The Genius Conjectures (via Bell Polynomials)

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February 4, 2020

Abstract

We present two related conjectures, arising in work on i-matchings in random r-regular bipartite graphs. The conjectures themselves are easily stated and involve only basic properties of convergent power series. One formulation involves Bell's polynomials. The conjectures name was chosen since we earnestly believe only a truly genius mathematician will prove them. We advise others not to try. A further belief is that the proof will arise from some deep properties of partitions.

We write the paper in a form so that the conjectures are reached as early as possible. Thus Section 1 presents the mathematical setting of the conjectures, and Section 2 contains the conjectures themselves. One need read only these two sections to see the task before one. Section 3 presents a reformulation of the setting using Bell's polynomials [1]. This section is entirely the work of my colleague David Williams. One may only read Section 3 and then Section 2 to arrive at a full treatment of the conjectures using Bell polynomials.

The final section, Section 4, on background provides the place of the conjectures in the development of mathematics, how they arose, and what follows from their truth. Thus, this section contains the contents of usual introductions and conclusions. I only say now they arose from certain equations of Mario Pernici [2] in his treatment of work of Ian Wanless on i-matchings, [3].

1 Setting

p is a fixed positive integer. Additionally we have:

- x, y variables
- $d_1, d_2, ..., d_p$ variables
- $u_1 = 1$
- $u_2, ..., u_p$ variables
- $F_1 = 1$
- $F_2, ..., F_p$ variables

We use the notation $[x^s]f$ to be the coefficient of x^s in a power series f in x. We require

$$[x^p]e^{\sum_{i=1}^{p} d_i x^i} = 0 \tag{1}$$

Theorem 1.1. *There are unique functions*

$$F_i = F_i(\{u_k\}, \{d_k\}), \qquad i = 2, ..., p$$

such that

$$[x^p]e^{\sum_{i=1}^{p}(yu_i + d_i)x^i} = [x^p]e^{\sum_{i=1}^{p} yF_i x^i} \tag{2}$$

holds

Notice that (2) becomes an equality of polynomials in y.

The proof is given by extracting a trivial inductive construction of $F_2, F_3, ..., F_p$.

2 Conjectures

The conjectures concern properties of the functions F_i, $i = 2, ..., p$.

Conjecture 1. *$F_i(\{u_k\}, \{d_k\})$ is linear in its dependence on the $\{u_k\}$.*

For the second conjecture we consider varying p over the positive integers, so F_i becomes a function of p: $F_i = F_i(p, \{u_k\}, \{d_k\})$.

Conjecture 2.

$$F_i = u_i + \sum_{j=1}^{s_i} r_{i,j} m_{i,j} \qquad i = 2, ..., p. \tag{3}$$

where $r_{i,j}$ is a rational function of p that goes to zero as p goes to infinity, and $m_{i,j}$ is a monomial in the variables $\{u_k\} \cup \{d_k\}$.

For our use of Conjecture 2 a weaker statement would be sufficient. But the form given has the advantage of containing more of the structure that might be useful in developing a proof.

3 Setting via Bell polynomials

In this section we follow David Williams in formulating the setting of Section 1 using Bell polynomials [1].

p is a fixed positive integer. Additionally we have:

- y variable
- $d_1, d_2, ..., d_p$ variables
- $u_1 = 1$
- $u_2, ..., u_p$ variables
- $F_1 = 1$
- $F_2, ..., F_p$ variables

We require

$$B_p(1!d_1, ..., p!d_p) = 0 \tag{4}$$

a (scaled) form of (1).

Theorem 3.1. *There are unique functions*

$$F_i = F_i(\{u_k\}, \{d_k\}), \qquad i = 2, ..., p$$

such that

$$B_p[1!(yu_1 + d_1), ..., p!(yu_p + d_p)] = B_p[1!yF_1, ..., p!yF_p]. \tag{5}$$

Equation (5) is a (scaled) form of (2).

The left side of equation (5) can be written as

$$\sum_{i=0}^{p} \binom{p}{i} B_{p-i}[1!yu_1, ..., (p-i)!yu_{p-i}] B_i(1!d_1, ..., i!d_i). \tag{6}$$

4 Background

The conjectures of this paper arose in the study of graph positivity which we now summarize.

We deal with r-regular bipartite graphs with $v = 2n$ vertices. We let m_i be the number of i-matchings. In [4], Butera, Pernici, and I introduced the quantity $d(i)$, in eq. (10) therein,

$$d(i) \equiv \ln\left(\frac{m_i}{r^i}\right) - \ln\left(\frac{\overline{m}_i}{(v-1)^i}\right) \tag{7}$$

where $\overline{m}_i$ is the number of i-matchings for the complete (not bipartite complete) graph on the same vertices,

$$\overline{m}_i = \frac{v!}{(v-2i)!\, i!\, 2^i} \tag{8}$$

We here have changed some of the notation from [4] to agree with notation in [2]. We then considered $\Delta^k d(i)$ where Δ is the finite difference operator, so

$$\Delta d(i) = d(i+1) - d(i) \tag{9}$$

A graph was defined to satisfy *graph positivity* if all the meaningful $\Delta^k d(i)$ were non-negative. That is

$$\Delta^k d(i) \geq 0 \tag{10}$$

for $k = 0, \ldots, v$ and $i = 0, \ldots, v-k$. We made the conjecture, the "graph positivity conjecture", supported by some computer evidence.

Conjecture. *For fixed r, as n goes to infinity the fraction of graphs that satisfy graph positivity approaches one.*

We note some of the impressive results of the numerical study of graph positivity in [4].

(1) All graphs $v < 14$ satisfy graph positivity.

(2) When $r = 4$ the first violations occur when $v = 22$ in 2 graphs out of the 2806490 graphs with $v = 22$.

(3) For $r = 3$ the fraction of graphs not satisfying graph positivity continuously decreases between $v = 14$ and $v = 30$. (There is a single violation at $v = 14$).

We have been working, [5], to prove the weaker result, weak graph positivity, the statement

Conjecture 3. *For fixed r and each i and k one has*

$$\text{Prob}\left(\Delta^k d(i) \geq 0\right) \xrightarrow[n\to\infty]{} 1. \tag{11}$$

In our effort toward proving this conjecture a central role is played by Pernici's work [2] systematizing results of Wanless, [3]. We now note some definitions from [2] in slightly modified notation.

$$T_r = \frac{2(r-1)}{2(r-1) - r + r\sqrt{1 - 4x(r-1)}}. \tag{12}$$

$$u_s(r) = 2[x^s]T_r. \tag{13}$$

$$M_j = [x^j]e^{nrx - \sum_{s\geq 2} \frac{nu_s(r)}{s}(-x)^s}. \tag{14}$$

$$\left(1 + \sum_{h=1}^{j-1} \frac{a_h(r,j)}{n^h}\right) = \frac{j!}{n^j r^j} M_j. \tag{15}$$

These incorporate equations (3), (10) and (12) of [2].

We note that M_j may be viewed as the number of j-matchings on the r-regular bipartite graph with $2n$ vertices, and having no closed loops, a non-existent ideal.

In [2] Pernici presents what I would call a formal derivation of the following equations:

$$[j^k n^{-h}] \ln \left(1 + \sum_{s=1}^{j-1} \frac{a_s(r,j)}{n^s}\right) = 0, \qquad k \geq h+2 \tag{16}$$

$$[j^{h+1} n^{-h}] \ln \left(1 + \sum_{s=1}^{j-1} \frac{a_s(r,j)}{n^s}\right) = \frac{1}{(h+1)h}\left(\frac{1}{r^h} - 2\right) \tag{17}$$

and did a significant numerical check of their validity. These equations are very important to us. As we discuss below we have found a rigorous proof of (16), in fact of a much stronger result than (16). As to (17), we think there should be a fussy technical upgrade of Pernici's formal argument that provides a rigorous proof.

From computer study we came to believe in a stronger form of (16). Namely eq. (16) holds if one computes the $a_s(r,j)$ instead of using (12) and (13) to compute the quantities $u_s(r)$, using any values of the $u_s(r)$! Assuming this stronger conjecture I arrived at a conjecture for Stirling numbers I put on the web, [6]. Robin Chapman has shown me a proof of this conjecture in a private communication. This conjecture now proved is presented in the Appendix. From this I was able to prove the stronger form of (16), [8]. This clever response of Robin Chapman has increased my expectation that someone will prove the Genius conjectures.

To complete the proof of weak graph positivity, we need a proof of the conjecture from Section 10 of [5]:

Conjecture 4. *The Awesome conjecture. Let z_i be positive integers. We set*

$$F = \sum_{s\geq 0} \frac{a_s(r,j)}{n^s} + \sum_i c_i j(j-1)\cdots(j-z_i+1)\frac{1}{n^{z_i} r^{z_i}} \sum_{s\geq 0} \frac{a_s(r,j-z_i)}{n^s}. \tag{18}$$

with $a_0 = 1$*. Then we conjecture:*

$$[j^k n^{-h}] \ln(F) = 0, \qquad k \geq h+2 \tag{19}$$

$$[j^{h+1} n^{-h}] \ln(F) = \frac{1}{(h+1)h} \left(\frac{1}{r^h} - 2 \right) \tag{20}$$

Note this includes (16) and (17) above. We will show in [8] that this Awesome conjecture follows from the validity of (17) and the Genius conjectures, and the work of Robin Chapman [7] (leading to the generalization of (16)). We find the interrelation of these different conjectures a beautiful context

APPENDIX Conjecture Proved by Chapman

The (unsigned) Stirling numbers of the first kind, $\begin{bmatrix} a \\ b \end{bmatrix}$, are defined by

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{21}$$

It is easy to show $\begin{bmatrix} n \\ n-w \end{bmatrix}$ is a polynomial in n of degree $2w$. So we may naturally define $\begin{bmatrix} x \\ x-w \end{bmatrix}$ for any number x by extending the domain of the polynomial. We set

$$P_w(x) \equiv \begin{bmatrix} x \\ x-w \end{bmatrix}. \tag{22}$$

Now we give ourself an integer $g \geq 2$, an integer w, $0 \leq w \leq g-2$, and a set of g distinct numbers,

$$S = \{c_1, ..., c_g\}. \tag{23}$$

We define a *configuration* as a sequence of non-empty subsets of S

$$S_1, S_2, ..., S_r \tag{24}$$

that are disjoint with union S, i.e.

$$S_i \neq \emptyset, \quad S_i \cap S_j = \emptyset \text{ if } i \neq j, \quad \bigcup_{i=1}^{r} S_i = S. \tag{25}$$

For a configuration we define

$$t_i = \sum_{c_k \in S_i} c_k, \qquad i = 1, ..., r. \tag{26}$$

A *weighted configuration* is a configuration as above for which each S_i is assigned a non-negative integer, w_i, its weight, with the restriction

$$\sum_{i=1}^{r} w_i = w. \tag{27}$$

Such a weighted configuration has an *evaluation* defined as

$$(-1)^r \frac{1}{r} \prod_i P_{w_i}(t_i). \tag{28}$$

Theorem 4.1. *The sum over all distinct weighted configurations of their evaluations is zero.*

References

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[5] Federbush, P., *A Near Proof of Weak Graph Positivity, A New Property of Regular Random Graphs,* arXiv:1710.00357.

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