On some exact solutions of heavenly equations in four dimensions

Ł. T. Stępień*

The Pedagogical University of Cracow,

ul. Podchorążych 2, 30-084 Kraków, Poland

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Abstract

Some new classes of exact solutions (so-called functionally-invariant solutions) of the elliptic and hyperbolic complex Monge-Ampère equations and of the second heavenly equation, mixed heavenly equation, asymmetric heavenly equation, evolution form of second heavenly equation, general heavenly equation, real general heavenly equation and one of the real sections of general heavenly equation, are found. Besides non-invariance of these found classes of solutions has been investigated. These classes of solutions determine the new classes of metrics without Killing vectors. A criterion of non-invariance of the solutions belonging to found classes, has been also formulated.

* sfstepie@cyf-kr.edu.pl, lukasz.stepien@up.krakow.pl

I. INTRODUCTION

This is well-known fact that Einstein equations of gravitational field do not appear to be generically integrable. In order to find exact solutions, one needs to consider spacetimes with symmetries. There in [1], is a survey including exact solutions of Einstein equations. One of such cases of symmetries of spacetime, is this, when the spacetime has two Killing vector, then the field equations reduce to so-called Ernst equation, [2]. This equation has been discussed in [3]. Another interesting and well-known case of reducing of Einstein equations, is the case of the so-called, second heavenly equation. It was derived by Plebański in [4]. The symmetries of this equation were investigated in, [5]. In [6] some hidden symmetry of type II and some exact solutions of the second heavenly equation have been obtained, by a reduction of this equation to the homogeneous Monge-Ampère equation in similarity variables. One of the very important classes of solutions of elliptic complex Monge-Ampère equation, are the solutions, which generate metrics, not possessing Killing vector, because such solutions are non-invariant and so they are the candidates for gravitational instantons, [7]. Gravitational instantons can be defined analogically to instantons in Yang-Mills theory, as solutions of Einstein equations, non-singular on a section of complexified spacetime, where the curvature decays at large distances, [8]. The most wanted gravitational instanton is so called, Kummer surface $K3$, [7], [9], [10]. $K3$ with a Ricci-flat Calabi-Yau metric and the complex torus with the flat metric, are the only compact 4-dimensional Riemannian hyper-Kähler manifolds, [11]. Finding of explicit form of the metric corresponding to Kummer surface $K3$ is a challenging problem, among others, because of the requirement of non-existence of Killing vector for such metric, which implicates the requirement of non-invariance of the solution of homogeneous elliptic complex Monge-Ampère equation. This stated the motivation of our searching for non-invariant solutions of homogeneous elliptic complex Monge-Ampère equation, however, as it has turned out, it is hard to find simultaneously non-invariant and real solutions in this case and just such solutions can describe Kummer surface. Some such exact solutions have been obtained in [9] and there also non-existence of Killing vector for these solutions has been checked. In [7] the real solutions of the hyperbolic complex Monge-Ampère equation was found and their non-invariance was also be checked. In [12] was showed that one can describe locally, non-degenerate complex surfaces by a solution for some Monge-Ampère type equation. Some exact solutions of multidimensional Monge-Ampère equation were found in [13], by using the subgroup of the generalized Poincarè group $P(1,4)$. In [14] contact transformations were appplied to Monge-Ampère

type equation. Some aspects of complex Monge-Ampère equation (among others, the existence and stability of its weak solutions on compact Kähler manifolds), were studied in [15]. In [16] the $sl(2n|2n)^1$ super-Toda-Lattices and the heavenly equations as continuum limit were investigated. The $N = 2$ heavenly equation is studied in [17]. In [18] the multikink solutions of second heavenly and asymmetric heavenly equations have been obtained. In [19] a classification of scalar partial differential equations of second order, non-invariant solutions of mixed heavenly equation and a connection between this equation and Husain equation have been presented. On the other hand, there in [20] it has been showed that every solution of Husain equation (related to chiral model of self-dual gravity, [21]) defines some solution of well-known Plebański first heavenly equation.
There in [19] also so-called asymmetric heavenly equation has been derived. This equation is connected to so-called evolution form of second heavenly equation, [22], [23], [24]. In [22] the evolution form of second heavenly equation has been derived by some symmetry reduction of Lie algebra of the area preserving group of diffeomorphisms of the 2-surface Σ^2 of self-dual Yang-Mills equations. The same result has been obtained in the case of second heavenly equation in [25]. In [26] it was showed that the so-called general heavenly equation governed anti-self-dual (ASD) gravity and also some exact solutions of this equations were presented.

In this paper we show that by applying so-called decomposition method, it is possible to find some new classes of exact non-invariant solutions (so-called functionally invariant solutions) of: the elliptic and hyperbolic complex Monge-Ampère equations, the second heavenly equation, mixed heavenly equation, asymmetric heavenly equations, evolution form of second heavenly equation, and general heavenly equation (also real general heavenly equation and one of real sections of general heavenly equation). The main version of the decomposition method, mentioned above, has been presented in [27].

This paper is organized as follows. In section 2 we briefly describe the procedure of the decomposition method (of course, one should not confuse this method with the Bogomolny decomposition, which was obtained for some nonlinear models in field theory, in the papers [28], [29] and [30]). Section 3 includes a short introduction of the heavenly equations mentioned above and to the non-invariance of the solutions of these equations. In section 4 we find, by using the mentioned decomposition method, the classes of exact solutions of the mentioned equations. Next, in this same section, we investigate non-invariance of the solutions belonging to the classes mentioned above.

We formulate also a criterion of non-invariance of the solutions belonging to found classes. In section 5 we give some conclusions. The current paper is a new version of the paper [31].

II. A SHORT DESCRIPTION OF THE DECOMPOSITION METHOD

Now we shortly describe the decomposition method, introduced for the first time in [27]. Let's assume that we have to solve some nonlinear partial differential equation

$$F(x^{\mu}, u_1, ..., u_m, u_{1,x^{\mu}}, ..., u_{m,x^{\mu}}, u_{1,x^{\mu},x^{\nu}}, ...) = 0, \tag{1}$$

where $u_{n,x^{\alpha}} = \frac{\partial u_n}{\partial x^{\alpha}}$ $etc.$

According to the assumptions of the decomposition method, which was presented first time in [27], firstly we check, whether it is possible to *decompose* the equation on the fragments, characterized by a *homogenity* of derivatives. For example, such decomposed investigated equation may be as follows, [27] :

$$G_1 \cdot [(u_{,x})^2 + (u_{,y})^2] + G_2 \cdot [u_{,xx} + u_{,xy}] = 0, \tag{2}$$

where $u = u(x,y)$ is some function of class $\mathscr{C}^2$ and the terms: G_1, G_2 may depend on $x^{\mu}, u, u_{x^{\mu}}, ...$ and $u \in \mathbb{R}$ or $u \in \mathbb{C}$, in dependence on investigated problem.

We see that the result of the checking is positive and then, we insert into (2), the ansatz, [27]:

$$u(x^{\sigma}) = \beta_1 + f(a_{\mu}x^{\mu} + \beta_2, b_{\nu}x^{\nu} + \beta_3, c_{\rho}x^{\rho} + \beta_4). \tag{3}$$

In the above ansatz we try to keep f as an *arbitrary* function (of class $\mathscr{C}^2$), so far as it is possible. The class of the solutions given by the ansatz of such kind, is called in the literature, as *functionally invariant* solution, [32]. The function f depends on the appropriate arguments, like this one: $a_{\mu}x^{\mu} + \beta_2$. In this paper: $a_{\mu}x^{\mu} = a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4$. The coefficients $a_{\mu}, b_{\nu}, c_{\rho}$ may be in general complex numbers, which are to be determined later, β_j may be in general complex constants, $j = 1, ..., 4$, and $\mu, \nu, \rho = 1, ..., 4$. In general, the set of values of μ, ν, ρ, depends on the investigated equation. We can decrease or increase the number of the arguments of the function f in (3) and also modify the form of the ansatz (3), in dependency on the situation.

We make such modification later in this section and in the section 4.

After inserting the ansatz (3) into the example equation (2), there, instead of partial derivatives of u, the derivatives of the function f appear: $D_1 f, D_2 f, ..., D_{1,1} f, D_{1,2} f, ...$, where the indices denote differentiating with respect to first and so far, arguments of the function f (like this one: $a_\mu x^\mu + \beta_2$).

For example, if we insert a two-dimensional version of ansatz (3) into (2) and collect appropriate terms by the derivatives $D_j f$ and $D_{j,k} f$, then, we get, [27]:

$$\begin{gathered}
G_1 \cdot [(a_1^2 + a_2^2)(D_1 f)^2 + (b_1^2 + b_2^2)(D_2 f)^2 + \\
(c_1^2 + c_2^2)(D_3 f)^2 + 2(a_1 b_1 + a_2 b_2) D_1 f D_2 f + \\
2(a_1 c_1 + a_2 c_2) D_1 f D_3 f + 2(b_1 c_1 + b_2 c_2) D_2 f D_3 f] + \\
G_2 \cdot [(a_1^2 + a_1 a_2) D_{1,1} f + (2a_1 b_1 + a_1 b_2 + a_2 b_1) D_{1,2} f \\
+ (2a_1 c_1 + a_1 c_2 + a_2 c_1) D_{1,3} f + (b_1^2 + b_1 b_2) D_{2,2} f + \\
(2b_1 c_1 + b_1 c_2 + b_2 c_1) D_{2,3} f + (c_1^2 + c_1 c_2) D_{3,3} f] = 0,
\end{gathered}$$

where $D_j f, D_{j,k} f$ denote correspondingly: a derivative of the function f with respect to $j-nary$ argument and the mixed derivative of this function with respect to $j-nary$ and $k-nary$ argument. Now, we require that all algebraic terms in the parenthesises must vanish. As a result we obtain a system of algebraic equations, which solutions are the parameters $a_1, a_2, ...$. We call such system of algebraic equations as determining algebraic system. Its solutions establish the relations between a_μ, b_ν, c_ρ and therefore they constitute, together with (3), some class of solutions of (2). In dependence on the situation, we may need to take into consideration additionally some other conditions, which must be satisfied by our class of solutions. These conditions implicate the requirement of satisfying of some algebraic equations, which we attach to the determining algebraic system. In this paper, one example of such additional condition, is the condition of non-invariance of the solutions. The ansatz (3) appears, as an effect of a generalization of some result, obtained in [28] and [29]. Namely, there in the mentioned papers, some classes of exact solutions of Bogomolny decomposition (Bogomolny equations) for Heisenberg model of ferromagnet have been obtained (but by applying some other method - so called, concept of strong necessary conditions):

$$\omega = \omega[(i\alpha + \beta\gamma)x_1 + (i\gamma - \alpha\beta)x_2 + (\beta^2 - 1)x_3], \quad c.c., \tag{4}$$

where ω is arbitrary holomorphic function of class $\mathscr{C}^2$, depending on its argument and $\omega = \frac{S^1+iS^2}{1+S^3}$, S^i $(i = 1,2,3)$ - components of classical Heisenberg spin, and $\alpha^2+\beta^2+\gamma^2 = 1$. So, the ansatz (3) is a generalization of (4). The solution (given by the ansatz (3)) has been obtained for: Heisenberg model, nonlinear σ model (or O(3) model) and scalar Born-Infeld-like equation in (3+1)-dimensions in [27] and for Skyrme-Faddeev model in [30].

One can easily show that this method can be extended for the class of the equations of *arbitrary* order:

$$\begin{aligned} &F(x^{\mu}, u_1, ..., u_m, u_{1,x^{\mu}}, \ldots, u_{m,x^{\mu}}, u_{1,x^{\mu}x^{\nu}}, \ldots, \\ &u_{1,x^{\alpha_1}\ldots x^{\alpha_n}}, \ldots, u_{m,x^{\mu}x^{\nu}}, \ldots, u_{m,x^{\alpha_1}\ldots x^{\alpha_n}}) = 0, \end{aligned} \tag{5}$$

obviously, if decomposition on the proper fragments, mentioned above, is possible.

Of course, this above decomposition method may be also applied for solving linear partial differential equations, homogeneous with respect to the derivatives.

Just now, in the order to find classes of non-invariant solutions, we make a modification of (3) and the basic form of (3), which be applied for all equations considered in this paper, is:

$$u(x^{\mu}) = \beta_1 + g_1(\Sigma_1) + g_2(\Sigma_2) + g_3(\Sigma_3) + g_4(\Sigma_4), \tag{6}$$

where:

$$\begin{aligned} &\Sigma_1 = a_{\mu}x^{\mu} + \beta_2, \quad \Sigma_2 = b_{\mu}x^{\mu} + \beta_3, \\ &\Sigma_3 = c_{\mu}x^{\mu} + \beta_4, \quad \Sigma_4 = d_{\mu}x^{\mu} + \beta_5, \end{aligned} \tag{7}$$

$g_k, (k = 1, ..., 4)$ are some functions, in dependence on situation, they can be complex or real and we wish they were *arbitrary* functions, but it can change in some cases, $a_{\mu}x^{\mu} = a_1x^1 + \ldots + a_4x^4$, x^{μ} are the independent variables. We assume that $g_k \in \mathscr{C}^2, (k = 1, ..., 4)$, (of course, we assume g_k are differentiable functions). In the cases of second heavenly equation and mixed heavenly equation we will extend the ansatz (6) to the functional series.

Of course, all equations, homogeneous with respect to the derivatives, can be solved by using decomposition method (if there exists at least one solution of the determining algebraic system). However, it is possible that found solutions of determining algebraic system will determine the classes of the solutions, which are useless from the physical viewpoint. So, the problem of finding of solutions of given equation is reduced to the problem of solving of the determining algebraic system.

It should be also mentioned here that the first method of finding of functionally invariant solutions, applied to the wave equation, comes from [32], but without the idea of decomposition method, introduced for the first time in [27] and applied in this paper. In [33] and [34] some extension (obtained by using a method, called also, as Erugin's method, [35]), of the results obtained in [32], has been presented. In [36], the extension of this above mentioned method, was presented and applied for nonlinear partial differential equations of second order and in [37] some analogical results were obtained for some kind of quasilinear partial differential equations of second order.

However, the method of searching of the solutions of the form (3), introduced, for the first time, in [27], "looks" at the investigated nonlinear partial differential equation, by the wievpoint of homogenity of some fragments of this equation, with respect to the derivatives and so it differs from these methods mentioned above. By comparison with the methods mentioned above, it seems to be more simple than they. Moreover, we have stated above that decomposition method can be applied for partial differential equation of arbitrary order, if this equation can be decomposed on proper fragments, mentioned above. In [38] some extension of this method (we apply its version in the current paper) was presented.

III. HEAVENLY EQUATIONS

A. Complex Monge-Ampère equations

The Einstein vacuum equation in the complex four-dimensional Riemann space together with the constraint of (anti-)self-duality can be reduced to the complex Monge-Ampère equation, [4]:

$$\Omega_{,pr}\Omega_{,qs} - \Omega_{,qr}\Omega_{,ps} = 1. \tag{8}$$

The metric, corresponding to this equation, is the following, [4]:

$$ds^2 = \Omega_{,pr}dpdr + \Omega_{,ps}dpds + \Omega_{,qr}dqdr + \Omega_{,qs}dqds, \tag{9}$$

where $p,q,r,s \in \mathbb{C}$ and $\Omega(p,q,r,s) \in \mathbb{C}$.

Because of physical requirements, we limit our considerations to the case: $\Omega(p,q,r,s) = v$, $v \in \mathbb{R}$, $(p,q,r,s \in \mathbb{C})$. If we choose: $p = z^1, q = z^2, r = \theta\bar{z}^1, s = \bar{z}^2$, then, the equation (8) becomes, [7]:

$$v_{,z^1\bar{z}^1}v_{,z^2\bar{z}^2} - v_{,z^1\bar{z}^2}v_{,z^2\bar{z}^1} = \theta, \tag{10}$$

where $\theta = \pm 1$, $\bar{z}^1$ is complex conjugation of z^1, $v_{,z^1} = \frac{\partial v}{\partial z^1}$ etc. The metric (9) has the form, [7]:

$$ds^2 = v_{,z^1\bar{z}^1}dz^1d\bar{z}^1 + v_{,z^1\bar{z}^2}dz^1d\bar{z}^2 + v_{,z^2\bar{z}^1}dz^2d\bar{z}^1 + v_{,z^2\bar{z}^2}dz^2d\bar{z}^2. \tag{11}$$

If $\theta = 1$, then the equation (10) is called as the elliptic complex Monge-Ampère equation and if $\theta = -1$, then the equation (10) is called as the hyperbolic complex Monge-Ampère equation.

1. *Elliptic complex Monge-Ampère equation*

As we stated it above, elliptic complex Monge-Ampère equation has the form:

$$v_{z^1\bar{z}^1}v_{z^2\bar{z}^2} - v_{z^1\bar{z}^2}v_{\bar{z}^1z^2} = 1 \tag{12}$$

This equation has many applications in mathematics and physics, among others, as we stated it in the previous section, equation (12) is strictly connected to instanton solutions of the Einstein equations of gravitational field. These solutions are desrcibed by 4-dimensional Kähler metrics, [9]:

$$ds^2 = v_{z^i\bar{z}^k}dz^id\bar{z}^k, \tag{13}$$

where we sum over the two values of both: unbarred and barred indices and $v_{z^i\bar{z}^k} = \frac{\partial^2 v}{\partial z^i \partial \bar{z}^k}$.

The metric satisfies the vacuum Einstein equations of gravitational field with Euclidean signature, provided that the Kähler potential is some solution of (12). We will look for non-invariant, real solutions of (12), which can be used for construction of hyper Kähler metrics, not possesing any Killing vectors. One of them is the $K3$ surface (Kummer surface), being the most important gravitational instanton, [9], [10]. In [9] some exact, non-invariant and real solution of (12) was found, by some reduction of the problem of solving (12) to solving some linear system of equations. Namely, this solution has the form:

$$
\begin{aligned}
w = \sum_{k=-\infty}^{\infty} & \exp\{2\Im([A_k^2(B_k^2+1)+1]z^2)\}\{\exp[2B_k\Re[A_k(p+\gamma z^2)]] \\
& \times\Re\{C_k\exp[2i[\Im(A_k(p+\gamma z^2))-2B_k\Re(A_k^2z^2)]]\} \\
& +\exp[-2B_k\Re[A_k(p+\gamma z^2)]]\Re\{H_k\exp[2i[\Im[A_k(p+\gamma z^2)]+2B_k\Re(A_k^2z^2)]]\}\},
\end{aligned}
\tag{14}
$$

where A_k, C_k, H_k are arbitrary complex constants, $B_k = \sqrt{1-1/\mid A_k\mid^2}$, γ is arbitrary real constant, $w = e^{-\psi}$, and ψ is a solution of Legendre transform of elliptic complex Monge-Ampère equation, [9]:

$$
\psi_{p\bar{p}}\psi_{z^2\bar{z}^2} - \psi_{p\bar{z}^2}\psi_{\bar{p}z^2} = \psi_{pp}\psi_{\bar{p}\bar{p}} - \psi_{p\bar{p}}^2,
\tag{15}
$$

and:

$$
v = \psi - p\psi_p - \bar{p}\psi_{,\bar{p}},\ \ v_{z^1} = p,\ \ v_{\bar{z}^1} = \bar{p}.
\tag{16}
$$

Some other solutions, functionally invariant ones, have been found in [39]:

$$
w = \int_{a_0}^{a_1} F(a, \beta_a + i\delta_a)da + \sum_k F_k(\beta_{a_k} + i\delta_{a_k}) + c.c.,
\tag{17}
$$

where $a \in \mathbb{R}$ and:

$$
\begin{aligned}
& v = w - pw_{,p} - \bar{p}w_{,\bar{p}} - rw_{,r},\ z^1 = -w_{,p},\ \bar{z}^1 = -w_{,\bar{p}}, \\
& \rho = \xi + \bar{\xi} = -w_{,r},\ \beta_{a_k} = p + \bar{p} + ia(\bar{p} - p), \\
& \delta_{a_k} = i\sqrt{\gamma}\left(r + \frac{a+i}{a-i}z + \frac{a-i}{a+i}\bar{z}\right),\ \gamma = a^2+1
\end{aligned}
\tag{18}
$$

and $\xi, \bar{\xi}$ are parameters of the symmetry group of (12).

2. Hyperbolic complex Monge-Ampère equation transformed by Legendre transformation

After applying Legendre transformation, [7]:

$$w=v-z^1v_{,z^1}-\bar{z}^1v_{,\bar{z}^1},\ p=v_{,z^1},\ \bar{p}=v_{,\bar{z}^1},\ z^1=-w_{,p},\ \bar{z}^1=-w_{,\bar{p}}\ , \tag{19}$$

the hyperbolic complex Monge-Ampère equation (10) becomes (if $\theta=-1$), [7]:

$$w_{,p\bar{p}}w_{,z^2\bar{z}^2}-w_{,p\bar{z}^2}w_{,\bar{p}z^2}-w^2_{,p\bar{p}}+w_{,pp}w_{,\bar{p}\bar{p}}=0. \tag{20}$$

The metric (11), governed by (10), after applying the transformation (19), has the form, [7]:

$$\begin{gathered}ds^2=\frac{1}{(w_{,pp}w_{,\bar{p}\bar{p}}-w^2_{,p\bar{p}})}\Big[w_{,pp}(w_{,p\bar{p}}dp+w_{,\bar{p}z^2}dz^2)^2+w_{,\bar{p}\bar{p}}(w_{,p\bar{p}}d\bar{p}+w_{,p\bar{z}^2}d\bar{z}^2)^2+\\ \frac{w_{,pp}w_{,\bar{p}\bar{p}}+w^2_{,p\bar{p}}}{w_{,p\bar{p}}}\mid w_{,p\bar{p}}dp+w_{,\bar{p}z^2}dz^2\mid^2\Big]-\frac{w_{,pp}w_{,\bar{p}\bar{p}}-w^2_{,p\bar{p}}}{w_{,p\bar{p}}}dz^2d\bar{z}^2.\end{gathered} \tag{21}$$

The condition of existence of Legendre transformation (19) has the form:

$$w_{,pp}w_{,\bar{p}\bar{p}}-w^2_{,p\bar{p}}\neq 0 \tag{22}$$

and it must be satisfied for the given solution or class of solutions of (20).

3. *Non-invariance of the solutions of the hyperbolic complex Monge-Ampère equation*

As it was showed in [7], the conditions of non-invariance of the solutions of the hyperbolic complex Monge-Ampère equation are strictly determined by Killing equations.

The condition equivalent to the Killing equation, has the form (after applying the invertible point transformation, generated by Legendre transformation (19)), [7]:

$$\begin{gathered}p\xi^1(-w_{,p},z^2)+w_{,z^2}\xi^2(-w_{,p},z^2)+\bar{p}\xi^{\bar{1}}(-w_{,\bar{p}},\bar{z}^2)+w_{,\bar{z}^2}\xi^{\bar{2}}(-w_{,\bar{p}},\bar{z}^2)=\\ h(-w_{,p},z^2)+\bar{h}(-w_{,\bar{p}},\bar{z}^2).\end{gathered} \tag{23}$$

The Killing vector exists for given solution of the hyperbolic complex Monge-Ampère equation, only if this solution satisfies (23) and then, such solution is invariant.

In [7], the searching of solutions of hyperbolic Monge-Ampère equation, has been reduced to solving some two systems of linear partial differential equations, by applying method of partner symmetries. The solutions of these mentioned systems of linear equation, found in [7], have the following form:

$$w = \sum_{j=1}^{n} \alpha_j e^{\Sigma_j}, \tag{24}$$

where $\Sigma_j = \gamma_j p + \bar{\gamma}_j \bar{p} + \delta_j z^2 + \bar{\delta}_j \bar{z}^2$. The coefficients $\alpha_j \in \mathbb{R}$ are arbitrary, but γ_j, δ_j must satisfy the following relations, [7]:

1. for the first system :

$$\mid \gamma_j \mid^2 = a\gamma_j + \bar{a}\bar{\gamma}_j, \qquad \delta_j = i\frac{\gamma_j^2 - (\bar{a} + i\bar{b})\gamma_j}{\bar{a}}, \tag{25}$$

2. for the second system:

$$\delta_j = \left(\nu + i - i\frac{\gamma_j}{\bar{\gamma}_j}\right)\gamma_j, \qquad c.c., \tag{26}$$

where $\nu = const$, and a, b - arbitrary complex constants.

So, there are two solutions of the hyperbolic complex Monge-Ampère equation and they are non-invariant, if $n \geq 4$, because they do not satisfy Killing equation (23), [7]. Namely, it is provided by the fact, that the matrix of coefficients, [7]:

$$M = \begin{pmatrix} 1 & e^{-2i\varphi_1} & e^{2i\varphi_1} & e^{-4i\varphi_1} \\ 1 & e^{-2i\varphi_2} & e^{2i\varphi_2} & e^{-4i\varphi_2} \\ 1 & e^{-2i\varphi_3} & e^{2i\varphi_3} & e^{-4i\varphi_3} \\ 1 & e^{-2i\varphi_4} & e^{2i\varphi_4} & e^{-4i\varphi_4} \end{pmatrix}, \tag{27}$$

where $\varphi_j = \arg(\gamma_j)$, $(j = 1, ..., 4)$, is non-singular. Hence:

$$\Sigma_j = \gamma_j p + \bar{\gamma}_j \bar{p} + \delta_j z^2 + \bar{\delta}_j \bar{z}^2 \tag{28}$$

are linearly independent and the transformations from $p, \bar{p}, z^2, \bar{z}^2$ to Σ_j are invertible, [7]. So, as it has been proved in [7], after inserting each of these above solutions into Killing equation (23), this equation becomes into:

$$F_1(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = 0. \tag{29}$$

Hence, as it has been proved in [7], the equation (23) cannot be satisfied identically for the solution of Legendre-transformed hyperbolic complex Monge-Ampère equation (20), found in [7].

As it has been pointed out in [7], the solutions, given by (24) and (25) can be generalized thanks to the functional invariance, established in a theorem proved in [7]. According to this theorem, we have that:

$$w = f\left(\sum_{j=1}^{n} \alpha_j e^{\Sigma_j}\right), \tag{30}$$

where the coefficients in Σ_j satisfy (25) and f is arbitrary function, ($f \in \mathscr{C}^2$), is also the class of the solutions of one of the mentioned above systems of linear PDE's and of course, of hyperbolic complex Monge-Ampère equation (20). So, (30) is some functionally invariant solution of hyperbolic complex Monge-Ampère equation (20).

B. Second heavenly equation of Plebański

The second heavenly equation of Plebański has the following form, [4], [6], [7]:

$$v_{,xx}v_{,yy} - v_{,xy}^2 + v_{,xw} + v_{,yz} = 0, \tag{31}$$

where $v(x,y,w,z) \in \mathbb{C}$ is a holomorphic function, $v_{xx} = \frac{\partial^2 v}{\partial x^2}$ etc. and $x,y,w,z \in \mathbb{C}$. The heavenly metric has the form, [6], [7]:

$$ds^2 = dwdx + dzdy - v_{,xx}dz^2 - v_{,yy}dw^2 + 2v_{,xy}dwdz. \tag{32}$$

In [7] the second heavenly equation (31) has been transformed by partial Legendre transformation:

$$\vartheta = v - wv_{,w} - yv_{,y}, \quad v_{,w} = t, \tag{33}$$

$$v_{,y} = r, \quad w = -\vartheta_{,t}, \quad y = -\vartheta_{,r}. \tag{34}$$

We need also to remember that this above transformation exists, if the following condition is satisfied, [7]:

$$\vartheta_{,tt}\vartheta_{,rr}-(\vartheta_{,rt})^2\neq 0. \tag{35}$$

Second heavenly equation, transformed by Legendre transformation (34), has the form, [7]:

$$\vartheta_{,tt}(\vartheta_{,xx}+\vartheta_{,rz})+\vartheta_{,xt}(\vartheta_{,rr}-\vartheta_{,xt})-\vartheta_{,rt}(\vartheta_{,rx}+\vartheta_{,tz})=0. \tag{36}$$

The metric (32) transformed by Legendre transformation, has the form, [7]:

$$\begin{gathered}
ds^2=\frac{1}{\vartheta_{,tt}(\vartheta_{,tt}\vartheta_{,rr}-\vartheta_{,tr}^2)}\\
(\vartheta_{,tt}(\vartheta_{,tt}dt+\vartheta_{,tr}dr+\vartheta_{,tx}dx+\vartheta_{,tz}dz)+\\
(\vartheta_{,tt}\vartheta_{,rx}-\vartheta_{,tr}\vartheta_{,tx})dz)^2-\frac{\vartheta_{,tt}\vartheta_{,xx}-\vartheta_{,tx}^2}{\vartheta_{,tt}}dz^2-\\
(\vartheta_{,tt}dt+\vartheta_{,tr}dr+\vartheta_{,tx}dx+\vartheta_{,tz}dz)dx\\
-(\vartheta_{,rt}dt+\vartheta_{,rr}dr+\vartheta_{,rx}dx+\vartheta_{,rz}dz)dz,
\end{gathered} \tag{37}$$

where $\vartheta(x,r,t,z)$ is the potential, which satisfies Legendre transformed second heavenly equation (36).

In the aim of linearization of the above equation, there in [7], translational symmetries have been applied. In the case of so called equal symmetries, instead of the equation (36), the following system of equations has been investigated, [7]:

$$\vartheta_{,rt}+\vartheta_{,rr}-\vartheta_{,xt}=0, \tag{38}$$

$$\vartheta_{,xx}+\vartheta_{,rz}=0, \tag{39}$$

$$\vartheta_{,rx}+\vartheta_{,xt}+\vartheta_{,tz}=0. \tag{40}$$

In the case of so called higher symmetry, the following system has been considered, instead of (36), [7]:

$$\vartheta_{,rr} - \vartheta_{,xt} = 0, \tag{41}$$

$$\vartheta_{,rx} + \vartheta_{,tz} = 0, \tag{42}$$

$$\vartheta_{,xx} + \vartheta_{,rz} = 0. \tag{43}$$

The appropriate form of the Killing equation for second-havenly equation, transformed by Legendre transformation was derived in [7].

In [7], a solution of the equations (38)-(40) and (41)-(43) have been obtained:

$$\vartheta = \sum_{j=1}^{n} m_j \exp\left(\alpha_j t + \gamma_j r + \zeta_j x + \lambda_j z\right), \tag{44}$$

where the coefficients must satisfy the relations, [7]:

1. for the system (38)-(40):

$$\alpha_j = \frac{\gamma_j^2}{\zeta_j - \gamma_j},\ \lambda_j = -\frac{\zeta_j^2}{\gamma_j}, \tag{45}$$

2. for the system (41)-(43):

$$\alpha_j = \frac{\gamma_j^2}{\zeta_j},\ \lambda_j = -\frac{\zeta_j^2}{\gamma_j}. \tag{46}$$

Hence, these above solutions are non-invariant, if $n \geq 4$, then, they generate metrics without Killing vector. It is provided by the fact that as in the case of the solutions of hyperbolic complex Monge-Ampere equation, the matrices of coefficients for solutions given either by (44) and (45) or by (44) and (46), are non-singular, [7].

C. Mixed heavenly equation

There in [19], has been derived and investigated, so called mixed heavenly equation, which, after symmetry reduction, has the form:

$$v_{,ty}v_{,xz} - v_{,tz}v_{,xy} + v_{,tt}v_{,xx} - v_{,tx}^2 = \theta, \tag{47}$$

where $\theta = \pm 1$.

After making Legendre transformation, [19]:

$$p = v_{,x}, \quad q = v_{,z}, \quad w(p,q,t,y) = v - xv_{,x} - zv_{,z},$$
$$x = -w_{,p}, \quad z = -w_{,q} \ , \tag{48}$$

the equation (47) becomes, [19]:

$$w_{,tq}w_{,py} - w_{,pq}w_{,ty} + w_{,tt}w_{,qq} - w_{,tq}^2 +$$
$$\theta(w_{,pp}w_{,qq} - (w_{,pq})^2) = 0. \tag{49}$$

The condition of existence of Legendre transformation (48) has the form, [19]:

$$w_{,pp}w_{,qq} - (w_{,pq})^2 \neq 0. \tag{50}$$

The problem of obtaining non-invariant solutions of (49), has been reduced in [19], to investigation of the following set of linear equations (for $\theta = 1$):

$$w_{,\eta\eta} + w_{,\xi\xi} - w_{,\xi q} = 0, \tag{51}$$

$$w_{,\xi q} - w_{,\eta q} + w_{,\xi y} = 0, \tag{52}$$

$$w_{,\xi q} + w_{,\eta q} - w_{,qq} + w_{,\eta y} = 0, \tag{53}$$

where $\eta = p+t, \xi = p-t$.

One of the solutions of the system (51)-(53), are, [19]:

$$w = \sum_j \exp\left(\pm\sqrt{A_j(A_j - B_j)}(\eta + \frac{B_j}{A_j}y)\right)$$
$$\times\{C_j \cos(A_j\xi + B_j(q-y)) + H_j \sin(A_j\xi + B_j(q-y))\}. \tag{54}$$

where: A_j, B_j, C_j, H_j are arbitrary constants and $\eta = p+t, \xi = p-t$.

This above solution is non-invariant solution, because it depends on four independent combinations of the variables η, ξ, q, y, [19].

D. Asymmetric heavenly equation

In [19], so called asymmetric heavenly equation has been derived:

$$u_{,tx}u_{,ty} - u_{,tt}u_{,xy} + Au_{,tz} + Bu_{,xz} + Cu_{,xx} = 0. \tag{55}$$

When $B = 0$, then, this above equation is called as evolution form of the second heavenly equation, [19], [22], [23], [24].

E. General heavenly equation

This equation was derived in [40] and it was investigated in [39] and in [26]. It has the form, [26]:

$$\begin{gathered}\alpha\omega_{,z^1z^2}\omega_{,z^3z^4}+\beta\omega_{,z^1z^3}\omega_{,z^2z^4}+\gamma\omega_{,z^1z^4}\omega_{,z^2z^3}=0,\\ \omega=\omega(z^1,z^2,z^3,z^4)\end{gathered}\tag{56}$$

where

$$\alpha+\beta+\gamma=0,\tag{57}$$

$$m_1=\omega_{,z^1z^3}\omega_{,z^2z^4}-\omega_{,z^1z^4}\omega_{,z^2z^3}\neq 0\tag{58}$$

There the following theorem was presented, [26], namely that if $Q(z^1,z^2,z^3,z^4)$ satisfies (56) and the differential constraint:

$$\begin{gathered}\alpha(Q_{,z^1}Q_{,z^2}Q_{,z^3z^4}+Q_{,z^3}Q_{,z^4}Q_{,z^1z^2})+\beta(Q_{,z^1}Q_{,z^3}Q_{,z^2z^4}+\\ Q_{,z^2}Q_{,z^4}Q_{,z^1z^3})+\gamma(Q_{,z^1}Q_{,z^4}Q_{,z^2z^3}+Q_{,z^2}Q_{,z^3}Q_{,z^1z^4})=0,\end{gathered}\tag{59}$$

then the function $\omega(z^1,z^2,z^3,z^4)$, determined *implicite* by $R(\omega,Q)=0$, where R - arbitrary smooth function, is also a solution of (56). Hence, it is functionally invariant solution.

In [26] real general heavenly equation has been derived:

$$\begin{gathered}\alpha\omega_{,z^1\bar{z}^1}\omega_{,z^2\bar{z}^2}+\beta\omega_{,z^1z^2}\omega_{,\bar{z}^1\bar{z}^2}+\gamma\omega_{,z^1\bar{z}^2}\omega_{,z^2\bar{z}^1}=0,\\ \omega=\omega(z^1,\bar{z}^1,z^2,\bar{z}^2),\end{gathered}\tag{60}$$

and

$$m_2=\omega_{,z^1z^2}\omega_{,\bar{z}^1\bar{z}^2}-\omega_{,z^1\bar{z}^2}\omega_{,z^2\bar{z}^1}\neq 0.\tag{61}$$

One can also consider so-called real cross-sections of (56), derived in [26]:

$$
\begin{gathered}
(\delta^2+1)\omega_{,z^1\bar{z}^1}\omega_{,z^2\bar{z}^2}-\delta^2\omega_{,z^1z^2}\omega_{,\bar{z}^1\bar{z}^2}-\omega_{,z^1\bar{z}^2}\omega_{,z^2\bar{z}^1}=0,\\
\frac{\beta}{\gamma}>0,\ \ \beta=\gamma\delta^2,\ \ \delta>0
\end{gathered}
\tag{62}
$$

and

$$
\begin{gathered}
(\delta^2-1)\omega_{,z^1\bar{z}^1}\omega_{,z^2\bar{z}^2}-\delta^2\omega_{,z^1z^2}\omega_{,\bar{z}^1\bar{z}^2}+\omega_{,z^1\bar{z}^2}\omega_{,z^2\bar{z}^1}=0,\\
\frac{\beta}{\gamma}<0,\ \ \beta=-\gamma\delta^2,\ \ \delta>0
\end{gathered}
\tag{63}
$$

IV. SOME NEW CLASSES OF EXACT SOLUTIONS AND THEIR NON-INVARIANCE

In this section we find the classes of exact solutions of the equations, (12), (20), (36), (49), (55), (60), (63), and we check the non-invariance of theese classes.

A. Class of exact solutions of elliptic complex Monge-Ampère equation

Now we want to find the class of exact solutions of elliptic complex Monge-Ampère equation (12). In contrary to the next subsections of this secion, we do not investigate Legendre transform of origin equation. Actually, at first sight, one can think that the simplest way of finding wanted class of exact solutions, is applying decomposition method to Legendre transform of (12).

However, in this case, after obtaining the corresponding determining algebraic system, it turns out that finding of the class of exact solutions, which satisfies three conditions of: existence of Legendre transformation, non-invariance and reality, simultaneously, is very hard and it seems that it is possible that there are no appropriate solutions of the determining algebraic system. Thus, we apply decomposition method directly to the elliptic complex Monge-Ampère equation (12). Obvioulsy, in this case we cannot use directly the ansatz (6) to this equation, because the main osbtacle, in applying of the original ansatz, is the presence of the free term "1" in (12). So, we apply some modification of the ansatz such that after inserting of it into (12), some possibility of balancing of the free term will appear (the necessity of balancing of the free term was took into consideration in decomposition method in [38]). We do it by choosing two functions in (6), as square functions, we choose the functions $g_k, (k=1,2)$ to be square functions and the functions

g_3 is arbitrary function of class $\mathscr{C}^2$, but $g_4 = \bar{g}_3$, in order to satisfy the condition of reality of the solution. Hence, we apply the ansatz:

$$v(x^\mu) = \beta_1 + (\Sigma_1)^2 + (\Sigma_2)^2 + g_3(\Sigma_3) + g_4(\Sigma_4), \tag{64}$$

where:

$$\begin{aligned} \Sigma_1 &= a_\mu x^\mu + \beta_2, \quad \Sigma_2 = b_\mu x^\mu + \beta_3, \\ \Sigma_3 &= c_\mu x^\mu + \beta_4, \quad \Sigma_4 = d_\mu x^\mu + \beta_5, \end{aligned} \tag{65}$$

$\Sigma_k \in \mathbb{R}, (k = 1,2)$, $g_3 \in \mathbb{C}$ is *arbitrary* function of class $\mathscr{C}^2$, $g_4 = \bar{g}_3$, $a_\mu x^\mu = a_1 x^1 + \ldots + a_4 x^4$ and $x^1 = z^1, x^2 = \bar{z}^1, x^3 = z^2, x^4 = \bar{z}^2$ are the independent variables. Owing to applying decomposition method directly to (12), not to its Legendre transform (16), we avoid the necessity of satisfying of the condition of existence of Legendre transformation. After inserting the ansatz (64) into (12) and collecting proper terms, we derive the determining algebraic system, which wanted solutions are:

$$a_1 = \bar{a}_2, a_3 = 0, a_4 = 0, b_1 = 1, b_2 = 1, b_3 = \frac{1}{2\bar{a}_2}, b_4 = \frac{1}{2a_2}, \tag{66}$$

$$c_1 = 0, c_2 = \frac{1}{2\bar{a}_2}, c_3 = 0, c_4 = \bar{d}_3, d_1 = \frac{1}{2a_2}, d_2 = 0, d_4 = 0, \tag{67}$$

$$\beta_1, \beta_2, \beta_3 \in \mathbb{R}, \ \beta_5 = \bar{\beta}_4. \tag{68}$$

Hence, the class of solutions of (12) has the form:

$$\begin{aligned} v(z^1, \bar{z}^1, z^2, \bar{z}^2) = \beta_1 + (\bar{a}_2 z^1 + a_2 \bar{z}^1 + \beta_2)^2 + \left(z^1 + \bar{z}^1 + \frac{1}{2\bar{a}_2} z^2 + \frac{1}{2a_2} \bar{z}^2 + \beta_3 \right)^2 + \\ g_3\left(\frac{1}{2\bar{a}_2} \bar{z}^1 + \bar{d}_3 \bar{z}^2 + \beta_4 \right) + \bar{g}_3\left(\frac{1}{2a_2} z^1 + d_3 z^2 + \bar{\beta}_4 \right). \end{aligned} \tag{69}$$

Now we need to check, whether the solutions, belonging to the found class, are non-invariant. We construct the matrix:

$$M = \begin{pmatrix} \Gamma_1 \bar{a}_2 & \Gamma_1 a_2 & 0 & 0 \\ \Gamma_2 & \Gamma_2 & \frac{\Gamma_2}{2\bar{a}_2} & \frac{\Gamma_2}{2a_2} \\ 0 & \frac{g_3'}{2\bar{a}_2} & 0 & g_3' \bar{d}_3 \\ \frac{\bar{g}_3'}{2a_2} & 0 & \bar{g}_3' d_3 & 0 \end{pmatrix}, \tag{70}$$

where $\Gamma_1 = 2(\bar{a}_2 z^1 + a_2 \bar{z}^1 + \beta_2), \Gamma_2 = 2z^1 + 2\bar{z}^1 + \frac{z^2}{\bar{a}_2} + \frac{\bar{z}^2}{a_2} + 2\beta_3$. In order to provide non-invariance of the solutions belonging to the class (69), the determinant of this above matrix must not vanish:

$$\det(M) = -\frac{1}{2\bar{a}_2^2 a_2^2}(\bar{a}_2 z^1 + a_2\bar{z}^1 + \beta_2)(2z^1\bar{a}_2 a_2 + 2\bar{z}^1\bar{a}_2 a_2 + z^2 a_2 + \bar{z}^2\bar{a}_2 + 2\beta_3\bar{a}_2 a_2)$$
$$(4d_3\bar{d}_3\bar{a}_2^2 a_2 - d_3\bar{a}_2 - 4\bar{a}_2 a_2^2 d_3\bar{d}_3 + \bar{d}_3 a_2)g_3'\bar{g}_3' \neq 0 \tag{71}$$

Let us assume that $a_2 \neq 0$. Hence, the solutions belonging to the class (69), depend on four variables and they are non-invariant, in the regions, where simultaneously the conditions: $\bar{a}_2 z^1 + a_2\bar{z}^1 + \beta_2 \neq 0$ and $2z^1\bar{a}_2 a_2 + 2\bar{z}^1\bar{a}_2 a_2 + z^2 a_2 + \bar{z}^2\bar{a}_2 + 2\beta_3\bar{a}_2 a_2 \neq 0$ and $g_3' \neq 0$, hold.

Of course, this above found class of solutions of the elliptic complex Monge-Ampère equation (12), is the class of *real* solutions. Obviously, these solutions are not differentiable in $\mathbb{C}$ sense. They are differentiable in $\mathbb{R}$ sense. If we repeat these above computations, but in the variables $x^1 = \Re(p), x^2 = \Im(p), x^3 = \Re(z^2), x^4 = \Im(z^2)$, $x^k \in \mathbb{R}, (k = 1,2,3,4)$, then it turns out that the function, given by (69), is still the class of exact solutions of elliptic complex Monge-Ampère equation (12) and these solutions are still non-invariant.

Hence, we have proved the following theorem:

Theorem 1 *The metric (13) with v, being some solution of elliptic complex Monge-Ampère equation (12), belonging to the class, given by by (69), when (71) and the conditions:* $\bar{a}_2 z^1 + a_2\bar{z}^1 + \beta_2 \neq 0$, $2z^1\bar{a}_2 a_2 + 2\bar{z}^1\bar{a}_2 a_2 + z^2 a_2 + \bar{z}^2\bar{a}_2 + 2\beta_3\bar{a}_2 a_2 \neq 0$ *and* $g_3' \neq 0$*, are simultaneously satisfied, does not possess Killing vector.*

B. Classes of exact solutions of hyperbolic complex Monge-Ampère equation

In this subsection we look for the class of exact solutions of hyperbolic complex Monge-Ampère equation (20). To this effect, we apply directly the ansatz (6) for (20), in contrast with [7], where this equation was linearized and the obtained systems of linear equations were solved. Here: $u \equiv w$ and the independent variables are: $x^1 = p, x^2 = \bar{p}, x^3 = z^2, x^4 = \bar{z}^2$, and g_j $(j = 1,2,3,4)$, are such functions of their arguments that $w(p,\bar{p},z^2,\bar{z}^2) \in \mathbb{R}$. We search the class of non-invariant solutions of (20), given by (6) and (7), which satisfy (22).

After making the procedure described in the section 2, we obtain some system of nonlinear algebraic equations, so called determining algebraic system. Apart from satisfying of it, we require also satisfying of the following conditions: the condition (22), the condition of non-singularity of Jacobian matrix and the condition that the solution must be *real*.

We found three classes of non-invariant, exact solutions of (20), satisfying the mentioned conditions. These classes are given by (6) and by the following sets of relations between the coefficients:

1. for the class I

$$\begin{gathered}
a_1 = a_2, a_3 = -\frac{a_2(d_1d_2 - d_2^2 - d_1d_4)}{d_1d_2}, a_4 = -\frac{a_2(d_1d_2 - d_2^2 - d_1d_4)}{d_1d_2},\\
b_1 = d_2, b_2 = d_1, b_3 = d_4, b_4 = -\frac{d_1^2 - d_2^2 - d_1d_4}{d_2},\\
c_1 = -iA, c_2 = iA, c_3 = -\frac{iA(d_1d_2 + d_2^2 + d_1d_4)}{d_1d_2}, c_4 = \frac{iA(d_1d_2 + d_2^2 + d_1d_4)}{d_1d_2},\\
d_3 = -\frac{d_1^2 - d_2^2 - d_1d_4}{d_2},
\end{gathered} \tag{72}$$

   in this case: $g_1, g_3 \in \mathbb{R}$, $g_2 \in \mathbb{C}, g_4 = \bar{g}_2$ (of course $\beta_5 = \bar{\beta}_3$),

2. for the class II

$$\begin{gathered}
a_1 = 0,\ a_2 = 0,\ a_3 = \bar{a}_4,\ b_1 = 0,\ b_2 = 0,\ b_3 = \bar{b}_4,\\
c_1 = 0,\ c_2 = d_4,\ c_3 = d_4,\ c_4 = d_3,\ d_1 = d_4,\ d_2 = 0,
\end{gathered} \tag{73}$$

   in this case: $g_1, g_2 \in \mathbb{R}$, $g_3 \in \mathbb{C}, g_4 = \bar{g}_3$ (of course $\beta_5 = \bar{\beta}_4$),

3. for the class III

$$\begin{gathered}
a_1 = A_2(1+i),\ a_2 = \bar{a}_1,\ a_3 = 2iA_2,\ a_4 = \bar{a}_3,\\
b_1 = \frac{\sqrt{B_3^2 + B_4^2}}{2}\left(1 + i\frac{B_3 + B_4 - \sqrt{B_3^2 + B_4^2}}{B_3 - B_4 + \sqrt{B_3^2 + B_4^2}}\right),\ b_2 = \bar{b}_1,\\
b_3 = B_3 + iB_4,\ b_4 = \bar{b}_3,\ c_1 = iC_2,\ c_2 = \bar{c}_1,\ c_3 = 0, c_4 = 0,\\
d_1 = H_2(-1+i),\ d_2 = \bar{d}_1,\ d_3 = 2iH_2,\ d_4 = \bar{d}_3
\end{gathered} \tag{74}$$

   in this case: $g_j, (j = 1,2,3,4) \in \mathbb{R}$.

In all these above cases, $g_j, (j = 1,2,3,4)$ are the functions of class $\mathscr{C}^2$.

Hence, the found classes of the exact solutions of (20) have the form:

1. class I

$$w(p,\bar{p},z^2,\bar{z}^2) = \beta_1 +$$
$$g_1\left(a_2 p + a_2\bar{p} - \frac{a_2(d_1d_2 - d_2^2 - d_1d_4)}{d_1d_2}z^2 - \frac{a_2(d_1d_2 - d_2^2 - d_1d_4)}{d_1d_2}\bar{z}^2 + \beta_2\right) +$$
$$g_2\left(d_2 p + d_1\bar{p} + d_4 z^2 - \frac{d_1^2 - d_2^2 - d_1d_4}{d_2}\bar{z}^2 + \beta_3\right) + \tag{75}$$
$$g_3\left(-iAp + iA\bar{p} - \frac{iA(d_1d_2 + d_2^2 + d_1d_4)}{d_1d_2}z^2 + \frac{iA(d_1d_2 + d_2^2 + d_1d_4)}{d_1d_2}\bar{z}^2 + \beta_4\right) +$$
$$\bar{g}_2\left(d_2 p + d_1\bar{p} + d_4 z^2 - \frac{d_1^2 - d_2^2 - d_1d_4}{d_2}\bar{z}^2 + \beta_3\right),$$

2. class II

$$w(p,\bar{p},z^2,\bar{z}^2) = \beta_1 + g_1(\bar{a}_4 z^2 + a_4\bar{z}^2 + \beta_2) + g_2(\bar{b}_4 z^2 + b_4\bar{z}^2 + \beta_3) +$$
$$g_3(d_4\bar{p} + d_4 z^2 + d_3\bar{z}^2 + \beta_4) + \bar{g}_3(d_4\bar{p} + d_4 z^2 + d_3\bar{z}^2 + \beta_4), \tag{76}$$

3. class III

$$w(p,\bar{p},z^2,\bar{z}^2) = \beta_1 + g_1\left(A_2(1+i)p + A_2(1-i)\bar{p} + 2iA_2 z^2 - 2iA_2\bar{z}^2 + \beta_2\right) +$$
$$g_2\left(\frac{\sqrt{B_3^2 + B_4^2}}{2}\left[1 + i\frac{B_3 + B_4 - \sqrt{B_3^2 + B_4^2}}{B_3 - B_4 + \sqrt{B_3^2 + B_4^2}}\right]p +\right.$$
$$\frac{\sqrt{B_3^2 + B_4^2}}{2}\left[1 - i\frac{B_3 + B_4 - \sqrt{B_3^2 + B_4^2}}{B_3 - B_4 + \sqrt{B_3^2 + B_4^2}}\right]\bar{p} + \tag{77}$$
$$\left.(B_3 + iB_4)z^2 + (B_3 - iB_4)\bar{z}^2 + \beta_3\right) +$$
$$g_3\left(iC_2 p - iC_2\bar{p} + \beta_4\right) +$$
$$g_4\left(H_2(-1+i)p + H_2(-1-i)\bar{p} + 2iH_2 z^2 - 2iH_2\bar{z}^2 + \beta_5\right),$$

where:

- for class I: g_1, g_3 are *arbitrary* real functions (of class $\mathscr{C}^2$) of their arguments, g_2 is some *arbitrary* complex function of its argument and $g_2 \in \mathscr{C}^2$, $\bar{g}_2$ is complex conjugation of g_2, i.e.: $g_2 = f(\Phi), \bar{g}_2 = f(\bar{\Phi})$, where $\Phi \in \mathbb{C}$ is the argument of g_2 given in (75) and $A, a_2, d_1, d_2, d_4, \beta_1, \beta_2, \beta_4 \in \mathbb{R}$ and $\beta_3 \in \mathbb{C}$.

- for class II: g_1, g_2 are *arbitrary* real functions (of class $\mathscr{C}^2$) of their arguments, g_3 is some *arbitrary* complex function of its argument and $g_3 \in \mathscr{C}^2$, $\bar{g}_3$ is complex conjugation of g_3, i.e.: $g_3 = f(\Phi), \bar{g}_3 = f(\bar{\Phi})$, where $\Phi \in \mathbb{C}$ is the argument of g_3 given in (76) and $a_4, b_4 \in \mathbb{C}$, $d_3, d_4, \beta_k \in \mathbb{R}, (k = 1, 2, 3)$ and $\beta_4 \in \mathbb{C}$.

- for class III: $g_j, (j = 1, 2, 3, 4)$ are *arbitrary* real functions (of class $\mathscr{C}^2$) of their arguments and $A_2, B_3, B_4, C_2, H_2, \beta_k \in \mathbb{R}, (k = 1, ..., 5)$.

Now, we check, whether the condition (22) is satisfied by the solutions belonging to the classes I, II and III. It turns out that it is satisfied, when:

1. for class I

$$a_2^2(d_1-d_2)^2 g_1''(\bar{g}_2''+g_2'') - A^2(d_1+d_2)^2 g_3''(\bar{g}_2''+g_2'') + (d_1^2-d_2^2)^2 g_2''\bar{g}_2'' - \\ 4a_2^2A^2 g_1'' g_3'' \neq 0, \tag{78}$$

2. for class II

$$d_4^4 g_3'' \bar{g}_3'' \neq 0, \tag{79}$$

3. for class III (some algebraic inequality, which we skip in this paper, due to its complicated structure),

where $g_j'', (j = 1, 2, 3, 4)$ denotes second derivative of the function g_j with respect to its argument.

Next, basing on the considerations included in the subsubsection 3.1.2, we make the analysis of non-invariance of the solutions, belonging to the found classes. The Jacobian matrices have the forms:

1. for class I

$$M = \begin{pmatrix} a_2 g_1' & a_2 g_1' & -a_2\frac{d_1d_2-d_2^2-d_1d_4}{d_1d_2} g_1' & -a_2\frac{d_1d_2-d_2^2-d_1d_4}{d_1d_2} g_1' \\ d_2 g_2' & d_1 g_2' & d_4 g_2' & -\frac{d_1^2-d_2^2-d_1d_4}{d_2} g_2' \\ -iAg_3' & iAg_3' & -\frac{iA(d_1d_2+d_2^2+d_1d_4)}{d_1d_2} g_3' & \frac{iA(d_1d_2+d_2^2+d_1d_4)}{d_1d_2} g_3' \\ d_1\bar{g}_2' & d_2\bar{g}_2' & -\frac{d_1^2-d_2^2-d_1d_4}{d_2}\bar{g}_2' & d_4\bar{g}_2' \end{pmatrix}, \tag{80}$$

2. for class II

$$M=\begin{pmatrix} 0 & 0 & \bar{a}_4 g_1' & a_4 g_1' \\ 0 & 0 & \bar{b}_4 g_2' & b_4 g_2' \\ 0 & d_4 g_3' & d_4 g_3' & d_3 g_3' \\ d_4 \bar{g}_3' & 0 & d_3 \bar{g}_3' & d_4 \bar{g}_3' \end{pmatrix}, \tag{81}$$

3. for class III

$$M=\begin{pmatrix} A_2(1+i)g_1' & A_2(1-i)g_1' & 2iA_2 g_1' & -2iA_2 g_1' \\ N_1 g_2' & \bar{N}_1 g_2' & (B_3+iB_4)g_2' & (B_3-iB_4)g_2' \\ iC_2 g_3' & -iC_2 g_3' & 0 & 0 \\ H_2(-1+i)g_4' & H_2(-1-i)g_4' & 2iH_2 g_4' & -2iH_2 g_4' \end{pmatrix}, \tag{82}$$

where $g_j',(j=1,2,3,4)$ denotes the first derivative of the function g_j with respect to its argument and $N_1=\frac{\sqrt{B_3^2+B_4^2}}{2}\left(1+i\frac{B_3+B_4-\sqrt{B_3^2+B_4^2}}{B_3-B_4+\sqrt{B_3^2+B_4^2}}\right)$. From the requirement of non-vanishing of the determinants of these above matrices, we have:

1. for the class I

$$\det M=\frac{2ia_2A}{d_1^2d_2^2}(d_1^6-d_2^6-3d_1^4d_2^2+3d_1^2d_2^4)g_1'g_2'\bar{g}_2'g_3'\neq 0, \tag{83}$$

2. for the class II

$$\det M=-(\bar{a}_4b_4-a_4\bar{b}_4)d_4^2g_1'g_2'g_3'\bar{g}_3'\neq 0, \tag{84}$$

3. for the class III

$$\det M=16A_2B_3C_2H_2g_1'g_2'g_3'g_4'\neq 0. \tag{85}$$

Let's assume additionally for class I:

$$d_1d_2\neq 0. \tag{86}$$

Let us **fix** now the functions $g_j,(j=1,...,4)$ in (75), (76) and (77), but such that the conditions (83) (together with (78) and (86)), (84) (together with (79)) and (85) (together with and $B_3-B_4+\sqrt{B_3^2+B_4^2}\neq 0$ and (A.1)) will be still satisfied, correspondingly.

We can now repeat from the subsubsection 3.1.2 (basing on [7]), that the equation (23) cannot be satisfied identically for any solution of Legendre-transformed hyperbolic complex Monge-Ampère equation (20) just by proper choice of the functions $\xi^1, \xi^{\bar{1}}, \xi^2, \xi^{\bar{2}}, h, \bar{h}$, because the variables $p, \bar{p}, w_{,z^2}, w_{,\bar{z}^2}$ explicitly enter into the coefficients of this equation.
$\Sigma_j,\ (j = 1,2,3,4)$ are linearly independent, for the three above classes of solutions. So, the transformations from $p, \bar{p}, z^2, \bar{z}^2$ to Σ_j are invertible and we can express $p, \bar{p}, z^2, \bar{z}^2$ through $\Sigma_j,\ (j = 1,2,3,4)$, so that $\Sigma_j,\ (j = 1,2,3,4)$, can be chosen as new independent variables in (23) and after inserting each of above classes of solutions into the equation (23), this equation becomes:

$$F_3(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = 0. \tag{87}$$

For example, if we choose $g_j = \exp(\Sigma_j)$ in the above found classes of solutions, especially in (77), we obtain form similar to the form of the ansatz (24), if $n = 4$. So, the solutions, belonging to the above classes: (75) or (76) or (77), with such fixed functions $g_k, (k = 1, ..., 4)$, do not have functional arbitrariness. Hence, after taking into account these above arguments, we see that these solutions cannot satisfy the first-order Killing equation (23).
Thus, we see that the Killing equation (23) cannot be satisfied identically for any solution, belonging to the classes of the form: (75) or (76) or (77).

Of course, all these above found classes of solutions of the hyperbolic complex Monge-Ampère equation (20), are the classes of *real* solutions. Obviously, these solutions are not differentiable in $\mathbb{C}$ sense. They are differentiable in $\mathbb{R}$ sense. If we repeat these above computations, but in the variables $x^1 = \Re(p), x^2 = \Im(p), x^3 = \Re(z^2), x^4 = \Im(z^2)$, $x^k \in \mathbb{R}, (k = 1,2,3,4)$, then it turns out that the functions, given either by (75) or by (75) or by (77), are still the classes of exact solutions of hyperbolic complex Monge-Ampère equation (20), these solutions are still non-invariant and the condition (22) is still satisfied.

Hence, we have proved the following theorem:

Theorem 2 *The metric (21) with w, being some solution of hyperbolic complex Monge-Ampère equation (20), belonging to any class, defined by:*

1. *(75) - class I, (when the relations: (78), (83) and (86) hold)*

2. *(76) - class II, (when the relations: (79), (84) hold)*

3. *(77) - class III, (when are satisfied the relations: (85),* $B_3 - B_4 + \sqrt{B_3^2 + B_4^2} \neq 0$ *and some algebraic inequality mentioned above),*

where the functions $g_j, (j = 1, ..., 4)$*, are fixed, does not possess Killing vector.*

C. Classes of exact solutions of second heavenly equation

Now, in order to find new classes of non-invariant solutions of heavenly equation, we use decomposition method, in the cases of equal symmetries and higher symmetries. We apply the ansatz (6) for the systems (38)-(40) and (41)-(43), but now $u \equiv \vartheta$, the independent variables are: $x^1 = x, x^2 = r, x^3 = t, x^4 = z$, and g_k, $(k = 1, 2, 3, 4)$, are *arbitrary* holomorphic, complex-valued functions of their arguments. The ansatz (6) presents the class of solutions of the systems (38)-(40), (41)-(43), when the relations, which must be satisfied by the coefficients, are, as follows:

1. for equations (38)-(40) - the case of equal symmetries:

$$\begin{aligned} a_1 &= \frac{a_2(a_2+a_3)}{a_3}, \quad & a_4 &= -\frac{a_2(a_2+a_3)^2}{a_3^2}, \\ b_1 &= \frac{b_2(b_2+b_3)}{b_3}, \quad & b_4 &= -\frac{b_2(b_2+b_3)^2}{b_3^2}, \\ c_1 &= \frac{c_2(c_2+c_3)}{c_3}, \quad & c_4 &= -\frac{c_2(c_2+c_3)^2}{c_3^2}, \\ d_1 &= \frac{d_2(d_2+d_3)}{d_3}, \quad & d_4 &= -\frac{d_2(d_2+d_3)^2}{d_3^2}, \end{aligned} \tag{88}$$

2. for equations (41)-(43) - the case of higher symmetry:

   - subclass I

$$\begin{aligned} a_3 &= \frac{a_2^2}{a_1}, \quad & a_4 &= -\frac{a_1^2}{a_2}, \\ b_3 &= \frac{b_2^2}{b_1}, \quad & b_4 &= -\frac{b_1^2}{b_2}, \\ c_3 &= \frac{c_2^2}{c_1}, \quad & c_4 &= -\frac{c_1^2}{c_2}, \\ d_3 &= \frac{d_2^2}{d_1}, \quad & d_4 &= -\frac{d_1^2}{d_2}, \end{aligned} \tag{89}$$

- subclass II

$$
\begin{aligned}
a_1 &= \frac{a_2^2}{a_3}, \quad a_4 = -\frac{a_2^3}{a_3^2}, \\
b_1 &= \frac{b_2^2}{b_3}, \quad b_4 = -\frac{b_2^3}{b_3^2}, \\
c_1 &= \frac{c_2^2}{c_3}, \quad c_4 = -\frac{c_2^3}{c_3^2}, \\
d_1 &= \frac{d_2^2}{d_3}, \quad d_4 = -\frac{d_2^3}{d_3^2}.
\end{aligned} \tag{90}
$$

The parameters $\beta_k, (k = 1, ..., 5)$, occuring in (6) and (7), are, in this case, arbitrary constants.

We check now, whether the condition (35) of existence of Legendre transformation (34) is satisfied for the ansatz (6) and for the three sets of the relations of the coefficients (88), (89) and (90). It turns out that this condition is satisfied for this ansatz and for these above three sets of relations of coefficients, if these below conditions are satisfied:

- for (88) and for (90), ($g_j, (j = 1, ..., 4)$, are the functions of the arguments, including the coefficients, which satisfy correspondingly (88) and (90))

$$
\begin{gathered}
(b_2d_3 - b_3d_2)^2 g_2'' g_4'' + (b_2c_3 - b_3c_2)^2 g_2'' g_3'' + \\
(a_2d_3 - a_3d_2)^2 g_1'' g_4'' + (c_2a_3 - c_3a_2)^2 g_1'' g_3'' + \\
(a_2b_3 - a_3b_2)^2 g_1'' g_2'' + (c_2d_3 - c_3d_2)^2 g_3'' g_4'' \neq 0,
\end{gathered} \tag{91}
$$

- for (89)

$$
\begin{gathered}
\frac{(b_2^4 a_1^2 c_1^2 d_1^2 a_2^2 - 2a_2^3 b_1 c_1^2 d_1^2 b_2^3 a_1 + a_2^4 b_1^2 c_1^2 d_1^2 b_2^2) g_1'' g_2''}{a_1^2 b_1^2 c_1^2 d_1^2} + \\
\frac{(-2a_2^3 b_1^2 c_1 d_1^2 c_2^3 a_1 + c_2^4 a_1^2 b_1^2 d_1^2 a_2^2 + a_2^4 b_1^2 c_1^2 d_1^2 c_2^2) g_1'' g_3''}{a_1^2 b_1^2 c_1^2 d_1^2} + \\
\frac{(-2a_2^3 b_1^2 c_1^2 d_1 d_2^3 a_1 + a_2^4 b_1^2 c_1^2 d_1^2 d_2^2 + d_2^4 a_1^2 b_1^2 c_1^2 a_2^2) g_1'' g_4''}{a_1^2 b_1^2 c_1^2 d_1^2} + \\
\frac{(c_2^4 a_1^2 b_1^2 d_1^2 b_2^2 + b_2^4 a_1^2 c_1^2 d_1^2 c_2^2 - 2b_2^3 a_1^2 c_1 d_1^2 c_2^3 b_1) g_2'' g_3''}{a_1^2 b_1^2 c_1^2 d_1^2} + \\
\frac{(-2b_2^3 a_1^2 c_1^2 d_1 d_2^3 b_1 + b_2^4 a_1^2 c_1^2 d_1^2 d_2^2 + d_2^4 a_1^2 b_1^2 c_1^2 b_2^2) g_2'' g_4''}{a_1^2 b_1^2 c_1^2 d_1^2} + \\
\frac{(d_2^4 a_1^2 b_1^2 c_1^2 c_2^2 + c_2^4 a_1^2 b_1^2 d_1^2 d_2^2 - 2c_2^3 a_1^2 b_1^2 d_1 d_2^3 c_1) g_3'' g_4''}{a_1^2 b_1^2 c_1^2 d_1^2} \neq 0,
\end{gathered} \tag{92}
$$

where $g''_j, (j = 1,2,3,4)$ denotes the second derivative of the function g_j with respect to its argument.

Now, basing on the considerations included in [7], we make the analysis of non-invariance of these above classes of exact solutions of second heavenly equation. Namely, we check now, whether $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ are linearly independent, i.e. the transformations from x,r,t,z to $\Sigma_j, (j = 1,2,3,4)$ are invertible. The Jacobian matrices are the following:

1. for the case of equal symmetries (when the relations (88) hold):

$$M = \begin{pmatrix} \frac{a_2(a_2+a_3)}{a_3}g'_1 & a_2g'_1 & a_3g'_1 & -\frac{a_2(a_2+a_3)^2}{a_3^2}g'_1 \\ \frac{b_2(b_2+b_3)}{b_3}g'_2 & b_2g'_2 & b_3g'_2 & -\frac{b_2(b_2+b_3)^2}{b_3^2}g'_2 \\ \frac{c_2(c_2+c_3)}{c_3}g'_3 & c_2g'_3 & c_3g'_3 & -\frac{c_2(c_2+c_3)^2}{c_3^2}g'_3 \\ \frac{d_2(d_2+d_3)}{d_3}g'_4 & d_2g'_4 & d_3g'_4 & -\frac{d_2(d_2+d_3)^2}{d_3^2}g'_4 \end{pmatrix}, \tag{93}$$

2. for the case of higher symmetry (when the relations (89) hold) :

$$M = \begin{pmatrix} a_1g'_1 & a_2g'_1 & \frac{a_2^2}{a_1}g'_1 & -\frac{a_1^2}{a_2}g'_1 \\ b_1g'_2 & b_2g'_2 & \frac{b_2^2}{b_1}g'_2 & -\frac{b_1^2}{b_2}g'_2 \\ c_1g'_3 & c_2g'_3 & \frac{c_2^2}{c_1}g'_3 & -\frac{c_1^2}{c_2}g'_3 \\ d_1g'_4 & d_2g'_4 & \frac{d_2^2}{d_1}g'_4 & -\frac{d_1^2}{d_2}g'_4 \end{pmatrix}, \tag{94}$$

3. for the case of higher symmetry (when the relations (90) hold) :

$$M = \begin{pmatrix} \frac{a_2^2}{a_3}g'_1 & a_2g'_1 & a_3g'_1 & -\frac{a_2^3}{a_3^2}g'_1 \\ \frac{b_2^2}{b_3}g'_2 & b_2g'_2 & b_3g'_2 & -\frac{b_2^3}{b_3^2}g'_2 \\ \frac{c_2^2}{c_3}g'_3 & c_2g'_3 & c_3g'_3 & -\frac{c_2^3}{c_3^2}g'_3 \\ \frac{d_2^2}{d_3}g'_4 & d_2g'_4 & d_3g'_4 & -\frac{d_2^3}{d_3^2}g'_4 \end{pmatrix}, \tag{95}$$

where g'_j $(j = 1,2,3,4)$, is the first derivative of the function g_j with respect to its argument. We require non-vanishing of the Jacobians:

- - the case of equal symmetries

$$
\begin{aligned}
\det M = \frac{g_1' g_2' g_3' g_4'}{a_3^2 b_3^2 d_3^2 c_3^2}\Big(& -a_2^2 a_3 b_2 b_3^2 c_3^3 d_2^3 + a_2^2 a_3 b_2 b_3^2 c_2^3 d_3^3 \\
& -a_2^2 a_3 c_2 c_3^2 d_3^3 b_2^3 + a_2^2 a_3 c_2 c_3^2 b_3^3 d_2^3 - a_2^2 a_3 d_2 d_3^2 b_3^3 c_2^3 + \\
& a_2^2 a_3 d_2 d_3^2 c_3^3 b_2^3 + b_2^2 b_3 a_2 a_3^2 c_3^3 d_2^3 - b_2^2 b_3 a_2 a_3^2 c_2^3 d_3^3 \\
& +b_2^2 b_3 c_2 c_3^2 d_3^3 a_2^3 - b_2^2 b_3 c_2 c_3^2 a_3^3 d_2^3 + b_2^2 b_3 d_2 d_3^2 a_3^3 c_2^3 - \\
& b_2^2 b_3 d_2 d_3^2 c_3^3 a_2^3 + c_2^2 c_3 a_2 a_3^2 d_3^3 b_2^3 - c_2^2 c_3 a_2 a_3^2 b_3^3 d_2^3 \\
& -c_2^2 c_3 b_2 b_3^2 d_3^3 a_2^3 + c_2^2 c_3 b_2 b_3^2 a_3^3 d_2^3 - c_2^2 c_3 d_2 d_3^2 a_3^3 b_2^3 + \\
& c_2^2 c_3 d_2 d_3^2 b_3^3 a_2^3 + d_2^2 d_3 a_2 a_3^2 b_3^3 c_2^3 - d_2^2 d_3 a_2 a_3^2 c_3^3 b_2^3 \\
& -d_2^2 d_3 b_2 b_3^2 a_3^3 c_2^3 + d_2^2 d_3 b_2 b_3^2 c_3^3 a_2^3 + d_2^2 d_3 c_2 c_3^2 a_3^3 b_2^3 - \\
& d_2^2 d_3 c_2 c_3^2 b_3^3 a_2^3 \Big) \neq 0,
\end{aligned}
\tag{96}
$$

- - the case of higher symmetry - subclass I

$$
\begin{aligned}
\det M = \frac{g_1' g_2' g_3' g_4'}{c_1 d_2 c_2 d_1 b_2 b_1 a_2 a_1}\Big(& -a_1^2 a_2 b_2^2 b_1 c_2^3 d_1^3 + \\
& a_1^2 a_2 b_2^2 b_1 c_1^3 d_2^3 - a_1^2 a_2 c_2^2 c_1 d_2^3 b_1^3 + \\
& a_1^2 a_2 c_2^2 c_1 b_2^3 d_1^3 - a_1^2 a_2 d_2^2 d_1 b_2^3 c_1^3 + \\
& a_1^2 a_2 d_2^2 d_1 c_2^3 b_1^3 + b_1^2 b_2 a_2^2 a_1 c_2^3 d_1^3 \\
& -b_1^2 b_2 a_2^2 a_1 c_1^3 d_2^3 + b_1^2 b_2 c_2^2 c_1 d_2^3 {a_1}^3 - \\
& b_1^2 b_2 c_2^2 c_1 a_2^3 d_1^3 + b_1^2 b_2 d_2^2 d_1 a_2^3 c_1^3 - \\
& b_1^2 b_2 d_2^2 d_1 c_2^3 a_1^3 + c_1^2 c_2 a_2^2 a_1 d_2^3 b_1^3 - \\
& c_1^2 c_2 a_2^2 a_1 b_2^3 d_1^3 - c_1^2 c_2 b_2^2 b_1 d_2^3 a_1^3 + \\
& c_1^2 c_2 b_2^2 b_1 a_2^3 d_1^3 - c_1^2 c_2 d_2^2 d_1 a_2^3 b_1^3 + \\
& c_1^2 c_2 d_2^2 d_1 b_2^3 a_1^3 + d_1^2 d_2 a_2^2 a_1 b_2^3 c_1^3 - \\
& d_1^2 d_2 a_2^2 a_1 c_2^3 b_1^3 - d_1^2 d_2 b_2^2 b_1 a_2^3 c_1^3 + \\
& d_1^2 d_2 b_2^2 b_1 c_2^3 a_1^3 + d_1^2 d_2 c_2^2 c_1 a_2^3 b_1^3 - \\
& d_1^2 d_2 c_2^2 c_1 b_2^3 a_1^3 \Big) \neq 0,
\end{aligned}
\tag{97}
$$

- - the case of higher symmetry - subclass II

$$
\begin{aligned}
\det M = \frac{g_1' g_2' g_3' g_4'}{a_3^2 b_3^2 d_3^2 c_3^2} \Big(& -a_2^2 a_3 b_2 b_3^2 c_3^3 d_2^3 + a_2^2 a_3 b_2 b_3^2 c_2^3 d_3^3 - \\
& a_2^2 a_3 c_2 c_3^2 d_3^3 b_2^3 + a_2^2 a_3 c_2 c_3^2 b_3^3 d_2^3 \\
& -a_2^2 a_3 d_2 d_3^2 b_3^3 c_2^3 + a_2^2 a_3 d_2 d_3^2 c_3^3 b_2^3 \\
& +b_2^2 b_3 a_2 a_3^2 c_3^3 d_2^3 - b_2^2 b_3 a_2 a_3^2 c_2^3 d_3^3 \\
& +b_2^2 b_3 c_2 c_3^2 d_3^3 a_2^3 - b_2^2 b_3 c_2 c_3^2 a_3^3 d_2^3 \\
& +b_2^2 b_3 d_2 d_3^2 a_3^3 c_2^3 - b_2^2 b_3 d_2 d_3^2 c_3^3 a_2^3 \\
& +c_2^2 c_3 a_2 a_3^2 d_3^3 b_2^3 - c_2^2 c_3 a_2 a_3^2 b_3^3 d_2^3 \\
& -c_2^2 c_3 b_2 b_3^2 d_3^3 a_2^3 + c_2^2 c_3 b_2 b_3^2 a_3^3 d_2^3 \\
& -c_2^2 c_3 d_2 d_3^2 a_3^3 b_2^3 + c_2^2 c_3 d_2 d_3^2 b_3^3 a_2^3 \\
& +d_2^2 d_3 a_2 a_3^2 b_3^3 c_2^3 - d_2^2 d_3 a_2 a_3^2 c_3^3 b_2^3 \\
& -d_2^2 d_3 b_2 b_3^2 a_3^3 c_2^3 + d_2^2 d_3 b_2 b_3^2 c_3^3 a_2^3 \\
& +d_2^2 d_3 c_2 c_3^2 a_3^3 b_2^3 - d_2^2 d_3 c_2 c_3^2 b_3^3 a_2^3 \Big) \neq 0.
\end{aligned}
\tag{98}
$$

We can now repeat the reasonings: from the [7], that the Killing equation cannot be satisfied identically for any solution of Legendre transformed second heavenly equation of Plebański (36) and from previous subsubsection.

In these above three cases: $\det(M) \neq 0$, when the corresponding polynomials, included in (96), (97), (98), do not possess zeroes and $g_1' g_2' g_3' g_4' \neq 0$. So, after assumption that $a_3 b_3 c_3 d_3 \neq 0$ (the case of equal symmetries) and $c_1 d_2 c_2 d_1 b_2 b_1 a_2 a_1 \neq 0$, $a_3 b_3 d_3 c_3 \neq 0$ (the case of higher symmetries), we may say that $\Sigma_1 = a_1 x + a_2 r + a_3 t + a_4 z + \beta_2, \ldots, \Sigma_4 = d_1 x + d_2 r + d_3 t + d_4 z + \beta_5$ (where the coefficients satisfy (88), (89) and (90), correspondingly), are linearly independent and the transformations from x, r, t, z to $\Sigma_j, (j = 1, 2, 3, 4)$, are invertible. Then, we can express x, r, t, z by $\Sigma_j, j = 1, 2, 3, 4$, so we can choose $\Sigma_j, j = 1, 2, 3, 4$, as new independent variables in Killing equation (derived in [7]). After inserting each of above classes of solutions into this equation, we obtain a relation of the form:

$$
F_4(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = 0. \tag{99}
$$

These above classes of exact solutions have been obtained, by solving systems of second-order linear equations together with Legendre-transformed second heavenly equation (36), they are determined up to arbitrary constants, because we may choose each of the function $g_j = f(\Sigma_j)$, as $f(\delta + \varepsilon\Sigma_j)$, where δ, ε are arbitrary constants. Let us **fix** now the functions $g_k, (k = 1, ..., 4)$ in found classes of solutions, but such that the conditions (96) (together with (91)), (97) (together with (92)) and (98) (together with (91)), are satisfied, correspondingly. For example, if we choose $g_j = \exp(\Sigma_j)$ in the above found classes of solutions, we obtain solutions similar to the solutions given either by (44) and (45) or by (44) and (46), if $n = 4$. So, having no functional arbitrariness, the solutions, belonging to these above classes with fixed functions $g_k, (k = 1, ..., 4)$, cannot satisfy in addition the first-order equation (Killing equation derived in [7]). Hence, this equation cannot be tautology for the solutions belonging to these above classes, for the Legendre-transformed potential $\Pi(x,r,t,z)$, satisfying corresponding equation (derived in [7]), and for suitable choice of the functions $q, c, e, \rho, \sigma, \psi$ and the constants a, k.

Thus, we have showed that the metric (37) with ϑ, being a solution, belonging to the classes, given by (6) and $\Sigma_1 = a_1x + a_2r + a_3t + a_4z + \beta_2, ..., \Sigma_4 = d_1x + d_2r + d_3t + d_4z + \beta_5$ (where in the case of the system (38)-(40), the coefficients satisfy the relations (88) and in the case of the system (41)-(43), the coefficients satisfy the relations (89) or (90)) and the functions $g_j, (j = 1, ..., 4)$ are fixed, does not possess Killing vector.

In order to obtain some extensions of classes of the solutions, given by (6) and (88), (89), (90) - the series of n (we assume as yet that n is a finite number) functions g_i, it is convenient to write the ansatz down in the convention applied in the formula (44) from [7]. Namely, the argument in each function g_j, is now: $\alpha_j x + \gamma_j r + \zeta_j t + \lambda_j z + \beta_j$. Hence, our ansatz has the form:

$$\vartheta(x,r,t,z) = \sum_{j=1}^{n} g_j(\Sigma_j), \tag{100}$$

where g_j are *arbitrary* holomorphic functions of:

$$\Sigma_j = \alpha_j x + \gamma_j r + \zeta_j t + \lambda_j z + \beta_j. \tag{101}$$

Obviously, now the notation of the coefficients changes. We give here this change for $j = 1, ..., 4$:

$$
\begin{aligned}
&a_1 = \alpha_1, \quad a_2 = \gamma_1, \quad a_3 = \zeta_1, \quad a_4 = \lambda_1, \\
&b_1 = \alpha_2, \quad b_2 = \gamma_2, \quad b_3 = \zeta_2, \quad b_4 = \lambda_2, \\
&c_1 = \alpha_3, \quad c_2 = \gamma_3, \quad c_3 = \zeta_3, \quad c_4 = \lambda_3, \\
&d_1 = \alpha_4, \quad d_2 = \gamma_4, \quad d_3 = \zeta_4, \quad d_4 = \lambda_4.
\end{aligned} \tag{102}
$$

Then, (100) is some class of solutions of the systems (38)-(40) and (41)-(43) (correspondingly) and in consequence, of second heavenly equation (36), if the coefficients satisfy the following relations:

1. for the case of equal symmetries:

$$
\alpha_j = \frac{\gamma_j(\gamma_j + \zeta_j)}{\zeta_j}, \quad \lambda_j = -\frac{\gamma_j(\gamma_j + \zeta_j)^2}{\zeta_j^2}, \tag{103}
$$

2. for the case of higher symmetry we found two subclasses:

- I subclass

$$
\zeta_j = \frac{\gamma_j^2}{\alpha_j}, \quad \lambda_j = -\frac{\alpha_j^2}{\gamma_j}, \tag{104}
$$

- II subclass

$$
\alpha_j = \frac{\gamma_j^2}{\zeta_j}, \quad \lambda_j = -\frac{\gamma_j^3}{\zeta_j^2}. \tag{105}
$$

In both cases: of equal symmetries and higher symmetry, β_j are arbitrary constants. It turns out that in the case of higher symmetry, the class of the solutions of second heavenly equation transformed by Legendre transformation, is given by functional series, which appear in (100), and this series can be infinite.

However, we have here the functional series (100). So, now we need to apply some properties of the functional series, [41]. Namely, from the requirement of differentiability of (100), we have the requirement that it needs to be uniformly convergent. Also, from the requirement of differentiability of (100), we see that the corresponding series (for the coefficients satisfying the relations (103)):

$$\sum_{j=1}^{n} \frac{\partial}{\partial x} g_j \ , \ \cdots \ , \ \sum_{j=1}^{n} \frac{\partial}{\partial z} g_j, \tag{106}$$

need to be uniformly convergent.

Further, from the requirement of differentiability of the series (106), the consecutive series, including the terms obtained by computing the derivatives in (106) (for the coefficients satisfying the relations (103)):

$$\sum_{j=1}^{n} \frac{\partial}{\partial x}\left(g'_j \frac{\gamma_j(\gamma_j+\zeta_j)}{\zeta_j}\right) \ , \ \sum_{j=1}^{n} \frac{\partial}{\partial t}\left(g'_j \frac{\gamma_j(\gamma_j+\zeta_j)}{\zeta_j}\right), \tag{107}$$

$$etc. \ \ etc. \tag{108}$$

need to be uniformly convergent, too, that the system (38)-(40), for (100), when the relations (103), is satisfied.

One can also check that the ansatz (100) with the parameters satisfying (103), satisfies the equation (36), too.

Analogically, in the case of higher symmetry or of the system (41)-(43), the series (100) needs also to be uniformly convergent. Also, (for the case of the relations (104) and (105)) from the requirement of differentiability of (100), we see that the corresponding series:

$$\sum_{j=1}^{n} \frac{\partial}{\partial x} g_j \ , \ \cdots \ , \ \sum_{j=1}^{n} \frac{\partial}{\partial z} g_j, \tag{109}$$

need to be uniformly convergent.

Further, from the requirement of differentiability of the series (109), the consecutive series, including the terms obtained by computing the derivatives in (109):

- when the relations (104) hold:

$$\sum_{j=1}^{n} \frac{\partial}{\partial x}\left(g'_j \alpha_j\right) \ , \ \sum_{j=1}^{n} \frac{\partial}{\partial x}\left(g'_j \gamma_j\right), \ etc. \ \ etc. \tag{110}$$

- when the relations (105) hold:

$$\sum_{j=1}^{n} \frac{\partial}{\partial x}\left(g'_j \frac{\gamma_j^2}{\zeta_j}\right) \ , \ \sum_{j=1}^{n} \frac{\partial}{\partial x}\left(g'_j \gamma_j\right), \ etc. \ \ etc. \tag{111}$$

need to be uniformly convergent, too. One can check that the system (41)-(43), is satisfied by (100), when the coefficients satisfy the relations (104) and similar situation is in the case, when the coefficients satisfy the relations (105).

As we see, the ansatz (100), with the relations (101), (104), is some direct generalization of the solution given by (44) and (46), found in [7]. We check now, whether the condition (35) is satisfied for these three classes of solutions given by the ansatz (100) and (103), (104), (105), correspondingly. It turns out that it is satisfied for these three classes, if the following relations hold:

1. - for (103)

$$\left(\sum_{j=1}^{n} g_j''\zeta_j^2\right)\left(\sum_{j=1}^{n} g_j''\gamma_j^2\right)-\left(\sum_{j=1}^{n} g_j''\gamma_j\zeta_j\right)^2 \neq 0, \tag{112}$$

2. - for (104)

$$\left(\sum_{j=1}^{n} \frac{g_j''\gamma_j^4}{\alpha_j^2}\right)\left(\sum_{j=1}^{n} g_j''\gamma_j^2\right)-\left(\sum_{j=1}^{n} \frac{g_j''\gamma_j^3}{\alpha_j}\right)^2 \neq 0, \tag{113}$$

3. - for (105)

$$\left(\sum_{j=1}^{n} g_j''\zeta_j^2\right)\left(\sum_{j=1}^{n} g_j''\gamma_j^2\right)-\left(\sum_{j=1}^{n} g_j''\gamma_j\zeta_j\right)^2 \neq 0. \tag{114}$$

The conditions (96), (97), (98) are satisfied also for (100) - (101) and (103), (104), (105), correspondingly, but, of course, the notation for the coefficients $a_j, b_j, c_j, d_j, (j = 1, ..., 4)$, changes according to (102).

Hence, now we may repeat the similar reasonings (included in [7]), as previously, and we may say that if $n \geq 4$, then $\Sigma_j = \alpha_j x + \gamma_j r + \zeta_j t + \lambda_j z + \beta_j, (j = 1,2,3,4)$, are linearly independent and the transformations from x, r, t, z to $\Sigma_j, (j = 1,2,3,4)$ are invertible, when the conditions: (96) - for the equal symmetries and (97), (98) - for the higher symmetry, are satisfied (after taking into consideration the relations (102)). Hence, we can express x, r, t, z by Σ_i and the same for $\Sigma_5, ..., \Sigma_n$, so we can choose $\Sigma_j, j = 1,2,3,4$, as new independent variables in Killing equation (derived in

[7]). Hence, after inserting any solution belonging to each of the classes, given either by (100), (101), (103) or by (100), (101), (104) or by (100), (101), (105), correspondingly, into the Killing equation mentioned above, this equation will possess the form like (99).

The solutions belonging to such classes, obtained by solving the systems of linear equations together with Legendre-transformed second heavenly equation (36), are determined up to arbitrary constants, because we may choose each of the function $g_j = f(\Sigma_j)$, as $f(\delta + \varepsilon\Sigma_j)$, where δ, ε are arbitrary constants. Now, let us **fix** the functions $g_j, (j = 1, ..., n)$ in (100) for each of obtained classes, but such that the conditions (96) (together with (112)), (97) (together with (113)) and (98) (together with (114)) are satisfied, correspondingly (of course, the notation for the coefficients $a_j, b_j, c_j, d_j, (j = 1, ..., 4)$, changes according to (102) and so, the relations: $\zeta_1\zeta_2\zeta_3\zeta_4 \neq 0$, $\alpha_1\gamma_1\alpha_2\gamma_2\alpha_3\gamma_3\alpha_4\gamma_4 \neq 0$ and $\zeta_1\zeta_2\zeta_3\zeta_4 \neq 0$, need to hold, correspondingly). For example, if we choose $g_j = \exp(\Sigma_j)$ in (100), then we obtain form similar to the form of the ansatz (44). So, after fixing functions $g_j, (j = 1, ..., n)$, these solutions, having no functional arbitrariness, cannot solve in addition, the first-order Killing equation (derived in [7]).
Hence, this equation cannot be tautology for the solutions belonging to these above three classes, for the Legendre-transformed potential $\Pi(x, r, t, z)$, satisfying appropriate equation (derived in [7]), for suitable choice of the functions $q, c, e, \rho, \sigma, \psi$ and the constants a, k.

Thus, we have proved the following theorem:

Theorem 3 *The metric (37) with ϑ, being exact solution of (36), belonging to any class, defined by (100) (where g_i are the functions of (101)), when n is an arbitrary natural number, and by the relations:*

1. *(103) - class I (the case of equal symmetries),*

2. *(104) - subclass I (the case I of higher symmetry),*

3. *(105) - subclass II (the case II of higher symmetry),*

where the functions $g_j, (j = 1, ..., n)$ are fixed, does not possess Killing vector, when $n \geq 4$ and the conditions: (96), $\zeta_1\zeta_2\zeta_3\zeta_4 \neq 0$, (112), n is finite - for the class I (the case of equal symmetries), (97), $\alpha_1\gamma_1\alpha_2\gamma_2\alpha_3\gamma_3\alpha_4\gamma_4 \neq 0$, (113) - for the subclass I (the case of higher symmetry) and (98), $\zeta_1\zeta_2\zeta_3\zeta_4 \neq 0$, (114) - for the subclass II (the case of higher symmetry), after taking into consideration the relations (102), are satisfied. In the case of higher symmetry, the class of the *solutions is given by an infinite functional series.*

In all these case, the series (100) some other corresponding series need to be uniformly convergent.

In the case of the higher symmetries, one can check by using Maple Waterloo Software that also the ansatz $\vartheta(x,r,t,z)=\sum_{j=1}^{\infty} g_j(\Sigma_j)$, (where g_j are arbitrary holomorphic functions of $\Sigma_j = \alpha_j x+\gamma_j r+\zeta_j t+\lambda_j z+\beta_j$), gives the class of the solutions of the system (41)-(43) and also of (36), when the relations (104) and (105) hold. One can also check that the condition (35) is satisfied in the case of this above ansatz and the relations (104) and (105). We can formulate the following theorem.

Theorem 4 *The second heavenly equation (36) (obtained by a Legendre transformation of (31)) and original version of the second heavenly equation (31),* not transformed by the Legendre transformation*, possess the class of solutions of the form of the functional series*

$$u(x^{\mu})=\beta_1+\sum_{j=1}^{n} g_j(\Sigma_j), \tag{115}$$

where $u=v$ for the case of (31) and $u=\vartheta$ for the case of (36), g_j are some arbitrary holomorphic functions of the arguments $\Sigma_m = a_{\nu}^{(m)}x^{\nu}+\beta_{m+1}$, (n can be any natural number, so the series $\sum_{j=1}^{n} g_j(\Sigma_j)$ can be finite or infinite), x^{ν} are proper independent variables, the series $\sum_{j=1}^{n} g_j(\Sigma_j)$ and its corresponding derivatives, are uniformly convergent, $a_{\nu}^{(m)}$ are some constants satisfying some relations following from satisfying of the system of algebraic equations, following from applying of the decomposition method to the versions of second heavenly equation: (36) and (31), and β_{m+1} are arbitrary constants.

Proof 1 *This is sufficient to prove that for any $n>0$ ($n\in 1,2,3,\ldots$), of course, afterproper changing of the independent variables in the ansatz (115), one can decompose the equation obtained after inserting this ansatz into (36) and (31), according to the idea of the decomposition method [27], and next, that according to this method, one can obtain a system of algebraic equations. The solutions of this system establish the relations between the coefficients, which occur in (115). If these relations are satisfied, then the ansatz (115) gives some class of the solutions of the equations (36) and (31) We prove this theorem by mathematical induction [42].*

1. *we check, whether this Theorem holds for* $n = 1$*, after substituting the ansatz (115) into (36) and (31), we obtain the equation*

$$g_1'' a_1^{(1)} a_3^{(1)} + g_1'' a_2^{(1)} a_4^{(1)} = 0, \tag{116}$$

so: $g_1''(a_1^{(1)} a_3^{(1)} + a_2^{(1)} a_4^{(1)}) = 0$ *and it suffices to find the solution of the algebraic equation* $a_1^{(1)} a_3^{(1)} + a_2^{(1)} a_4^{(1)} = 0$.

2. *we prove that if this Theorem holds for* $n = k$*, then this holds also for* $n = k + 1$

After substituting the ansatz $u(x^\mu) = \beta_1 + \sum_{j=1}^{n=k} g_j(\Sigma_j)$*, into (36) and (31), and collecting the algebraic terms by* $g_i'' g_j''$, g_i''*, one obtains the following differential equation (it follows from the fact that the equations (36) and (31) satisfy the assumptions of the decomposition method)*

$$g_1''^2 X_1 + g_1'' g_2'' X_2 + g_1'' g_3'' X_3 + \ldots + g_1'' g_m'' X_m + \tag{117}$$

$$g_2''^2 X_2 + \ldots + g_{k-1}'' g_k'' X_{k-1} + (g_k'')^2 X_k = 0, \tag{118}$$

where $X_i, (i = 1, \ldots, k)$ *are some polynomials including the constants* $a_\nu^{(k)}$*. If we demand vanishing of these polynomials, we obtain some system of algebraic equations. Its solutions establish the relations between* $a_\nu^{(n)}$*. Next, after substituting the ansatz* $u(x^\mu) = \beta_1 + \sum_{n=1}^{n=k+1} g_j(\Sigma_j)$*, into (36) and (31), and collecting the algebraic terms by* $g_i'' g_j''$, g_i''*, one obtains the following differential equation (it also follows from the fact that the equations (36) and (31) satisfy the assumptions of the decomposition method)*

$$g_1''^2 X_1 + g_1'' g_2'' X_2 + g_1'' g_3'' X_3 + \ldots + g_1'' g_m'' X_m + \tag{119}$$

$$g_2''^2 X_2 + \ldots + g_k'' g_{k+1}'' X_k + (g_{k+1}'')^2 X_{k+1} = 0, \tag{120}$$

where $X_i, i = 1, \ldots, k + 1$ *are some polynomials including the constants* $a_\nu^{(k+1)}$*. If we demand vanishing of these polynomials, we obtain again some system of algebraic equations. Similarly, as previously, its solutions establish the relations between* $a_\nu^{(n)}$.

q.e.d.

In all these above considerations, the conditions: $g_j'' \neq 0$ need to be satisfied.

D. Classes of exact solutions of mixed heavenly equation

In order to find more general solutions of mixed heavenly equation (47), instead of solving equation (49), obtained from (47) by Legendre transformation (48) , we apply decomposition method to the system (51)-(53).

Similarly to the case of second heavenly equation, we look for the class of solutions, given by the ansatz (6), but here: $u \equiv w$, $x^1 = \eta, x^2 = \xi, x^3 = q, x^4 = y$, and g_j $(j = 1,2,3,4)$, are *arbitrary* functions of their arguments. The ansatz (6) presents a class of solutions of the system (51)-(53), when the following relations are satisfied:

$$\begin{aligned}
a_3 &= \frac{a_1^2+a_2^2}{a_2}, \quad a_4 = -\frac{a_1^2 a_2 + a_2^3 - a_1^3 - a_1 a_2^2}{a_2^2},\\
b_3 &= \frac{b_1^2+b_2^2}{b_2}, \quad b_4 = -\frac{b_1^2 b_2 + b_2^3 - b_1^3 - b_1 b_2^2}{b_2^2},\\
c_3 &= \frac{c_1^2+c_2^2}{c_2}, \quad c_4 = -\frac{c_1^2 c_2 + c_2^3 - c_1^3 - c_1 c_2^2}{c_2^2},\\
d_3 &= \frac{d_1^2+d_2^2}{d_2}, \quad d_4 = -\frac{d_1^2 d_2 + d_2^3 - d_1^3 - d_1 d_2^2}{d_2^2}.
\end{aligned} \tag{121}$$

It turns out that the condition (50) is satisfied for the above class of solutions, when relation (A1) holds (see Appendix A).
So, now we check, whether the class (6), is a class of non-invariant solutions or it depends on four independent combinations of the variables η, ξ, q, y, for these relations between the coefficients (121). Analogously, as in the case of second heavenly euqation, we write down the Jacobian matrix:

$$M = \begin{pmatrix}
a_1 g_1' & a_2 g_1' & \frac{a_1^2+a_2^2}{a_2} g_1' & -\frac{a_1^2 a_2 + a_2^3 - a_1^3 - a_1 a_2^2}{a_2^2} g_1' \\
b_1 g_2' & b_2 g_2' & \frac{b_1^2+b_2^2}{b_2} g_2' & -\frac{b_1^2 b_2 + b_2^3 - b_1^3 - b_1 b_2^2}{b_2^2} g_2' \\
c_1 g_3' & c_2 g_3' & \frac{c_1^2+c_2^2}{c_2} g_3' & -\frac{c_1^2 c_2 + c_2^3 - c_1^3 - c_1 c_2^2}{c_2^2} g_3' \\
d_1 g_4' & d_2 g_4' & \frac{d_1^2+d_2^2}{d_2} g_4' & -\frac{d_1^2 d_2 + d_2^3 - d_1^3 - d_1 d_2^2}{d_2^2} g_4'
\end{pmatrix}. \tag{122}$$

We compute its determinant and we require non-vanishing of it:

$$
\begin{aligned}
\det M = -\frac{g_1'g_2'g_3'g_4'}{a_2^2b_2^2c_2^2d_2^2}\Big(& -a_1a_2^2b_2^3c_2c_1^2d_1^3 + a_1a_2^2b_2^3d_2c_1^3d_1^2 \\
& -a_1a_2^2c_2^3d_2d_1^2b_1^3 + a_1a_2^2c_2^3b_2b_1^2d_1^3 - a_1a_2^2d_2^3b_2b_1^2c_1^3 + \\
& a_1a_2^2d_2^3c_2c_1^2b_1^3 + b_1b_2^2a_2^3c_2c_1^2d_1^3 - b_1b_2^2a_2^3d_2c_1^3d_1^2 \\
& +b_1b_2^2c_2^3d_2d_1^2a_1^3 - b_1b_2^2c_2^3a_2a_1^2d_1^3 + b_1b_2^2d_2^3a_2a_1^2c_1^3 - \\
& b_1b_2^2d_2^3c_2c_1^2a_1^3 + c_1c_2^2a_2^3d_2d_1^2b_1^3 - c_1c_2^2a_2^3b_2b_1^2d_1^3 \\
& -c_1c_2^2b_2^3d_2d_1^2a_1^3 + c_1c_2^2b_2^3a_2a_1^2d_1^3 - c_1c_2^2d_2^3a_2a_1^2b_1^3 + \\
& c_1c_2^2d_2^3b_2b_1^2a_1^3 + d_1d_2^2a_2^3b_2b_1^2c_1^3 - d_1d_2^2a_2^3c_2c_1^2b_1^3 \\
& -d_1d_2^2b_2^3a_2a_1^2c_1^3 + d_1d_2^2b_2^3c_2c_1^2a_1^3 + d_1d_2^2c_2^3a_2a_1^2b_1^3 - \\
& d_1d_2^2c_2^3b_2b_1^2a_1^3\Big) \neq 0.
\end{aligned} \tag{123}
$$

As we see, it is nonzero, if $g_1'g_2'g_3'g_4' \neq 0$ and the polynomial present in (123) does not possess zeroes. So, if additionally: $a_2b_2c_2d_2 \neq 0$, the ansatz (6) and the relations between the coefficients (121), give some class of non-invariant solutions of mixed heavenly equation.

Analogically to the case of second heavenly equation, we extend the ansatz (6):

$$
w(\eta,\xi,q,y) = \sum_{j=1}^{n} g_j(\Sigma_j), \tag{124}
$$

where g_j are *arbitrary* functions of their arguments and:

$$
\Sigma_j = \alpha_j\eta + \gamma_j\xi + \zeta_j q + \lambda_j y + \beta_j. \tag{125}
$$

Obviously, now the notation of the coefficients changes. We give here this change for $j = 1,...,4$:

$$
\begin{aligned}
a_1 &= \alpha_1, \quad a_2 = \gamma_1, \quad a_3 = \zeta_1, \quad a_4 = \lambda_1, \\
b_1 &= \alpha_2, \quad b_2 = \gamma_2, \quad b_3 = \zeta_2, \quad b_4 = \lambda_2, \\
c_1 &= \alpha_3, \quad c_2 = \gamma_3, \quad c_3 = \zeta_3, \quad c_4 = \lambda_3, \\
d_1 &= \alpha_4, \quad d_2 = \gamma_4, \quad d_3 = \zeta_4, \quad d_4 = \lambda_4.
\end{aligned} \tag{126}
$$

The ansatz (124) presents some class of solutions of the system (51)-(53) and consequently of (49), when the following relations between coefficients are satisfied:

$$\zeta_j = \frac{\alpha_j^2 + \gamma_j^2}{\gamma_j}, \quad \lambda_j = -\frac{\alpha_j^2\gamma_j + \gamma_j^3 - \alpha_j^3 - \alpha_j\gamma_j^2}{\gamma_j^2} \tag{127}$$

and $\theta = 1, \eta = p+t, \xi = p-t$, ($\beta_j$ are arbitrary constants).

Of course, analogically, as in the case of second heavenly equation, some properties of functional series need to be satisfied, [41]. Thus, (124) needs to be uniformly convergent. Also here, from the requirement of differentiability of (124), we see that the corresponding series:

$$\sum_{j=1}^{n} \frac{\partial}{\partial \eta} g_j \ , \ ..., \ \sum_{j=1}^{n} \frac{\partial}{\partial y} g_j, \tag{128}$$

need to be uniformly convergent.

Further, from the requirement of differentiability of the series (128), the consecutive series, including the terms obtained by computing the derivatives in (128) (for the coefficients satisfying the relations (127)):

$$\sum_{j=1}^{n} \frac{\partial}{\partial \eta} \left(g'_j \alpha_j \right) \ , \ \sum_{j=1}^{n} \left(\frac{\partial}{\partial q} g'_j \alpha_j \right), \ etc.\ etc., \tag{129}$$

need also to be uniformly convergent. One can check that the system (51)-(53), is satisfied by (124), when the relations (127) hold.

One can check also that (124), for the relations (127), satisfies the mixed-heavenly equation (49).

However, from the other hand, it turns out that the condition (50) is satisfied for the class of exact solutions given by (124) and (127), when the following relations hold:

$$\left(\sum_{j=1}^{n} g''_j (\alpha_j + \gamma_j)^2 \right) \left(\sum_{j=1}^{n} \frac{g''_j (\alpha_j^2 + \gamma_j^2)^2}{\gamma_j^2} \right) - \tag{130}$$

$$\left(\sum_{j=1}^{n} \frac{g''_j (\alpha_j^2 + \gamma_j^2)(\alpha_j + \gamma_j)}{\gamma_j} \right)^2 \neq 0. \tag{131}$$

The condition (123) holds also for (124) - (125) and (127), but of course, the notation for the coefficients $a_j, b_j, c_j, d_j, (j = 1, ..., 4)$, changes according to (126).

Hence, the ansatz (124) with (125), when n is an arbitrary natural number, and the relations between the parameters are given by (127), gives the class of the exact solutions of (51)-(53) and consequently of (49) and these solutions depend on four variables, if $n \geq 4$, and the conditions: (123) (after taking into consideration the relations (126)), $\gamma_1\gamma_2\gamma_3\gamma_4 \neq 0$ and (131), are satisfied. Thus, these solutions are non-invariant.

One can check by using Maple Waterloo Software that the ansatz $\vartheta = \beta_1 + \sum_{j=1}^{n} g_j(\Sigma_j)$, (where g_j are arbitrary holomorphic functions of $\Sigma_j = a_j^{(n)} x^\mu + \beta_j$), satisfies the system (51)-(53) and the Legendre transformed mixed heavenly equation (49) and also of , when the coefficients satisfy the relations (127) One can also check that the condition (50) is satisfied for this above ansatz and relations (127).

Theorem 5 *The Legendre transformed mixed heavenly equation (49) (obtained by a Legendre transformation of (47)), possess the class of solutions of the form of the functional series (when* $\theta = 1$*):*

$$u(x^\mu) = \beta_1 + \sum_{j=1}^{n} g_j(\Sigma_j), \tag{132}$$

where $u = w$ *,* g_j *are some arbitrary holomorphic functions of the arguments* $\Sigma_m = a_\nu^{(m)} x^\nu + \beta_{m+1}$*, (n can be any natural number, so the series* $\sum_{j=1}^{n} g_j(\Sigma_j)$ *can be finite or infinite),* x^ν *are proper independent variables, the series* $\sum_{j=1}^{n} g_j(\Sigma_j)$ *and its corresponding derivatives, are uniformly convergent, and* $a_\nu^{(m)}$ *are some constants satisfying some relations following from satisfying of the system of algebraic equations, following from applying of the decomposition method to the Legendre transformed mixed heavenly equation (49) .*

Proof 2 *Analogical, as in the case of the second heavenly equation.*

E. Classes of exact solutions of asymmetric heavenly equation and evolution form of second heavenly equation

Now we will solve the asymmetric heavenly equation (55), by applying decomposition method.

We insert also the ansatz (6), but now: $x^1 = x, x^2 = y, x^3 = z, x^4 = t$, and $g_j \in \mathscr{C}^2$ $(j = 1,2,3,4)$, are *arbitrary* functions of their arguments. This ansatz gives the class of non-invariant solutions of (55), when:

$$
\begin{aligned}
C &= -\frac{c_3(Bc_1+Ac_4)}{c_1^2},\\
a_2 &= \frac{a_1c_2(-Ba_3c_1^2+a_1c_1c_3B+a_1c_3c_4A)}{Aa_3c_1^2c_4},\\
a_4 &= \frac{a_1(-Ba_3c_1^2+a_1c_1c_3B+a_1c_3c_4A)}{Aa_3c_1^2}, b_1=0, b_4=0,\\
d_1 &= 0, d_4=0,
\end{aligned}
\tag{133}
$$

and $\beta_k, (k=1,...,5)$ are arbitrary constants.

The Jacobian matrix has the form:

$$
M = \begin{pmatrix} a_1g_1' & N_1g_1' & a_3g_1' & N_2g_1' \\ 0 & b_2g_2' & b_3g_2' & 0 \\ c_1g_3' & c_2g_3' & c_3g_3' & c_4g_3' \\ 0 & d_2g_4' & d_3g_4' & 0 \end{pmatrix},
\tag{134}
$$

where $N_1 = \frac{a_1c_2}{Aa_3c_1^2c_4}(-Ba_3c_1^2+a_1c_1c_3B+Aa_1c_3c_4), N_2 = \frac{a_1}{Aa_3c_1^2}(-Ba_3c_1^2+a_1c_1c_3B+Aa_1c_3c_4)$.

We require non-vanishing of determinant of this above Jacobian matrix:

$$
\begin{aligned}
\det M = \frac{g_1'g_2'g_3'g_4'}{c_1Aa_3}\Big(&a_1(b_2d_3-b_3d_2)(-c_4Aa_3c_1-\\
&Ba_3c_1^2+a_1c_1c_3B+a_1c_3c_4A)\Big) \neq 0.
\end{aligned}
\tag{135}
$$

This above determinant is non-zero, when $g_1'g_2'g_3'g_4' \neq 0$ and the polynomial, present in (135), does not possess zeroes.

In the case $B=0$ we obtain the class of solutions of the evolution form of second heavenly equation. Let us notice that also in this case: $B=0$, the ansatz (6) with the set of modified relations (133) (after putting $B=0$), remains the class of non-invariant solutions.

Hence, the ansatz (6) and the relations (133), give the class of exact solutions of assymetric heavenly equation (if $B\neq 0$), and of the evolution form of the second heavenly equation (if $B=0$) and these solutions depend on four variables (when (135) and $Ac_1a_3c_4 \neq 0$ hold). Hence, these solutions are non-invariant.

F. Classes of exact solutions of general heavenly equation and of real section of this equation

1. *The case of general heavenly equation*

We insert the ansatz (6) into (56), but now: $x^1 = z^1, x^2 = z^2, x^3 = z^3, x^4 = z^4$, and we wish $g_j \in \mathscr{C}^2$ $(j = 1,2,3,4)$, are as *arbitrary* functions of their arguments, as it is possible.

The decomposition method gives a chance of obtaining several classes of exact solutions - they have the form of (6), where $a_i, b_i, c_i, d_i (i = 1,...,4)$ are the solutions of corresponding system of algebraic equations.

I class:

$$\begin{gathered} a_3 = \frac{a_2 b_3 c_3 (b_2 c_4 - b_4 c_2)(\beta + \gamma)}{b_2 c_2 \beta (b_3 c_4 - b_4 c_3)}, \\ a_4 = -\frac{a_2 b_4 c_4 (b_2 c_3 - b_3 c_2)(\beta + \gamma)}{b_2 c_2 \gamma (b_3 c_4 - b_4 c_3)}, \\ b_1 = 0, c_1 = 0, d_1 = 0, d_2 = 0, d_4 = 0, \\ \beta \neq 0, \gamma \neq 0 \end{gathered} \tag{136}$$

II class:

$$\begin{gathered} a_3 = \frac{c_3 (a_2 c_4 \beta + (a_2 c_4 - a_4 c_2)\gamma)}{\beta c_2 c_4},\ b_1 = 0,\ b_2 = 0, \\ b_3 = 0,\ c_1 = 0,\ c_3 = c_2,\ d_1 = 0,\ d_3 = 0,\ d_4 = 0. \end{gathered} \tag{137}$$

We have checked that the condition (58) is satisfied for these both classes.

The conditions of non-invariance of these classes are, correspondingly

$$\det(M) = -a_1 d_3 (b_2 c_4 - b_4 c_2) g_1' g_2' g_3' g_4' \neq 0, \tag{138}$$

$$\det(M) = -a_1 b_4 c_2 d_2 g_1' g_2' g_3' g_4' \neq 0. \tag{139}$$

Next, the classes of exact solutions of real general heavenly equation (60) are given by (6), however, in this case: $g_1 = \bar{g}_2, g_k \in \mathbb{C}(k = 1,2), g_n \in \mathbb{R}(n = 3,4), c_1 = \bar{c}_2, c_3 = \bar{c}_4, d_1 = \bar{d}_2, d_3 = \bar{d}_4$, and the solutions of corresponding system of algebraic equations:

I class:

$$a_2=0, b_1=0, c_1=0, c_2=0, d_1=0, d_2=0,$$
$$\text{and } \beta=-\frac{\gamma b_3\bar{b}_3}{b_4\bar{b}_4}, \tag{140}$$
$$g_1=\bar{g}_2, g_k\in\mathbb{C}\ \ (k=1,2),\ \ g_n\in\mathbb{R}\ \ (n=3,4)$$

II class

$$a_1=0, a_2=0, a_3=0, b_1=0, b_2=0, b_4=0$$
$$\text{and}$$
$$\beta=-\frac{\gamma(\bar{c}_2c_2\bar{d}_4d_4-c_4\bar{c}_2d_2\bar{d}_4-c_2\bar{c}_4d_4\bar{d}_2+\bar{c}_4c_4d_2\bar{d}_2)}{\bar{c}_2c_2\bar{d}_4\bar{d}_4-\bar{c}_2\bar{c}_4d_2d_4-c_2c_4\bar{d}_2\bar{d}_4+\bar{d}_2\bar{c}_4c_4d_2}, \tag{141}$$
$$g_1=\bar{g}_2, g_k\in\mathbb{C}\ \ (k=1,2),\ \ g_n\in\mathbb{R}\ \ (n=3,4).$$

Obviously, we require the condition (61) was satisfied. It holds, when at least, some derivatives of $g_i, i=1,2,3,4$ are non-zero and at least one of the polynomials appearing in the numerators of the fractions, included in the determinant of Jacobian, does not possess zeroes:

$$m_2=b_2\bar{b}_2(b_4\bar{b}_4-b_3\bar{b}_3)g_1''g_3''\neq 0\ \ (\text{I class}), \tag{142}$$
$$m_2=(\bar{c}_4d_4-c_4\bar{d}_4)(\bar{c}_2d_2-c_2\bar{d}_2)g_3''g_4''\neq 0(\text{II class}). \tag{143}$$

The conditions of non-invariance of these classes of solutions are correspondingly

$$\det M=-b_2\bar{b}_2(\bar{c}_3d_3-c_3\bar{d}_3)g_1'g_2'g_3'g_4'\neq 0,\ \ (\text{I class}) \tag{144}$$
$$\det M=-b_3\bar{b}_3(c_2\bar{d}_2-\bar{c}_2d_2)g_1'g_2'g_3'g_4'\neq 0\ \ \ (\text{ II class}). \tag{145}$$

We stress here that we have obtained these above classes of solutions of (56), i.e. functionally invariant solutions of this equation, without imposing the differential constraint (59).

2. *The case of one of the real sections of general heavenly equation*

We insert the ansatz (6) into (63), but now: $x^1=z^1, x^2=\bar{z}^1, x^3=z^3, x^4=\bar{z}^2, \omega\in\mathbb{R}$, and, as previously, we wish $g_j\in\mathscr{C}^2$ $(j=1,2,3,4)$, are as *arbitrary* functions of their arguments, as it is possible. It turns out that certain non-invariant class of the solutions for (63), for $m\neq 0$, is given by the ansatzes (6), when respectively:

$$
\begin{gathered}
g_1 = \bar{g}_2, g_3 = \bar{g}_4, g_n \in \mathbb{C} \ \ (n = 1,2,3,4), \\
\delta = \sqrt{\frac{b_4\bar{b}_4}{a_4\bar{a}_4}}, a_1 = 0, a_3 = \bar{b}_4, b_1 = \bar{a}_2, b_2 = 0, \\
b_3 = \bar{a}_4, c_1 = 0, c_2 = 0, c_3 = \bar{d}_4, c_4 = \bar{d}_3, d_1 = 0, \\
d_2 = 0, \beta_k \in \mathbb{C} \ \ (k = 1,2,3,4), \\
\beta_1 = \bar{\beta}_2, \beta_3 = \bar{\beta}_4.
\end{gathered} \tag{146}
$$

The condition of non-invariance of these solutions, is:

$$
\det(M) = a_2\bar{a}_2(d_3\bar{d}_3 - d_4\bar{d}_4)g'_1g'_2g'_3g'_4 \neq 0. \tag{147}
$$

The condition (61) is satisfied, when

$$
m_2 = a_2\bar{a}_2(a_4\bar{a}_4 - b_4\bar{b}_4)g''_1g''_2 \neq 0. \tag{148}
$$

G. The criterion for non-invariance of found solutions

From the results obtained in previous sections of this paper, especially from the forms of the Jacobians for the found classes of exact solutions, the criterion for non-invariance of the solutions, belonging to the above mentioned classes, follows immediately:

Corollary 1 *Let the ansatz:*

$$
u(x^k) = \sum_{j=1}^{n} g_j(\Sigma_j), \quad k = 1,2,3,4, \tag{149}
$$

where: $\Sigma_j = \alpha_j x^1 + \gamma_j x^2 + \zeta_j x^3 + \lambda_j x^4 + \beta_j$, the coefficients $\alpha_j, \gamma_j, \zeta_j, \lambda_j, \beta_j$ satisfy some relations, and $n = 4$ (for the equations: elliptic and hyperbolic complex Monge-Ampère, asymmetric heavenly, evolution form of heavenly equation, general heavenly equation, real general heavenly equation and one of the real sections of general heavenly equation) or $n \geq 4$ (for second heavenly equation and Legendre transformed mixed heavenly equation), $g_j \in \mathscr{C}^2$ are arbitrary *functions (however: in the case of elliptic complex Monge-Ampère equation, g_1, g_2 are the square functions of their arguments and in the case of higher symmetry for the second heavenly equation, the ansatz*

(149) needs to be convergent series, some series including terms obtained after computing derivatives of (149), need to be uniformly convergent, moreover, in the case of elliptic and hyperbolic complex Monge-Ampère equation, the solutions, belonging to the corresponding classes, given by (149), need to be real), gives the class of exact solutions of the equations: elliptic and hyperbolic complex Monge-Ampère one, second heavenly, mixed heavenly, asymmetric heavenly, evolution form of second heavenly equation, general heavenly equation, real general heavenly equation and one of the real sections of general heavenly equation, when corresponding conditions of existence of Legendre transformation are satisfied (in the case of the equations: hyperbolic complex Monge-Ampère, second heavenly, mixed heavenly).
These solutions are non-invariant, if first derivatives of the all functions $g_n, (n = 1, ..., 4)$, are non-zero, the polynomials included in Jacobians, corresponding to each of these above mentioned classes of solutions, do not possess zeroes and the coefficients $\alpha_j, \gamma_j, \zeta_j, \lambda_j$ satisfy some additional relations (in the case of the equations: second heavenly and mixed, also uniform convergence of the series and their proper derivatives, is required).

V. CONCLUSIONS

We applied decomposition method for finding of classes of exact solutions (functionally invariant solutions) of heavenly equations: second heavenly, mixed heavenly, asymmetric heavenly, evolution form of second heavenly equation, general heavenly equation, real general heavenly equation and one of the real sections of general heavenly equation. For each of these equations, we have obtained the algebraic determining system, following from inserting the ansatz into investigated equation. Apart from satisfying of such algebraic determining system and the condition of non-invariance of wanted solutions, belonging to our classes (which implicated non-vanishing of the Jacobians), in the cases of the equations: second heavenly, mixed heavenly, general heavenly equation, real general heavenly equation and one of the real sections of general heavenly equation, some additional conditions must be satisfied by the functions included in ansatz and by the coefficients. It depends on the investigated equation:

1. elliptic complex Monge-Ampère equation - the condition of reality of the solutions,
2. hyperbolic complex Monge-Ampère equation - the condition of existence of Legendre transformation and the condition of reality of the solutions,

3. second heavenly equation - the condition of existence of Legendre transformation; the class of exact solutions is given by general functional series (for any n); obviously, this series needs to be uniformly convergent, some series obtained by twice differentiating of this series, need to be also uniformly convergent,

4. mixed-heavenly equation - the class of exact solutions is given by general functional series (for any n); the condition of existence of Legendre transformation, of course, the condition of unifom convergence is analogical to the case of the second heavenly equation,

5. asymmetric heavenly equation - the condition (135)

6. general heavenly equation - the condition (58)

7. real general heavenly equation and one of the real sections of general heavenly equation - the condition (61) and the condition of reality of the solutions.

Obviously, in the case of the equations: hyperbolic complex Monge-Ampère, second heavenly equation, mixed heavenly equation, the corresponding conditions of existence of Legendre transformation should also be satisfied.

The subclasses I and II (for the case of higher symmetry) was applied to construct in [43], some exact solutions of self-dual Yang-Mills (SDYM) equations, in the non-R gauge case, owing to the reduction of SDYM equations to second heavenly equation, done in [44].

We tried to keep the generality of the functions including in the ansatz, as it was possible, too.

To sum up, we have found some new classes of exact, non-invariant solutions of each of all above mentioned equations and we have established also the criterion for the non-invariance of the solutions, belonging to these classes.

Moreover, although there are several methods of solving of nonlinear partial differential equations [45], [35], [46], [47], [48], and the decomposition method, applied in this paper, is not general method, one can say that this method can offer sometimes more easy way of finding of exact solutions of some nonlinear partial differential equations, like heavenly equations, in comparison with other exact methods.

In some cases, this method can give a possibility of finding classes of exact solutions of given non-linear PDE, by applying this method directly to the equation, without the necessity of linearization of this PDE. A good example can be here

Of course, mentioned above solutions, of the equations investigated in this paper, are not first found functionally invariant solutions of these equations. Actually, some functionally invariant solutions were found to the equations: second heavenly in [49] . However, they possess different form, than the solutions presented in the section 4 of the current paper.

It is easy to check that the functionally-invariant solutions found in the current paper for the second heavenly equation, have more general functional form than the multikink solutions found in [18].

Appendix A: Condition (50) for the class of solutions, given by (6) and (121), of mixed heavenly equation

$$\frac{1}{a_2^2b_2^2c_2^2d_2^2}(N_1g_1''g_2''+N_2g_1''g_3''+N_3g_1''g_4''+N_4g_2''g_3''+N_5g_2''g_4''+N_6g_3''g_4'')\neq 0, \tag{A1}$$

where $N_1 = b_2^4c_2^2d_2^2a_1^4+a_2^4c_2^2d_2^2b_1^4+a_1^2a_2^2c_2^2d_2^2b_1^4+2\,a_1a_2^3c_2^2d_2^2b_1^4+a_2^4c_2^2d_2^2b_1^2b_2^2+a_1^2a_2^2c_2^2d_2^2b_2^4-$
$2b_2c_2^2d_2^2a_1a_2^3b_1^3-2b_2^3c_2^2d_2^2a_1a_2^3b_1+2b_1b_2^3c_2^2d_2^2a_1^2a_2^2-2b_2c_2^2d_2^2a_1^3a_2b_1^3-2b_2^2c_2^2d_2^2a_1^3a_2b_1^2-2b_2^3c_2^2d_2^2a_1^3a_2b_1+$
$2a_1^2a_2^2c_2^2d_2^2b_1^2b_2^2+2a_1a_2^3c_2^2d_2^2b_1^2b_2^2+b_1^2b_2^2c_2^2d_2^2a_1^4+2b_1b_2^3c_2^2d_2^2a_1^4-2b_2^4c_2^2d_2^2a_1^3a_2-2b_2c_2^2d_2^2a_2^4b_1^3-$
$2b_2c_2^2d_2^2a_1^2a_2^2b_1^3$,
$N_2=a_1^2a_2^2b_2^2d_2^2c_2^4+2a_1a_2^3b_2^2d_2^2c_1^4+a_2^4b_2^2d_2^2c_1^4+c_2^4b_2^2d_2^2a_1^4+2c_1c_2^3b_2^2d_2^2a_1^4-2b_2^2c_2^4d_2^2a_1^3a_2+a_2^4b_2^2d_2^2c_1^2c_2^2+$
$c_1^2b_2^2c_2^2d_2^2a_1^4+a_1^2a_2^2b_2^2d_2^2c_1^4-2b_2^2c_2d_2^2a_1^2a_2^2c_1^3-2b_2^2c_2d_2^2a_2^4c_1^3+2a_1a_2^3b_2^2d_2^2c_1^2c_2^2+2c_1c_2^3b_2^2d_2^2a_1^2a_2^2-$
$2b_2^2c_2d_2^2a_1^3a_2c_1^3-2b_2^2c_2d_2^2a_1a_2^3c_1^3-2b_2^2c_2^3d_2^2a_1a_2^3c_1+2a_1^2a_2^2b_2^2d_2^2c_1^2c_2^2-2b_2^2c_2^2d_2^2a_1^3a_2c_1^2-2b_2^2c_2^3d_2^2a_1^3a_2c_1$,
$N_3=d_1^2b_2^2c_2^2d_2^2a_1^4+2d_1d_2^3b_2^2c_2^2a_1^4+2a_1a_2^3b_2^2c_2^2d_1^2d_2^2+d_2^4b_2^2c_2^2a_1^4+a_1^2a_2^2b_2^2c_2^2d_2^4+2d_1d_2^3b_2^2c_2^2a_1^2a_2^2-$
$2b_2^2c_2^2d_2^4a_1^3a_2-2b_2^2c_2^2d_2a_2^4d_1^3+a_2^4b_2^2c_2^2d_1^4+a_2^4b_2^2c_2^2d_1^2d_2^2+2a_1^2a_2^2b_2^2c_2^2d_1^2d_2^2-2b_2^2c_2^2d_2^3a_1^3a_2d_1-2b_2^2c_2^2d_2a_1^2a_2^2d_1^3-$
$2b_2^2c_2^2d_2a_1a_2^3d_1^3-2b_2^2c_2^2d_2^2a_1^3a_2d_1^2-2b_2^2c_2^2d_2a_1^3a_2d_1^3+a_1^2a_2^2b_2^2c_2^2d_1^4-2b_2^2c_2^2d_2^3a_1a_2^3d_1+2a_1a_2^3b_2^2c_2^2d_1^4$,
$N_4=2b_1b_2^3a_2^2d_2^2c_1^4+b_1^2a_2^2b_2^2d_2^2c_2^4+b_1^2a_2^2b_2^2d_2^2c_1^4-2a_2^2c_2^3d_2^2b_1^3b_2c_1-2a_2^2c_2d_2^2b_1^2b_2^2c_1^3-2a_2^2c_2d_2^2b_2^4c_1^3-$
$2a_2^2c_2^4d_2^2b_1^3b_2-2a_2^2c_2d_2^2b_1^3b_2c_1^3-2a_2^2c_2^2d_2^2b_1^3b_2c_1^2+2c_1c_2^3a_2^2d_2^2b_1^2b_2^2+b_2^4a_2^2d_2^2c_1^2c_2^2+b_2^4a_2^2d_2^2c_1^4+c_2^4a_2^2d_2^2b_1^4+$
$2c_1c_2^3a_2^2d_2^2b_1^4+2b_1b_2^3a_2^2d_2^2c_1^2c_2^2+2b_1^2a_2^2b_2^2d_2^2c_1^2c_2^2+c_1^2a_2^2c_2^2d_2^2b_1^4-2a_2^2c_2d_2^2b_1b_2^3c_1^3-2a_2^2c_2^3d_2^2b_1b_2^3c_1$,
$N_5=b_1^2a_2^2b_2^2c_2^2d_2^4-2a_2^2c_2^2d_2b_2^4d_1^3+2b_1b_2^3a_2^2c_2^2d_1^2d_2^2+b_1^2a_2^2b_2^2c_2^2d_1^4-2a_2^2c_2^2d_2^4b_1^3b_2+b_2^4a_2^2c_2^2d_1^2d_2^2+$
$2d_1d_2^3a_2^2c_2^2b_1^2b_2^2+2d_1d_2^3a_2^2c_2^2b_1^4+2b_1b_2^3a_2^2c_2^2d_1^4+2b_1^2a_2^2b_2^2c_2^2d_1^2d_2^2+d_1^2a_2^2c_2^2d_2^2b_1^4+d_2^4a_2^2c_2^2b_1^4-2a_2^2c_2^2d_2^3b_1b_2^3d_1-$
$2a_2^2c_2^2d_2b_1^3b_2d_1^3+b_2^4a_2^2c_2^2d_1^4-2a_2^2c_2^2d_2^3b_1^3b_2d_1-2a_2^2c_2^2d_2b_1^2b_2^2d_1^3-2a_2^2c_2^2d_2b_1b_2^3d_1^3-2a_2^2c_2^2d_2^2b_1^3b_2d_1^2$,

$$N_6 = 2c_1c_2^3a_2^2b_2^2d_1^4 + c_2^4a_2^2b_2^2d_1^2d_2^2 + 2d_1d_2^3a_2^2b_2^2c_1^4 - 2a_2^2b_2^2d_2^3c_1^3c_2d_1 - 2a_2^2b_2^2d_2c_1^2c_2^2d_1^3 + c_1^2a_2^2b_2^2c_2^2d_2^4 + d_1^2a_2^2b_2^2d_2^2c_1^4 - 2a_2^2b_2^2d_2^2c_1^3c_2d_1^2 - 2a_2^2b_2^2d_2c_2^4d_1^3 + c_1^2a_2^2b_2^2c_2^2d_1^4 - 2a_2^2b_2^2d_2c_1^3c_2d_1^3 - 2a_2^2b_2^2d_2^4c_1^3c_2 - 2a_2^2b_2^2d_2c_1c_2^3d_1^3 - 2a_2^2b_2^2d_2^3c_1c_2^3d_1 + c_2^4a_2^2b_2^2d_1^4 + d_2^4a_2^2b_2^2c_1^4 + 2d_1d_2^3a_2^2b_2^2c_1^2c_2^2 + 2c_1^2a_2^2b_2^2c_2^2d_1^2d_2^2 + 2c_1c_2^3a_2^2b_2^2d_1^2d_2^2.$$

Appendix B: Acknowledgements

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Appendix C: Computational resources

The computations were carried out, by using Waterloo MAPLE Software on the computers: "mars" and "saturn" (No. of grants: MNiI/IBM BC HS21/AP/057/2008 and MNiI/Sun6800/WSP/008/2005, correspondingly) in ACK-Cyfronet AGH in Kraków (Poland). Some part of the computations was done by using Waterloo Maple Software, owing to the financial support, provided by The Pedagogical University of Cracow, within some research project (the leader of this project: Dr K. Rajchel). This research was supported also by PL-Grid Infrastructure. The computations were carried out also in Interdisciplinary Centre for Mathematical and Computer Modelling (ICM), within the grant No. G31-6.

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