequences with two potentials

Proof of Theorem 1.2. Let us first establish the uniqueness of solutions for equation (1.2). Suppose it admits two solutions $u_1(x,y), u_2(x,y) \in L^2(\mathbb{R}^6)$. Then their difference $w(x,y) := u_1(x,y) - u_2(x,y) \in L^2(\mathbb{R}^6)$ solves the problem

$$H_{U,\,V} w = 0.$$

Since operator (1.3) has no nontrivial square integrable zero modes in the whole space as discussed above, $w(x,y)$ vanishes in $\mathbb{R}^6$.

Let us apply the generalized Fourier transform (1.8) to both sides of problem (1.2). This yields

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^{s_1} + \{k^2+q^2\}^{s_2}} \chi_{\left\{\sqrt{k^2+q^2} \le 1\right\}} + \tag{2.14}$$

$$+\frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}>1\right\}}$$

with $k,q\in\mathbb{R}^3$. Here and further down χ_A will denote for the characteristic function of a set A. Evidently, the second term in the right side of (2.14) can be bounded from above in the absolute value by $\frac{|\tilde{\tilde{f}}(k,q)|}{2}\in L^2(\mathbb{R}^6)$ due to the one of our assumptions. By means of Corollary 2.2 of [23] (see also [22]) under the stated conditions for $k,q\in\mathbb{R}^3$ we have $\varphi_k(x),\ \eta_q(y)\in L^\infty(\mathbb{R}^3)$ and

$$\|\varphi_k(x)\|_{L^\infty(\mathbb{R}^3)}\leq\frac{1}{1-I(V)}\frac{1}{(2\pi)^{\frac{3}{2}}},\quad \|\eta_q(y)\|_{L^\infty(\mathbb{R}^3)}\leq\frac{1}{1-I(U)}\frac{1}{(2\pi)^{\frac{3}{2}}}. \tag{2.15}$$

This enables us to estimate the first term in the right side of (2.14) from above in the absolute value by by

$$\frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f\|_{L^1(\mathbb{R}^6)}\frac{\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}}{\{k^2+q^2\}^{s_1}}.$$

Hence,

$$\left\|\frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}\right\|_{L^2(\mathbb{R}^6)}\leq$$

$$\leq\frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f\|_{L^1(\mathbb{R}^6)}\sqrt{\frac{|S^6|}{6-4s_1}}<\infty$$

by virtue of the conditions of our theorem. Here $|S^6|$ denotes the Lebesgue measure of the unit sphere in the space of six dimensions. Therefore, for the unique solution of equation (1.2) we have $u(x,y)\in L^2(\mathbb{R}^6)$. ∎

Let us turn our attention to the solvability in the sense of sequences for our equation in the case of two scalar potentials.

Proof of Theorem 1.3. Problems (1.2) and (1.9) possess unique square integrable solutions $u(x,y), u_n(x,y)$ respectively in $\mathbb{R}^6$, where $n\in\mathbb{N}$ via the result of Theorem 1.2 above. Let us apply the generalized Fourier transform (1.8) to both sides of equations (1.2) and (1.9). This yields

$$\tilde{\tilde{u}}(k,q)=\frac{\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}},\quad \tilde{\tilde{u}}_n(k,q)=\frac{\tilde{\tilde{f}}_n(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}$$

with $0<s_1<s_2<1$ and $n\in\mathbb{N}$. Thus, $\tilde{\tilde{u}}_n(k,q)-\tilde{\tilde{u}}(k,q)$ can be written as

$$\frac{\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}+ \tag{2.16}$$

$$+\frac{\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}>1\right\}}.$$

Clearly, the second term in (2.16) can be trivially bounded from above in the absolute value by $\frac{|\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)|}{2}$. Hence,

$$\left\|\frac{\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}>1\right\}}\right\|_{L^2(\mathbb{R}^6)}\leq$$

$$\leq\frac{\|f_n(x,y)-f(x,y)\|_{L^2(\mathbb{R}^6)}}{2}\to 0, \quad n\to\infty$$

as assumed. The first term in (2.16) can be estimated from above in the absolute value by means of inequalities (2.15) by

$$\frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f_n(x,y)-f(x,y)\|_{L^1(\mathbb{R}^6)}\frac{\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}}{\{k^2+q^2\}^{s_1}},$$

so that

$$\left\|\frac{\tilde{\tilde{f}}_n(k,q)-\tilde{\tilde{f}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}\right\|_{L^2(\mathbb{R}^6)}\leq$$

$$\leq\frac{1}{(2\pi)^3}\frac{1}{1-I(V)}\frac{1}{1-I(U)}\|f_n(x,y)-f(x,y)\|_{L^1(\mathbb{R}^6)}\sqrt{\frac{|S^6|}{6-4s_1}}\to 0, \quad n\to\infty$$

via our assumptions. Therefore, $u_n(x,y)\to u(x,y)$ in $L^2(\mathbb{R}^6)$ as $n\to\infty$, which completes the proof of our theorem. ∎

The final section of the article deals with the situation when the free negative Laplacian is added to the three dimensional Schrödinger operator.

3. Solvability in the sense of sequences with Laplacian and a single potential

Proof of Theorem 1.4. To demonstrate the uniqueness of solutions for our equation, let us suppose that (1.10) admits two solutions $u_1(x,y), u_2(x,y)\in L^2(\mathbb{R}^{d+3})$. Then their difference $w(x,y):=u_1(x,y)-u_2(x,y)\in L^2(\mathbb{R}^{d+3})$ satisfies the equation

$$L_U w=0.$$

Since operator (1.11) considered in the whole space does not have any nontrivial square integrable zero modes as discussed above, $w(x,y)$ vanishes in $\mathbb{R}^{d+3}$.

We apply the generalized Fourier transform (1.12) to both sides of equation (1.10). This yields

$$\tilde{\tilde{u}}(k,q)=\frac{\tilde{\tilde{\phi}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}+ \tag{3.17}$$

$$+\frac{\hat{\tilde{\tilde{\phi}}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}>1\right\}}$$

with $k\in\mathbb{R}^d,\ q\in\mathbb{R}^3$. Clearly, the second term in the right side of (3.17) can be bounded from above in the abosolute value by $\frac{|\hat{\tilde{\tilde{\phi}}}(k,q)|}{2}\in L^2(\mathbb{R}^{d+3})$ due to the one of our assumptions. Using (2.15), we easily derive

$$|\hat{\tilde{\tilde{\phi}}}(k,q)|\leq\frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}.$$

Thus, the first term in the right side of (3.17) can be easily estimated from above in the abosolute value by

$$\frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}\frac{\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}}{\{k^2+q^2\}^{s_1}}.$$

Hence,

$$\left\|\frac{\hat{\tilde{\tilde{\phi}}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}\right\|_{L^2(\mathbb{R}^{d+3})}\leq$$

$$\leq\frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}\sqrt{\frac{|S^{d+3}|}{d+3-4s_1}},$$

which is finite due to the given conditions. Here $|S^{d+3}|$ stands for the Lebesgue measure of the unit sphere in the space of $d+3$ dimensions. Therefore, $u(x,y)\in L^2(\mathbb{R}^{d+3})$, which completes the proof of our theorem. ■

We conclude the article with demonstrating the solvability in the sense of sequences for our problem when the free negative Laplacian is added to a three dimensional Schrödinger operator.

Proof of Theorem 1.5. Equations (1.10) and (1.13) have unique square integrable in $\mathbb{R}^{d+3}$ solutions $u(x,y)$ and $u_n(x,y)$ respectively for $n\in\mathbb{N}$ via Theorem 1.4 above. We apply the generalized Fourier transform (1.12) to both sides of problems (1.10) and (1.13). This gives us

$$\tilde{\hat{u}}(k,q)=\frac{\hat{\tilde{\tilde{\phi}}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}},\quad \tilde{\hat{u}}_n(k,q)=\frac{\hat{\tilde{\tilde{\phi}}}_n(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}$$

with $0<s_1<s_2<1$ and $n\in\mathbb{N}$. Let us express $\tilde{\hat{u}}_n(k,q)-\tilde{\hat{u}}(k,q)$ as

$$\frac{\hat{\tilde{\tilde{\phi}}}_n(k,q)-\hat{\tilde{\tilde{\phi}}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}+\tag{3.18}$$

$$+\frac{\hat{\tilde{\tilde{\phi}}}_n(k,q)-\hat{\tilde{\tilde{\phi}}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}>1\right\}}.$$

Obviously, the second term in (3.18) can be trivially estimated from above in the absolute value by $\dfrac{|\hat{\tilde{\tilde{\phi}}}_n(k,q)-\hat{\tilde{\tilde{\phi}}}(k,q)|}{2}$. Thus,

$$\left\|\frac{\hat{\tilde{\tilde{\phi}}}_n(k,q)-\hat{\tilde{\tilde{\phi}}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}>1\right\}}\right\|_{L^2(\mathbb{R}^{d+3})}\leq$$

$$\leq\frac{1}{2}\|\phi_n(x,y)-\phi(x,y)\|_{L^2(\mathbb{R}^{d+3})}\to 0,\quad n\to\infty$$

as assumed. Let us obtain the upper bound in the the absolute value for the first term in (3.18) using (2.15). It is given by

$$\frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi_n(x,y)-\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}\frac{\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}}{\{k^2+q^2\}^{s_1}}.$$

Therefore,

$$\left\|\frac{\hat{\tilde{\tilde{\phi}}}_n(k,q)-\hat{\tilde{\tilde{\phi}}}(k,q)}{\{k^2+q^2\}^{s_1}+\{k^2+q^2\}^{s_2}}\chi_{\left\{\sqrt{k^2+q^2}\leq 1\right\}}\right\|_{L^2(\mathbb{R}^{d+3})}\leq$$

$$\leq\frac{1}{(2\pi)^{\frac{d+3}{2}}}\frac{1}{1-I(U)}\|\phi_n(x,y)-\phi(x,y)\|_{L^1(\mathbb{R}^{d+3})}\sqrt{\frac{|S^{d+3}|}{d+3-4s_1}}\to 0,\quad n\to\infty$$

by means of our assumptions. This gives us that $u_n(x,y)\to u(x,y)$ in $L^2(\mathbb{R}^{d+3})$ as $n\to\infty$. ■

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