Existence of solutions for some non-Fredholm integro-differential equations with mixed diffusion

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Abstract. We establish the existence in the sense of sequences of solutions for certain integro-differential type equations in two dimensions involving the normal diffusion in one direction and the anomalous diffusion in the other direction in $H^2(\mathbb{R}^2)$ via the fixed point technique. The elliptic equation contains a second order differential operator without the Fredholm property. It is proved that, under the reasonable technical conditions, the convergence in $L^1(\mathbb{R}^2)$ of the integral kernels implies the existence and convergence in $H^2(\mathbb{R}^2)$ of the solutions.

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1 Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. As a consequence, the equation $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. These properties of Fredholm operators are widely used in many methods of linear and nonlinear analysis.

Elliptic problems in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Shapiro-Lopatinskii conditions are fulfilled (see e.g. [1], [6], [16], [20]). This is the main result of the theory of linear elliptic equations. In the case of unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For instance, the Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ does not satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic problems in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions cited above, limiting operators are invertible (see [21]). In some simple cases, limiting operators can be explicitly constructed. For instance, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinity,

$$a_\pm = \lim_{x\to\pm\infty} a(x), \quad b_\pm = \lim_{x\to\pm\infty} b(x), \quad c_\pm = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are:

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u.$$

Since the coefficients are constants, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ fails to satisfy the Fredholm property, can be explicitly found by virtue of the Fourier transform:

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}.$$

Invertibility of limiting operators is equivalent to the condition that the essential spectrum does not contain the origin.

In the case of general elliptic problems, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, these conditions may not be explicitly written.

In the case of non-Fredholm operators the usual solvability conditions may not be applicable and solvability conditions are, in general, not known. There are some classes of operators for which solvability conditions are derived. We illustrate them with the following example. Consider the problem

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where a is a positive constant. The operator L coincides with its limiting operators. The homogeneous equation has a nonzero bounded solution. Hence the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can apply the Fourier transform and find the solution explicitly. Solvability relations can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exists a solution of this problem in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [26]). Here and further down S^d_r stands for the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Thus, though the operator does not satisfy the Fredholm property, solvability relations are formulated similarly. However, this similarity is only formal since the range of the operator is not closed.

In the case of the operator with a potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

Fourier transform is not directly applicable. Nevertheless, solvability relations in $\mathbb{R}^3$ can be derived by a rather sophisticated application of the theory of self-adjoint operators (see [24]). As before, solvability conditions are formulated in terms of orthogonality to solutions of the homogeneous adjoint equation. There are several other examples of linear elliptic non Fredholm operators for which solvability conditions can be obtained (see [11], [21], [22], [23], [24], [26]).

Solvability relations play a crucial role in the analysis of nonlinear elliptic problems. In the case of non-Fredholm operators, in spite of some progress in understanding of linear equations, there exist only few examples where nonlinear non-Fredholm operators are analyzed (see [5], [9], [25], [26], [29]). The large time behavior of solutions of a class of fourth-order parabolic equations defined on unbounded domains using the Kolmogorov ε-entropy as a measure was studied in [8]. The work [7] deals with the finite and infinite dimensional attractors for evolution equations of mathematical physics. The attractor for a nonlinear reaction-diffusion system in an unbounded domain in $\mathbb{R}^3$ was investigated in [12]. The articles [13] and [19] are devoted to the understanding of the Fredholm and properness properties of quasilinear elliptic systems of second order and of operators of this kind on $\mathbb{R}^N$. Exponential decay and Fredholm properties in second-order quasilinear elliptic systems were addressed in [14]. In the present article we treat another class of stationary nonlinear problems, for which the Fredholm property may not be satisfied:

$$\frac{\partial^2 u}{\partial x_1^2} - \Big(-\frac{\partial^2}{\partial x_2^2}\Big)^s u + \int_{\mathbb{R}^2} G(x-y)F(u(y), y)dy = 0, \quad 0 < s < 1, \tag{1.2}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$. Here the operator

$$L_s := -\frac{\partial^2}{\partial x_1^2} + \Big(-\frac{\partial^2}{\partial x_2^2}\Big)^s : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad 0 < s < 1 \tag{1.3}$$

is defined via the spectral calculus. The novelty of the present work is that in the diffusion term we add the standard minus Laplacian in the x_1 variable with the negative Laplace operator in x_2 raised to a fractional power. Such model is new and not much is understood about it, especially in the context of the integro-differential equations. The difficulty we have to overcome is that such problem becomes anisotropic and it is more technical to obtain the desired estimates when dealing with it. In population dynamics in the Mathematical Biology the integro-differential problems describe models with intra-specific competition and nonlocal consumption of resources (see e.g. [2], [3]). It is important to study the equations of this kind in unbounded domains from the point of view of the understanding of the spread of the viral infections, since many countries have to deal with the pandemics. We use the explicit form of the solvability conditions and establish the existence of solutions of such nonlinear equation. In the case of the standard Laplacian instead of (1.3), the problem analogous to (1.2) was considered in [25] and [29]. The solvability of the integro-differential equations involving in the diffusion term only the negative Laplace operator raised to a fractional power was actively studied in recent years in the context of the anomalous diffusion (see e.g. [10], [27], [28]). The anomalous diffusion can be described as a random process of particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite in the case of the normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at infinity of the probability density function determines the value of the power of the Laplacian (see [18]). In article [17] the authors prove the imbedding theorems and study the spectrum of certain pseudodifferential operators.

2 Formulation of the results

The nonlinear part of equation (1.2) will satisfy the following regularity conditions.

Assumption 1. *Function $F(u,x) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is satisfying the Caratheodory condition (see [15]), such that*

$$|F(u,x)| \le k|u| + h(x) \quad for \quad u \in \mathbb{R},\ x \in \mathbb{R}^2 \tag{2.1}$$

with a constant $k > 0$ and $h(x) : \mathbb{R}^2 \to \mathbb{R}^+$, $h(x) \in L^2(\mathbb{R}^2)$. Moreover, it is a Lipschitz continuous function, such that

$$|F(u_1,x) - F(u_2,x)| \le l|u_1 - u_2| \quad for \quad any \quad u_{1,2} \in \mathbb{R}, \quad x \in \mathbb{R}^2 \tag{2.2}$$

with a constant $l > 0$.

The solvability of a local elliptic equation in a bounded domain in $\mathbb{R}^N$ was considered in [4], where the nonlinear function was allowed to have a sublinear growth. In order to study of the existence of solutions of (1.2), we introduce the auxiliary equation

$$-\frac{\partial^2 u}{\partial x_1^2} + \left(-\frac{\partial^2}{\partial x_2^2}\right)^s u = \int_{\mathbb{R}^2} G(x-y)F(v(y),y)dy, \quad 0 < s < 1. \tag{2.3}$$

We denote

$$(f_1(x), f_2(x))_{L^2(\mathbb{R}^2)} := \int_{\mathbb{R}^2} f_1(x)\bar{f}_2(x)dx, \tag{2.4}$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f_1(x) \in L^1(\mathbb{R}^2)$ and $f_2(x)$ is bounded, like for instance those involved in orthogonality relation (4.4) below, the integral in the right side of (2.4) makes sense. In the the article we work in the space of the two dimensions, such that the appropriate Sobolev space is equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^2)} := \|u\|^2_{L^2(\mathbb{R}^2)} + \|\Delta u\|^2_{L^2(\mathbb{R}^2)}. \tag{2.5}$$

In the equation above we are dealing with the operator L_s defined in (1.3). By virtue of the standard Fourier transform (4.1), it can be easily checked that its essential spectrum is given by

$$\lambda_s(p) = p_1^2 + |p_2|^{2s}, \quad p = (p_1, p_2) \in \mathbb{R}^2. \tag{2.6}$$

Since set (2.6) contains the origin, our operator L_s fails to satisfy the Fredholm property, which is the obstacle to solve our equation.

The similar situations but in linear problems, both self- adjoint and non self-adjoint involving non Fredholm differential operators have been studied extensively in recent years (see [21], [22], [24], [26]). Our present work is related to our article [11] since we also deal with the non Fredholm operator, now involved in the problem, which is not linear anymore and contains the nonlocal terms. Currently, as distinct from [11], the space dimension is restricted to $d = 2$ to avoid the extra technicalities.

In the present work we manage to establish that under the reasonable technical assumptions problem (2.3) defines a map $T_{2,\ s} : H^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$, $0 < s < 1$, which is a strict contraction.

Theorem 1. *Let Assumption 1 hold,* $0 < s < 1$, *the function* $G(x) : \mathbb{R}^2 \to \mathbb{R}$, *such that* $G(x) \in L^1(\mathbb{R}^2)$ *and* $x^2 G(x) \in L^1(\mathbb{R}^2)$. *Moreover,* $(-\Delta)^{1-s} G(x) \in L^1(\mathbb{R}^2)$.

We also assume that orthogonality conditions (4.4), (4.5) hold if $0 < s \leq \frac{1}{2}$ *and relations (4.4), (4.5) and (4.6) are valid for* $\frac{1}{2} < s < 1$ *and that* $2\sqrt{2}\pi N_{2,\ s} l < 1$. *Then the map* $T_{2,s} v = u$ *on* $H^2(\mathbb{R}^2)$ *defined by problem (2.3) admits a unique fixed point* $v_{2,s}$, *which is the only solution of equation (1.2) in* $H^2(\mathbb{R}^2)$.

This fixed point $v_{2,s}$ *is nontrivial provided the intersection of supports of the Fourier transforms of functions* $supp\widehat{F(0,x)} \cap supp\widehat{G}$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}^2$.

Related to equation (1.2) in the space of two dimensions, we study the sequence of approximate equations with $m \in \mathbb{N}$

$$\frac{\partial^2 u_m}{\partial x_1^2} - \left(- \frac{\partial^2}{\partial x_2^2} \right)^s u_m + \int_{\mathbb{R}^2} G_m(x-y) F(u_m(y), y) dy = 0, \quad 0 < s < 1. \tag{2.7}$$

The sequence of kernels $\{G_m(x)\}_{m=1}^{\infty}$ tends to $G(x)$ as $m \to \infty$ in the appropriate function spaces discussed below. We will show that, under the appropriate technical conditions, each of equations (2.7) has a unique solution $u_m(x) \in H^2(\mathbb{R}^2)$, the limiting problem (1.2) admits a unique solution $u(x) \in H^2(\mathbb{R}^2)$, and $u_m(x) \to u(x)$ in $H^2(\mathbb{R}^2)$ as $m \to \infty$, which is the so-called *existence of solutions in the sense of sequences.* In this case, the solvability relations can be formulated for the iterated kernels G_m. They yield the convergence of the kernels in terms of the Fourier transforms (see the Appendix) and, as a consequence, the convergence or the solutions (Theorem 2 below). Similar ideas in the context of the standard Schrödinger type operators were exploited in [23]. Our second main proposition is as follows.

Theorem 2. *Let Assumption 1 hold,* $0 < s < 1$, $m \in \mathbb{N}$, *the functions* $G_m(x) : \mathbb{R}^2 \to \mathbb{R}$ *are such that* $G_m(x) \in L^1(\mathbb{R}^2)$, $x^2 G_m(x) \in L^1(\mathbb{R}^2)$ *and* $(-\Delta)^{1-s} G_m(x) \in L^1(\mathbb{R}^2)$. *Moreover,* $G_m(x) \to G(x)$ *in* $L^1(\mathbb{R}^2)$, $x^2 G_m(x) \to x^2 G(x)$ *and* $(-\Delta)^{1-s} G_m(x) \to (-\Delta)^{1-s} G(x)$ *in* $L^1(\mathbb{R}^2)$ *as* $m \to \infty$.

We also assume that for all $m \in \mathbb{N}$ *orthogonality conditions (4.21), (4.22) hold if* $0 < s \le \frac{1}{2}$ *and relations (4.21), (4.22) and (4.23) are valid for* $\frac{1}{2} < s < 1$. *Furthermore, we suppose that (4.24) holds for all* $m \in \mathbb{N}$ *with a certain* $0 < \varepsilon < 1$.

Then each problem (2.7) admits a unique solution $u_m(x) \in H^2(\mathbb{R}^2)$, *limiting equation (1.2) possesses a unique solution* $u(x) \in H^2(\mathbb{R}^2)$ *and* $u_m(x) \to u(x)$ *in* $H^2(\mathbb{R}^2)$ *as* $m \to \infty$.

The unique solution $u_m(x)$ *of each problem (2.7) is nontrivial provided that the intersection of supports of the Fourier transforms of functions* $supp\widehat{F(0,x)} \cap supp\widehat{G}_m$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}^2$. *Similarly, the unique solution* $u(x)$ *of limiting equation (1.2) does not vanish identically if* $supp\widehat{F(0,x)} \cap supp\widehat{G}$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}^2$.

Remark 1. *In the article we work with real valued functions by virtue of the assumptions on* $F(u,x), G_m(x)$ *and* $G(x)$ *involved in the nonlocal terms of the iterated and limiting problems discussed above.*

Remark 2. *The importance of Theorem 2 above is the continuous dependence of solutions with respect to the integral kernels.*

3 Proofs Of The Main Results

Proof of Theorem 1. Let us first suppose that for a certain $v(x) \in H^2(\mathbb{R}^2)$ there exist two solutions $u_{1,2}(x) \in H^2(\mathbb{R}^2)$ of problem (2.3). Then their difference $w(x) := u_1(x) - u_2(x) \in H^2(\mathbb{R}^2)$ will be a solution of the homogeneous equation

$$-\frac{\partial^2 w}{\partial x_1^2} + \left(-\frac{\partial^2}{\partial x_2^2}\right)^s w = 0.$$

Because the operator $L_s : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined in (1.3) does not have any nontrivial zero modes, the function $w(x)$ vanishes in the space of two dimensions.

We choose arbitrarily $v(x) \in H^2(\mathbb{R}^2)$. Let us apply the standard Fourier transform (4.1) to both sides of (2.3) and arrive at

$$\widehat{u}(p) = 2\pi \frac{\widehat{G}(p)\widehat{f}(p)}{p_1^2 + |p_2|^{2s}}, \quad p^2\widehat{u}(p) = 2\pi \frac{p^2\widehat{G}(p)\widehat{f}(p)}{p_1^2 + |p_2|^{2s}}, \tag{3.1}$$

where $\widehat{f}(p)$ denotes the Fourier image of $F(v(x), x)$. Evidently, we have the estimates from above

$$|\widehat{u}(p)| \le 2\pi N_{2,\, s}|\widehat{f}(p)| \quad and \quad |p^2\widehat{u}(p)| \le 2\pi N_{2,\, s}|\widehat{f}(p)|.$$

Note that $N_{2,\ s} < \infty$ by means of Lemma 3 of the Appendix under the given conditions. This enables us to obtain the upper bound on the norm

$$\|u\|^2_{H^2(\mathbb{R}^2)} = \|\widehat{u}(p)\|^2_{L^2(\mathbb{R}^2)} + \|p^2\widehat{u}(p)\|^2_{L^2(\mathbb{R}^2)} \leq 8\pi^2 N^2_{2,\ s}\|F(v(x), x)\|^2_{L^2(\mathbb{R}^2)},$$

which is finite by virtue of (2.1) of Assumption 1 because $v(x) \in L^2(\mathbb{R}^2)$. Clearly, $v(x) \in H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ due to the Sobolev embedding. Thus, for an arbitrary $v(x) \in H^2(\mathbb{R}^2)$ there exists a unique solution $u(x) \in H^2(\mathbb{R}^2)$ of problem (2.3), such that its Fourier image is given by (3.1). Hence, the map $T_{2,s} : H^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$ is well defined. This allows us to choose arbitrary functions $v_{1,2}(x) \in H^2(\mathbb{R}^2)$, such that their images $u_{1,2} := T_{2,s}v_{1,2} \in H^2(\mathbb{R}^2)$. Clearly, (2.3) yields

$$-\frac{\partial^2 u_1}{\partial x_1^2} + \left(-\frac{\partial^2}{\partial x_2^2}\right)^s u_1 = \int_{\mathbb{R}^2} G(x-y)F(v_1(y), y)dy, \tag{3.2}$$

$$-\frac{\partial^2 u_2}{\partial x_1^2} + \left(-\frac{\partial^2}{\partial x_2^2}\right)^s u_2 = \int_{\mathbb{R}^2} G(x-y)F(v_2(y), y)dy, \tag{3.3}$$

where $0 < s < 1$. Let us apply the standard Fourier transform (4.1) to both sides of the equations of system (3.2), (3.3) above. We arrive at

$$\widehat{u}_1(p) = 2\pi\frac{\widehat{G}(p)\widehat{f}_1(p)}{p_1^2 + |p_2|^{2s}}, \quad \widehat{u}_2(p) = 2\pi\frac{\widehat{G}(p)\widehat{f}_2(p)}{p_1^2 + |p_2|^{2s}}. \tag{3.4}$$

Here $\widehat{f}_1(p)$ and $\widehat{f}_2(p)$ stand for the Fourier images of $F(v_1(x), x)$ and $F(v_2(x), x)$ respectively. By means of (3.4) we derive the upper bounds

$$|\widehat{u}_1(p) - \widehat{u}_2(p)| \leq 2\pi N_{2,\ s}|\widehat{f}_1(p) - \widehat{f}_2(p)|, \quad p^2|\widehat{u}_1(p) - \widehat{u}_2(p)| \leq 2\pi N_{2,\ s}|\widehat{f}_1(p) - \widehat{f}_2(p)|,$$

such that

$$\|u_1 - u_2\|^2_{H^2(\mathbb{R}^2)} = \|\widehat{u}_1(p) - \widehat{u}_2(p)\|^2_{L^2(\mathbb{R}^2)} + \|p^2[\widehat{u}_1(p) - \widehat{u}_2(p)]\|^2_{L^2(\mathbb{R}^2)} \leq$$

$$\leq 8\pi^2 N^2_{2,\ s}\|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(\mathbb{R}^2)}.$$

Evidently, $v_{1,2}(x) \in H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ via the Sobolev embedding. Condition (2.2) above implies that

$$\|T_{2,s}v_1 - T_{2,s}v_2\|_{H^2(\mathbb{R}^2)} \leq 2\sqrt{2}\pi N_{2,\ s} l\|v_1 - v_2\|_{H^2(\mathbb{R}^2)}$$

and the constant in the right side of this inequality is less than one via the one of our assumptions. Thus, by means of the Fixed Point Theorem, there exists a unique function $v_{2,s} \in H^2(\mathbb{R}^2)$ with the property $T_{2,s}v_{2,s} = v_{2,s}$, which is the only solution of problem (1.2) in $H^2(\mathbb{R}^2)$. Suppose $v_{2,s}(x) = 0$ identically in the space of two dimensions. This will contradict to our assumption that the Fourier images of $G(x)$ and $F(0, x)$ do not vanish on a set of nonzero Lebesgue measure in $\mathbb{R}^2$. ∎

Let us proceed to establishing the solvability in the sense of sequences for our integro-differential problem in the space of two dimensions.

Proof of Theorem 2. By virtue of the result of Theorem 1 above, each problem (2.7) has a unique solution $u_m(x) \in H^2(\mathbb{R}^2),\ m \in \mathbb{N}$. Limiting equation (1.2) admits a unique solution $u(x) \in H^2(\mathbb{R}^2)$ by means of Lemma 4 below along with Theorem 1. Let us apply the standard Fourier transform (4.1) to both sides of (1.2) and (2.7). This yields

$$\widehat{u}(p) = 2\pi\frac{\widehat{G}(p)\widehat{\varphi}(p)}{p_1^2 + |p_2|^{2s}}, \quad \widehat{u}_m(p) = 2\pi\frac{\widehat{G}_m(p)\widehat{\varphi}_m(p)}{p_1^2 + |p_2|^{2s}}, \tag{3.5}$$

$$p^2\widehat{u}(p) = 2\pi\frac{p^2\widehat{G}(p)\widehat{\varphi}(p)}{p_1^2 + |p_2|^{2s}}, \quad p^2\widehat{u}_m(p) = 2\pi\frac{p^2\widehat{G}_m(p)\widehat{\varphi}_m(p)}{p_1^2 + |p_2|^{2s}}, \quad m \in \mathbb{N}, \tag{3.6}$$

where $\widehat{\varphi}(p)$ and $\widehat{\varphi}_m(p)$ stand for the Fourier images of $F(u(x), x)$ and $F(u_m(x), x)$ respectively. Apparently,

$$|\widehat{u}_m(p) - \widehat{u}(p)| \leq 2\pi\left\|\frac{\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}} - \frac{\widehat{G}(p)}{p_1^2 + |p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}|\widehat{\varphi}(p)| +$$

$$+2\pi\left\|\frac{\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}|\widehat{\varphi}_m(p) - \widehat{\varphi}(p)|.$$

Hence

$$\|u_m - u\|_{L^2(\mathbb{R}^2)} \leq 2\pi\left\|\frac{\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}} - \frac{\widehat{G}(p)}{p_1^2 + |p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}\|F(u(x), x)\|_{L^2(\mathbb{R}^2)} +$$

$$+2\pi\left\|\frac{\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}\|F(u_m(x), x) - F(u(x), x)\|_{L^2(\mathbb{R}^2)}.$$

Upper bound (2.2) of Assumption 1 gives us

$$\|F(u_m(x), x) - F(u(x), x)\|_{L^2(\mathbb{R}^2)} \leq l\|u_m(x) - u(x)\|_{L^2(\mathbb{R}^2)}. \tag{3.7}$$

Note that $u_m(x),\ u(x) \in H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ due to the Sobolev embedding. Thus, we arrive at

$$\|u_m(x) - u(x)\|_{L^2(\mathbb{R}^2)}\left\{1 - 2\pi\left\|\frac{\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)} l\right\} \leq$$

$$\leq 2\pi\left\|\frac{\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}} - \frac{\widehat{G}(p)}{p_1^2 + |p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}\|F(u(x), x)\|_{L^2(\mathbb{R}^2)}.$$

Using (4.24), we derive

$$\|u_m(x) - u(x)\|_{L^2(\mathbb{R}^2)} \leq \frac{2\pi}{\varepsilon}\left\|\frac{\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}} - \frac{\widehat{G}(p)}{p_1^2 + |p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}\|F(u(x), x)\|_{L^2(\mathbb{R}^2)}.$$

Upper bound (2.1) of Assumption 1 gives us $F(u(x), x) \in L^2(\mathbb{R}^2)$ for $u(x) \in L^2(\mathbb{R}^2)$. Hence, we obtain that under the given conditions

$$u_m(x) \to u(x), \quad m \to \infty \tag{3.8}$$

in $L^2(\mathbb{R}^2)$ due to the result of Lemma 4 of the Appendix. By virtue of (3.6), we arrive at

$$|p^2\widehat{u}_m(p) - p^2\widehat{u}(p)| \leq 2\pi \left\| \frac{p^2\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}} - \frac{p^2\widehat{G}(p)}{p_1^2 + |p_2|^{2s}} \right\|_{L^\infty(\mathbb{R}^2)} |\widehat{\varphi}(p)| +$$

$$+2\pi \left\| \frac{p^2\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}} \right\|_{L^\infty(\mathbb{R}^2)} |\widehat{\varphi}_m(p) - \widehat{\varphi}(p)|.$$

Therefore,

$$\|\Delta u_m(x) - \Delta u(x)\|_{L^2(\mathbb{R}^2)} \leq 2\pi \left\| \frac{p^2\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}} - \frac{p^2\widehat{G}(p)}{p_1^2 + |p_2|^{2s}} \right\|_{L^\infty(\mathbb{R}^2)} \|F(u(x), x)\|_{L^2(\mathbb{R}^2)} +$$

$$+2\pi \left\| \frac{p^2\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}} \right\|_{L^\infty(\mathbb{R}^2)} \|F(u_m(x), x) - F(u(x), x)\|_{L^2(\mathbb{R}^2)}.$$

Inequality (3.7) enables us to obtain the upper bound

$$\|\Delta u_m(x) - \Delta u(x)\|_{L^2(\mathbb{R}^2)} \leq 2\pi \left\| \frac{p^2\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}} - \frac{p^2\widehat{G}(p)}{p_1^2 + |p_2|^{2s}} \right\|_{L^\infty(\mathbb{R}^2)} \|F(u(x), x)\|_{L^2(\mathbb{R}^2)} +$$

$$+2\pi \left\| \frac{p^2\widehat{G}_m(p)}{p_1^2 + |p_2|^{2s}} \right\|_{L^\infty(\mathbb{R}^2)} l \|u_m(x) - u(x)\|_{L^2(\mathbb{R}^2)}.$$

By means of the result of Lemma 4 of the Appendix along with (3.8), we derive $\Delta u_m(x) \to \Delta u(x)$ in $L^2(\mathbb{R}^2)$ as $m \to \infty$. Definition (2.5) of the norm gives us $u_m(x) \to u(x)$ in $H^2(\mathbb{R}^2)$ as $m \to \infty$.

Suppose the solution $u_m(x)$ of problem (2.7) studied above vanishes in the space of two dimensions for a certain $m \in \mathbb{N}$. This will contradict to the given condition that the Fourier transforms of $G_m(x)$ and $F(0, x)$ are nontrivial on a set of nonzero Lebesgue measure in $\mathbb{R}^2$. The analogous argument is valid for the solution $u(x)$ of limiting equation (1.2). ■

4 Appendix

Let $G(x)$ be a function, $G(x) : \mathbb{R}^2 \to \mathbb{R}$, for which we denote its standard Fourier transform using the hat symbol as

$$\widehat{G}(p) := \frac{1}{2\pi} \int_{\mathbb{R}^2} G(x) e^{-ipx} dx, \quad p \in \mathbb{R}^2, \tag{4.1}$$

such that

$$\|\widehat{G}(p)\|_{L^\infty(\mathbb{R}^2)} \le \frac{1}{2\pi}\|G\|_{L^1(\mathbb{R}^2)} \tag{4.2}$$

and $G(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \widehat{G}(q)e^{iqx}dq,\ \ x \in \mathbb{R}^2$. For the technical purposes we introduce the auxiliary quantities

$$N_{2,\ s} := max\left\{\left\|\frac{\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)},\quad \left\|\frac{p^2\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}\right\},\quad 0<s<1. \tag{4.3}$$

Lemma 3. *Let $0 < s < 1$, the function $G(x) : \mathbb{R}^2 \to \mathbb{R}$, such that $G(x) \in L^1(\mathbb{R}^2)$ and $x^2G(x) \in L^1(\mathbb{R}^2)$. We also assume that $(-\Delta)^{1-s}G(x) \in L^1(\mathbb{R}^2)$.*

a) If $0 < s \le \frac{1}{2}$ then $N_{2,\ s} < \infty$ if and only if

$$(G(x), 1)_{L^2(\mathbb{R}^2)} = 0, \tag{4.4}$$

$$(G(x), x_1)_{L^2(\mathbb{R}^2)} = 0. \tag{4.5}$$

b) Suppose $\frac{1}{2} < s < 1$. Then $N_{2,\ s} < \infty$ if and only if orthogonality conditions (4.4) and (4.5) along with

$$(G(x), x_2)_{L^2(\mathbb{R}^2)} = 0 \tag{4.6}$$

hold.

Proof. Let us first observe that in both cases a) and b) of our lemma the boundedness of $\frac{\widehat{G}(p)}{p_1^2+|p_2|^{2s}}$ yields that $\frac{p^2\widehat{G}(p)}{p_1^2+|p_2|^{2s}}$ is bounded as well. We easily express

$$\frac{p^2\widehat{G}(p)}{p_1^2+|p_2|^{2s}} = \frac{p^2\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\chi_{\{|p|\le 1\}} + \frac{p^2\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\chi_{\{|p|>1\}}. \tag{4.7}$$

Here and further down χ_A will denote the characteristic function of a set $A \subseteq \mathbb{R}^2$. Clearly, the first term in the right side of (4.7) can be estimated from above in the absolute value by

$$\left\|\frac{\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)} < \infty$$

as assumed. Inequality (4.2) gives us

$$\||p|^{2(1-s)}\widehat{G}(p)\|_{L^\infty(\mathbb{R}^2)} \le \frac{1}{2\pi}\|(-\Delta)^{1-s}G(x)\|_{L^1(\mathbb{R}^2)}. \tag{4.8}$$

The right side of (4.8) is finite due to the one of our assumptions. In the polar coordinates we have

$$p = (|p|cos\theta, |p|sin\theta) \in \mathbb{R}^2,$$

where θ denotes the angle variable. Apparently, the second term in the right side of (4.7) can be bounded from above in the absolute value as

$$\frac{|p|^{2(1-s)}|\widehat{G}(p)|}{|p|^{2(1-s)}cos^2\theta+|sin\theta|^{2s}}\chi_{\{|p|>1\}}\leq\frac{||p|^{2(1-s)}\widehat{G}(p)|}{cos^2\theta+|sin\theta|^{2s}}\leq C||p|^{2(1-s)}\widehat{G}(p)|. \tag{4.9}$$

Here and throughout the article C will stand for a finite, positive constant. By means of (4.8), the right side of (4.9) can be estimated from above by

$$\frac{C}{2\pi}\|(-\Delta)^{1-s}G(x)\|_{L^1(\mathbb{R}^2)}<\infty.$$

Therefore, $\dfrac{p^2\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\in L^\infty(\mathbb{R}^2)$ as well. Evidently,

$$\frac{\widehat{G}(p)}{p_1^2+|p_2|^{2s}}=\frac{\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\chi_{\{|p|\leq1\}}+\frac{\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\chi_{\{|p|>1\}}. \tag{4.10}$$

The second term in the right side of (4.10) can be trivially bounded from above in the absolute value using (4.2) as

$$\frac{|\widehat{G}(p)|\chi_{\{|p|>1\}}}{p^2cos^2\theta+|p|^{2s}|sin\theta|^{2s}}\leq\frac{\|G(x)\|_{L^1(\mathbb{R}^2)}}{2\pi(cos^2\theta+|sin\theta|^{2s})}\leq C\|G(x)\|_{L^1(\mathbb{R}^2)}<\infty$$

via the one of our assumptions. Let us express

$$\widehat{G}(p)=\widehat{G}(0)+|p|\frac{\partial\widehat{G}}{\partial|p|}(0,\theta)+\int_0^{|p|}\left(\int_0^s\frac{\partial^2\widehat{G}(|q|,\theta)}{\partial|q|^2}d|q|\right)ds. \tag{4.11}$$

Identity (4.11) allows us to write the first term in the right side of (4.10) as

$$\frac{\widehat{G}(0)}{p_1^2+|p_2|^{2s}}\chi_{\{|p|\leq1\}}+\frac{|p|\frac{\partial\widehat{G}}{\partial|p|}(0,\theta)}{p_1^2+|p_2|^{2s}}\chi_{\{|p|\leq1\}}+\frac{\int_0^{|p|}(\int_0^s\frac{\partial^2\widehat{G}(|q|,\theta)}{\partial|q|^2}d|q|)ds}{p_1^2+|p_2|^{2s}}\chi_{\{|p|\leq1\}}. \tag{4.12}$$

Using the definition of the standard Fourier transform (4.1), we easily derive

$$\left|\frac{\partial^2\widehat{G}(p)}{\partial|p|^2}\right|\leq\frac{1}{2\pi}\|x^2G(x)\|_{L^1(\mathbb{R}^2)}<\infty \tag{4.13}$$

as assumed. Let us verify the following trivial statement

$$\frac{p^2}{p_1^2+|p_2|^{2s}}\chi_{\{|p|\leq1\}}\leq1,\quad p\in\mathbb{R}^2. \tag{4.14}$$

Indeed, the left side of (4.14) can be easily estimated as

$$\frac{|p|^2}{|p|^2cos^2\theta+|p|^{2s}|sin\theta|^{2s}}\chi_{\{|p|\leq1\}}=\frac{|p|^{2(1-s)}}{|p|^{2(1-s)}-|p|^{2(1-s)}sin^2\theta+|sin\theta|^{2s}}\chi_{\{|p|\leq1\}}\leq$$

$$\leq \frac{|p|^{2(1-s)}}{|p|^{2(1-s)} + |sin\theta|^{2s} - sin^2\theta} \chi_{\{|p|\leq 1\}} \leq 1,$$

such that (4.14) is valid. By means of (4.13) along with (4.14), we obtain the upper bound in the absolute value for the third term in (4.12) as

$$\frac{\|x^2 G(x)\|_{L^1(\mathbb{R}^2)} |p|^2}{4\pi(p_1^2 + |p_2|^{2s})} \chi_{\{|p|\leq 1\}} \leq \frac{\|x^2 G(x)\|_{L^1(\mathbb{R}^2)}}{4\pi} < \infty$$

via the one of the assumptions of the lemma. Definition (4.1) of the standard Fourier transform gives us

$$\frac{\partial \widehat{G}}{\partial |p|}(0, \theta) = -\frac{i}{2\pi} \int_{\mathbb{R}^2} G(x)|x| cos\beta dx, \tag{4.15}$$

where β denotes the angle between the vectors p and x in the plane. Let us introduce the auxiliary expressions

$$Q_1 := \int_{\mathbb{R}^2} G(x) x_1 dx, \quad Q_2 := \int_{\mathbb{R}^2} G(x) x_2 dx. \tag{4.16}$$

Clearly, under the given conditions $Q_{1,2}$ are well defined, since we can trivially estimate the norm

$$\|xG(x)\|_{L^1(\mathbb{R}^2)} = \int_{|x|\leq 1} |x||G(x)|dx + \int_{|x|>1} |x||G(x)|dx \leq \|G(x)\|_{L^1(\mathbb{R}^2)} + \|x^2 G(x)\|_{L^1(\mathbb{R}^2)} < \infty.$$

By virtue of (4.15) along with (4.16) we easily derive

$$\frac{\partial \widehat{G}}{\partial |p|}(0, \theta) = -\frac{i}{2\pi} \{Q_1 cos\theta + Q_2 sin\theta\}. \tag{4.17}$$

Using (4.17), we can write the sum of the first two terms in (4.12) as

$$\frac{\widehat{G}(0)}{|p|^2 cos^2\theta + |p|^{2s}|sin\theta|^{2s}} \chi_{\{|p|\leq 1\}} - \frac{i|p|\{Q_1 cos\theta + Q_2 sin\theta\}}{2\pi(|p|^2 cos^2\theta + |p|^{2s}|sin\theta|^{2s})} \chi_{\{|p|\leq 1\}}. \tag{4.18}$$

Let us fix the polar angle $\theta = 0$ and suppose that $|p|$ tends to zero. Obviously, expression (4.18) will be unbounded unless $\widehat{G}(0)$ and Q_1 vanish in both cases a) and b) of our lemma. This is equivalent to orthogonality conditions (4.4) and (4.5). Therefore, it remains to analyze the term

$$-\frac{i|p|Q_2 sin\theta}{2\pi(|p|^2 cos^2\theta + |p|^{2s}|sin\theta|^{2s})} \chi_{\{|p|\leq 1\}}. \tag{4.19}$$

Let us first consider the situation when $\frac{1}{2} < s < 1$. We fix the polar angle $\theta = \frac{\pi}{2}$ and let $|p| \to 0$. Then (4.19) will be unbounded unless $Q_2 = 0$. This is equivalent to orthogonality relation (4.6) and completes the proof of the part b) of our lemma.

Finally, we treat the case when $0 < s \le \dfrac{1}{2}$. Then (4.19) can be trivially estimated from above in the absolute value as

$$\frac{|p||Q_2||sin\theta|}{2\pi(|p|^2cos^2\theta+|p|^{2s}|sin\theta|^{2s})}\chi_{\{|p|\le 1\}} \le \frac{1}{2\pi}|p|^{1-2s}|sin\theta|^{1-2s}|Q_2|\chi_{\{|p|\le 1\}} \le \frac{|Q_2|}{2\pi} < \infty,$$

such that in the case a) of the lemma no any further orthogonality conditions than (4.4) and (4.5) are needed. ■

For the purpose of the study of problems (2.7), we define the following technical expressions

$$N_{2,\ s,\ m} := max\left\{\left\|\frac{\widehat{G}_m(p)}{p_1^2+|p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}, \quad \left\|\frac{p^2\widehat{G}_m(p)}{p_1^2+|p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}\right\}, \tag{4.20}$$

where $0 < s < 1$ and $m \in \mathbb{N}$. Our final statement is as follows.

Lemma 4. *Let $0 < s < 1$, $m \in \mathbb{N}$, the functions $G_m(x) : \mathbb{R}^2 \to \mathbb{R}$, such that $G_m(x) \in L^1(\mathbb{R}^2)$, $G_m(x) \to G(x)$ in $L^1(\mathbb{R}^2)$ as $m \to \infty$. Similarly, $x^2G_m(x) \in L^1(\mathbb{R}^2)$, $x^2G_m(x) \to x^2G(x)$ in $L^1(\mathbb{R}^2)$ as $m \to \infty$. Moreover, $(-\Delta)^{1-s}G_m(x) \in L^1(\mathbb{R}^2)$, $(-\Delta)^{1-s}G_m(x) \to (-\Delta)^{1-s}G(x)$ in $L^1(\mathbb{R}^2)$ as $m \to \infty$. We also assume that for all $m \in \mathbb{N}$*

$$(G_m(x), 1)_{L^2(\mathbb{R}^2)} = 0, \tag{4.21}$$

$$(G_m(x), x_1)_{L^2(\mathbb{R}^2)} = 0 \tag{4.22}$$

if $0 < s \le \dfrac{1}{2}$ and for $\dfrac{1}{2} < s < 1$ orthogonality conditions (4.21), (4.22) along with

$$(G_m(x), x_2)_{L^2(\mathbb{R}^2)} = 0, \quad m \in \mathbb{N} \tag{4.23}$$

hold. Finally, let us suppose that

$$2\sqrt{2}\pi N_{2,\ s,\ m}l \le 1-\varepsilon \tag{4.24}$$

is valid for all $m \in \mathbb{N}$ with some $0 < \varepsilon < 1$.

Then

$$\frac{\widehat{G}_m(p)}{p_1^2+|p_2|^{2s}} \to \frac{\widehat{G}(p)}{p_1^2+|p_2|^{2s}}, \quad m \to \infty, \tag{4.25}$$

$$\frac{p^2\widehat{G}_m(p)}{p_1^2+|p_2|^{2s}} \to \frac{p^2\widehat{G}(p)}{p_1^2+|p_2|^{2s}}, \quad m \to \infty \tag{4.26}$$

in $L^\infty(\mathbb{R}^2)$, such that

$$\left\|\frac{\widehat{G}_m(p)}{p_1^2+|p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)} \to \left\|\frac{\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}, \quad m \to \infty, \tag{4.27}$$

$$\left\|\frac{p^2\widehat{G}_m(p)}{p_1^2+|p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)} \to \left\|\frac{p^2\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}, \quad m\to\infty. \tag{4.28}$$

Furthermore,

$$2\sqrt{2}\pi N_{2,\ s} l \le 1-\varepsilon \tag{4.29}$$

holds.

Proof. By means of inequality (4.2) along with the one of our assumptions we arrive at

$$\|\widehat{G}_m(p)-\widehat{G}(p)\|_{L^\infty(\mathbb{R}^2)} \le \frac{1}{2\pi}\|G_m(x)-G(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad m\to\infty. \tag{4.30}$$

Note that under the given conditions by means of the result of Lemma 3 above we have $N_{2,\ s,\ m}<\infty$. Let us use (4.21) to estimate

$$|(G(x),1)_{L^2(\mathbb{R}^2)}| = |(G(x)-G_m(x),1)_{L^2(\mathbb{R}^2)}| \le \|G_m(x)-G(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad m\to\infty$$

as assumed, such that orthogonality condition (4.4) holds in the limit. By virtue of (4.22) and the assumptions of the lemma we easily derive

$$|(G(x),x_1)_{L^2(\mathbb{R}^2)}| = |(G(x)-G_m(x),x_1)_{L^2(\mathbb{R}^2)}| \le \int_{\mathbb{R}^2}|G_m(x)-G(x)||x_1|dx \le$$

$$\le \int_{|x|\le 1}|G_m(x)-G(x)||x|dx + \int_{|x|>1}|G_m(x)-G(x)||x|dx \le$$

$$\le \|G_m(x)-G(x)\|_{L^1(\mathbb{R}^2)} + \|x^2G_m(x)-x^2G(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad m\to\infty.$$

Hence, orthogonality relation (4.5) is valid in the limit. When $\frac{1}{2}<s<1$, by the similar reasoning we can show that orthogonality condition (4.6) holds in the limit as well. The result of Lemma 3 above gives us that in both cases $0<s\le\frac{1}{2}$ and $\frac{1}{2}<s<1$ we have $N_{2,\ s}<\infty$.

Let us establish that (4.25) implies (4.26). Clearly, we have the identity

$$\frac{p^2[\widehat{G}_m(p)-\widehat{G}(p)]}{p_1^2+|p_2|^{2s}} = \frac{p^2[\widehat{G}_m(p)-\widehat{G}(p)]}{p_1^2+|p_2|^{2s}}\chi_{\{|p|\le 1\}} + \frac{p^2[\widehat{G}_m(p)-\widehat{G}(p)]}{p_1^2+|p_2|^{2s}}\chi_{\{|p|>1\}}. \tag{4.31}$$

Apparently, the second term in the right side of (4.31) can be estimated in the absolute value as

$$\frac{|p|^{2(1-s)}|\widehat{G}_m(p)-\widehat{G}(p)|}{|p|^{2(1-s)}cos^2\theta+|sin\theta|^{2s}}\chi_{\{|p|>1\}} \le \frac{|p|^{2(1-s)}|\widehat{G}_m(p)-\widehat{G}(p)|}{cos^2\theta+|sin\theta|^{2s}} \le$$

$$\le C\||p|^{2(1-s)}[\widehat{G}_m(p)-\widehat{G}(p)]\|_{L^\infty(\mathbb{R}^2)} \le \frac{C}{2\pi}\|(-\Delta)^{1-s}[G_m(x)-G(x)]\|_{L^1(\mathbb{R}^2)}$$

due to (4.8). Thus,

$$\left\| \frac{p^2[\widehat{G}_m(p)-\widehat{G}(p)]}{p_1^2+|p_2|^{2s}}\chi_{\{|p|>1\}}\right\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{2\pi}\|(-\Delta)^{1-s}[G_m(x)-G(x)]\|_{L^1(\mathbb{R}^2)} \to 0, \quad m\to\infty$$

as assumed. Evidently, the first term in the right side of (4.31) can be bounded from above in the norm as

$$\left\| \frac{p^2[\widehat{G}_m(p)-\widehat{G}(p)]}{p_1^2+|p_2|^{2s}}\chi_{\{|p|\leq 1\}}\right\|_{L^\infty(\mathbb{R}^2)} \leq \left\| \frac{\widehat{G}_m(p)}{p_1^2+|p_2|^{2s}} - \frac{\widehat{G}_m(p)}{p_1^2+|p_2|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)} \to 0, \quad m\to\infty$$

assuming that (4.25) holds. Therefore, (4.26) will be valid as well. Obviously,

$$\frac{\widehat{G}_m(p)}{p_1^2+|p_2|^{2s}} - \frac{\widehat{G}(p)}{p_1^2+|p_2|^{2s}} = \frac{\widehat{G}_m(p)-\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\chi_{\{|p|\leq 1\}} + \frac{\widehat{G}_m(p)-\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\chi_{\{|p|>1\}}. \tag{4.32}$$

The second term in the right side of (4.32) can be estimated from above in the absolute value using (4.30) as

$$\frac{|\widehat{G}_m(p)-\widehat{G}(p)|}{p^2cos^2\theta+|p|^{2s}|sin\theta|^{2s}}\chi_{\{|p|>1\}} \leq \frac{|\widehat{G}_m(p)-\widehat{G}(p)|}{cos^2\theta+|sin\theta|^{2s}} \leq C\|\widehat{G}_m(p)-\widehat{G}(p)\|_{L^\infty(\mathbb{R}^2)} \leq$$

$$\leq \frac{C}{2\pi}\|G_m(x)-G(x)\|_{L^1(\mathbb{R}^2)}.$$

Hence, we obtain

$$\left\| \frac{\widehat{G}_m(p)-\widehat{G}(p)}{p_1^2+|p_2|^{2s}}\chi_{\{|p|>1\}}\right\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{2\pi}\|G_m(x)-G(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad m\to\infty$$

as assumed. Similarly to (4.11), we express for $m \in \mathbb{N}$

$$\widehat{G}_m(p) = \widehat{G}_m(0) + |p|\frac{\partial \widehat{G}_m}{\partial |p|}(0,\theta) + \int_0^{|p|}\left(\int_0^s \frac{\partial^2 \widehat{G}_m(|q|,\theta)}{\partial |q|^2}d|q|\right)ds. \tag{4.33}$$

Orthogonality conditions (4.4) and (4.21) yield

$$\widehat{G}(0)=0, \quad \widehat{G}_m(0)=0, \quad m\in\mathbb{N}. \tag{4.34}$$

By means of (4.33) along with (4.11) and (4.34) the first term in the right side of (4.32) can be written as

$$\frac{|p|\left[\frac{\partial \widehat{G}_m}{\partial |p|}(0,\theta) - \frac{\partial \widehat{G}}{\partial |p|}(0,\theta)\right]}{p_1^2+|p_2|^{2s}}\chi_{\{|p|\leq 1\}} + \frac{\int_0^{|p|}\left(\int_0^s\left[\frac{\partial^2 \widehat{G}_m(|q|,\theta)}{\partial |q|^2} - \frac{\partial^2 \widehat{G}(|q|,\theta)}{\partial |q|^2}\right]d|q|\right)ds}{p_1^2+|p_2|^{2s}}\chi_{\{|p|\leq 1\}}. \tag{4.35}$$

By virtue of the definition of the standard Fourier transform (4.1), we easily derive

$$\left|\frac{\partial^2 \widehat{G}_m(|p|,\theta)}{\partial |p|^2} - \frac{\partial^2 \widehat{G}(|p|,\theta)}{\partial |p|^2}\right| \le \frac{1}{2\pi}\|x^2 G_m(x) - x^2 G(x)\|_{L^1(\mathbb{R}^2)}. \tag{4.36}$$

Inequalities (4.36) and (4.14) enable us to obtain the upper bound in the absolute value for the second term in (4.35) given by

$$\frac{p^2\|x^2 G_m(x) - x^2 G(x)\|_{L^1(\mathbb{R}^2)}}{4\pi(p_1^2 + |p_2|^{2s})}\chi_{\{|p|\le 1\}} \le \frac{1}{4\pi}\|x^2 G_m(x) - x^2 G(x)\|_{L^1(\mathbb{R}^2)}.$$

Therefore,

$$\left\|\frac{\int_0^{|p|}\left(\int_0^s\left[\frac{\partial^2 \widehat{G}_m(|q|,\theta)}{\partial |q|^2} - \frac{\partial^2 \widehat{G}(|q|,\theta)}{\partial |q|^2}\right]d|q|\right)ds}{p_1^2 + |p_2|^{2s}}\chi_{\{|p|\le 1\}}\right\|_{L^\infty(\mathbb{R}^2)} \le \frac{1}{4\pi}\|x^2 G_m(x) - x^2 G(x)\|_{L^1(\mathbb{R}^2)} \to 0$$

as $m \to \infty$ due to the one of the assumptions of the lemma. A trivial calculation yields

$$\frac{\partial \widehat{G}_m}{\partial |p|}(0,\theta) = -\frac{i}{2\pi}[(G_m(x), x_1)_{L^2(\mathbb{R}^2)} cos\theta + (G_m(x), x_2)_{L^2(\mathbb{R}^2)} sin\theta], \tag{4.37}$$

$$\frac{\partial \widehat{G}}{\partial |p|}(0,\theta) = -\frac{i}{2\pi}[(G(x), x_1)_{L^2(\mathbb{R}^2)} cos\theta + (G(x), x_2)_{L^2(\mathbb{R}^2)} sin\theta]. \tag{4.38}$$

Let us consider first the situation when $\frac{1}{2} < s < 1$. By means of orthogonality relations (4.22) and (4.23) along with (4.37), we obtain that $\frac{\partial \widehat{G}_m}{\partial |p|}(0,\theta) = 0,\ m \in \mathbb{N}$. Similarly, (4.5) and (4.6) along with (4.38) imply that $\frac{\partial \widehat{G}}{\partial |p|}(0,\theta) = 0$. Hence, in the case of $\frac{1}{2} < s < 1$, the first term in (4.35) vanishes.

Then we turn our attention to the situation when $0 < s \le \frac{1}{2}$. Let us estimate the norm

$$\||x|G_m(x)\|_{L^1(\mathbb{R}^2)} = \int_{|x|\le 1}|x||G_m(x)|dx + \int_{|x|>1}|x||G_m(x)|dx \le$$

$$\le \|G_m(x)\|_{L^1(\mathbb{R}^2)} + \|x^2 G_m(x)\|_{L^1(\mathbb{R}^2)} < \infty$$

due to the conditions of our lemma. Thus, $|x|G_m(x) \in L^1(\mathbb{R}^2)$. By the similar reasoning, we derive

$$\||x|G_m(x) - |x|G(x)\|_{L^1(\mathbb{R}^2)} = \int_{|x|\le 1}|x||G_m(x) - G(x)|dx + \int_{|x|>1}|x||G_m(x) - G(x)|dx \le$$

$$\le \|G_m(x) - G(x)\|_{L^1(\mathbb{R}^2)} + \|x^2 G_m(x) - x^2 G(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad m \to \infty$$

as assumed, such that $|x|G_m(x) \to |x|G(x)$ in $L^1(\mathbb{R}^2)$ as $m \to \infty$. Orthogonality relation (4.22) and formula (4.37) yield

$$\frac{\partial \widehat{G}_m}{\partial |p|}(0,\theta) = -\frac{i}{2\pi} sin\theta \int_{\mathbb{R}^2} G_m(x) x_2 dx.$$

Analogously, by virtue of (4.5) and (4.38) we arrive at

$$\frac{\partial \widehat{G}}{\partial |p|}(0,\theta) = -\frac{i}{2\pi} sin\theta \int_{\mathbb{R}^2} G(x) x_2 dx.$$

This allows us to estimate the first term in (4.35) from above in the absolute value as

$$\frac{|p||sin\theta| \int_{\mathbb{R}^2} |x||G_m(x) - G(x)| dx}{2\pi(p^2 cos^2\theta + |p|^{2s}|sin\theta|^{2s})} \chi_{\{|p| \le 1\}} \le$$

$$\le \frac{|p|^{1-2s}|sin\theta|^{1-2s}}{2\pi} \||x|G_m(x) - |x|G(x)\|_{L^1(\mathbb{R}^2)} \chi_{\{|p|\le 1\}} \le \frac{1}{2\pi} \||x|G_m(x) - |x|G(x)\|_{L^1(\mathbb{R}^2)}.$$

Hence, for $0 < s \le \frac{1}{2}$ we obtain

$$\left\| \frac{|p|[\frac{\partial \widehat{G}_m}{\partial |p|}(0,\theta) - \frac{\partial \widehat{G}}{\partial |p|}(0,\theta)]}{p_1^2 + |p_2|^{2s}} \chi_{\{|p|\le 1\}} \right\|_{L^\infty(\mathbb{R}^2)} \le \frac{1}{2\pi} \||x|G_m(x) - |x|G(x)\|_{L^1(\mathbb{R}^2)} \to 0, \quad m \to \infty$$

as discussed above. Therefore, by virtue of the argument above (4.25) holds in both cases when $0 < s \le \frac{1}{2}$ and for $\frac{1}{2} < s < 1$. Evidently, by means of the standard triangle inequality (4.27) and (4.28) follow easily from (4.25) and (4.26) respectively. Finally, (4.29) is valid via a simple limiting argument using (4.27) and (4.28). ■

Remark 3. *Note that in Lemmas 3 and 4 above when* $0 < s \le \frac{1}{2}$ *only two orthogonality conditions for our integral kernels are required, as distinct from the result of the part b) of Lemma A2 of [25].*

Remark 4. *The existence in the sense of sequences of the solutions of our equation (1.2) involving the transport term will be addressed in our consecutive article.*

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