ON THE SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON FREDHOLM OPERATORS WITH DRIFT AND ANOMALOUS DIFFUSION

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Abstract: We study the solvability of certain linear nonhomogeneous elliptic equations and establish that under some technical conditions the convergence in L^2 of their right sides yields the existence and the convergence in the appropriate Sobolev space of the solutions. The problems involve the differential operators with or without Fredholm property, in particular the one dimensional negative Laplacian in a fractional power, on the whole real line or on a finite interval with periodic boundary conditions. We prove that the transport term contained in these equations provides the regularization of the solutions.

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1. Introduction

Consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant, and the function $V(x)$ converges to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$, which corresponds to the left side of problem (1.1) contains the origin. Consequently, such operator fails satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. In the present work we will study certain properties of the operators of this kind. Note that elliptic equations with non-Fredholm operators were treated extensively in recent years (see [15], [18], [19], [20], [21], also [3]) along with their potential applications to the theory of reaction-diffusion problems (see [7], [8]). In the particular case when $a=0$, the operator A satisfies the Fredholm property in some properly chosen weighted spaces (see [1], [2], [3], [5], [6]). However, the case of $a \neq 0$ is significantly different and the approach developed in these articles cannot be applied.

One of the important issues about equations with non-Fredholm operators concerns their solvability. We will address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, such that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n, \; n \in \mathbb{N}.$$

Since the operator A fails to satisfy the Fredholm property, the sequence u_n may not be convergent. Let us call a sequence u_n such that $Au_n \to f$ a solution in the sense of sequences of problem $Au = f$ (see [14]). If this sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this problem. Solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of non-Fredholm operators this convergence may not hold or it can occur in some weaker sense. In this case, the solution in the sense of sequences may not imply the existence of the usual solution. In the present work we will find sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent.

In the first part of the article we study the problem with the transport term

$$\Big(-\frac{d^2}{dx^2}\Big)^s u - b\frac{du}{dx} - au = f(x), \quad x \in \mathbb{R}, \quad 0 < s < 1, \tag{1.2}$$

where $a \geq 0$ and $b \in \mathbb{R}, \; b \neq 0$ are constants and the right side belongs to $L^2(\mathbb{R})$. The operator $\Big(-\frac{d^2}{dx^2}\Big)^s$ can be defined by means of the spectral calculus and is extensively used, for instance in the studies of the anomalous diffusion and related problems (see [22] and the references therein). Anomalous diffusion can be described as a random process of particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite in the case of normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at the infinity of the probability density function determines the value of the power of the Laplace operator (see [13]). The form boundedness criterion for the relativistic Schrödinger operator was proved in [12]. The article [11] deals with establishing the embedding theorems and the studies of the spectrum of a certain pseudodifferential operator. The equation with drift in the context of the Darcy's law describing the fluid motion in the porous medium was treated in [20]. The transport term is significant when studying the emergence and propagation of patterns arising in the theory of speciation (see [16]). Nonlinear propagation phenomena for the reaction-diffusion type equations including the drift term was studied in [4]. Weak solutions of the Dirichlet and Neumann problems with drift were considered in [10]. Apparently, the operator involved in the left side of (1.2)

$$L_{a,\ b,\ s} := \Big(-\frac{d^2}{dx^2}\Big)^s - b\frac{d}{dx} - a : \quad H^1(\mathbb{R}) \to L^2(\mathbb{R}), \quad 0 < s \leq \frac{1}{2}, \tag{1.3}$$

$$L_{a,\ b,\ s} := \Big(-\frac{d^2}{dx^2}\Big)^s - b\frac{d}{dx} - a : \quad H^{2s}(\mathbb{R}) \to L^2(\mathbb{R}), \quad \frac{1}{2} < s < 1 \tag{1.4}$$

is non-selfadjoint. By means of the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ipx}dx, \quad p \in \mathbb{R} \tag{1.5}$$

it can be easily derived that the essential spectrum of the operator $L_{a,\ b,\ s}$ is given by

$$\lambda_{a,\ b,\ s}(p) := |p|^{2s} - a - ibp, \quad p \in \mathbb{R}.$$

Evidently, when $a > 0$ the operator $L_{a,\ b,\ s}$ is Fredholm, since the origin does not belong to its essential spectrum. But when a vanishes, the operator $L_{0,\ b,\ s}$ does not satisfy the Fredholm property because its essential spectrum contains the origin.

Note that in the absence of the transport term we are dealing with the self-adjoint operator

$$\Big(-\frac{d^2}{dx^2}\Big)^s - a : \quad H^{2s}(\mathbb{R}) \to L^2(\mathbb{R}), \quad a > 0,$$

which fails to satisfy the Fredholm property (see [23]). Let us write down the corresponding sequence of approximate equations with $m \in \mathbb{N}$ as

$$\Big(-\frac{d^2}{dx^2}\Big)^s u_m - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in \mathbb{R}, \quad 0 < s < 1, \tag{1.6}$$

where the right sides tend to the right side of (1.2) in $L^2(\mathbb{R})$ as $m \to \infty$. The inner product of two functions

$$(f(x), g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx, \tag{1.7}$$

with a slight abuse of notations when these functions are not square integrable. Indeed, if $f(x) \in L^1(\mathbb{R})$ and $g(x) \in L^\infty(\mathbb{R})$, then clearly the integral in the right side of (1.7) makes sense, like for example in the case of functions involved in the orthogonality conditions (1.10) and (1.11) of Theorems 1.1 and 1.2 below. For our problems on the finite interval $I := [0, 2\pi]$ with periodic boundary conditions, we will use the inner product analogous to (1.7), replacing the real line with I. In the first part of the present work we will consider the spaces $H^1(\mathbb{R})$ and $H^{2s}(\mathbb{R}),\ 0 < s < 1$ equipped with the norms

$$\|u\|^2_{H^1(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \left\|\frac{du}{dx}\right\|^2_{L^2(\mathbb{R})} \tag{1.8}$$

and

$$\|u\|_{H^{2s}(\mathbb{R})}^2 := \|u\|_{L^2(\mathbb{R})}^2 + \left\| \Big(-\frac{d^2}{dx^2} \Big)^s u \right\|_{L^2(\mathbb{R})}^2 \tag{1.9}$$

respectively. When using the norms $H^1(I)$ and $H^{2s}(I),\ 0 < s < 1$ in the second part of the article, we will replace $\mathbb{R}$ with I in formulas (1.8) and (1.9) respectively. Our first main proposition is as follows.

Theorem 1.1. *Let $f(x) : \mathbb{R} \to \mathbb{R}$ and $f(x) \in L^2(\mathbb{R})$.*

a) Suppose $a > 0$ and $0 < s \leq \frac{1}{2}$. Then equation (1.2) admits a unique solution $u(x) \in H^1(\mathbb{R})$.

b) If $a > 0$ and $\frac{1}{2} < s < 1$, then problem (1.2) possesses a unique solution $u(x) \in H^{2s}(\mathbb{R})$.

c) Suppose $a = 0$ and $0 < s < \frac{1}{4}$. Let in addition $f(x) \in L^1(\mathbb{R})$. Then equation (1.2) has a unique solution $u(x) \in H^1(\mathbb{R})$.

d) If $a = 0$ and $\frac{1}{4} \leq s \leq \frac{1}{2}$, we also assume that $xf(x) \in L^1(\mathbb{R})$. Then problem (1.2) admits a unique solution $u(x) \in H^1(\mathbb{R})$ if and only if the orthogonality condition

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{1.10}$$

holds.

e) Suppose $a = 0$ and $\frac{1}{2} < s < 1$. Let in addition $xf(x) \in L^1(\mathbb{R})$. Then equation (1.2) possesses a unique solution $u(x) \in H^{2s}(\mathbb{R})$ if and only if the orthogonality relation (1.10) holds.

Evidently, the expression in the left side of (1.10) is well defined by virtue of the simple argument analogous to the proof of Fact 1 of [18]. We turn our attention to establishing the solvability in the sense of sequences for our equation on the whole real line.

Theorem 1.2. *Let $m \in \mathbb{N},\ f_m(x) : \mathbb{R} \to \mathbb{R}$ and $f_m(x) \in L^2(\mathbb{R})$. Moreover, $f_m(x) \to f(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$.*

a) If $a > 0$ and $0 < s \leq \frac{1}{2}$, then equations (1.2) and (1.6) have unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R})$ respectively, such that $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$.

b) Suppose $a > 0$ and $\frac{1}{2} < s < 1$. Then problems (1.2) and (1.6) possess unique solutions $u(x) \in H^{2s}(\mathbb{R})$ and $u_m(x) \in H^{2s}(\mathbb{R})$ respectively, such that $u_m(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $m \to \infty$.

c) If $a=0$ and $0<s<\frac{1}{4}$, let in addition $f_m(x)\in L^1(\mathbb{R})$ and $f_m(x)\to f(x)$ in $L^1(\mathbb{R})$ as $m\to\infty$. Then equations (1.2) and (1.6) admit unique solutions $u(x)\in H^1(\mathbb{R})$ and $u_m(x)\in H^1(\mathbb{R})$ respectively, such that $u_m(x)\to u(x)$ in $H^1(\mathbb{R})$ as $m\to\infty$.

d) Suppose that $a=0$ and $\frac{1}{4}\le s\le\frac{1}{2}$. We also assume that $xf_m(x)\in L^1(\mathbb{R})$ and $xf_m(x)\to xf(x)$ in $L^1(\mathbb{R})$ as $m\to\infty$. Furthermore,

$$(f_m(x),1)_{L^2(\mathbb{R})}=0,\quad m\in\mathbb{N} \tag{1.11}$$

holds. Then problems (1.2) and (1.6) have unique solutions $u(x)\in H^1(\mathbb{R})$ and $u_m(x)\in H^1(\mathbb{R})$ respectively, such that $u_m(x)\to u(x)$ in $H^1(\mathbb{R})$ as $m\to\infty$.

e) Suppose that $a=0$ and $\frac{1}{2}<s<1$. Let in addition $xf_m(x)\in L^1(\mathbb{R})$ and $xf_m(x)\to xf(x)$ in $L^1(\mathbb{R})$ as $m\to\infty$. Moreover, orthogonality relations (1.11) hold. Then equations (1.2) and (1.6) possess unique solutions $u(x)\in H^{2s}(\mathbb{R})$ and $u_m(x)\in H^{2s}(\mathbb{R})$ respectively, such that $u_m(x)\to u(x)$ in $H^{2s}(\mathbb{R})$ as $m\to\infty$.

Note that in the parts a) and b) of Theorems 1.1 and 1.2 above the orthogonality conditions are not used, as distinct from the situation without a drift term considered in the parts e) of Theorems 1.1 and 1.2 of [23]. Another issue here is that in Theorems 1.1 and 1.2 of the present article we establish the solvability of our equations in $H^1(\mathbb{R})$ for $0<s\le\frac{1}{2}$ but in the cases a) and e) of Theorems 1.1 and 1.2 of [23] we show the solvability of our problems without a transport only in $H^{2s}(\mathbb{R})$. Finally, we observe that in the parts e) of Theorems 1.1 and 1.2 above only a single orthogonality condition is needed, as distinct from the cases a) of Theorems 1.1 and 1.2 of [23], where the second orthogonality relation is required for $s\in\left[\frac{3}{4},1\right)$ along with the assumption that $x^2f(x),\ x^2f_m(x)\in L^1(\mathbb{R}),\ m\in\mathbb{N}$. Hence, the introduction of the transport term provides the regularization for the solutions of our equations.

In the second part of the work we study our equation on the finite interval with periodic boundary conditions, i.e. $I:=[0,\ 2\pi]$, namely

$$\Big(-\frac{d^2}{dx^2}\Big)^s u-b\frac{du}{dx}-au=f(x),\quad x\in I, \tag{1.12}$$

where $a\ge 0$ and $b\in\mathbb{R},\ b\neq 0$ are constants and the right side of (1.12) is bounded and periodic. Obviously,

$$\|f\|_{L^1(I)}\le 2\pi\|f\|_{L^\infty(I)}<\infty,\quad \|f\|_{L^2(I)}\le\sqrt{2\pi}\|f\|_{L^\infty(I)}<\infty. \tag{1.13}$$

Thus $f(x)\in L^1(I)\cap L^2(I)$ as well. We use the Fourier transform

$$f_n:=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}f(x)e^{-inx}dx,\quad n\in\mathbb{Z}, \tag{1.14}$$

such that

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{inx}}{\sqrt{2\pi}}.$$

Evidently, the non-selfadjoint operator involved in the left side of (1.12)

$$\mathcal{L}_{a,\ b,\ s} := \Big(-\frac{d^2}{dx^2}\Big)^s - b\frac{d}{dx} - a: \quad H^1(I) \to L^2(I), \quad 0 < s \leq \frac{1}{2}, \tag{1.15}$$

$$\mathcal{L}_{a,\ b,\ s} := \Big(-\frac{d^2}{dx^2}\Big)^s - b\frac{d}{dx} - a: \quad H^{2s}(I) \to L^2(I), \quad \frac{1}{2} < s < 1 \tag{1.16}$$

is Fredholm. By means of (1.14), it can be easily verified that the spectrum of $\mathcal{L}_{a,\ b,\ s}$ is given by

$$\lambda_{a,\ b,\ s}(n) := |n|^{2s} - a - ibn, \quad n \in \mathbb{Z}$$

and the corresponding eigenfunctions are the Fourier harmonics $\frac{e^{inx}}{\sqrt{2\pi}},\ n \in \mathbb{Z}$. The eigenvalues of the operator $\mathcal{L}_{a,\ b,\ s}$ are simple, as distinct from the situation without the transport term, when the eigenvalues corresponding to $n \neq 0$ are double-degenerate. The appropriate function spaces here $H^1(I)$ and $H^{2s}(I)$ are

$$\{u(x) : I \to \mathbb{R} \mid u(x),\ u'(x) \in L^2(I),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi)\}.$$

and

$$\left\{u(x) : I \to \mathbb{R} \mid u(x),\ \left(-\frac{d^2}{dx^2}\right)^s u(x) \in L^2(I),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi)\right\}$$

respectively. For the technical purposes, we introduce the following auxiliary constrained subspaces

$$H_0^1(I) = \{u(x) \in H^1(I) \mid (u(x), 1)_{L^2(I)} = 0\} \tag{1.17}$$

and

$$H_0^{2s}(I) = \{u(x) \in H^{2s}(I) \mid (u(x), 1)_{L^2(I)} = 0\} \tag{1.18}$$

which are Hilbert spaces as well (see e.g. Chapter 2.1 of [9]). Clearly, for $a > 0$, the kernel of the operator $\mathcal{L}_{a,\ b,\ s}$ is trivial. When $a = 0$, we consider

$$\mathcal{L}_{0,\ b,\ s}: \quad H_0^1(I) \to L^2(I), \quad 0 < s \leq \frac{1}{2},$$

$$\mathcal{L}_{0,\ b,\ s}: \quad H_0^{2s}(I) \to L^2(I), \quad \frac{1}{2} < s < 1.$$

Evidently, such operator has the trivial kernel as well. We write down the corresponding sequence of the approximate equations with $m \in \mathbb{N}$, namely

$$\Big(-\frac{d^2}{dx^2}\Big)^s u_m - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in I, \tag{1.19}$$

where the right sides are bounded, periodic and converge to the right side of (1.12) in $L^\infty(I)$ as $m \to \infty$. The purpose of Theorems 1.3 and 1.4 below is to demonstrate the formal similarity of the results on the finite interval with periodic boundary conditions to the ones derived for the whole real line situation in Theorems 1.1 and 1.2 above.

Theorem 1.3. *Let $f(x) : I \to \mathbb{R}$, such that $f(0) = f(2\pi)$ and $f(x) \in L^\infty(I)$.*

a) If $a > 0$ and $0 < s \le \frac{1}{2}$, then equation (1.12) admits a unique solution $u(x) \in H^1(I)$.

b) Suppose $a > 0$ and $\frac{1}{2} < s < 1$. Then problem (1.12) has a unique solution $u(x) \in H^{2s}(I)$.

c) If $a = 0$ and $0 < s \le \frac{1}{2}$, then equation (1.12) possesses a unique solution $u(x) \in H_0^1(I)$ if and only if the orthogonality condition

$$(f(x), 1)_{L^2(I)} = 0 \tag{1.20}$$

holds.

d) Suppose $a = 0$ and $\frac{1}{2} < s < 1$. Then problem (1.12) admits a unique solution $u(x) \in H_0^{2s}(I)$ if and only if the orthogonality relation (1.20) holds.

Our final main statement deals with the solvability in the sense of sequences for our problem on the finite interval I.

Theorem 1.4. *Let $m \in \mathbb{N}$, $f_m(x) : I \to \mathbb{R}$, such that $f_m(0) = f_m(2\pi)$. Furthermore, $f_m(x) \in L^\infty(I)$ and $f_m(x) \to f(x)$ in $L^\infty(I)$ as $m \to \infty$.*

a) Suppose $a > 0$ and $0 < s \le \frac{1}{2}$. Then equations (1.12) and (1.19) possess unique solutions $u(x) \in H^1(I)$ and $u_m(x) \in H^1(I)$ respectively, such that $u_m(x) \to u(x)$ in $H^1(I)$ as $m \to \infty$.

b) If $a > 0$ and $\frac{1}{2} < s < 1$, then problems (1.12) and (1.19) admit unique solutions $u(x) \in H^{2s}(I)$ and $u_m(x) \in H^{2s}(I)$ respectively, such that $u_m(x) \to u(x)$ in $H^{2s}(I)$ as $m \to \infty$.

c) Suppose that $a=0, \quad 0<s\le \frac{1}{2}$ *and*

$$(f_m(x),1)_{L^2(I)}=0, \quad m\in \mathbb{N}. \tag{1.21}$$

Then equations (1.12) and (1.19) have unique solutions $u(x)\in H_0^1(I)$ *and* $u_m(x)\in H_0^1(I)$ *respectively, such that* $u_m(x)\to u(x)$ *in* $H_0^1(I)$ *as* $m\to\infty$.

d) If $a=0, \quad \frac{1}{2}<s<1$ *and orthogonality relations (1.21) hold. Then problems (1.12) and (1.19) admit unique solutions* $u(x)\in H_0^{2s}(I)$ *and* $u_m(x)\in H_0^{2s}(I)$ *respectively, such that* $u_m(x)\to u(x)$ *in* $H_0^{2s}(I)$ *as* $m\to\infty$.

Note that in the cases a) and b) of Theorems 1.3 and 1.4 above the orthogonality relations are not needed. When there is no transport term in our problems, the situation is more singular (see formulas (3.2) and (3.8) below with $a=n_0^{2s},\ n_0\in\mathbb{N}$).

2. The whole real line case

Proof of Theorem 1.1. Let us first demonstrate that it would be sufficient to solve our equation in $L^2(\mathbb{R})$. Indeed, if $u(x)$ is a square integrable solution of (1.2), directly from this equation under the stated assumptions we obtain

$$\Big(-\frac{d^2}{dx^2}\Big)^s u - b\frac{du}{dx}\in L^2(\mathbb{R})$$

as well. By means of the standard Fourier transform (1.5), we derive $(|p|^{2s}-ibp)\widehat{u}(p)\in L^2(\mathbb{R})$, such that

$$\int_{-\infty}^{\infty}(|p|^{4s}+b^2p^2)|\widehat{u}(p)|^2dp<\infty. \tag{2.1}$$

Let $0<s\le\frac{1}{2}$. From (2.1) we easily deduce that $\int_{-\infty}^{\infty}p^2|\widehat{u}(p)|^2dp<\infty$. Hence, $\frac{du}{dx}\in L^2(\mathbb{R})$ and $u(x)\in H^1(\mathbb{R})$ as well.

Suppose $\frac{1}{2}<s<1$. Then (2.1) yields

$$\int_{-\infty}^{\infty}|p|^{4s}|\widehat{u}(p)|^2dp<\infty.$$

Therefore, $\Big(-\frac{d^2}{dx^2}\Big)^s u\in L^2(\mathbb{R})$, such that $u(x)\in H^{2s}(\mathbb{R})$ as well.

Let us establish the uniqueness of solutions of (1.2) in the case when $0 < s \leq \frac{1}{2}$. For $\frac{1}{2} < s < 1$ the argument will be similar. Suppose that $u_1(x),\ u_2(x) \in H^1(\mathbb{R})$ satisfy (1.2). Then their difference $w(x) := u_1(x) - u_2(x) \in H^1(\mathbb{R})$ solves the homogeneous problem

$$\Big(-\frac{d^2}{dx^2}\Big)^s w - b\frac{dw}{dx} - aw = 0.$$

Since the operator $L_{a,\ b,\ s}$ defined in (1.3) does not possess any nontrivial zero modes in $H^1(\mathbb{R})$, the function $w(x) = 0$ identically on $\mathbb{R}$.

We apply the standard Fourier transform (1.5) to both sides of equation (1.2) and arrive at

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s} - a - ibp}, \quad p \in \mathbb{R}, \quad 0 < s < 1. \tag{2.2}$$

Thus,

$$\|u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{|\widehat{f}(p)|^2}{(|p|^{2s} - a)^2 + b^2 p^2} dp. \tag{2.3}$$

Let us first consider the cases a) and b) of the theorem. (2.3) implies that

$$\|u\|^2_{L^2(\mathbb{R})} \leq \frac{1}{C}\|f\|^2_{L^2(\mathbb{R})} < \infty$$

due to the one of our assumptions. Here and further down C will denote a finite, positive constant. By virtue of the argument above, in the situation when $a > 0$, equation (1.2) admits a unique solution $u(x) \in H^1(\mathbb{R})$ for $0 < s \leq \frac{1}{2}$ and $u(x) \in H^{2s}(\mathbb{R})$ if $\frac{1}{2} < s < 1$.

We conclude the argument by treating the cases when $a = 0$. Formula (2.2) gives us

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|\leq 1\}} + \frac{\widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|>1\}}. \tag{2.4}$$

Here and throughout the article χ_A will denote the characteristic function of a set $A \subseteq \mathbb{R}$. Evidently, the second term in the right side of (2.4) can be estimated from above in the absolute value by $\frac{|\widehat{f}(p)|}{\sqrt{1+b^2}} \in L^2(\mathbb{R})$ since $f(x)$ is square integrable as assumed. Clearly, the inequality

$$\|\widehat{f}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}}\|f(x)\|_{L^1(\mathbb{R})} \tag{2.5}$$

holds. When $0 < s < \frac{1}{4}$, we use (2.5) to derive

$$\Big|\frac{\widehat{f}(p)}{|p|^{2s} - ibp}\chi_{\{|p|\leq 1\}}\Big| \leq \frac{|\widehat{f}(p)|}{|p|^{2s}}\chi_{\{|p|\leq 1\}} \leq \frac{\|f(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|p|^{2s}}\chi_{\{|p|\leq 1\}}.$$

This allows us to obtain the upper bound on the norm

$$\left\| \frac{\widehat{f}(p)}{|p|^{2s} - ibp} \chi_{\{|p|\le 1\}} \right\|^2_{L^2(\mathbb{R})} \le \frac{\|f(x)\|^2_{L^1(\mathbb{R})}}{\pi(1-4s)} < \infty$$

since $f(x) \in L^1(\mathbb{R})$ as assumed. By means of the argument above, in the case c) of the theorem equation (1.2) possesses a unique solution $u(x) \in H^1(\mathbb{R})$.

To establish the statements d) and e) of our theorem, we express

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^p \frac{d\widehat{f}(s)}{ds} ds.$$

Hence, the first term in the right side of (2.4) can be written as

$$\frac{\widehat{f}(0)}{|p|^{2s} - ibp} \chi_{\{|p|\le 1\}} + \frac{\int_0^p \frac{d\widehat{f}(s)}{ds} ds}{|p|^{2s} - ibp} \chi_{\{|p|\le 1\}}. \tag{2.6}$$

By virtue of definition (1.5) of the standard Fourier transform, we easily arrive at

$$\left| \frac{d\widehat{f}(p)}{dp} \right| \le \frac{1}{\sqrt{2\pi}} \|xf(x)\|_{L^1(\mathbb{R})}.$$

This enables us to estimate the second term in (2.6) from above in the absolute value by

$$\frac{1}{\sqrt{2\pi}} \frac{\|xf(x)\|_{L^1(\mathbb{R})}}{|b|} \chi_{\{|p|\le 1\}} \in L^2(\mathbb{R})$$

due to our assumptions. Let us analyze the first term in (2.6), which is given by

$$\frac{\widehat{f}(0)}{|p|^{2s} - ibp} \chi_{\{|p|\le 1\}}. \tag{2.7}$$

Clearly, if $\frac{1}{4} \le s \le \frac{1}{2}$, expression (2.7) can be bounded below in the absolute value by

$$\frac{|\widehat{f}(0)|}{|p|^{2s}\sqrt{1+b^2}} \chi_{\{|p|\le 1\}},$$

which does not belong to $L^2(\mathbb{R})$ unless $\widehat{f}(0)$ vanishes. This gives us orthogonality relation (1.10). In the case d) of the theorem, according to the argument above, the square integrability of the solution $u(x)$ of equation (1.2) will be equivalent to $u(x) \in H^1(\mathbb{R})$.

Evidently, for $\frac{1}{2} < s < 1$, expression (2.7) can be estimated below in the absolute value by

$$\frac{|\widehat{f}(0)|}{|p|\sqrt{1+b^2}} \chi_{\{|p|\le 1\}},$$

which is not square integrable unless orthogonality condition (1.10) holds. In the case e) of our theorem, by virtue of the argument above, the square integrability of the solution $u(x)$ of problem (1.2) will be equivalent to $u(x) \in H^{2s}(\mathbb{R})$. ∎

Let us proceed to establishing the solvability in the sense of sequences for our problem on the whole real line.

Proof of Theorem 1.2. First we suppose that equations (1.2) and (1.6) admit unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R}),\ m \in \mathbb{N}$ respectively if $0 < s \leq \frac{1}{2}$, similarly $u(x) \in H^{2s}(\mathbb{R})$ and $u_m(x) \in H^{2s}(\mathbb{R}),\ m \in \mathbb{N}$ for $\frac{1}{2} < s < 1$, such that $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. This will imply that $u_m(x)$ also converges to $u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ for $0 < s \leq \frac{1}{2}$ and analogously $u_m(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $m \to \infty$ when $\frac{1}{2} < s < 1$. Indeed, from (1.2) and (1.6) we easily deduce

$$\left\| \left(-\frac{d^2}{dx^2}\right)^s (u_m - u) - b\frac{d(u_m - u)}{dx} \right\|_{L^2(\mathbb{R})} \leq \|f_m - f\|_{L^2(\mathbb{R})} + a\|u_m - u\|_{L^2(\mathbb{R})}. \quad (2.8)$$

The right side of upper bound (2.8) tends to zero as $m \to \infty$ due to our assumptions. Using the standard Fourier transform (1.5), we easily arrive at

$$\int_{-\infty}^{\infty} (|p|^{4s} + b^2 p^2)|\widehat{u}_m(p) - \widehat{u}(p)|^2 dp \to 0, \quad m \to \infty. \quad (2.9)$$

Let $0 < s \leq \frac{1}{2}$. By means of (2.9)

$$\int_{-\infty}^{\infty} p^2 |\widehat{u}_m(p) - \widehat{u}(p)|^2 dp \to 0, \quad m \to \infty,$$

such that $\frac{du_m}{dx} \to \frac{du}{dx}$ in $L^2(\mathbb{R})$ as $m \to \infty$. Therefore, if $0 < s \leq \frac{1}{2}$, we have $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ as well.

Suppose $\frac{1}{2} < s < 1$. By virtue of (2.9)

$$\int_{-\infty}^{\infty} |p|^{4s} |\widehat{u}_m(p) - \widehat{u}(p)|^2 dp \to 0, \quad m \to \infty.$$

Hence $\left(-\frac{d^2}{dx^2}\right)^s u_m \to \left(-\frac{d^2}{dx^2}\right)^s u$ in $L^2(\mathbb{R})$ as $m \to \infty$. This implies that for $\frac{1}{2} < s < 1$, we obtain $u_m(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $m \to \infty$ as well.

We apply the standard Fourier transform (1.5) to both sides of (1.6), such that

$$\widehat{u}_m(p) = \frac{\widehat{f}_m(p)}{|p|^{2s} - a - ibp}, \quad m \in \mathbb{N}, \quad p \in \mathbb{R}, \quad 0 < s < 1. \tag{2.10}$$

Let us first discuss the cases a) and b) of the theorem. By means of the parts a) and b) of Theorem 1.1, when the constant $a > 0$ equations (1.2) and (1.6) admit unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R}),\ m \in \mathbb{N}$ respectively if $0 < s \leq \frac{1}{2}$, similarly $u(x) \in H^{2s}(\mathbb{R})$ and $u_m(x) \in H^{2s}(\mathbb{R}),\ m \in \mathbb{N}$ for $\frac{1}{2} < s < 1$. Formulas (2.10) and (2.2) give us

$$\|u_m - u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{|\widehat{f}_m(p) - \widehat{f}(p)|^2}{(|p|^{2s} - a)^2 + b^2p^2} dp.$$

Hence,

$$\|u_m - u\|_{L^2(\mathbb{R})} \leq \frac{1}{C} \|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty$$

via the one of our assumptions. Therefore, in the cases of $a > 0$ we have $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ for $0 < s \leq \frac{1}{2}$ and $u_m(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $m \to \infty$ if $\frac{1}{2} < s < 1$ by virtue of the argument above.

Let us finish the proof of the theorem by treating the situations when the parameter $a = 0$. By means of the result of the part a) of Lemma 3.3 of [17], under our assumptions

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{2.11}$$

holds in the cases d) and e). Then by virtue of the parts c), d) and e) of Theorem 1.1, problems (1.2) and (1.6) with $a = 0$ have unique solutions $u(x) \in H^1(\mathbb{R})$ and $u_m(x) \in H^1(\mathbb{R}),\ m \in \mathbb{N}$ respectively when if $0 < s \leq \frac{1}{2}$, similarly $u(x) \in H^{2s}(\mathbb{R})$ and $u_m(x) \in H^{2s}(\mathbb{R}),\ m \in \mathbb{N}$ for $\frac{1}{2} < s < 1$. Formulas (2.10) and (2.2) imply that

$$\widehat{u}_m(p) - \widehat{u}(p) = \frac{\widehat{f}_m(p) - \widehat{f}(p)}{|p|^{2s} - ibp} \chi_{\{|p| \leq 1\}} + \frac{\widehat{f}_m(p) - \widehat{f}(p)}{|p|^{2s} - ibp} \chi_{\{|p| > 1\}}. \tag{2.12}$$

Apparently, the second term in the right side of (2.12) can be bounded from above in the $L^2(\mathbb{R})$ norm by

$$\frac{1}{\sqrt{1 + b^2}} \|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty$$

via the one of our assumptions. Let $0 < s < \frac{1}{4}$. Using the analog of inequality (2.5), we derive

$$\left| \frac{\widehat{f}_m(p) - \widehat{f}(p)}{|p|^{2s} - ibp} \chi_{\{|p| \leq 1\}} \right| \leq \frac{|\widehat{f}_m(p) - \widehat{f}(p)|}{|p|^{2s}} \chi_{\{|p| \leq 1\}} \leq \frac{\|f_m - f\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|p|^{2s}} \chi_{\{|p| \leq 1\}},$$

such that

$$\left\| \frac{\widehat{f_m}(p) - \widehat{f}(p)}{|p|^{2s} - ibp} \chi_{\{|p| \le 1\}} \right\|_{L^2(\mathbb{R})} \le \frac{\|f_m - f\|_{L^1(\mathbb{R})}}{\sqrt{\pi(1-4s)}} \to 0, \quad m \to \infty$$

due to the one of the given assumptions. By means of the argument above, $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ in the situation when the constant a vanishes and $0 < s < \frac{1}{4}$.

To address the cases d) and e) of our theorem, we use the orthogonality conditions (2.11) and (1.11). Definition (1.5) of the standard Fourier transform yields

$$\widehat{f}(0) = 0, \quad \widehat{f_m}(0) = 0, \quad m \in \mathbb{N}.$$

Thus

$$\widehat{f}(p) = \int_0^p \frac{d\widehat{f}(s)}{ds} ds, \quad \widehat{f_m}(p) = \int_0^p \frac{d\widehat{f_m}(s)}{ds} ds, \quad m \in \mathbb{N}. \tag{2.13}$$

Therefore, the first term in the right side of (2.12) in the cases d) and e) of the theorem can be written as

$$\frac{\int_0^p [\frac{d\widehat{f_m}(s)}{ds} - \frac{d\widehat{f}(s)}{ds}] ds}{|p|^{2s} - ibp} \chi_{\{|p| \le 1\}}.$$

By means of the definition of the standard Fourier transform (1.5), we easily derive

$$\left| \frac{d\widehat{f_m}(p)}{dp} - \frac{d\widehat{f}(p)}{dp} \right| \le \frac{1}{\sqrt{2\pi}} \|x f_m(x) - x f(x)\|_{L^1(\mathbb{R})}.$$

Hence

$$\left| \frac{\int_0^p [\frac{d\widehat{f_m}(s)}{ds} - \frac{d\widehat{f}(s)}{ds}] ds}{|p|^{2s} - ibp} \chi_{\{|p| \le 1\}} \right| \le \frac{\|x f_m(x) - x f(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|b|} \chi_{\{|p| \le 1\}},$$

such that

$$\left\| \frac{\int_0^p [\frac{d\widehat{f_m}(s)}{ds} - \frac{d\widehat{f}(s)}{ds}] ds}{|p|^{2s} - ibp} \chi_{\{|p| \le 1\}} \right\|_{L^2(\mathbb{R})} \le \frac{\|x f_m(x) - x f(x)\|_{L^1(\mathbb{R})}}{\sqrt{\pi}|b|} \to 0$$

as $m \to \infty$ by means of the one of our assumptions. Therefore, $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. By virtue of the argument above in the situation when $a = 0$ we have $u_m(x) \to u(x)$ in $H^1(\mathbb{R})$ as $m \to \infty$ if $\frac{1}{4} \le s \le \frac{1}{2}$ and $u_m(x) \to u(x)$ in $H^{2s}(\mathbb{R})$ as $m \to \infty$ for $\frac{1}{2} < s < 1$. ∎

3. The problem on the finite interval

Proof of Theorem 1.3. Let us first establish that it would be sufficient to solve our problem in $L^2(I)$. Indeed, if $u(x)$ is a square integrable solution of (1.12), periodic on I along with its first derivative, directly from our equation under the given conditions we derive

$$\Big(-\frac{d^2}{dx^2}\Big)^s u - b\frac{du}{dx} \in L^2(I).$$

(1.14) implies $(|n|^{2s} - ibn)u_n \in l^2$, such that

$$\sum_{n=-\infty}^{\infty} (|n|^{4s} + b^2n^2)|u_n|^2 < \infty. \tag{3.1}$$

Suppose $0 < s \leq \frac{1}{2}$. Then by means of (3.1) we have $\sum_{n=-\infty}^{\infty} n^2|u_n|^2 < \infty$, which yields $\frac{du}{dx} \in L^2(I)$. Hence, $u(x) \in H^1(I)$ as well.

Let $\frac{1}{2} < s < 1$. By virtue of (3.1) we obtain $\sum_{n=-\infty}^{\infty} |n|^{4s}|u_n|^2 < \infty$, which gives us $\Big(-\frac{d^2}{dx^2}\Big)^s u(x) \in L^2(I)$. Thus, $u(x) \in H^{2s}(I)$ as well.

To show the uniqueness of solutions of (1.12), we discuss the situation when $a > 0$ and $0 < s \leq \frac{1}{2}$. If $a > 0$ and $\frac{1}{2} < s < 1$, the similar ideas can be exploited in $H^{2s}(I)$. For $a = 0,\ 0 < s \leq \frac{1}{2}$ and when $a = 0,\ \frac{1}{2} < s < 1$ our argument can be generalized using the constrained subspaces $H_0^1(I)$ and $H_0^{2s}(I)$ respectively defined above. Let us suppose that $u_1(x),\ u_2(x) \in H^1(I)$ solve (1.12). Then their difference $w(x) := u_1(x) - u_2(x) \in H^1(I)$ satifies the homogeneous equation

$$\Big(-\frac{d^2}{dx^2}\Big)^s w - b\frac{dw}{dx} - aw = 0.$$

Since the operator $\mathcal{L}_{a,\ b,\ s}$ introduced in (1.15) does not have any nontrivial $H^1(I)$ zero modes, the function $w(x) \equiv 0$ on I.

We apply the Fourier transform (1.14) to both sides of problem (1.12), which yields

$$u_n = \frac{f_n}{|n|^{2s} - a - ibn}, \quad n \in \mathbb{Z}. \tag{3.2}$$

Hence

$$\|u\|^2_{L^2(I)} = \sum_{n=-\infty}^{\infty} \frac{|f_n|^2}{(|n|^{2s} - a)^2 + b^2n^2}. \tag{3.3}$$

First we deal with the cases a) and b) of our theorem. By virtue of (3.3), we arrive at

$$\|u\|^2_{L^2(I)} \leq \frac{1}{C}\|f\|^2_{L^2(I)} < \infty$$

via the one of our assumptions along with (1.13). By means of the argument above, in the situation when $a > 0$, equation (1.12) admits a unique solution $u(x) \in H^1(I)$ if $0 < s \leq \frac{1}{2}$ and $u(x) \in H^{2s}(I)$ for $\frac{1}{2} < s < 1$.

In order to conclude the proof of the theorem, we consider the situation when $a = 0$. Then (3.2) yields

$$u_n = \frac{f_n}{|n|^{2s} - ibn}, \quad n \in \mathbb{Z}. \tag{3.4}$$

Evidently, the right side of (3.4) belongs to l^2 if and only if

$$f_0 = 0, \tag{3.5}$$

such that

$$\|u\|^2_{L^2(I)} = \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{|f_n|^2}{n^{4s} + b^2 n^2} \leq \frac{1}{1+b^2}\|f\|^2_{L^2(I)} < \infty,$$

due to the one of the given conditions and (1.13). By virtue of the argument above in the cases c) and d) of the theorem $u(x) \in H^1_0(I)$ and $u(x) \in H^{2s}_0(I)$ respectively as well. Obviously, (3.5) is equivalent to orthogonality condition (1.20). ■

Let us proceed to establishing the solvability in the sense of sequences for our problem on the interval I with periodic boundary conditions.

Proof of Theorem 1.4. Under the given assumptions, we obtain

$$|f(0) - f(2\pi)| \leq |f(0) - f_m(0)| + |f_m(2\pi) - f(2\pi)| \leq 2\|f_m - f\|_{L^\infty(I)} \to 0$$

as $m \to \infty$. Hence, $f(0) = f(2\pi)$. By means of (1.13) for $f_m(x),\ f(x)$ bounded on the interval I, we obtain $f_m(x),\ f(x) \in L^1(I) \cap L^2(I),\ m \in \mathbb{N}$. The analog of (1.13) also implies

$$\|f_m(x) - f(x)\|_{L^1(I)} \leq 2\pi\|f_m(x) - f(x)\|_{L^\infty(I)} \to 0, \quad m \to \infty. \tag{3.6}$$

Hence, $f_m(x) \to f(x)$ in $L^1(I)$ as $m \to \infty$. Similarly, (1.13) yields

$$\|f_m(x) - f(x)\|_{L^2(I)} \leq \sqrt{2\pi}\|f_m(x) - f(x)\|_{L^\infty(I)} \to 0, \quad m \to \infty. \tag{3.7}$$

Hence, $f_m(x) \to f(x)$ in $L^2(I)$ as $m \to \infty$ as well. We apply the Fourier transform (1.14) to both sides of (1.19) and derive

$$u_{m,n} = \frac{f_{m,n}}{|n|^{2s} - a - ibn}, \quad m \in \mathbb{N}, \quad n \in \mathbb{Z}. \tag{3.8}$$

First we consider the cases a) and b) of our theorem. By virtue of the parts a) and b) of Theorem 1.3, when $a > 0$ problems (1.12) and (1.19) have unique solutions $u(x) \in H^1(I)$ and $u_m(x) \in H^1(I),\ m \in \mathbb{N}$ respectively if $0 < s \le \frac{1}{2}$, similarly $u(x) \in H^{2s}(I)$ and $u_m(x) \in H^{2s}(I),\ m \in \mathbb{N}$ for $\frac{1}{2} < s < 1$. (3.8) along with (3.2) and (3.7) imply that

$$\|u_m - u\|^2_{L^2(I)} = \sum_{n=-\infty}^{\infty} \frac{|f_{m,n} - f_n|^2}{(|n|^{2s} - a)^2 + b^2 n^2} \le \frac{1}{C} \|f_m - f\|^2_{L^2(I)} \to 0, \quad m \to \infty.$$

Thus, $u_m(x) \to u(x)$ in $L^2(I)$ as $m \to \infty$. By means of (1.12) and (1.19) we derive

$$\left\| \left(-\frac{d^2}{dx^2} \right)^s (u_m - u) - b\frac{d(u_m - u)}{dx} \right\|_{L^2(I)} \le \|f_m - f\|_{L^2(I)} + a\|u_m - u\|_{L^2(I)}.$$

The right side of this inequality converges to zero as $m \to \infty$ due to (3.7). The Fourier transform (1.14) gives us

$$\sum_{n=-\infty}^{\infty} (|n|^{4s} + b^2 n^2)|u_{m,n} - u_n|^2 \to 0, \quad m \to \infty. \tag{3.9}$$

Suppose $0 < s \le \frac{1}{2}$. Then (3.9) yields

$$\sum_{n=-\infty}^{\infty} n^2 |u_{m,n} - u_n|^2 \to 0, \quad m \to \infty.$$

Therefore, $\frac{du_m}{dx} \to \frac{du}{dx}$ in $L^2(I)$ as $m \to \infty$, which implies that $u_m(x) \to u(x)$ in $H^1(I)$ as $m \to \infty$ as well in the case a) of our theorem.

Let $\frac{1}{2} < s < 1$. By means of (3.9), we have

$$\sum_{n=-\infty}^{\infty} |n|^{4s} |u_{m,n} - u_n|^2 \to 0, \quad m \to \infty,$$

such that

$$\left(-\frac{d^2}{dx^2} \right)^s u_m \to \left(-\frac{d^2}{dx^2} \right)^s u$$

in $L^2(I)$ as $m \to \infty$. Therefore, $u_m(x) \to u(x)$ in $H^{2s}(I)$ as $m \to \infty$ as well in the case b) of the theorem.

Finally, let us turn our attention to the situation when the constant a vanishes. (1.21) along with (3.6) imply

$$|(f(x),1)_{L^2(I)}| = |(f(x)-f_m(x),1)_{L^2(I)}| \le \|f_m - f\|_{L^1(I)} \to 0, \quad m\to\infty.$$

Hence, the limiting orthogonality condition

$$(f(x),1)_{L^2(I)} = 0 \tag{3.10}$$

holds. By virtue of the parts c) and d) of Theorem 1.3, when $a=0$ equations (1.12) and (1.19) possess unique solutions $u(x)\in H_0^1(I)$ and $u_m(x)\in H_0^1(I),\ m\in\mathbb{N}$ respectively for $0<s\le\frac{1}{2}$, analogously $u(x)\in H_0^{2s}(I)$ and $u_m(x)\in H_0^{2s}(I),\ m\in\mathbb{N}$ if $\frac{1}{2}<s<1$. Formulas (3.2) and (3.8) yield

$$u_{m,n}-u_n = \frac{f_{m,n}-f_n}{|n|^{2s}-ibn}, \quad m\in\mathbb{N}, \quad n\in\mathbb{Z}. \tag{3.11}$$

Orthogonality relations (3.10) and (1.21) give us

$$f_0 = 0, \quad f_{m,0}=0, \quad m\in\mathbb{N}.$$

We obtain the upper bound on the norm

$$\|u_m-u\|_{L^2(I)} = \sqrt{\sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{|f_{m,n}-f_n|^2}{|n|^{4s}+b^2n^2}} \le \frac{\|f_m-f\|_{L^2(I)}}{\sqrt{1+b^2}} \to 0, \quad m\to\infty$$

via (3.7). Hence, $u_m(x)\to u(x)$ in $L^2(I)$ as $m\to\infty$. Therefore, when a vanishes and $0<s\le\frac{1}{2}$, we have $u_m(x)\to u(x)$ in $H_0^1(I)$ as $m\to\infty$ as well by means of the argument analogous to the one above in the proof of the part a) of the theorem. When $a=0$ and $\frac{1}{2}<s<1$, we derive $u_m(x)\to u(x)$ in $H_0^{2s}(I)$ as $m\to\infty$ as well by virtue of the argument analogical to the one in the proof of the case b) of our theorem. ■

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