An Introduction to the Equatorial Orbitals of Toy Neutron Stars

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Abstract

In modelling the behaviour of ring systems for neutron stars in a non-relativistic setting one has to combine Newton's inverse square law of force of gravity with the Lorentz force due to the magnetic dipole moment in what we refer to as the KLMN equation. Here we investigate this problem using analytical dynamics revealing a strong link between Weierstrass functions and equatorial orbitals.

The asymptotics of our solutions reveal some new physics involving unstable circular orbits with possible long term effects - doubly asymptotic spirals, the hyperbolic equivalent of Keplerian ellipses in this setting. The corresponding orbitals can only arise when the discriminant of a certain quartic, $Q, \Delta_4 = 0$. When this occurs simultaneously with g_3, its quartic invariant, changing sign, the physics changes dramatically. Here we introduce dimensionless variables Z and W for this problem, $\Delta_4(Z, W) = 0$, is then an algebraic plane curve of degree 6 whose singularities reveal such changes in the physics. The use of polar coordinates displays this singularity structure very clearly.

Our first result shows that a time-change in the equations of motion leads to a complete solution of this problem in terms of Weierstrass functions. Since Weierstrass's results underlie this entire work we present this result first. Legendre's work leads to a complete solution. Both results reinforce the physical importance of the condition,

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$\Delta_4 = 0$, our final flourish being provided by Jacobi with his theta function.
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1 Introduction

1.1 Background

The KLMN (Kepler/Lorentz/Maxwell/Newton) equation for P, a unit mass, unit charged particle in the gravitational field of a μ unit point mass and electromagnetic field due to a constant magnetic dipole moment, $\mathbf{m}$, representing a neutron star centred at the origin O, for $\vec{OP} = \mathbf{r}$ and time t reads:

$$\frac{d^2\mathbf{r}}{dt^2} = -\mu r^{-3}\mathbf{r} + \frac{d\mathbf{r}}{dt} \wedge \mathbf{H},$$

with $\mathbf{H} = 3(\mathbf{m}.\mathbf{r})r^{-5}\mathbf{r} - \mathbf{m}r^{-3}, r = |\mathbf{r}|$, where we assume perfect alignment of the dipole moment and rotational axis of star.

This equation has been around for over 100 years and not attracted the attention of the mathematical community because it ignores the two greatest discoveries in mathematical physics - relativity and quantum mechanics. Nevertheless, it does contain interesting physics as it is the limiting case of these two theories. This will be our main concern herein where just for now we concentrate on the equatorial case.

As we shall see in the next section, using polar coordinates (r, θ), $r = \frac{1}{u}$, in equatorial plane, the plane of the motion for zero dipole moment, $\mathbf{m}.\mathbf{r} = 0$, $\mathbf{m} = (0, 0, B)$ in cartesians,

$$\dot{r}^2 = f(u),$$

where $f(u)$ is the quartic

$$f(u) = 2\{E + \mu u - \frac{u^2}{2}(C - Bu)^2\} = 2Q(u), \qquad \text{E and C constants.}$$

Borrowing a well-known result from plane algebraic curve theory known as "the uniformisation of curves of genus unity", c.f. [3, Whittaker & Watson]: if x and y are variables connected by the equation,

$$y^2 = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4,$$

they can be expressed as one-valued functions of the variable $z = \int_{x_0}^{x} \{f(t)\}^{-\frac{1}{2}}$, x_0 a zero of $f(.) = 0$, in the form,

$$x - x_0 = \frac{1}{4}f'(x_0)[\wp(z) - \frac{1}{24}f''(x_0)]^{-1},$$

$$y = \frac{1}{4}f'(x_0)\wp'(z)[\wp(z) - \frac{1}{24}f''(x_0)]^{-2},$$

where $f(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$, Weierstrass $\wp$ function being formed with the invariants of f, $g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2$, and the determinant, $g_3 = a_0(a_2a_4 - a_3^2) - a_1(a_1a_4 - a_2a_3) + a_2(a_1a_3 - a_2^2)$.

This suggest that we introduce the time-change,

$$T = \int_0^t \frac{ds}{r^2(s)},$$

the origin of T is

$$T = \int_{u_0}^{u} \frac{du}{\sqrt{f(u)}} = -\int_{u_0}^{u} \frac{du}{\dot{r}} = \int_{u_0}^{u} \frac{du}{\frac{1}{u^2}\dot{u}} = \int_0^t u^2(s)ds,$$

where we have chosen $\dot{r} = -\sqrt{f(u)}$.

Theorem 1. *Assume that f has no repeated roots and denote its invariants by g_2, g_3. Assume $Q(u_0) = 0, u = \frac{1}{r}, u_0 = \frac{1}{r_0}, r_0$ initial value of r. Then, if $\wp(z; g_2, g_3) = \wp(z)$,*

$$u - u_0 = \frac{f'(u_0)}{4[\wp(T) - \frac{1}{24}f''(u_0)]},$$

and

$$\dot{r} = \frac{dr}{dt} = +\frac{f'(u_0)\wp'(T)}{4[\wp(T) - \frac{1}{24}f''(u_0)]^2}.$$

To complete this we need $\theta(T)$.

Corollary 1.

$$\theta(T) - \theta_0 = (C - B\mu u_0)T + \frac{\mu B\zeta(T_0)T}{\wp'(T_0)} - \frac{\mu B f'(u_0)}{2\wp'(T_0)} \ln\left(\frac{\sigma(T - T_0)}{\sigma(T + T_0)}\right),$$

where

$$T_0 = \int_{\frac{f''(u_0)}{12}}^{\infty} \frac{dw}{\sqrt{4w^3 - g_2w - g_3}}, \qquad \wp'(T_0) \neq 0 \textit{ by assumption,}$$

σ and ζ being (non-elliptic) Weierstrass functions, $\zeta'(z) = -\wp(z), \frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$.

The proof is given in Section(2). It uses energy conservation and the conserved quantity $(r^2\dot\theta + Br^{-1})$, which follows from axial symmetry:

$$\frac{\dot{\mathbf{r}}^2}{2} = (E+\mu u),\ C = (r^2\dot\theta + Br^{-1}),\ \frac{\left(\frac{du}{d\theta}\right)^2}{2} = \frac{(E+\mu u)}{(C-Bu)^2} - \frac{u^2}{2},\ u = \frac{1}{r}.\ (\Diamond)$$

This is the Weierstrass connection. If we rewrite the above in terms of the Jacobi theta function we see explicitly how the sign of the discriminant, $\Delta(f)$, affects this answer (see last section). A related constant of the motion is, $D = r^2\dot\theta + BT_K/\mu$, T_K being kinetic energy, $C = D - BE/\mu$. We set $D_0 = r^2\dot\theta$ when $B = 0$.

1.2 Keplerian Asymptotics

Since it is easy to reinstate μ we set $\mu = 1$. If we assume $E < 0$, from energy conservation we see that the motion is bounded, $r < \frac{-1}{E}$. Setting

$$Q(u) = E + u - \frac{u^2}{2}(C-Bu)^2, \qquad F_B(u) = \frac{Q(u)}{(C-Bu)^2},$$

we can use Ferrari's method to give somewhat intractable formulae for the 4 roots of Q, u_1, u_2, u_3, u_4. We concentrate on the case when 2 of these roots u_1 and u_2 are positive $0 < u_1 < u_2$, the other pair could be a complex conjugate pair or a real pair. We find conditions for this later. Here we concentrate on the asymptotics of the solution of:

$$2^{-1}\left(\frac{du}{d\theta}\right)^2 = F_B(u), \qquad u \in (u_1, u_2), \qquad \theta \in (\theta_1, \theta_2),$$

as $B \sim 0$, for $E \in (\frac{-1}{2D^2}, 0)$. We avoid some of the algebraic complications of Ferrari by finding asymptotics of $u_i(B)$ as $B \sim 0, i = 1, 2, 3, 4$.

Theorem 2. *Let $u = u(\theta)$ be a solution of above polar equation, with $u(\theta_0) = u_0 = \frac{1}{r_0}$ and $u_1 < \frac{D}{B} - E, (u_0, u_1) \subset (\tilde u_1, \tilde u_2), \tilde u_{1,2}$ being smallest positive roots of quartic $\tilde Q$,*

$$\tilde Q(u) = \frac{E+u}{D}\left(1 + \frac{2B(E+u)}{D} + \frac{3B^2(E+u)^2}{D^2} + \frac{4B^3(E+u)^3}{D^3}\right) - \frac{u^2}{2},$$

with invariants $\tilde g_2, \tilde g_3$. Then as $B \sim 0$, the polar equation of the orbital reads:

$$\wp(\sqrt{2}(\theta-\theta_0)) = \left(\frac{(\tilde Q(u)\tilde Q(u_0))^{\frac12} + \tilde Q(u_0)}{2(u-u_0)^2} + \frac{\tilde Q'(u_0)}{4(u-u_0)} + \frac{\tilde Q''(u_0)}{24}\right)(1+\varepsilon), (\Diamond\Diamond)$$

where $\tilde{\wp}(z) = \wp(z; \tilde{g}_2, \tilde{g}_3), \varepsilon$ *is* $O(B^4)$ *with coefficient depending on* u_0. *Sadly it blows up as* $u_0 \to u_{1,2}$. *When* $B \to 0$ *last equation reduces to classical Keplerian case,*

$$\frac{l}{r} = 1 + \tilde{e}\cos(\theta - \theta_0), \qquad l = D_0^2 \text{ and } \tilde{e} = \sqrt{1 + 2D_0^2 E}, \qquad E < 0.$$

An elementary underlying lemma is:

Lemma 1. *For* $E < 0$, $\frac{B}{D} > 0$,

$$\int \left(\frac{E+u}{(D - B(E+u))^2} - \frac{u^2}{2} \right)^{-\frac{1}{2}} du \longrightarrow \int \left(\frac{E+u}{D_0^2} - \frac{u^2}{2} \right)^{-\frac{1}{2}} du, \text{ as } B \to 0.$$

The proofs are given in Section(2).

Modulo our assumptions above the last theorem tells us that as $B \sim 0$, if $u \in (u_0, u_1)$, a proper subset of $(\tilde{u}_1, \tilde{u}_2), \tilde{u}_{1,2}$ being the two postive roots of $\tilde{Q}$, u has an asymptotic series in B. However, if $u_{0,1} \to \tilde{u}_{1,2}$, the first term is inevitably $O(B^{\frac{1}{2}})$ and Puiseux series have to be used. This explains the limitation of the next lemma.

Lemma 2. *When* $u = u(\theta)$ *has an asymptotic expansion:*

$$u(\theta) = u^{(0)}(\theta) + Bu^{(1)}(\theta) + B^2 u^{(2)}(\theta) + \dots$$

where $u^{(0)}(\theta) = D_0^{-2}(1 + \hat{e}\cos\theta)$, $\quad \tilde{e} = \sqrt{1 + 2D_0^2 E}$, $\quad \theta \in (\theta_1, \theta_2) \subset (0, \pi)$,

$$u^{(1)}(\theta) = \frac{BE}{\tilde{e}D_0^3(\tilde{e}^2 - 1)}((\tilde{e}^4 + 6\tilde{e}^2 + 1)\cos\theta + 4\tilde{e}^2\theta\sin\theta),$$

i.e. correct to first order in $B, \theta \in (\theta_1, \theta_2)$,

$$u = \frac{1}{D_0^2}\left(1 + \tilde{e}\cos\left[\theta - \frac{BE}{D_0^3(\tilde{e}^2 - 1)}((\tilde{e}^2 + 6 + \tilde{e}^{-2})\cot\theta + 4\theta) \right] \right).$$

Bracketed term blows up at $\theta = n\pi, n = 0, \pm 1, \pm 2,$ *as expected.*

The proof is given in Section(2).

1.3 On Cubic and Quartic Curves

The practical signifcance of $(\Diamond\Diamond)$ is not easy to justify but it does emphasise the important link between the KLMN orbitals and quartic and cubic curves, in particular with the Weierstrass cubic curve with equation,

$$y^2 = x^3 + px + q.$$

The corresponding cubic equation, $x^3 + px + q = 0$, has discriminant, Δ_3, where

$$-\frac{\Delta_3}{27} = q^2 + \frac{4p^3}{27}.$$

This cubic has one real root and a complex conjugate pair if $\Delta_3 < 0$, and, if $\Delta_3 > 0$, three real roots; $\Delta_3 = 0$ if and only if the cubic has a repeated root. It turns out that for any quartic,

$$Q = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4, \qquad (a_0 \neq 0),$$

Q has a repeated root if and only if the cubic,

$$4x^3 - g_2x - g_3 = 0,$$

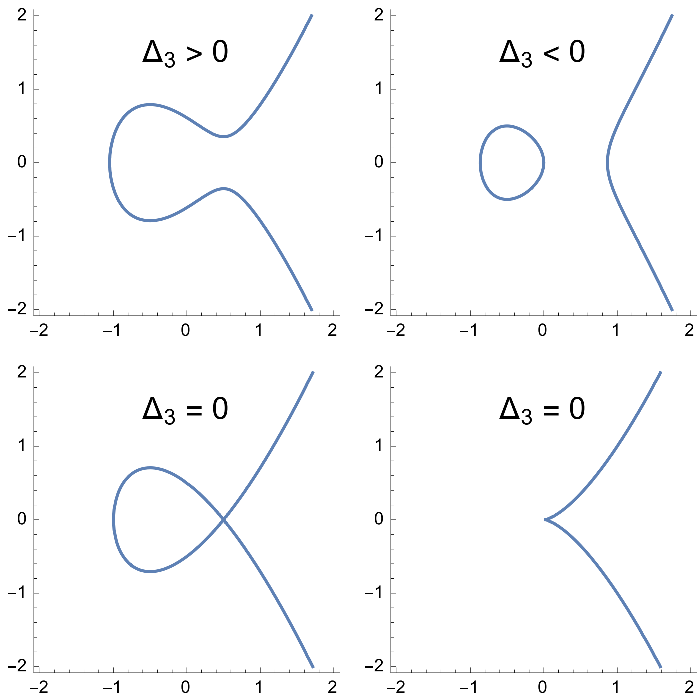
has a repeated root, where $g_2 = (a_0a_4 - 4a_1a_3 + 3a_2^2)$ and the determinant, $g_3 = a_0(a_2a_4 - a_3^2) - a_1(a_1a_4 - a_2a_3) + a_2(a_1a_3 - a_2^2)$, c.f. [3, 8, 14].

This result is implicit in Ferrari's work but was not spelled out in these terms. We shall refer to the cubic $(4x^3 - g_2x - g_3)$ as our reducing cubic, with discriminant zero, if and only if $27g_3^2 = g_2^3$. However, it is important to remember that

$$\Delta_4 = 256(g_2^3 - 27g_3^2), \qquad \Delta_4 \text{ being the discriminant of } Q,$$

$$\Delta_3 = \frac{1}{16}(g_2^3 - 27g_3^2), \qquad \Delta_3 \text{ above.}$$

Figure 1. Some Cubic Curves



When $B \sim 0$, the first term in our asymptotic expansion corresponds to the classical orbit with polar equation,

$$u = \frac{1}{D_0^2}(1 + \tilde{e}\cos\theta), \qquad \tilde{e} = \sqrt{1 + 2D_0^2 E}, \qquad E < 0,$$

and associated quadratic curve,

$$y^2 = 2\left(\frac{E + x}{D_0^2} - \frac{x^2}{2}\right).$$

See Fig(2). What about the infinite B limit?

Reinstating the neutron star gravitational mass μ, after some obvious minor amendments the governing quartic:

$$(E + u) - \frac{u^2}{2}(D - B(E + u))^2 \longrightarrow (E + \mu u) - \frac{u^2}{2}(C - Bu)^2.$$

Letting $B \to \infty$, $C \to \infty$, $\frac{\mu}{B^2} \to 0$, $\frac{E}{B^2} \to E_0$, $\frac{C}{B} \to \frac{C_0}{B_0}$, we arrive at the limiting quartic:

$$E_0 - \frac{u^2}{2}(C_0 - B_0 u)^2,$$

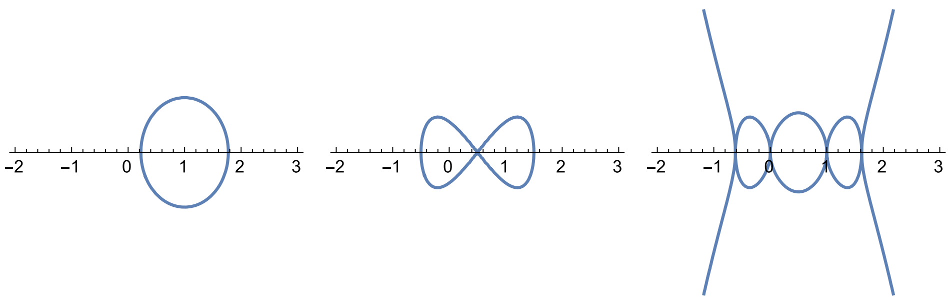
i.e. a symmetric double well potential, with $\Delta = 0$ and $g_3 < 0$.

The corresponding quartic curve has equation:

$$y^2 = -\frac{x^4}{2} + \frac{x^2}{4}.$$

The corresponding curves are sketched below. The reflectional symmetry in x axis is blindingly obvious in these pictures and will be important later. The slightly mysterious middle diagram and its connection with the above will be explained later. For the cognoscenti, it is a 'classical instanton', as we shall see. The first is the classical ellipse of Kepler.

Figure 2. Some Quartic Curves



1.4 The Weierstrass Elliptic Function $\wp$

What can we say about our original equation ($\Diamond$). Firstly, Legendre [2] showed that it cannot be solved in terms of simple periodic functions; what is needed, as we have seen, is a doubly periodic function such as the elliptic functions of Weierstrass and Jacobi. Legendre's work will be crucial in establishing the full solution as we shall see in the last section. For now we are less ambitious.

For motivational reasons we give a brief account of the Weierstrass elliptic function and the main properties we need in what follows. We start with the lattice, $\mathbb{Z}^2$, contained in the real plane $\mathbb{R}^2$ and its quotient space $\mathbb{R}^2/\mathbb{Z}^2$, formed by identifying opposite sides of the fundamental cell folded over to form a torus.

Now identify the real plane $\mathbb{R}^2$ with the Argand diagram $\mathbb{C}$. Evidently $\mathbb{C}$ has different embedded lattices, Λ_τ, generated by $1 \in \mathbb{R}\,\text{and}\,\tau \in U$, the upper half plane of $\mathbb{C}$; in some conventions, 2ω and $2\omega'$ are the two fundamental periods c.f. [1].

Define $\wp(z;g_2, g_3)$ by

$$\wp_\tau(z) = \frac{1}{z^2} + \Sigma_{\tilde{\omega}\in\Lambda_\tau\searrow\{0\}} \left(\frac{1}{(z-\tilde{\omega})^2} - \frac{1}{\tilde{\omega}^2}\right),$$

then $\wp(\cdot)$ is analytic at all points, save the points $(2m\omega + 2n\omega')$, (m, n), $\in \mathbb{Z}^2, m, n = 0, \pm1, \pm2, \dots$ where it has double poles. Evidently one can prove:

$$\wp_\tau(z) = \wp_\tau(z + \tilde{\omega}), \qquad \wp'_\tau(z) = \wp'_\tau(z + \tilde{w}), \qquad \tilde{w} \in \Lambda_\tau.$$

The amazing result proved by Weierstrass is that,

$$(\wp'_\tau(z))^2 = 4\wp^3_\tau(z) - g_2\wp_\tau(z) - g_3,$$

with $g_2 = 60\Sigma_{\tilde{\omega}\in\Lambda_\tau\searrow\{0\}}\ \tilde{\omega}^{-4}, \qquad g_3 = 140\Sigma_{\tilde{\omega}\in\Lambda_\tau\setminus\{0\}}\ \tilde{\omega}^{-6}$, both being real valued.

This leads to:

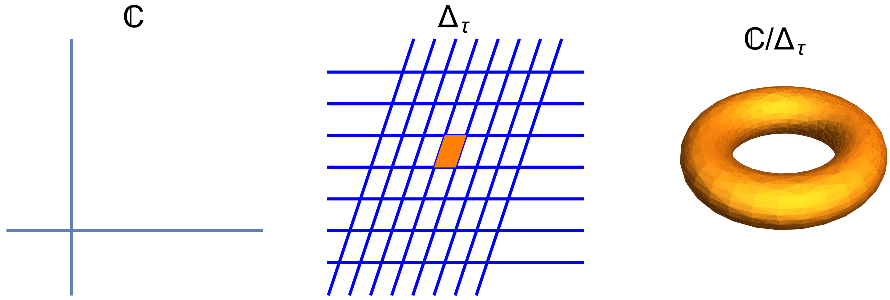
$$z = \int_\zeta^\infty \frac{dt}{(4t^3 - g_2t - g_3)^{\frac{1}{2}}}, \qquad \text{if and only if } \zeta = \wp(z; g_2, g_3),$$

when $\Delta > 0$, defining $\omega_1 = \omega, \omega_2 = \omega + \omega', \omega_3 = \omega', e_i$ succesive zeros of $(4z^3 - g_2z - g_3), i = 1, 2, 3$,

$$e_i = \wp(w_i), \qquad w_i = \int_{e_i}^\infty \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}.$$

(See table at end of paper and c.f. [1])

Figure 3. Quotient Space Mapping



$$\pi: \mathbb{C} \nearrow \Lambda_\tau \to \mathbb{C} \times \mathbb{C}: \qquad \pi(z) = (\wp_\tau(z), \wp'_\tau(z))$$
$$\longrightarrow \mathbb{C}_\tau = \{y^2 = 4x^2 - g_2 x - g_3\}, \text{an elliptic curve.}$$

We need another elementary property of $\wp(z; g_2, g_3)$, when $27g_3^2 = g_2^3$ i.e. $\Delta = 0$ since g_3 is real, $g_2 \geq 0$ but g_3 can have either sign; in this connection:

if $g_2 = 12\alpha^2, \qquad g_3 = 8\alpha^3, \qquad \alpha > 0$, then for $\Delta = 0$,

$$\wp(z; g_2, g_3) = -\alpha + \frac{3\alpha}{\sin^2((3\alpha)^{\frac{1}{2}} z)} \text{ (equal roots } -\alpha, -\alpha, \text{ remaining root } 2\alpha),$$

if $g_2 = 12\alpha^2, \qquad g_3 = -8\alpha^3, \qquad \alpha > 0$, then

$$\wp(z; g_2, g_3) = \alpha + \frac{3\alpha}{\sinh^2((3\alpha)^{\frac{1}{2}} z)} \text{ (equal roots } \alpha, \alpha, \text{ remaining root } -2\alpha),$$

c.f.[1, Abramowitz & Stegun].

Since $\Delta \sim 0$ as $B \sim 0$, in our case above result is relevant for classical limit. Second result is relevant when the $\mathbb{E} \sim 0$ orbital centres on an unstable circular orbit. Both of these occur at the same singular point in the (Z, W) plane on the algebraic curve on which $\Delta = 0, Z = -E^3B^2, W = E^2BD$, classical eccentricity $e = \sqrt{1 - 2W^2/Z}$.

1.5 Discriminants

The quartic invariants of, $Q(u) = E + u - \frac{u^2}{2}(D - B(E+u))^2$, with $a_0 = -\frac{B^2}{2}, a_1 = \frac{BC}{4}, a_2 = -\frac{C^2}{12}, a_3 = \frac{1}{4}, a_4 = E$, are $g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2$ and $g_3 = a_0(a_2a_4 - a_3^2) - a_1(a_1a_4 - a_2a_3) + a_2(a_1a_3 - a_2^2)$.

i.e. $g_2 = -\frac{EB^2}{2} - \frac{BC}{4} + \frac{C^4}{48}, \quad g_3 = -\frac{EB^2C^2}{48} + \frac{B^2}{32} - \frac{BC^3}{96} + \frac{C^6}{1728}$.

$$16\Delta_3 = g_2^3 - 27g_3^2, \text{ underlying cubic, } 4z^3 - g_2 z - g_3 = 0.$$

i.e.

$$16\Delta_3 = \frac{1}{8}\left(\frac{C^4}{24} - \frac{BC}{2} - EB^2\right)^3 - \frac{27}{16^2}\left(-\frac{EB^2C^2}{3} + \frac{B^2}{2} - \frac{BC^3}{6} + \frac{C^6}{108}\right)^2,$$

where the first term on the right hand side is

$$\frac{1}{8}\frac{C^{12}}{(24)^3} + \frac{3}{8}\left(\frac{C^4}{24}\right)^2\left(-\frac{BC}{2} - EB^2\right) + \frac{3}{8}\frac{C^4}{24}\left(\frac{BC}{2} + EB^2\right)^2 - \frac{1}{8}\left(\frac{BC}{2} + EB^2\right)^3,$$

and the second term on the right hand side is

$$\frac{27}{16^2}\left(\frac{C^{12}}{(108)^2} + \frac{2C^6}{108}\left(-\frac{EB^2C^2}{3} + \frac{B^2}{2} - \frac{BC^3}{6}\right) + \left(-\frac{EB^2C^2}{3} + \frac{B^2}{2} - \frac{BC^3}{6}\right)\right)^2.$$

The terms in C^{12}, BC^9 and EB^2C^8 agree. The first interesting term is in EB^3C^5. We clearly need a different approach. We base this on a dimensional analysis of our equation.

1.6 Dimensional Analysis

Fundamental equation reads in plane, $\mathbf{m}.\mathbf{r} = 0, \mathbf{m} = (0,0,B)$, where we shall eventually assume mass dimensionless, $\mu = 1$, $\ddot{\mathbf{r}} = -\mu r^{-3}\mathbf{r} + \dot{\mathbf{r}} \wedge (-\mathbf{m}r^{-3})$, $\mathbf{m} = (0,0,B)$, giving $[\mu] = L^3T^{-2}$, $[B] = L^3T^{-2}$, $[E] = [\mu]L^{-1}$, by comparing terms. Hence, $[E] = L^2T^{-2}$. It follows that $[E^3B^2] = L^{12}T^{-8} = [\mu^4]$ i.e. E^3B^2/μ^4 is dimensionless. Recall that $C = D - \frac{BE}{\mu}$ so $[D] = [\frac{BE}{\mu}]$.

So

$$[E^2BD] = [\mu^3] \text{ i.e. } E^2BD/\mu^3 \text{ is dimensionless.}$$

Set $\mu = 1$ then $Z = -E^3B^2/\mu^4$ and $W = E^2BD/\mu^3$ are dimensionless and numerically equal to $-E^3B^2$, E^2BD, still being denoted by Z and W. We rewrite equation $\Delta_3 = 0$ in terms of Z and W. It is then easy to recover μ, since,

$$E + \mu u - \frac{u^2}{2}(C - Bu)^2 = \mu\left(\frac{E}{\mu} + u - \frac{u^2}{2}\left(\frac{C}{\sqrt{\mu}} - \frac{Bu}{\sqrt{\mu}}\right)^2\right),$$

so $E \to \frac{E}{\mu}$, $C \to \frac{C}{\sqrt{\mu}}$, $B \to \frac{B}{\sqrt{\mu}}$.

A tedious calculation for $\mu = 1$ yields the condition for $\Delta_4 = 0$ in the form:-

$$((W+Z)^4 + 12(W-Z)Z^2)^3 = ((W+Z)^6 + 54Z^4 + 18(Z-W)Z^2)^2.$$

So, in the (Z,W) plane, the condition for equal roots $\Delta = 0$ is an algebraic plane curve of degree 12. Fortunately the above cancellations reduce the complexity to give an algebraic curve of degree 5 in the form:-

$$4W(W+Z)^4 + 2Z(Z-W)(Z^2+34ZW+W^2) + 27Z^3 = 0, \qquad \text{or } Z = 0.$$

This curve has some hidden symmetry leading to a very simple expansion in polar coordinates for the vanishing of discriminant Δ.

Theorem 3. *The polar equation of the discriminant curve reads in the* (Z,W) *plane*

$$r = \cos^2\theta \left\{ \frac{(\sin\theta - \cos\theta)(1+17\sin 2\theta) \pm (1-7\sin 2\theta)^{\frac{3}{2}}}{2\sin 2\theta (1+\sin 2\theta)^2} \right\}, \qquad r > 0.$$

The proof is given in Section (3).
In dimensionless λ, Z, W,

$$E^2B^2Q(\lambda E) = -\left\{(1+\lambda)Z + \frac{\lambda^2}{2}(W+(1+\lambda)Z)^2\right\}.$$

So we can consider the quartic in λ directly for general μ,

$$1+\lambda)Z + \frac{\lambda^2}{2}\left(W+(1+\lambda)Z\right)^2 = 0,$$

$$Z \to Z_\mu = \frac{-E^3B^2}{\mu^4}, \quad W \to W_\mu = \frac{E^2BD}{\mu^3}, \quad u = \frac{\lambda E}{\mu}.$$

2 Analytical Dynamics, Some Proofs and Examples

2.1 Equations of Motion via Lagrange and Hamilton

Our Hamiltonian reads: $\mathcal{H} = 2^{-1}(\mathbf{p} - (\mathbf{m}\wedge\mathbf{q})q^{-3})^2 - \frac{1}{q}$, where $\mathbf{q} = \mathbf{r} = \vec{OP}$ and $q = |\mathbf{q}|$. Here vector potential of magnetic dipole $\mathbf{m}$ at origin O is given by,

$$\mathbf{A} = (\mathbf{m}\wedge\mathbf{q})q^{-3},$$

and scalar potential due to gravitational attraction of unit mass at origin O is

$$V = -\frac{1}{q}.$$

This is the simplest representation of a 'toy' neutron star incorporating the basic physics for a charged unit mass particle P carrying unit charge in full field of a star. In this set up the equations of motion are the Hamilton equations:-

$$\dot{\mathbf{q}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \qquad \text{and} \qquad \dot{\mathbf{p}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}}.$$

Since $\mathbf{m}$ is constant by hypothesis, $\mathcal{H} = E$, a constant. So energy is conserved,

$$\dot{\mathbf{q}} = (\mathbf{p} - \mathbf{A}),$$

and equivalent Lagrangian is, $\mathcal{L} = (\mathbf{p}.\dot{\mathbf{q}} - \mathcal{H})$, giving

$$\mathcal{L} = 2^{-1}\dot{\mathbf{q}}^2 + \dot{\mathbf{q}}.(\mathbf{m} \wedge \mathbf{q})q^{-3} + \frac{1}{q}.$$

However, $\dot{\mathbf{q}}.(\mathbf{m} \wedge \mathbf{q}) = (\mathbf{q} \wedge \dot{\mathbf{q}}).\mathbf{m} = h_3 B$ if $\mathbf{m} = (0, 0, B)$ in cartesians, h_3 being third component of angular momentum. This suggests that we use cylindrical polars (ρ, ϕ, z) and

$$\mathcal{L} = 2^{-1}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \frac{B\rho^2\dot{\phi}}{(\rho^2 + z^2))^{\frac{3}{2}}} + \frac{1}{(\rho^2 + z^2)^{\frac{1}{2}}},$$

with equations of motion:-

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}, \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\rho}}\right) = \frac{\partial \mathcal{L}}{\partial \rho}, \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) = \frac{\partial \mathcal{L}}{\partial z},$$

c.f. [4, Whittaker].

Theorem 4. *The equations of motion reduce to:*

$$C = \rho^2\dot{\phi} + \frac{B\rho^2}{(\rho^2 + z^2)^{\frac{3}{2}}}, \; C\, constant, \tag{2.1}$$

$$\begin{aligned} \ddot{\rho} - \rho\dot{\phi}^2 &= -\frac{\rho}{(\rho^2 + z^2)^{\frac{3}{2}}} \\ &\quad + \frac{B}{(\rho^2 + z^2)^{\frac{3}{2}}}\left(C - \frac{B\rho^2}{(\rho + z^2)^{\frac{3}{2}}}\right)\left(\frac{2}{\rho} - \frac{3\rho}{(\rho^2 + z^2)}\right), \end{aligned} \tag{2.2}$$

$$\ddot{z} = -\frac{z}{(\rho^2 + z^2)^{\frac{3}{2}}} - \frac{3zB}{(\rho^2 + z^2)^{\frac{3}{2}}}\left(C - \frac{B\rho^2}{(\rho^2 + z^2)^{\frac{3}{2}}}\right). \tag{2.3}$$

These are just the cylindrical polar versions of the KLMN equation,

$$\ddot{\boldsymbol{r}} = -\boldsymbol{r}r^{-3} + \dot{\boldsymbol{r}} \wedge \mathcal{H},$$

where $\mathcal{H} = 3(\boldsymbol{m}.\boldsymbol{r})\boldsymbol{r}r^{-5} - \boldsymbol{m}r^{-3}$*, magnetic field of dipole* $\boldsymbol{m} = (0, 0, B)$.

Proof. A simple consequence of above equations, axial symmetry, $\frac{\partial \mathcal{L}}{\partial \phi} = 0$ and above expression for vector potential of magnetic dipole $\mathbf{m}$. $\square$

Corollary 2. *The motion is bounded if $E < 0$.*

Proof. Energy conservation: $2^{-1}\dot{\mathbf{r}}^2 - \frac{1}{r} = E < 0$ and $\dot{\mathbf{r}}^2 \geq 0$ so $r \leq -\frac{1}{E}$. The reader should note that the above equation can be used to model the behaviour of normal star auroras. For now we concentrate on $\mathbf{m}.\mathbf{r} \equiv 0$ case i.e. $z \equiv 0$. $\square$

Theorem 5. *The equations (2.1), (2.2) and (2.3) when $z \equiv 0$ reduce to:*

$$2^{-1}(C - Bu)^2 \left(\frac{du}{d\theta}\right)^2 = E + u - \frac{u^2}{2}(C - Bu)^2, \qquad u = \frac{1}{r},$$

E being the total energy $(2^{-1}\dot{\mathbf{r}}^2 - 1/r)$ and$r^2\dot{\theta} + B/r = C$, for polar coordinates (r, θ).

Proof. A consequence of identity:

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -r^2\dot{\theta}\frac{du}{d\theta} = -(C - Bu)\frac{du}{d\theta},$$

and energy conservation: $2^{-1}(\dot{r}^2 + r^2\dot{\theta}^2) - 1/r = E$. $\square$

We focus on this equatorial orbital for now.

2.2 Some Proofs

Proof. (Theorem 1 and Corollary 1)
The uniformisation of curves of genus unity gives a proof of the Theorem. Referring to the constant C defined in ($\Diamond$).

$$r^2\frac{d\theta}{dt} = (C - Bu)$$

and as we have seen $\frac{dT}{dt} = u^2$, gives

$$\frac{d\theta}{dT} = \frac{(C - Bu)u^2}{u^2} = (C - Bu),$$

$$\textit{i.e.} \qquad \frac{d\theta}{dT} = C - B\left[u_0 + \frac{1}{4}\frac{f'(u_0)}{[\wp(T) - \frac{1}{24}f''(u_0)]}\right].$$

Recalling [1, 18.7.3, p641, Abramowitz & Stegun], c.f. [5, Tannery & Molk].

$$\wp'(\alpha)\int^{z}\frac{dz}{[\wp(z)-\wp(\alpha)]}=2\zeta(\alpha)z+\ln\left[\frac{\sigma(z-\alpha)}{\sigma(z+\alpha)}\right].$$

In our case we define α by $\wp(\alpha)=\frac{Q''(u_0)}{12}$ i.e.

$$\alpha=\int_{\frac{Q''(u_0)}{12}}^{\infty}\frac{dw}{\sqrt{4w^3-g_2w-g_3}}.$$

We set limits of z integration, $z=0$, $z=T$ and $\alpha=T_0$.

The desired result follows by integration, since σ is an entire symmetric function and $\zeta(z)$ has a simple pole at $z=0$, with principal part z^{-1},

$$\zeta'(z)=-\wp(z),\qquad\frac{\sigma'(z)}{\sigma(z)}=\zeta(z).$$

□

2.3 Periodicity

From Theorem 1, u is periodic in τ with period $2w_1$; as long as cubic, $4z^3-g_2z-g_3=0$, has 3 real roots e_1,e_2,e_3, the real $\frac{1}{2}$ period w being given by

$$w_1=\int_{e_1}^{\infty}\frac{dz}{\sqrt{4z^3-g_2z-g_3}},\quad g_2^3>27g_3^2\qquad\text{i.e. discriminant}\quad\Delta_3>0.$$

In this case the orbit is periodic if and only if

$$\frac{w}{\pi}=\frac{p}{q},\ \text{for coprime}\,p,q\in\mathcal{N}.$$

Proof. (Lemma 1)

We consider the limiting B-orbit, with polar equation: $2^{-1}\left(\frac{du}{d\theta}\right)^2=F_B(u)$,

$$F_B(u)=\frac{E+u}{(D-B(E+u))^2}-\frac{u^2}{2},\qquad\text{as } B\sim 0, E<0,\text{ fixed}, \frac{B}{D}>0,$$

passing through the fixed point (u_0,θ_0). To take the limit we set $f_n=F_{B_n}^{-\frac{1}{2}}, n=1,2,\ldots$ where $\{B_n\}_{n=1,2,\ldots}$ is a monotonic decreasing sequence with $B_n\to 0$ as $n\to\infty$.

Since on B-orbit by energy conservation, $E+u>0, f_{n+1}>f_n, n=1,2,\ldots$ and $f_n\to q^{-\frac{1}{2}}$, with $q=\frac{E+u}{D^2}-\frac{u^2}{2}$, so by monotone convergence,

$$\sqrt{2}\Delta\theta_n\to\int_{u_0}^{u}\frac{du}{q^{\frac{1}{2}}(u)},\qquad\Delta\theta_n=\int_{\theta_0}^{\theta}d\theta_n.$$

□

To evaluate the integral, assuming our orbit is bounded, $u \in (u_-, u_+) \subset \mathcal{R}_+, u_\pm$ roots of $q(u) = 0$ i.e. one can we obtain the equation for limiting orbit in the form:

$$\frac{l}{r} = 1 + \tilde{e}\cos\theta, \quad l = D_0^2, \quad \tilde{e} = \sqrt{(1 + 2D_0^2 E)}, \quad E < 0.$$

This is the equation of the classical Keplerian ellipse found by Newton.

An alternative way to evaluate the integral is to use the benefit of hindsight and to define new integration variable ϕ by, $u = \frac{1}{D_0^2}(1 + e\cos\phi)$. If we choose $\tilde{e} = \sqrt{(1 + 2D_0^2 E)}, q(u)$ simplifies to read:-

$$q(u) = \frac{\tilde{e}^2 \sin^2\phi}{2D_0^4} \qquad \frac{du}{d\phi} = -\frac{\tilde{e}\sin\phi}{D_0^2}, \text{ and } u_\pm \text{ have correct values,}$$

so $\sqrt{2}\Delta\theta_n \to \sqrt{2}\int_{\phi_0}^{\phi} d\phi$, yielding the last result if $\theta_0 = \phi_0 = 0$. This piece of 'jiggery pokery' will prove useful in in trying to shed some light on the physics of this situation.

Proof. (Theorem 2)

Fix $E < 0$ and $\frac{B}{D} > 0$. If $\tilde{Q}$ is the quartic,

$$\tilde{Q} = \frac{(E+u)}{D^2}\left(1 + \frac{2B}{D}(E+u) + \frac{3B^2}{D^2}(E+u)^2 + \frac{4B^3}{D^3}(E+u)^3\right) - \frac{u^2}{2},$$

on any interval $I = (u_r, u_l)$ on which $\left|\frac{B}{D}(E+u)\right| < 1, u \in I \in (\tilde{u}_1, \tilde{u}_2)$, $\tilde{u}_{1,2}$ the smallest 2 positive zeros of $\tilde{Q}$,

$$F_B(u) = \tilde{Q}(u) + O\left(\frac{B^4(E+u)^5}{D^6}\right) = \tilde{Q}(u)\left(1 + O\left(\frac{B^4}{m}\right)\right),$$

uniformly; if $m = min(\tilde{Q}) > \delta > 0$, independent of B. The error term is $O(B^4)$; we note however that if $u_r \sim \tilde{u}_1$ or $u_l \sim \tilde{u}_2, m^{-1} = O(B)$ so error term is $O(B^3)$.

Denoting error term by ε, if B is sufficiently small that $\left|\frac{B}{D}(E+u)\right) < 1$, i.e. $u = r^{-1} < \frac{B}{D} - E < \tilde{u}_2$,

$$\sqrt{2}\int_{\theta_l}^{\theta_r} d\theta = \int_{u_l}^{u_r} \frac{du}{\sqrt{\tilde{Q}(u)}}(1+\varepsilon).$$

A beautiful result due to Weierstrass, c.f.[1, Abramowitz & Stegun, p447] is that if quartic $\tilde{Q}$ has no repeated factors and

$$z = \int_{u_0}^{u} \frac{du}{\sqrt{\tilde{Q}(u)}},$$

then

$$\wp(z;\tilde{g}_2,\tilde{g}_3) = \frac{(\tilde{Q}(u)\tilde{Q}(u_0))^{\frac{1}{2}} + \tilde{Q}(u_0)}{2(u-u_0)^2} + \frac{\tilde{Q}'(u_0)}{4(u-u_0)} + \frac{\tilde{Q}''(u_0)}{24},$$

$\tilde{g}_2, \tilde{g}_3$ invariants of $\tilde{Q}$. Then, using the scaling formula,

$$\wp(tz;g_2;g_3) = t^{-2}(\wp(z;t^4 g_2, t^6 g_3)),$$

the equation of the orbit satisfying,

$$2^{-1}\left(\frac{du}{d\theta}\right)^2 = F_B(u), \quad \theta = \theta_0, \quad u = u_0, \quad u \in I,$$

as $B \sim 0$ has the form

$$\wp(\sqrt{2}(\theta-\theta_0);\tilde{g}_2,\tilde{g}_3) = \left(\frac{(\tilde{Q}(u)\tilde{Q}(u_0))^{\frac{1}{2}} + \tilde{Q}(u_0)}{2(u-u_0)^2} + \frac{\tilde{Q}'(u_0)}{4(u-u_0)} + \frac{\tilde{Q}''(u_0)}{24}\right)(1+\varepsilon),$$

$u = r^{-1} < \frac{D}{B} - E, \quad \frac{D}{B} > 0, \quad E < 0, \quad r^{-1} \in (\tilde{u}_1, \tilde{u}_2).$ □

Proof. (Lemma 2)

Recall that:- $\left(\frac{du}{d\theta}\right)^2/2 = Q(u)/(D - B(E+u))^2$, where $Q(u) = Q_0(u) + BQ_1(u) + O(B^2)$, with $Q_0(u) = E + u - \dfrac{D_0^2}{2}u^2$,

$Q_1(u) = D_0 u^2(E+u), \qquad E < 0.$

$(D - B(E+u))^2 = D^2 - 2BD(E+u) + O(B^2).$

So correct to first order in B,

$$\left(\frac{du}{d\theta}\right)^2 = \frac{Q_0(u) + BQ_1(u)}{D_0^2\left(1 - \frac{2B}{D_0}(E+u)\right)} = \frac{Q_0(u) + BQ_1(u))}{D_0^2}\left(1 + \frac{2B}{D_0}(E+u)\right),$$

$$i.e. \qquad 2^{-1}\left(\frac{du}{d\theta}\right)^2 = \frac{Q_0(u) + BQ_1(u)}{D_0^2} + \frac{2(E+u)Q_0(u)}{D_0^3}.$$

Substitute $u = u^{(0)} + Bu^{(1)}$ to arrive at:-

$$\frac{\left(\frac{du^{(0)}}{d\theta}\right)^2}{2} + B\frac{du^{(0)}}{d\theta}\frac{du^{(1)}}{d\theta}$$
$$= \frac{Q_0(u^{(0)}) + Bu^{(1)}Q_0'(u^0) + BQ_1(u^{(0)})}{D^2} + \frac{2B}{D^3}(E + u^{(0)})Q_0(u^{(0)})$$

$u^0 = (1 + \tilde{e}\sin\theta)/D_0^2$ gives,

$$\frac{du^{(1)}}{d\theta} - \cot\theta u^{(1)} = \frac{(E + u^{(0)})}{(-\tilde{e}\sin\theta)}\left(D_0(u^{(0)})^2 + \frac{1}{D_0^3}\tilde{e}^2\sin\theta\right).$$

A simple integration gives for eccentricity $\tilde{e}$,

$$u^{(1)} = \frac{E}{\tilde{e}D_0^3(\tilde{e}^2 - 1)}(((\tilde{e}^2 + 1)^2 + 4\tilde{e}^2)\cos\theta + 4\tilde{e}^2\theta\sin\theta).$$

□

Observe that last result can be rewritten:-

$$u = \frac{1}{D_0^2}\left(1 + \tilde{e}\cos\left(\theta - \frac{BE}{D_0^3(\tilde{e}^2 - 1)}\left((\tilde{e}^2 + 6 + \tilde{e}^{-2})\cot\theta + 4\tilde{e}^2\right)\right)\right),$$

revealing the angular correction correct to first order in B, $\tilde{e}$ the eccentricity of B=0 elliptical orbit, $\tilde{e} = \sqrt{1 + 2D_0^2E}$, D_0 angular momentum, $E < 0$, the energy.

2.4 Some Examples including Cusped, Loopy and Sinusoidal orbits and Doubly Asymptotic Spirals.

When $B = 0$ and $E < 0$ our first example corresponds to bounded rectilinear motion and to unbounded motion when $E > 0$. When $B \neq 0$, $D = 0$, $E < 0$ in this case we obtain quasielliptic orbitals confined to an annular region, the orbit having cusps on the outer boundary.

Example 1 (Cusped, Quasi-elliptic orbit)($D = 0, E < 0$)

Since $E < 0, r < -E^{-1}$ gives an upper bound for r. To find the lower bound when D=0 we must solve,

$$E + u = \frac{B^2u^2(E + u)^2}{2}, \qquad r = u^{-1}.$$

As expected this gives one solution $u = -E$. Alternatively,

$$u^3 + Eu^2 - \frac{2}{B^2} = 0.$$

Setting $u = \left(v - \frac{E}{3}\right)$ gives the cubic

$$v^3 - \frac{E^2v}{3} + \frac{2E^3}{27} - \frac{2}{B^2} = 0,$$

working in dimensionless variables $Z = -E^3B^2$ and $v = E\lambda$ gives,

$$\lambda^3 - \frac{\lambda}{3} + 2\left(\frac{1}{27} + \frac{1}{Z}\right) = 0,$$

with discriminant

$$\Delta_3 = -\frac{4}{(27)}\left(\left(1 + \frac{27}{Z}\right)^2 - 1\right) < 0, \qquad \text{for } Z > 0.$$

In this case our cubic in λ has only one real root and a complex conjugate pair. If this gives a positive root u of original equation, motion is confined to an annulus and is quasi-elliptic. A simple computation yields, $u = u_0$, where

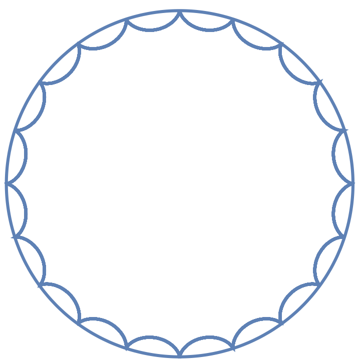
$$u_0 + \frac{E}{3}$$
$$= -\frac{E}{3}\left(1 + \frac{27}{Z}\right)^{\frac{1}{3}}\left[\left(1 + \sqrt{1 - \frac{1}{(1 + \frac{27}{Z})^2}}\right)^{\frac{1}{3}} + \left(1 - \sqrt{1 - \frac{1}{(1 + \frac{27}{Z})^2}}\right)^{\frac{1}{3}}\right],$$

when $Z = -E^3B^2 > 0$.

So $r = u^{-1} \in (u_0^{-1}, -E^{-1})$, motion being quasi elliptic. Since, when $u + E = 0$, both $\dot{r}$ and $\frac{d\theta}{du}$ are zero, the orbit must have cusps on $r = -E^{-1}$ as long as $D = 0$.

Remark.
1. To prove that $u_0^{-1} < -E^{-1}$ is a simple consequence of AM/GM inequality.
2. Orbit is quasielliptic for $Z > -\frac{27}{2}$, root of $\Delta_3 = 0$, if $u_0 > 0$, but no motion is possible for $\frac{-27}{2} < Z < 0$, as u_0 is negative here.
3. We leave the reader to compute the orbit in the case when $Z = -\frac{27}{2}$. In this case the cubic in λ has a repeated root $\lambda = -\frac{1}{3}$ and a simple root $\lambda = \frac{2}{3}$.

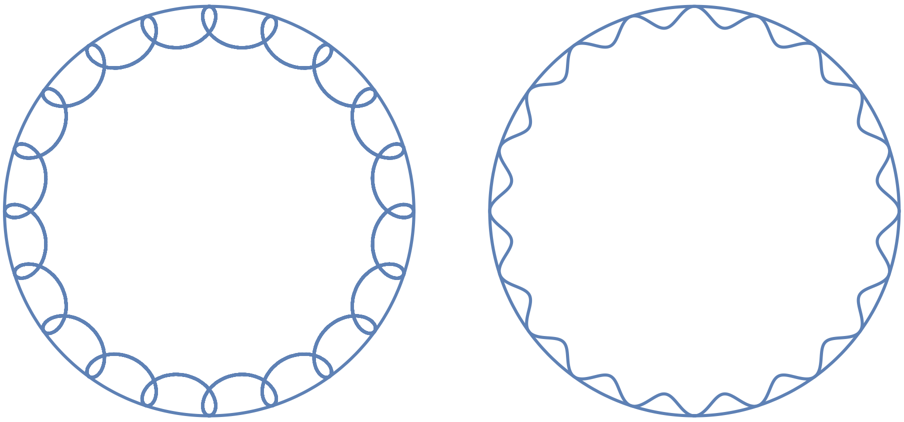
Figure 4. Cusped Motion



Example 2 ($|D| \sim 0$, $\frac{B}{D} \gtrless 0$)

Correct to first order in D the quartic Q still has the root, $u + E = 0$, where the radial motion reverses at $\dot{r} = 0$. However, the sense of rotation reverses when, $C - Bu = 0$, i.e. $u = \frac{D}{B} - E$. Since $-\frac{1}{E - \frac{D}{B}} \gtrless -\frac{1}{E}$ if $\frac{D}{B} \gtrless 0$ the motion is loopy/sinusoidal in these cases. So the motion switches between loopy and sinusoidal as we cross $W = 0$.

Figure 5. Loopy and Sinusoidal



We delay giving the full equations of the orbits in Example 1 until we present the powerful methods of Legendre in this subject, c.f.[2, Legendre]

Example 3 ($E \sim 0$, escape energy case where $D^3/B = 27/2$ and the case when $B = 0$)($\Delta_4 = 0$ singular point orbitals, at the origin 0)

The bounds for u are given by

$$u = \frac{u^2}{2}(D - Bu)^2,$$

i.e. the cubic in $u^{\frac{1}{2}}$,

$$\pm\sqrt{2} = u^{\frac{1}{2}}(D - Bu),$$

$$i.e. \qquad u^{\frac{3}{2}} - \frac{D}{B}u^{\frac{1}{2}} \pm \frac{\sqrt{2}}{B} = 0.$$

Any real root $u^{\frac{1}{2}}$ will give a positive root of our original equation. Since above is a simple cubic in $u^{\frac{1}{2}}$, with

$$p = -\frac{D}{B}, \qquad q = \pm\frac{\sqrt{2}}{B},$$

the discriminant

$$-\frac{27}{16}\Delta_3 = q^2 + \frac{4p^3}{27} = \frac{2}{B^2}\left(1 - \frac{2}{27}\frac{D^3}{B}\right),$$

so as long as $\left(1 - \frac{2}{27}\frac{D^3}{B}\right) > 0$, we have a unique positive u bound in addition to $u = 0$, and a complex conjugate pair of roots of $Q(u) = 0$. This motion is not quasi-elliptic in Whittaker's sense since it is not even bounded. We leave the reader to compute the actual u bound as in previous example. Let us pause to see what happens when $\frac{D^3}{B} = \frac{27}{2}$, the critical case when $\Delta = 0$.

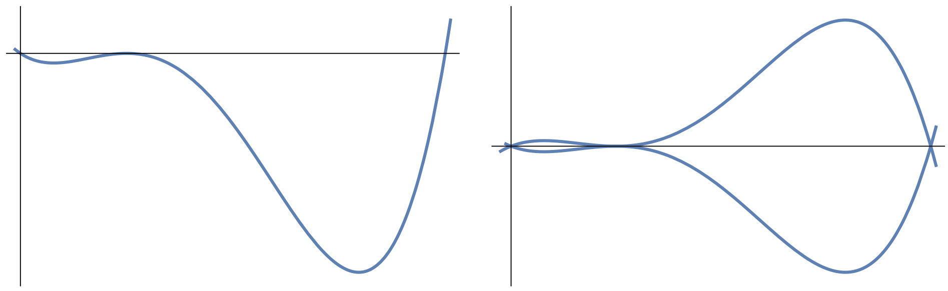
In this case it is easy to see the cubic in $\sqrt{u}$ has a repeated root at $\pm\sqrt{\frac{D}{3B}}$ and a further root at $-2\sqrt{\frac{D}{3B}}$. So the roots of u equation are at:-

$$u = 0, \quad \frac{D}{3B}, \quad \frac{D}{3B}, \quad \frac{4D}{3B}, \quad \text{the quartic Q being}$$

$$Q(u) = u - \frac{u^2}{2}\left(\left(\frac{27B}{2}\right)^{\frac{1}{3}} - Bu\right)^2, \qquad \frac{D^3}{B} = \frac{27}{2}.$$

The effective potential here for the radial motion and the corresponding quartic curve is in Fig(6).

Figure 6. Effective Potential



Evidently we have an unstable circular orbit at $r = \frac{3B}{D}$ which can be perturbed to give an unstable quasi-elliptic orbit or to shoot off to infinity. Perturbing the motion so that $\frac{D^3}{B} \sim \frac{27}{2}$ could just separate these 2 possible motions.

Recall that

$$g_3 = \frac{1}{16}\left(-EB^2C^2 + \frac{B^2}{2} - \frac{BC^3}{6} + \frac{C^6}{108}\right), \qquad g_2 = \frac{1}{2}\left(\frac{C^4}{24} - \frac{BC}{2} - EB^2\right),$$

and when $E \sim 0$,

$$16g_3 \sim \frac{B^2}{2} - \frac{BC^3}{6} + \frac{C^6}{108}, \quad 2g_2 \sim \frac{C^4}{24} - \frac{BC}{2}, \quad C = D - BE,$$

as $E \sim 0$

$$16g_3 \simeq \frac{1}{108}(C^3 - \alpha)(C^3 - \beta),$$

α, β being roots of the quadratic: $t^2/108 - Bt/6 + B^2/2 = 0$. Hence, we obtain

$$\alpha, \beta = 9B\left(1 \pm \frac{1}{\sqrt{3}}\right) = 9B \pm 3\sqrt{3}B.$$

So for $g_3 < 0$ as $E \sim 0$, we require, for $B > 0$,

$$\frac{D^3}{B} \in \left(9 - 3\sqrt{3}, 9 + 3\sqrt{3}\right).$$

The $\Delta = 0$ root as $E \sim 0$ is $\frac{D^3}{B} = \frac{27}{2} \in \left(9 - 3\sqrt{3}, 9 + 3\sqrt{3}\right)$, so in above case $g_3 < 0$ and there exists a small positive $\bar{\alpha}$ s.t. $g_2 \sim 12\tilde{\alpha}^2$, $g_3 \sim -8\tilde{\alpha}^3$ and so in our equations,

$$\wp(z; g_2, g_3) \sim \tilde{\alpha} + \frac{3\tilde{\alpha}}{\sinh^2\left(\sqrt{3\alpha}z\right)}.$$

In terms of dimensionless variables, $Z = -E^3B$, $W = E^2BD$, our algebraic plane curve $\Delta = 0$ has a singularity at origin O. So letting $E \sim 0$ and simultaneously $\Delta \sim 0$, we are approaching O on the semi-cubical parabola, $Z^2 = \frac{B}{D^3}W^3$, and it is only if $\frac{D^3}{B} \in (9 - 3\sqrt{3}, 9 + 3\sqrt{3})$ that we get the hyperbolic trigonometric behaviour above. Another way to approach the origin O in the (Z, W) plane is to let $B \sim 0$, by converging to O on the parabola, $W^2 = -ED^2Z$, $E < 0$, giving $\Delta \sim 0$ and $g_3 > 0$. In this case we obtain,

$$\wp(z; g_2, g_3) \sim -\tilde{\alpha} + \frac{3\tilde{\alpha}}{\sin^2\left(\sqrt{3\alpha}z\right)}, \quad g_2 \sim 12\tilde{\alpha}^2, \quad g_3 \sim 8\tilde{\alpha}^3.$$

The next section shows that this yields the classical Keplerian ellipses since $-ED^2 = (1-\tilde{e}^2)/2$, $\tilde{e}$ being the eccentricity, the former case asymptotic spirals, both at the origin O of (Z, W) plane; on the curve $\Delta = 0$.

Examples of Orbital Equations when $\Delta = 0$

(a) $B \sim 0$, $\quad Q(u) = E + u - D^2u^2/2$, $\quad g_3 = D^6/16.108 > 0$, $\quad g_2 = D^4/48$, $\tilde{\alpha} = D^2/24$, $\quad E < 0$.

Figure 7. Keplerian Illustrations (Parabolas)

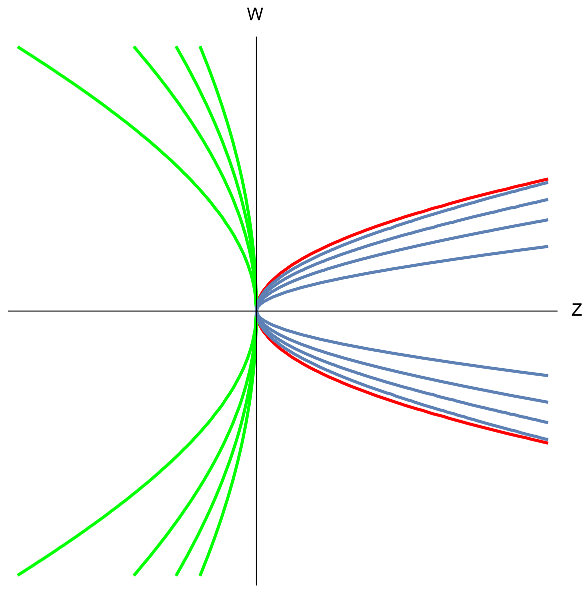
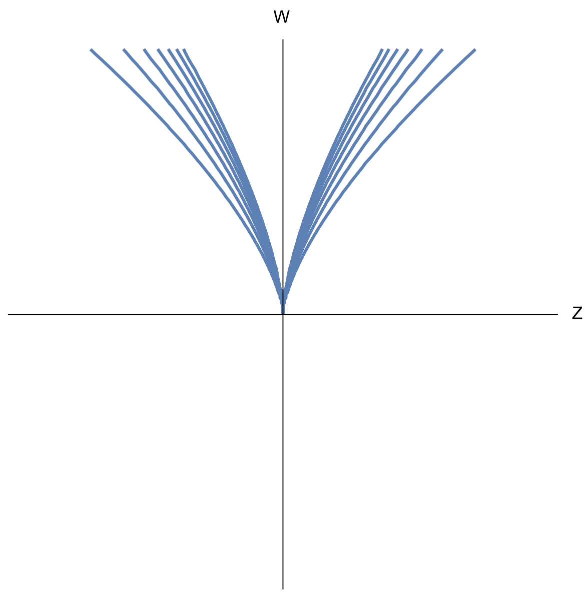


Figure 8. Detail at origin (Semi - cubical Parabolas)



Taking the limit as $B \to 0$ for fixed $D^2E = -\frac{W^2}{Z}$, on above parabolas, those on the right in Figure 7 give classical Keplerian ellipses, on the left classical hyperbolic orbits, with eccentricities, $\tilde{e} = \sqrt{\left(1 - \frac{2W^2}{Z}\right)}$. In a similar sense, letting $E \to 0$, for fixed $\frac{D^2}{B} = \frac{W^3}{Z^2}$, above semi-cubical parabolas in

Figure 8 gives zero energy orbits depending critically upon whether on not $\frac{D^3}{B} \in (9 - 3\sqrt{3}, 9 + 3\sqrt{3})$.

Weierstrass formula gives

$$\frac{1}{r} - \frac{1}{r_0} = \frac{\left(1 - \frac{D_0^2}{r_0}\right)}{4\left[\wp\left(\frac{\sqrt{2}}{D_0}(\theta - \theta_0)\right) + \frac{D_0^2}{24}\right]},$$

$$\wp(z) = -\tilde{\alpha} + \frac{3\tilde{\alpha}}{\sin^2(\sqrt{3\tilde{\alpha}}z)},$$

r_0 being the initial value of $r = 1/u, u = u_0$ being a root of $Q(u) = 0$, i.e. $r_0 = D^2/(1 \pm \tilde{e})$, $\quad \tilde{e} = \sqrt{1 + 2D^2E}$ being the eccentricity of the Keplerian ellipse. Taking $\theta_0 = 0$ and the plus sign

$$\frac{1}{r} - \frac{1}{r_0} = -\frac{2\tilde{e}\sin^2\left(\frac{\theta}{2}\right)}{D_0^2}$$

or

$$\frac{D_0^2}{r} = 1 + \tilde{e}\cos\theta, \text{ the more usual form.}$$

(b) $E \sim 0, \quad \frac{D^3}{B} = \frac{27}{2}, \quad Q(u) = u - \frac{D^2}{2}u^2 + BDu^3 - \frac{B^2}{2}u^4,$

$$g_3 = \frac{1}{16}\left(\frac{B^2}{2} - \frac{BD^3}{6} + \frac{D^6}{108}\right) < 0, \qquad g_2 = \frac{1}{2}\left(\frac{D^4}{24} - \frac{BD}{2}\right), \tilde{\alpha} = \frac{D^2}{72}.$$

Weierstrass formula gives when $E = 0$:

$$\frac{1}{r} - \frac{1}{r_0} = \frac{-9}{4\left[\frac{D^2}{72} + \frac{D^2}{24\sinh^2\left(\frac{Dz}{\sqrt{24}}\right)} + \frac{11D^2}{72}\right]},$$

where

$$z = \frac{3}{\sqrt{2D}}\left(\Delta\theta + \arcsin\left(\frac{3B}{2Dr} - 1\right) - \frac{\pi}{2}\right),$$

$$\Delta\theta = \theta - \theta_0, \quad r_0 = \frac{3B}{4D}, \quad u = \frac{1}{r} \in \left(\frac{D}{3B}, \frac{4D}{3B}\right).$$

This is a doubly asymptotic spiral converging to unstable circular orbit at $r = \frac{3B}{D}$, internally and externally.

These equations can be simplified as we show below. We first give another limiting example for $B \sim \infty$.

Example 4 (Instanton Asymptotic Spiral) ($\mu \sim 0,\ B \sim \infty$)

Orbital equation reads

$$\pm\sqrt{2}\,\frac{d\theta}{du} = \frac{(C_0 - B_0 u)}{\sqrt{E_0 - \frac{u^2}{2}(C_0 - B_0 u)^2}}.$$

Set $u = \frac{C_0 v}{B_0}$, giving

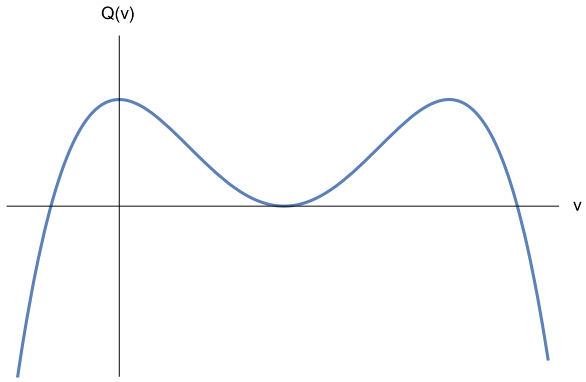
$$\pm\sqrt{2}\,\frac{d\theta}{du} = \frac{1-v}{\sqrt{\left(\tilde{E} - \frac{v^2}{2}(1-v)^2\right)}}, \qquad \tilde{E} = \frac{B_0^2 E_0}{C_0^4}.$$

Double asymptotics correspond to $\tilde{E} = \frac{1}{32}$ giving,

$$Q(v) = \frac{1}{32} - \frac{v^2}{2}(1-v)^2.$$

Q has a repeated root at $v = \frac{1}{2}$ and simple root at $v = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$. It is easy to check that $\Delta = 0$ and $g_3 = -\frac{1}{4.12^3}$.

Figure 9. $\underline{Q(v)}$



There is clearly an unstable circular orbit at $v = \frac{1}{2}$. Now, if we choose $v_0 = \frac{1}{2} + \frac{1}{\sqrt{2}}$ and $v \in \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{\sqrt{2}}\right)$,

$$\int_{v_0}^{v} \frac{(1-v)dv}{\sqrt{\tilde{E} - \frac{v^2}{2}(1-v)^2}} \sim \int_{v_0}^{v} \frac{(1-v)dv}{\sqrt{\frac{1}{32} - \frac{v^2}{2}(1-v)^2}}, \qquad as\ \tilde{E} \sim \frac{1}{32}.$$

This can be proved by calculating the asymptotics of $\tilde{E} - \frac{v^2}{2}(1-v)^2 = 0$. If $Q(v)$ has no repeated root and v_0 is a root of $Q(v) = 0$, from Weierstrass, if $z = \int_{v_0}^{v} \frac{dv}{\sqrt{Q(v)}}$,

$$v - v_0 = \frac{1}{4}\frac{Q'(v_0)}{\left[\wp(z) - \frac{Q''(v_0)}{24}\right]},$$

where $\wp(z) = \wp(z; g_2, g_3), g_2, g_3$ invariants of Q.

Now for $Q(v) = \frac{1}{32} - \frac{v^2}{2}(1-v)^2$, we write:

$$\int \frac{(1-v)dv}{\sqrt{Q(v)}} = \frac{1}{2}\int \frac{dv}{\sqrt{Q(v)}} - \sqrt{2}\int \frac{dv}{\sqrt{\frac{1}{2} - (v-\frac{1}{2})^2}}$$

However, for $g_3 < 0, \alpha = \sqrt{\frac{g_2}{12}}$ our $\wp(z)$ is given by

$$\wp(z) \sim \alpha + \frac{3\alpha}{\sinh^2 \sqrt{3\alpha} z} \quad \text{as } \Delta \sim 0$$

and if initially $\Delta\theta < 0$,

$$-2\sqrt{2}\Delta\theta + 2\sqrt{2}\Delta \arcsin\left(\sqrt{2}v - \frac{1}{\sqrt{2}}\right) = z = \int_{v_0}^{v} \frac{dv}{\sqrt{Q(v)}},$$

giving

$$v - v_0 \sim \frac{1}{4}\frac{Q'(v_0)}{\left[\alpha + \frac{3\alpha}{\sinh^2\sqrt{3\alpha}z} - \frac{Q''(v_0)}{24}\right]}, \qquad \alpha = \sqrt{\frac{g_2}{12}} = \frac{1}{48},$$

for above z and $v_0 = \frac{1}{2} + \frac{1}{\sqrt{2}}$.

Evidently, as $v \sim \frac{1}{2}, z \sim \infty$, so if we denote $v = \frac{1}{2}$ by the repeated root v_r, we must have:-

$$v_r - v_0 = \frac{1}{4}\frac{Q'(v_0)}{\left[\alpha - \frac{Q''(v_0)}{24}\right]},$$

if so, the orbital equation can be written

$$\sinh^2(\sqrt{3\alpha}z) = \frac{12\alpha}{Q'(v_0)}\frac{(v-v_0)(v_r-v_0)}{(v_r-v)},$$

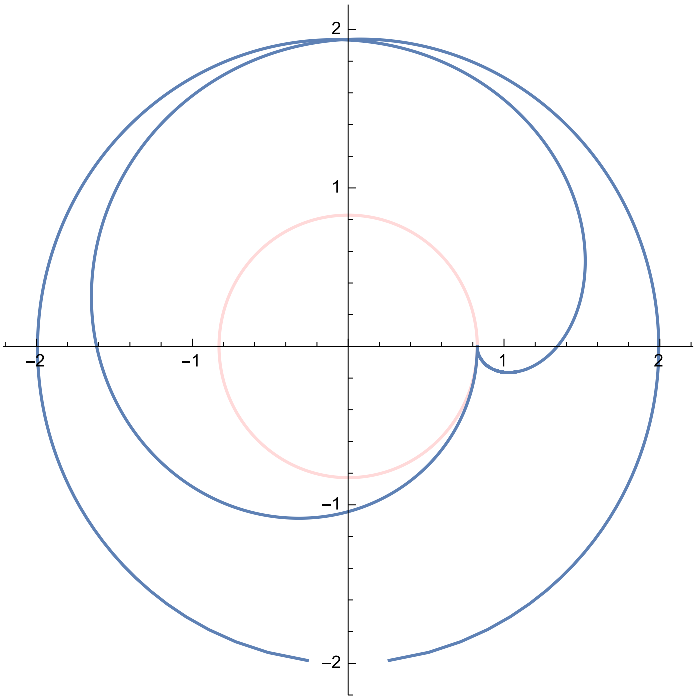
$z = -2\sqrt{2}(\theta - \theta_0) + 2\sqrt{2}\arcsin(\sqrt{2}v - \frac{1}{\sqrt{2}}) - \sqrt{2}\pi, \quad \alpha = \frac{1}{48}, \quad v = \frac{B_0 u}{C_0}.$

Above version of equation for orbital is very simple and practical in the form:-

$$z = \frac{1}{\sqrt{3\alpha}}\sinh^{-1}\left(\sqrt{\frac{12\alpha}{Q'(v_0)}\frac{(v-v_0)(v_r-v_0)}{(v_r-v)}}\right), \qquad \text{z above.}$$

Above parameters can also give an 'instanton' in the sense that there is a solution which at time $-\infty$ is at $\frac{1}{2}$ and at some time t_0 is at $\frac{1}{2} + \frac{1}{\sqrt{2}}$, returning to $\frac{1}{2}$ at time $+\infty$.

Figure 10. Asymptotic Spiral and Reflection



2.5 The Lacuna and a Quartic Symmetry

The observant reader will have noticed the gap in above arguments deriving the simplified equation of the doubly asymptotic spirals. Here we remedy this.

Lemma 3. *For any quartic polynomial $Q(u)$ with repeated root u_r and simple roots $u_i, i = 1, 2,$*

$$u_r - u_i = \frac{Q'(u_i)}{4\left[\tilde{\alpha} - \frac{Q''(u_i)}{24}\right]}, \qquad i = 1, 2,$$

where $\tilde{\alpha}$ is the repeated root of the reduced cubic:

$$4t^3 - g_2 t - g_3 = 0,$$

g_2, g_3 being the quartic invariants of Q.

Proof. By Taylor's Theorem

$$Q(u) = \frac{Q''(u_r)}{2}(u - u_r)^2 + \frac{Q'''(u_r)}{6}(u - u_r)^3 + \frac{Q^{iv}(u_r)}{24}(u - u_r)^4.$$

And in our nomenclature,

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2 = \frac{3Q''(u_r)^2}{144} = \frac{(Q''(u_r))^2}{48}$$

and $\tilde{\alpha} = \sqrt{\frac{g_2}{12}} = \frac{Q''(u_r)}{24}$.

So all we have to prove is that:-

$$u_r - u_i = \frac{6Q'(u_i)}{Q''(u_r) - Q''(u_i)}, i = 1, 2.$$

But we can assume without loss of generality that

$$Q(u) = u^2(u-\alpha)(u-\beta),$$

The quartic symmetry above is then easy to check. □

There are other quartic symmetries given in the last chapter. We conclude this section with a Puiseux series. (c.f.[10, B. Simon]). However, we note last lemma makes it easy to prove:-

Theorem 6. *When the quartic Q has a repeated root u_r and simple roots u_1, and $u_0, u_r \in (u_1, u_0)$ for $u \in (u_r, u_0)$ the equation of the orbit reads:*

$$\theta - \theta_0 = \frac{2(C - Bu_r)}{(Q''(u_r))^{\frac{1}{2}}} \sinh^{-1} \sqrt{\frac{Q''(u_r)(u_0-u)(u_0-u_r)}{2Q'(u_0)(u_r-u)}} - \arcsin\left(\frac{2u - u_0 - u_1}{u_0 - u_1}\right),$$

all roots being positive, for orbit starting at $(r, \theta) = (\frac{1}{u_0}, \theta_0)$. A similar result holds for orbit starting at $(\frac{1}{u_1}, \theta_0)$.

Above is the equation of doubly asymptotic spiral. We return to this topic in Section (3.2).

2.6 Example Asymptotics of Roots in the case $\frac{D^3}{B} = \frac{27}{2}$ as $E \sim 0$ (Puiseux Series)

$$Q(u) = E + u - \frac{u^2}{2}(D - B(E+u))^2 = 0$$

Correct to first order in E,

$$\left(u - \frac{u^2}{2}(D - Bu)^2\right) + E(1 - u^3B^2 + DBu^2) = 0,$$

second term being $E\frac{\partial Q}{\partial E}|_{E=0}$

When $\frac{D^3}{B} = \frac{27}{2}$, zeroth order term has simple roots at $u = 0$ and $u = 4D/3B$. We first find their asymptotics as $E \sim 0$ by writing $u = u^{(0)} + Eu^{(1)}$, so

$$Eu^{(1)}(1 - D^2u - 2B^2u^3 + 3DBu^2)\big|_{u=u^{(0)}} + E(1 - u^3B^2 + DBu^2)\big|_{u=u^{(0)}} = 0,$$

where first bracketed term is $\frac{\partial}{\partial u}(u - \frac{u^2}{2}(D - Bu)^2)$, we get, when $u^{(0)} = 0$,

$$Eu^{(1)} + E = 0, \qquad u^{(0)} + Eu^{(1)} = -E.$$

When $u^{(0)} = 4D/3B$, we get

$$u^{(1)} = -\left(1 - \frac{4}{3}\frac{D^2}{B}\right) \Big/ \left(1 - \frac{20}{27}\frac{D^3}{B}\right)$$

$$u^{(0)} + Eu^{(1)} = \frac{4D}{3B} - E\left(1 - \frac{4}{3}\frac{D^3}{B}\right) \Big/ \left(1 - \frac{20}{27}\frac{D^3}{B}\right)$$

It is easy to see that the sum of roots of quartic is $(2D/B - 2E)$ and the product $(-2E/B^2)$. An easy calculation now yields the leading term in the Puiseux series for the repeated root as,

$$u \cong \frac{D}{3B} \pm \frac{3}{\sqrt{2}D}(-E)^{\frac{1}{2}} + \frac{4E}{9}$$

as $E \sim 0$, c.f.[10, Reed & Simon]

3 Symmetries and Discriminants

3.1 Golden Ratios and 5-fold symmetry

We begin with a simple lemma, c.f. [11, 12, 13].

Lemma 4. *For the graph of any quartic polynomial f, $\quad y = f(x)$, with 2 real simple points of inflexion, Q and R, if the secant line through Q and R cuts the graph at P, Q, R, S; $\quad P(x_P, y_P), Q(x_Q, y_Q), R(x_R, y_R), S(x_S, y_S)$, with abscissae $x's$ arranged in ascending order, then*

$$\frac{(x_R - x_Q)}{(x_Q - x_P)} = g = \frac{(x_R - x_Q)}{(x_S - x_R)}, \qquad g \textit{ the golden ratio } \frac{(1 + \sqrt{5})}{2}.$$

Proof. Without loss of generality, we can assume $f''(x) = x(x - 1)$, so that

$$f(x) = \frac{x^4}{12} - \frac{x^3}{6} + ax + b, \text{ for real constants } a \text{ and } b.$$

$Q(0,b), R(1, -\frac{1}{12}+a+b)$, slope of secant line $= (-\frac{1}{12}+a)$ so comparing equation of secant line where it cuts the quartic, we obtain

$$\frac{x^4}{12} - \frac{x^3}{6} + ax + b = b + x(a - \frac{1}{12}) \qquad \text{i.e. } \frac{x^4}{12} - \frac{x^3}{6} + \frac{x}{12} = 0.$$

i.e. $x(x-1)(\frac{x^2}{12} - \frac{x}{12} - \frac{1}{12}) = 0$ i.e. $x = 0, 1, \frac{1\pm\sqrt{5}}{2}$. $\square$

Corollary 3. *Let the area bounded by $y = f(x)$, and segment QR of secant line be denoted $A(QR)$ etc. Then*

$$\frac{A(QR)}{A(PQ)} = \frac{A(QR)}{A(RS)} = 2.$$

Proof. From above, $A(QR) = \int_0^1 \left(\frac{x^4}{12} - \frac{x^3}{6} + \frac{x}{12}\right) dx = \frac{1}{60}$, so we have to prove

$$-\frac{1}{120} = \int_{\frac{1}{2}-\frac{\sqrt{5}}{2}}^{0} = \int_{1}^{\frac{1}{2}+\frac{\sqrt{5}}{2}}$$

which reduces to

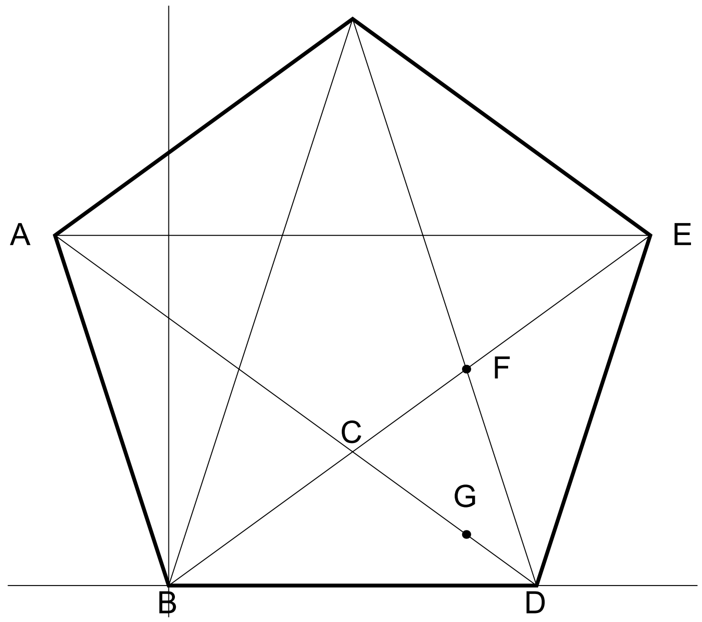
$$\frac{g^5}{60} - \frac{g^4}{24} + \frac{g^2}{60} - \frac{1}{120} = 0 \text{ etc.}$$

This is a simple exercise. $\square$

If we try to use above formulation to obtain pentagonal symmetry of quartics we have to work with pentagons of side length $\sqrt{3}$. It is simpler to use regular pentagons with unit side length. This makes g gymnastics a lot easier. To this end we consider

$$y = f(v) = \mathcal{A}v^2(v-1)^2, \quad \text{for a constant } \mathcal{A} > 0.$$

Figure 11. Pentagonal Symmetries



3.2 Full Pentagonal Symmetry

To get full pentagonal symmetry for our quartic f we have to choose $\mathcal{A} = 8\tan(\pi/5) = 8.5^{\frac{1}{4}}g^{\frac{-3}{2}}$. This is the value of $\mathcal{A}$ which ensures that not only does f have 2 local minima, one at B, and one at D but also a local maximum at C i.e. $f(\frac{1}{2}) = \frac{1}{2}\tan\frac{\pi}{5}$. However, it also ensures that $y = f(v)$ passes through A and E as we see when we consider the intersection of $y = \tan\left(\frac{\pi}{5}\right)v$ with $y = \mathcal{A}v^2(v-1)^2$
i.e. $v = 0$, or $\tan\pi/5 = \mathcal{A}v(v^2 - 2v + 1)$
i.e. $\frac{1}{8} = v^3 - 2v^2 + v$.

Since $v_E = \frac{(1+g)}{2}$ all we have to prove is that:

$$\frac{1}{8} = \frac{1}{8}(g+1)^3 - \frac{2}{4}(g+1)^2 + \frac{g+1}{2},$$

which follows from $g^2 = g+1$. More is true: let F be the point of intersection of the 2 diameters of pentagon passing through D and G, the point of intersection of DA with the quartic $y = 8.5^{\frac{1}{4}}g^{\frac{-3}{2}}v^2(v-1)^2$, then $v_F = v_G = g/2$. This is a simple result of g gymnastics which yields the coordinates:

$$F\left(\frac{g}{2}, \frac{5^{\frac{1}{4}}g^{\frac{-1}{2}}}{2}\right) \text{ and } G\left(\frac{g}{2}, 5^{\frac{1}{4}}g^{\frac{-3}{2}}\left(1 - \frac{g}{2}\right)\right).$$

The infinite B limit of our $Q(u)$ quartic ($u = 1/r$) described in section 1 has full pentagonal symmetry in the sense that for the radial kinetic energy $\frac{\dot{r}^2}{2}$ satisfies:

Lemma 5. *($B \sim \infty$ limit)*

$$\frac{\mathcal{A}B_0^2}{C_0^4}\left(E_0 - \frac{\dot{r}^2}{2}\right) = f(v) = \mathcal{A}v^2(v-1)^2,$$

where $v = C_0 u/B_0$, for $\mathcal{A} = 8.5^{\frac{1}{4}} g^{\frac{-3}{2}}$, g the golden ratio.

What is so striking in the above is the first result that the secant line of the points of inflexion has such a simple relationship with g the golden ratio. The points of inflexion have a very important role in the study of stability of circular orbits as we see next.

3.3 Stability of Circular Orbits

Recall that if our KLMN equation gives rise to a circular orbit with radius $r = r_0$ then $u = u_0 = \frac{1}{r_0}$ must be a repeated root of quartic,

$$Q(u) = E + u - \frac{u^2}{2}(C - Bu)^2,$$

where for simplicity we assume for now that $E < 0$. Setting $u = Cv/B$ gives

$$\frac{B^2E}{C^4} + \frac{Bv}{C^3} = \frac{v^2}{2}(v-1)^2.$$

So we are interested in when the straight line: $y = B^2E/C^4 + Bv/C^3$, coincides with the tangent to, $y = v^2(v-1)^2/2$ at $v = v_r$ i.e. when

$$y - \frac{v_r^2}{2}(v_r - 1)^2 = m(v - v_r), \quad m = \frac{d}{dv}(v^2(v-1)^2)\Big|_{v=v_r}$$

coincides with $y = B^2E/C^4 + Bv/C^3$ i.e.

$$\frac{B}{C^3} = v_r(v_r - 1)(2v_r - 1), \tag{3.1}$$

$$\frac{B^2E}{C^4} = -\frac{3v_r^2}{2}(v_r - \frac{1}{3})(v_r - 1). \tag{3.2}$$

Now the point of inflexion of our graph of the double well occurs when

$$\frac{d}{dv}(v(2v^2 - 3v + 1)) = 0 \quad \text{i.e} \quad v = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right).$$

From a simple sketch, for a stable circular orbit for slight increases in energy E with radius $r_0 = u_0^{-1}, u_0 = Cv_0/B$, we want a solution of (4) and (5) e.g. with $v_0 \in \left(0, \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right)$, for $E < 0$, with $E \to E + \delta E$, ensuring that our straight line meets double well graph at v points close to v_0 with

$$\frac{B^2E}{C^4} + \frac{Bv}{C^3} \geq \frac{v^2}{2}(v-1)^2.$$

Overleaf we find the corresponding E values. For now we work in dimensionless variables $Z = -E^3B^2$ and $W = E^2BD$. Above values of B/C^3 and E correspond to Z, W variables satisfying $\Delta_4(Z, W) = 0$. Evidently,

$$\frac{B}{C^3} = \frac{Z^2}{(Z+W)^3}, \qquad \frac{B^2E}{C^4} = -\frac{Z^3}{(Z+W)^4}.$$

The physics changes at singularities where $g_3 = 0$ and $\Delta_4(Z, W) = 0$ i.e. where $g_2 = 0$ and $\Delta_4(Z, W) = 0$. However, we know that

$$g_2 = \frac{1}{2}\left(\frac{C^4}{24} - \frac{BC}{2} - EB^2\right), \qquad C = D - EB,$$

so $g_2 = 0$ if $C = 0$ or $\left(1 - \frac{12B}{C^3} - 24\frac{EB^2}{C^4}\right) = 0$ i.e. $W + Z = 0$ or

$$1 - \frac{12Z^2}{(W+Z)^3} - \frac{24Z^3}{(W+Z)^4} = 0.$$

In first case $W + Z = 0, \Delta = 0 \implies Z = W = 0$ a singularity we have already encountered. The second case reduces to

$$3v_r^4 - 6v_r^3 + 4v_r^2 - v_r + \frac{1}{2} = 0,$$

or

$$\left(v_r - \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)\right)^2 \left(v_r - \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right)^2 = 0.$$

So we have proved the lemma:

Lemma 6. $\Delta = 0, g_3 = 0$ *corresponds to* $v_r = 0$, *or* $v_r = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right)$, *the* v *values at points of inflexion of double well graph, or of quartic* Q.

These are the physical singularities on $\Delta(Z, W) = 0$ as Weierstrass's result shows. We have the $((Z, W)$ map:

$$\text{Dividing (3.1) and (3.2) gives:} \quad \frac{Z}{W+Z} = \frac{v_r(3v_r - 1)}{2(2v_r - 1)}$$

Multiplying (3.1) and (3.2) gives: $Z^2 = \dfrac{v_r^4(3v_r-1)^6}{2^6(2v_r-1)^8(v_r-1)^2}$

These lead to:

$$\begin{aligned} Z &= \pm\frac{v_r^2}{8}\frac{(3v_r-1)^3}{(2v_r-1)^4(v_r-1)}, \\ W &= \pm\frac{3}{8}\frac{v_r\left(v_r-\frac{2}{3}\right)(3v_r-1)^2}{(2v_r-1)^4}, \\ \frac{W}{Z} &= \frac{3(1-v_r)\left(\frac{2}{3}-v_r\right)}{v_r(1-3v_r)}. \end{aligned}$$

If we define θ by $Z = r\cos\theta$, $\quad W = r\sin\theta$, the above corresponds to the points of inflexion and it is easy to prove that $\sin(2\theta) = \frac{1}{7}$. So we obtain:

Lemma 7. *The physical singularities are on the radius vectors in (Z, W) plane determined by $\sin(2\theta) = \frac{1}{7}$ and the corresponding Z and W are given by:*

$$Z = \pm\frac{9}{32}\left(15 \pm \frac{26}{\sqrt{3}}\right), \qquad W = \pm\frac{9}{32}\left(1 \pm \frac{2}{\sqrt{3}}\right).$$

These are crunodes on $\Delta = 0$.

Now returning to (3.1) and (3.2) we have:

Theorem 7. *For $B \neq 0$, the condition for the 2 polynomials in (3.1) and (3.2) to have a common root reduces to the cubic:*

$$\Delta_4(E) = 128B^3E^3 + (4BC^4 - 192B^2C)E^2 + (60BC^2 - 4C^5)E - 27B + 2C^3 = 0,$$

which is equivalent to the cubic:

$$t^3 + pt + q = 0,$$

$$\textit{where}\, p = \frac{-9C^2}{32B^2} - \frac{C^8}{3(32)^2B^4}, \qquad q = \frac{2}{27(32)^2}\frac{C^{12}}{B^6} - \frac{5}{(32)^2}\frac{C^6}{B^4} - \frac{27}{128B^2}.$$

Proof. The first result is obtained at the expense of evaluating the resultant of the two polynomials in (4) and (5), a (7×7) determinant. For the polynomial

$$ax^3 + bx^2 + cx + d = 0,$$

$$\text{setting} \quad x = t - \frac{b}{3a} \text{ gives } \quad p = \frac{3ac - b^2}{3a^2}, \quad q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}.$$

Simple algebra then gives desired result. Alternatively one can simply complete the calculation at the beginning of Section 1.5. $\square$

The last result shows there is always a real E value for which (4) and (5) have a common root. In the next section we investigate the applications to the stability of circular orbits. Such orbits will have radius $r_0 = u_0^{-1}, u_0 = Cv_0/B$, where

$$v_0^2 - 2v_0\left(\frac{1}{6} - \frac{2BE}{3C}\right) - \frac{2BE}{3C} = 0.$$

Lemma 8. *(Local Stability)*

If at v_0, $\frac{d^2}{dv^2}\left(\frac{v^2}{2}(v-1)^2\right) > 0$ *and if* $\left(\delta\left(\frac{B^2E}{C^4}\right) + \delta\left(\frac{B}{C^2}\right)v_0\right) > 0$, *the circular orbit with radius* Bv_0/C *is locally stable.*

Proof. $\frac{d^2}{dv^2}\left(\frac{v^2}{2}(v-1)^2\right)\Big|_{v=v_0} > 0$ guarantees that the double-well curve is upwardly convex at v_0, so all we need is that for $m = \frac{BE}{C}$ and $c = \frac{B^2E}{C^4}$ in a neighbourhood of v_0:

$$mv + c - \frac{v^2}{2}(v-1)^2 > 0, \quad \text{so that } \dot{r}^2 > 0,$$

for sufficiently small δm and δc. This is guaranteed, if

$$(m_0 + \delta m)(v_0 + \delta v) + c_0 + \delta c - \delta v\frac{d}{dv}\left(\frac{v^2}{2}(1-v)^2\right)\Big|_{v=v_0} > 0$$

for sufficiently small δv.

But $m_0 = v(v-1)(2v-1)|_{v=v_0}$ reduces the last inequality to above condition. □

Exercise

Check that at $C = 1, \quad B = 1/6\sqrt{3}, \quad q^2 + \frac{4p^3}{27} = 0.$

i.e. above identity is satisfied when

$$p = -\frac{9(108)}{32} - \frac{(108)^2}{3(32)^2},$$

and

$$q = \frac{2(108)^2}{27(32)^3} - \frac{5(108)^2}{(32)^2} - \frac{27}{128}\cdot 108.$$

We conclude with the energy values computed from above in different cases arising. We begin with the roots of the cubic in t.

3.3.1 3 real roots $(4p^3 + 27q^2 < 0)$

$$t_0 = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right)\right) = E(p,q), \quad t_2 = -E(p,-q),$$
$$t_1(p,q) = -t_0(p,q) - t_2(p,q), \quad t_0 \geq t_1 \geq t_2.$$

3.3.2 1 real root $(4p^3 + 27q^2 > 0)$

$$p < 0, \quad t_0 = -\frac{2|q|}{q}\sqrt{\frac{-p}{3}}\cosh\left(\frac{1}{3}\cosh^{-1}\left(\frac{-3|q|}{2p}\sqrt{\frac{-3}{p}}\right)\right),$$
$$p > 0, \quad t_0 = -2\sqrt{\frac{p}{3}}\sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{3q}{2p}\sqrt{\frac{3}{p}}\right)\right).$$

In both of above cases, needless to say, for corresponding E values $E_0E_1E_2 = (27B - 2C^3)/128B^3$, as we shall see.

3.4 Weird Symmetry of Double Well Potential

We begin with a simple but striking result about the number of possible energies for circular motion. We assume $B \neq 0$ and $C \neq 0$,

Theorem 8. *For the E-equation above, the corresponding depressed cubic reads:*

$$t^3 + pt + q = 0,$$

where

$$p = \frac{-C^8}{2^6B^4}\left(\frac{1}{48} + 18\lambda\right), \qquad q = \frac{C^{12}}{2^9B^6}\left(\frac{1}{864} - \frac{5\lambda}{2} - 108\lambda^2\right),$$

with $\lambda = C^{-6}B^2$, and

$$q^2 + \frac{4p^3}{27} = \frac{1}{1082^{18}B^{10}}(108B^2 - C^6)^3,$$

so the discriminant of the E-equation reduces to

$$\Delta(\Delta_E) = -3276B^5(108B^2 - C^6)^3.$$

Proof. Simple algebraic computations in above polynomials in λ. $\square$

The possible energies for a given circular motion $\Delta_E > 0$ are E_k, where

$$t_k = E_k + \frac{(4B^2C - BC^4)}{8}, \quad k = 0, 1, 2,$$

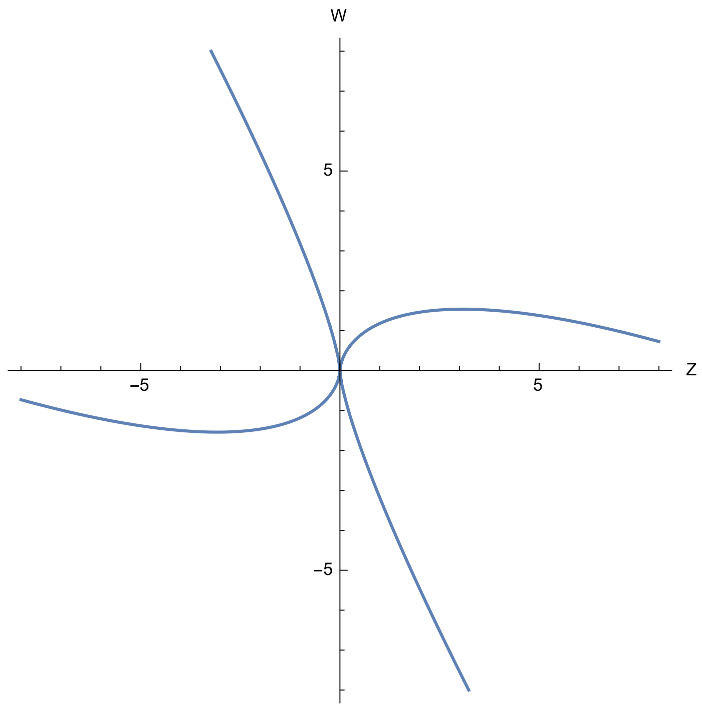
with a similar formula when $\Delta_E \geq 0$.

The simplicity of the condition for $\Delta_E \gtrless 0$, the sign of the above perfect cube, is striking. As we shall see, there are other perfect cubes in this model. We refer to this fact as weird symmetry. The most likely explanation is the double well symmetry discussed below.

Lemma 9. *The unique energy curve,* $108B^2 = C^3$ *i.e.* $\frac{Z^4}{(Z+W)^6} = \frac{1}{108}$ *has the symmetry* $(Z, W) \to (-Z, -W)$ *and is such that as* $Z \sim 0, W(Z) = 0(Z^{\frac{2}{3}})$. *The behaviour near the origin* O *is given by the semicubical parabolas,*

$$W^3 = \pm 6\sqrt{3}Z^2$$

Figure 12. Energy Bifurcation Curve



Recall:

Theorem 3

The polar equation of the plane algebraic curve, $\Delta_4(Z, W) = 0$, can be written in the form

$$r = \cos^2\theta \left\{ \frac{(\sin\theta - \cos\theta)(1 + 17\sin 2\theta) \pm (1 - 7\sin 2\theta)^{\frac{3}{2}}}{2\sin(2\theta)(1 + \sin 2\theta)^2} \right\}.$$

Proof. In polar coordinates the equation of the curve, $\Delta_4(Z,W)=0$, reduces to the quadratic,

$$Ar^2+Br+C=0,$$

$A=4\sin\theta(\cos\theta+\sin\theta)^4, B=2\cos\theta(\cos\theta-\sin\theta)(\cos^2\theta+34\sin\theta\cos\theta+\sin^2\theta),\quad C=27\cos^3\theta$. The condition for the discriminant to be zero reduces to $\sin 2\theta=\frac{1}{7}$, or $\cos\theta=0$. In point of fact,

$$\Delta(\Delta_4(r))=2(1-7\sin 2\theta)^3\cos^2\theta,$$

revealing another perfect cube. $\square$

This explains the physical significance of the radius vectors on which $\sin 2\theta=\frac{1}{7}$.

Corollary 4. *In polar coordinates the curves,* $\Delta_4(Z,W)=0$ *and* $\Delta_E(Z,W)=0$ *meet where* $\sin(2\theta)=\frac{1}{7}$ *and*

$$r=\frac{7^{\frac{3}{2}}3^{\frac{3}{2}}}{2^{\frac{7}{2}}}\cos^2\theta=\frac{3^{\frac{3}{2}}7^{\frac{1}{2}}}{2^{\frac{9}{2}}}(7\pm\sqrt{48}).$$

$$i.e\qquad (Z,W)=\mp\frac{9}{32}\left(15\pm\frac{26}{\sqrt{3}},1\pm\frac{2}{\sqrt{3}}\right),$$

two of the crunodes on $\Delta=0$*, a third being at the origin* O*.*

Further the other possible intersections are given by the quartic in $S=\sin(2\theta)$*,*

$$\begin{aligned}
f(S)&=(\mathcal{C}(S))^2-12^3(1-S^2)S^2(1+17S)^2,\ \textit{where}\\
f(S)&=f''\frac{(1/7)}{2}+f'''\frac{(1/7)}{6}(S-1/7)+f^{iv}\frac{(1/7)}{24}(S-1/7)^2\\
&\quad+f^{v}\frac{(1/7)}{120}(S-1/7)^3+f^{vi}\frac{(1/7)}{720}(S-1/7)^4,\\
\mathcal{C}(S)&=143S^3+687S^2+33S+1-(1-7S)^3.
\end{aligned}$$

Proof. This involves only simple arithmetic and algebra. The point is that $f(S)=0$ has a repeated root at $S=1/7$. $\square$

As one can see, the potentially horrendous algebraic complications involved with quartics seem to disappear as if by magic in this physically important case. We attribute this to the weird double well geometry.

Corollary 5. *For definiteness assume that $B/C > 0$ and $E \neq 0$. Set $h(v) = v^2(1-v)^2/2$ and choose as putative radius for unstable circular orbit (B/Cv_r), where $v_r \in (v_0, v_1)$, $0 < v_0 < v_r < v_1$, the corresponding u's being roots of $Q(u) = 0$, u_r a repeated root, u_0, u_1 simple roots. Then v_0, v_1 are roots of the quadratic,*

$$\frac{1}{2}h''(v_r) + \frac{v - v_r}{6}h'''(v_r) + \frac{(v - v_r)^2}{24}h^{\mathrm{iv}} = 0,$$

and it is necesary and sufficient that $v_r \in (\frac{\sqrt{3}-1}{2\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}})$. This unstable circular orbit will give rise to a doubly asymptotic spiral iff $v_r \in (\frac{\sqrt{3}-1}{2\sqrt{3}}, \frac{1}{2})$.

Proof. We have to find where the tangent to $y = h(v)$ at $v = v_r$ meets the curve $y = h(v)$ i.e. the roots of $H(v) = 0$, where

$$H(v) = h(v_r) + h'(v_r)(v - v_r)/6 - v^2(1-v)^2/2.$$

The latter equation has a repeated root at $v = v_r$. Taylor's theorem gives the quadratic for v_0, v_1,

$$\frac{1}{2}h''(v_r) + \frac{v - v_r}{6}h'''(v_r) + \frac{(v - v_r)^2}{24}h^{\mathrm{iv}} = 0,$$

i.e. $v - v_r = (1 - 2v_r) \pm \sqrt{2v_r(1 - v_r))}$, being real for $0 < v_r < 1$.

The product of the roots in the above has the same sign as $6(v_r^2 - v_r + 1/6)$ and

$$(v_r^2 - v_r + 1/6) = (v_r - \frac{\sqrt{3}+1}{2\sqrt{3}})(v_r - \frac{\sqrt{3}-1}{2\sqrt{3}}),$$

proving the desired result, since for $B/C > 0$ the gradient has to be positive. □

Remark. For a reversal of direction, i.e. $C/B \in (u_r, u_1)$, we would need $1 \in (v_r, 1 - v_r \pm \sqrt{2v_r(1 - v_r)})$. Since $v_r < 1/2$, by assumption, we require $1 < 1 - v_r + \sqrt{2v_r(1 - v_r)}$ or $2v_r(1 - v_r) \geq v_r^2$. This is just $0 < v_r < 2/3$ but since $\frac{\sqrt{3}-1}{2\sqrt{3}} < v_r < 1/2$, by assumption, there is a reversal of direction. We note that other cases can be treated similarly.

We now return to Theorem 5 , explaining how the above analysis enables us to findu_0, u, and u_r for our doubly asymptotic spirals, where $v = Bu/C$.

3.5 On Double Well Symmetry

The KLMN equation leads to the differential equation for the orbit in the form:

$$\frac{\dot{r}^2}{2} = E + u - \frac{u^2}{2}(C - Bu)^2, \qquad u = \frac{1}{r},$$

for real constants B, C, E; the perfect square arising from the constancy of energy and the simple constant of motion associated with axial symmetry. Needless to say the roots of r.h.s are given by the intersection of a straight line with a symmetric double well. These roots are all important in determining the nature of the complete solution to this problem.

More generally, if we consider the equation:

$$\frac{\dot{r}^2}{2} = au^4 + bu^3 + cu^2 + du + e, \qquad a < 0,$$

then above form of our quartic Q will only obtain , if $b^2 = 4ac$, which we refer to as our weird symmetry, since it drives all our algebraic simplifications. If we consider the nature of the roots, the discriminant,

$$\Delta = 256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + 144ab^2ce^2 - 6ab^2d^2e$$

$$-80abc^2de + 18abcd^3 + \cancel{16ac^4e} - \cancel{4ac^3d^2} - 27b^4e^2 + 18b^3cde - 4b^3d^3 - \cancel{4b^2c^3e} + \cancel{b^2c^2d^2}$$

and four other polynomials are all important P, Q, Δ_0, D, where

$$\frac{P}{8a^2} = \frac{(8ac - 3b^2)}{8a^2} \text{ (2^{nd} degree coefficient of depressed quartic)}$$

$$\frac{Q}{8a^3} = \frac{(b^3 + 8da^2 - 4abc)}{8a^3} \text{ (1^{st} degree coefficient of associated depressed quartic)}$$

with $\Delta_0 = (c^2 - 3bd + 12ae)$ (which is zero if quartic has a triple root) and

$$D = (64a^3c - \cancel{16a^2c^2} + \cancel{16ab^2c} - 16a^2bd - \cancel{3b^4})$$

(which is zero if quartic has 2 double roots).

<u>Exercise</u> Prove that if $\Delta_4 = 0$ a necessary and sufficient condition for all 4 roots of Q to be real is $P < 0$ and $D < 0$, i.e. $2Z - W < 0$.

In the first and last of these we have indicated cancellations which occur if our quartic has weird symmetry. It is our contention that this is all that is required for the appearance of the perfect cubes above. At the expense of repeating some heavy algebra it would not be too difficult to prove. At this stage it is more important to show that this symmetry will of course affect the quartic invariants g_2 and g_3, so our orbitals will automatically inherit it, restricting Legendre's general solution as we can see next. In any case in dimensionless variables:

$$g_2 = \frac{Z^2(Z - W)}{4} + \frac{(Z + W)^4}{48}, \qquad g_3 = \frac{(Z + W)^2 Z^2 (W - Z)}{96} - \frac{(Z + W)^6}{1728} - \frac{Z^4}{32}.$$

3.6 Legendre and Elliptic Integrals

Recall that

$$\pm\sqrt{2}\int d\theta = \int \frac{(C-Bu)}{\sqrt{Q(u)}}du,$$

where $Q(u) = E + u - \frac{C^2}{2}u^2 + BCu^3 - \frac{B^2}{2}u^4$. We assume $\Delta_4 > 0$ and $P < 0, D < 0, E < 0$.

Assume that $Q(u) = 0$ has 4 roots: $a < b < 0 < c < d$, respectively, arranged in ascending order. So there is no interlacing of zeros of

$$q_1 = x^2 - (a+b)x + ab, \qquad q_2 = x^2 - (c+d)x + cd,$$

i.e. $(b-d)(a-c)(b-c)(a-d) > 0$.

Write $(q_1q_2)(x) = q_4(x)$, where $Q(x) = -\frac{B^2}{2}q_4(x)$,

$$q_1 + \lambda q_2 = (1+\lambda)x^2 - (a+b+\lambda(c+d))x + ab + \lambda cd,$$

which is a perfect square if

$$(a+b+\lambda(c+d))^2 = 4(1+\lambda)(ab+\lambda cd)$$

i.e. if Λ equation is satisfied

$$\lambda^2(c-d)^2 + 2\lambda((a+b)(c+d) - 2(ab+cd)) + (a-b)^2 = 0$$

i.e.

$$A\lambda^2 + 2B\lambda + C = 0.$$

The latter equation has 2 real roots λ, μ if $B^2 - AC > 0$ which simplifies to:-

$$(c-a)(d-b)(d-a)(c-b) > 0$$

i.e. there is no interlacing of roots of q_1 and q_2. So in our case λ, μ are real and distinct, with

$$q_1 + \lambda q_2 = S_1, \qquad q_1 + \mu q_2 = S_2,$$

S_1, S_2 being the perfect squares:

$$S_1 = (1+\lambda)\left(x - \frac{1}{2(1+\lambda)}(a+b+\lambda(c+d))\right)^2,$$

$$S_2 = (1+\mu)\left(x - \frac{1}{2(1+\mu)}(a+b+\mu(c+d))\right)^2.$$

It follows that

$$q_1 = \frac{(\mu S_1 - \lambda S_2)}{(\mu - \lambda)}, \qquad q_2 = \frac{(S_1 - S_2)}{(\lambda - \mu)}.$$

Set $\alpha = \dfrac{a+b+\lambda(c+d)}{2(1+\lambda)}$, $\beta = \dfrac{a+b+\mu(c+d)}{2(1+\mu)}$ and $t = \dfrac{(x-\alpha)}{(x-\beta)}$.

This gives for $q_4 = q_1 q_2$,

$$\int \frac{dx}{\sqrt{q_4(x)}} = \int \frac{(\alpha-\beta)^{-1}dt}{\sqrt{(A_1 t^2 + B_1)(A_2 t^2 + B_2)}},$$

where

$$A_1 = \frac{\mu(1+\lambda)}{(\mu-\lambda)}, \quad B_1 = \frac{-\lambda(1+\mu)}{(\mu-\lambda)}; \quad A_2 = -\frac{(1+\lambda)}{(\mu-\lambda)}, \quad B_2 = \frac{(1+\mu)}{(\mu-\lambda)}.$$

However,

$$t = (x-\alpha)/(x-\beta) \text{ gives } x = (t\beta - \alpha)/(t-1).$$

Defining

$$x_e(t) = \frac{x(t) + x(-t)}{2}, \qquad x_o(t) = \frac{x(t) - x(-t)}{2},$$

even and odd parts,

$$x_e(t) = \frac{(t^2\beta - \alpha)}{(t^2-1)}, \qquad x_o(t) = \frac{(\beta-\alpha)t}{(t^2-1)},$$

evidently,

$$x(t) = \frac{(t^2\beta - \alpha)}{(t^2-1)} + \frac{(\beta-\alpha)t}{(t^2-1)}.$$

We now separate these terms in evaluating: $\int \frac{(C-Bx)dx}{\sqrt{(Q(x))}}$. Firstly, $\int \frac{dx}{\sqrt{q_4(x)}}$ is by definition an elliptic integral of the first kind. Secondly, we obtain

$$\int \frac{xdx}{\sqrt{q_4(x)}} = \int \frac{(t\beta - \alpha)}{(t-1)} \frac{(\alpha-\beta)^{-1}dt}{\sqrt{(A_1 t^2 + B_1)(A_2 t^2 + B_2)}},$$

where

$$\text{r.h.s.} = \int \left(\beta + \frac{(\beta-\alpha)}{(t^2-1)}\right) \frac{(\alpha-\beta)^{-1}}{\sqrt{(A_1 t^2 + B_1)(A_2 t^2 + B_2)}} dt$$
$$+ \int \frac{tdt}{(t^2-1)\sqrt{(A_1 t^2 + B_1)(A_2 t^2 + B_2)}}.$$

The substitution $v = t^2$ enables one to integrate last term in terms of simple functions giving a simple function $f(t)$ i.e. we have the decomposition:

$$\int \frac{xdx}{\sqrt{q_4(x)}} = \frac{\beta}{(\alpha-\beta)} \int \frac{dt}{\sqrt{(A_1t^2+B_1)(A_2t^2+B_2)}}$$
$$- \int \frac{dt}{(t^2-1)\sqrt{(A_1t^2+B_1)(A_2t^2+B_2)}} + f(t).$$

The second term is an elliptic integral of the third kind, first term an elliptic integral of first kind.
We need only calculate $\alpha, \beta, f(t), A_1, B_1, A_2, B_2$. In particular we need:-

$$\alpha-\beta = \frac{(\mu-\lambda)}{2(\lambda\mu+\lambda+\mu+1)}, \lambda+\mu = \frac{-2((a+b)(c+d)-2(ab+cd))}{(c-d)^2}, \lambda\mu = \frac{(a-b)^2}{(c-d)^2},$$

so $$\lambda+\mu+\lambda\mu+1 = \frac{(a+b-c-d)^2}{(c-d)^2}, \mu-\lambda = \pm\frac{2\sqrt{(b-d)(a-c)(b-c)(a-d)}}{(c-d)^2},$$

$$\alpha-\beta = \pm\frac{\sqrt{(b-d)(a-c)(b-c)(a-d)}}{(a+b-c-d)^2} \neq 0,$$

by non-interlacing condition.

It is easy now to see $A_1B_1 < 0$ and $A_2B_2 < 0$ and $A_1A_2 < 0$ and

$$(A_1t^2+B_1)(A_2t^2+B_2) = A_1A_2\left(t^2 - \left|\frac{B_1}{A_1}\right|\right)\left(t^2 - \left|\frac{B_2}{A_2}\right|\right).$$

In the case of our integral for $(\theta-\theta_0)$ we have a contribution in the form of an elliptical integral of first kind, plus,

$$\int_1 = \frac{B}{|B|\,|A_1B_1A_2B_2|}(\alpha-\beta)^{-1}\left|\frac{B_2}{A_2}\right| \int \frac{dv'}{\sqrt{(1-mv'^2)(1-v'^2)}},$$

where $t = \left|\frac{B_2}{A_2}\right| v'$ and $m = \left|\frac{A_1B_2}{B_1A_2}\right|$ and an elliptical integral of third kind:

$$\int_3 = \frac{-B}{|B|}\frac{|B_2/A_2|}{\sqrt{|A_1A_2B_1B_2|}} \int \frac{dv'}{(1-\left|\frac{B_2}{A_2}\right|v'^2)\sqrt{(1-mv'^2)(1-v'^2)}},$$

giving N and m in Legendre's notation, $N = -\left|\frac{B_2}{A_2}\right|$. It is easy to calculate $f(t)$.

We conclude here with the result of our calculation for m,

$$\pi_4 = (c-a)^2(d-b)^2(d-a)^2(c-b)^2(c-d)^2(a-b)^2,$$

$$m = \left(\sqrt{\left(1 + \frac{\sqrt{\pi_4}}{(c-d)^3(a-b)^3}\right)} - \sqrt{\left(\frac{\sqrt{\pi_4}}{(c-d)^3(a-b)^3}\right)} \right)^2 = (\surd_1 - \surd_2)^2$$

which follows from

$$\left|\frac{\lambda}{\mu}\right| = m, \qquad \lambda\mu = \frac{(a-b)^2}{(c-d)^2}, \qquad \lambda + \mu = \frac{-2((a+b)(c+d) - 2(ab+cd))}{(c-d)^2},$$

middle coeff B in Λ-equation being

$$B = \sqrt{(c-d)^2(a-b)^2 + (b-d)(a-c)(b-c)(a-d)},$$

$$\lambda = \frac{(b-a)}{(c-b)}\left(\surd_1 + \surd_2\right), \qquad \mu = \frac{(d-c)}{(b-a)}\left(\surd_1 - \surd_2\right).$$

This provides a link with the results in the previous part of the paper via the link between Δ_4 and π_4.

We conclude with a classical result embodied in the next theorem, using the nomenclature of [1, Abramowitz & Stegun]:-

Theorem 9. *The elliptic integral of third kind,*

$$\begin{aligned}\Pi(N;\phi\backslash\alpha) &= \int_0^{\phi}(1 - N\sin^2\theta)^{-1}(1 - \sin^2\alpha\sin^2\theta)^{-\frac{1}{2}}d\theta \\ &= \delta\Big[v\cot\beta + 4v\sin(2\beta)\Sigma_{s=1}^{\infty}q^{2s}\left(1 - 2q^{2s}\cos 2\beta + q^{4s}\right)^{-1} \\ &\quad -2\Sigma_{s=1}^{\infty}s^{-1}q^3\left(1 - q^{2s}\right)^{-1}\sin(2sv)\sin(2s\beta)\Big],\end{aligned}$$

where in our case $m = \sin^2\alpha, \qquad q = e^{i\pi\alpha}$ *and*

$$\begin{aligned}\delta &= \left[N(1-N)^{-1}\left(\sin^2\alpha - N\right)^{-1}\right]^{\frac{1}{2}} \\ v &= \pi F(\phi\backslash\alpha)/2K(\alpha) \\ \beta &= \pi F\left(\sin^{-1}\left(\frac{N}{\sin^2\alpha}\right)^{\frac{1}{2}}\backslash\alpha\right)\Big/ 2K(\alpha),\end{aligned}$$

with

$$\begin{aligned}K(\alpha) = K(\alpha, m) &= \int_0^{\frac{\pi}{2}}(1 - m\sin^2\theta)^{-\frac{1}{2}}d\theta, \\ F(\phi\backslash\alpha) &= \int_0^{\phi}(1 - m\sin^2\theta)^{-\frac{1}{2}}d\theta, \\ K'(\alpha, m) &= \int_0^{\frac{\pi}{2}}(1 - m'\sin^2\theta)^{-\frac{1}{2}}d\theta, \qquad m + m' = 1.\end{aligned}$$

Finally in connection with the time-changed solution, referring to [1, Abramowitz & Stegun], θ_1 Jacobi's theta function, z unphysical time in our case,

$$\theta_1(z,q) = 2q^{\frac{1}{2}}\Sigma_0^{\infty}(-1)^n q^{n(n+1)}\sin((2n+1)z),$$

nome q,

$$\wp(z) = e_1 + \frac{d^2}{dv^2}\ln[\theta_1(v,q)]\Big|_{v=\frac{\pi}{2}} - \frac{d^2}{dv^2}\ln(\theta_1(v,q)),$$

v and q below depending upon sign of Δ. (c.f.[14, McKean & Moll])

$$\underline{\Delta > 0}, \quad \underline{w_1 = w, w_2 = w + w', w_3 = w', w_2' = w' - w} \quad , \underline{\Delta < 0}$$

$\Delta > 0$	$\Delta < 0$
$e_1 > 0 \geq e_2 > e_3$, q real, $q = e^{-\pi K'/K}$	$e_2 \geq 0$, equality when $g_3 = 0$; e_1, e_3 complex conjugate pair, $q = ie^{-\pi\lvert w_2'\rvert/2w_2}$,
$\tau = w'/w$,	$\tau_2 = w_2'/2w_2$,
$q = e^{i\pi\tau}$,	$q = iq_2 = ie^{i\pi\tau_2}$,
$v = \pi z/2w$,	$v = \pi z/2w_2$,
$\zeta(z) = \frac{\eta z}{w} + \pi\frac{\theta_1'(v)}{2w\theta_1(v)}$,	$\zeta(z) = \frac{\eta_2 z}{w_2} + \frac{\pi\theta_1'(v)}{2w_2\theta_1(v)}$,
$\sigma(z) = \frac{2w}{\pi}\exp\left(\frac{\eta z^2}{2w}\right)\frac{\theta_1(v)}{\theta_1'(0)}$,	$\sigma(z) = \frac{2w_2}{\pi}\exp\left(\frac{\eta_2 z^2}{2w_2}\right)\frac{\theta_1(v)}{\theta_1'(0)}$,
$\eta = \zeta(w) = -\frac{\pi^2}{12w}\frac{\theta_1'''(0)}{\theta_1'(0)}$,	$\eta_2 = \zeta(w_2) = -\frac{\pi^2}{12w_2}\frac{\theta_1'''(0)}{\theta_1'(0)}$,
$\Delta^{\frac{1}{4}} = \frac{\pi^3}{4w^3}\theta_1'^2(0)$,	$(-\Delta)^{\frac{1}{4}} = \frac{\pi^3}{4w_2^3}\theta_1'^2(0)e^{-i\pi/4}$,

c.f.[1]

This emphasises the importance of the curve $\Delta = 0$ in the Z, W plane, especially in view of the result below in connection with Theorem 1 and the above table.

Corollary 6. *For* $\Delta > 0$

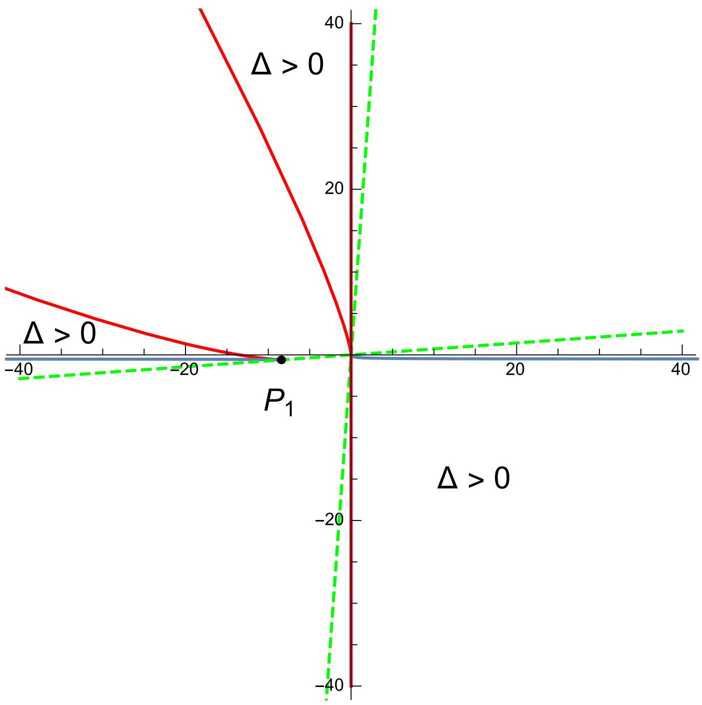
$$\frac{d^2}{dz^2}\ln\theta_1(z,q) = \frac{d^2}{dz^2}\ln\left(\Sigma_0^\infty(-1)^n|q|^{n(n+1)}\sin(\overline{2n+1}z)\right),$$

for $\Delta < 0$

$$\frac{d^2}{dz^2}\ln\theta_1(z,q) = \frac{d^2}{dz^2}\ln\left(\Sigma_0^\infty(-1)^{n(n+3)/2}|q|^{n(n+1)}\sin(\overline{2n+1}z)\right).$$

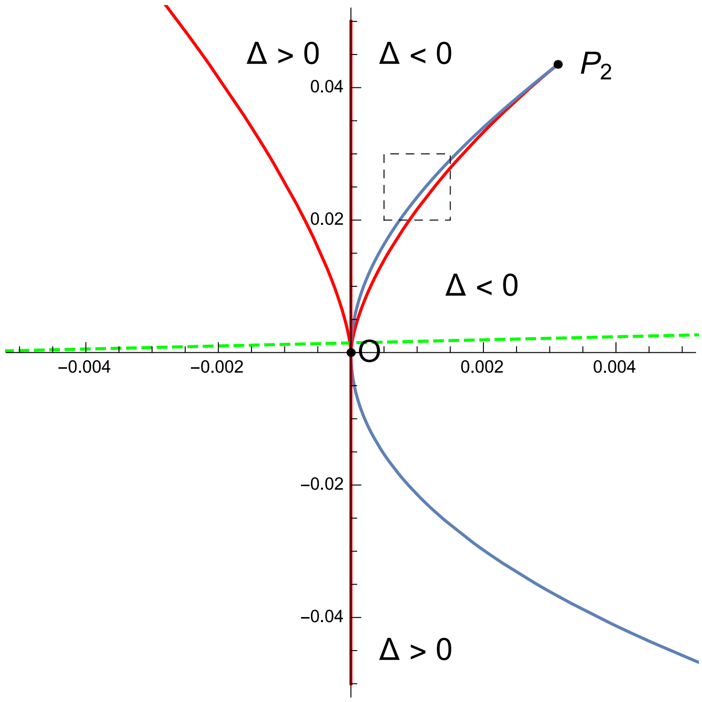
The reader should note that in the second series the signs of the coefficeients of $\sin(\overline{2n+1}z)$ alternate in pairs $+1,+1$ and then $-1,-1$; whilst in first series the signs simply alternate between $+1,-1$. This change comes about when we cross the curve $\Delta = 0$, making the diagrams below seminal in characterising the equatorial orbitals for a given Z and W. The above gives the difference in our solution as we cross the curve $\Delta = 0$ and corresponding physical changes, c.f. [9, Jacobi].

Figure 13. Discriminant plot



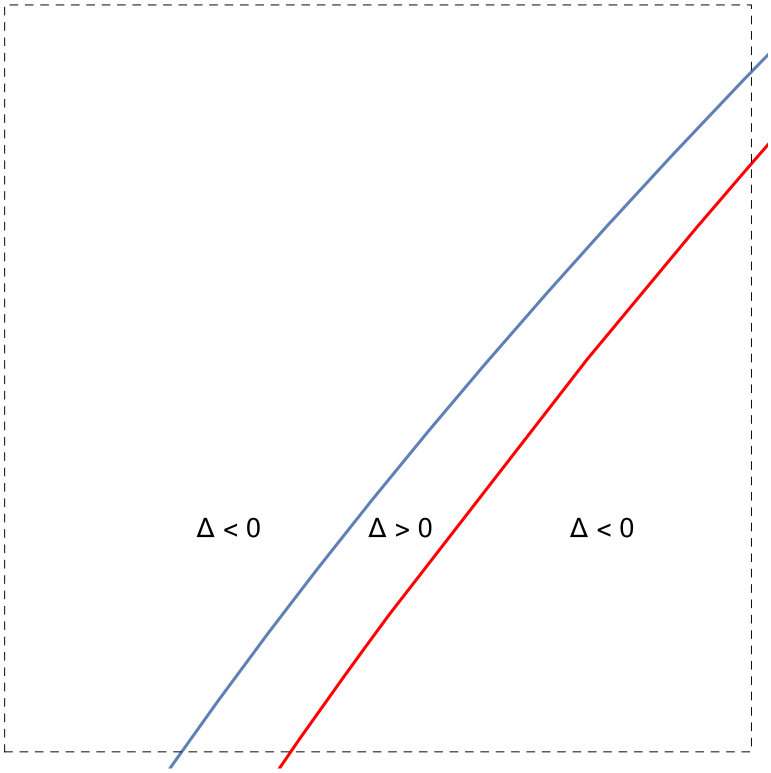
The point P_1 corresponds to $v_r = (1+1/\sqrt{3})/2$, being a cusp.

Figure 14. Discriminant plot - near origin



The point P_2 corresponds to $v_r = (1 - 1/\sqrt{3})/2$, being a cusp, and the point O corresponds to $v_r = 1/3$, being a triple cusp. Recall that $v = (1 \pm 1/\sqrt{3})/2$ corresponds to points of inflexion of our quartic $Q(u)$, $u = Cv/B$.

Figure 15. YFS stability detail



Expressing the polar equation of the discriminant curve $r(\theta)$ of Theorem 3 in the form $r(\theta) = r_0(\theta) \pm \delta r(\theta)$ allows us to approximate the transverse width of YFS as $2\delta r(\theta) r_0(\theta)/\sqrt{r_0(\theta)^2 + r_0'(\theta)^2}$.

4 Conclusion

We have investigated the KLMN equation for neutron star orbitals invoking the constancy of energy E and a new constant of the motion, C, a linear combination of angular momentum and potential energy. By so doing we have seen that the equatorial orbitals can be described by exploiting a connection with Weierstrass' and Legendre's work on elliptic functions and integrals. Modulo a time change, Weierstrass functions give a complete solution to the KLMN equation and, as we have seen, Legendre's integrals give a complete solution for the polar equation of orbit, both indicate the importance of the vanishing of the discriminant Δ_4, especially in connection with Jacobi's theta functions.

Several specific examples have been discussed herein including cusped, loopy and sinusoidal motion as well as doubly asymptotic spiral orbitals associated with unstable circular orbits. If we ignore radiation effects, the latter can persist over infinite times exhibiting quite bizarre behavour in which the apparently stable circular motion can recur after doubly infinite time.

These examples point up the importance of the dimensionless variables, $Z = -E^3B^2/\mu^4$, $W = E^2BD/\mu^3$, with $C = D - BE/\mu$, B being the dipole moment of star, μ its gravitational mass, and the discriminant of a certain quartic, $Q(u), u = r^{-1}, \Delta_4(Z, W)$, where $\dot{r}^2 = 2Q(u)$. It turns out that the quartic Q has within it the symmetry of a double well coming from the form of C, which dominates the large B behaviour of the star. In a limiting case this symmetry involves the golden ratio g. More generally, considering this double well symmetry graphically, we discussed the stability of circular orbits giving a simple criterion for local stability. This analysis leads naturally to a cubic, $\Delta_4(E) = \Delta_4(Z, W)$, for the energy E, and the striking result $\Delta(\Delta_4(E)) \propto (108B^2 - C^6)^3$.

Needless to say the discriminant of Q is one of the main ingredients in Legendre's elliptical integral expessions for the full solution of our problem. Also, our simple formula for the discriminant of the discriminant tells us that when $B = \pm C^3/6\sqrt{3}$, the number of possible energies for circular orbits changes. These are bifurcation singularities.

A similar result occurs when one considers the plane algebraic curve, $\Delta_4(Z, W) = 0$. If we take polar coordinates (r, θ) in (Z, W) plane then $\Delta_4(Z, W) = 0$ reduces to $\Delta_4(r) = 0$, a quadratic in r with discriminant, $\Delta(\Delta_4(r)) \propto \cos^2\theta(1 - 7\sin 2\theta)^3 = 0$. The Weierstrass connection then enables one to establish that physical singularities characterised by, $\Delta_4(Z, W) = g_3(Z, W) = 0, g_3$ quartic invariant of Q, can only occur at points of inflexion of Q. We have included a preliminary study of the singularity at the origin O of (Z, W) plane. We hope to give a detailed investigation of all singularities in the near future. Here we are content with the simple map above.

The appearance of another perfect cube in the last formula depends on the double well symmetry which can be traced back to the form of the constant C. This is quite intriguing, especially since C is not a constant in the quantum mechanical treatment of this problem. We refer to this as the weird symmetry in the text. We hope to give a relativistic treatment of this problem by taking the semiclassical limit of the Dirac equation. See [6, 7, Maslov] to give a physically more realistic treatment. In connection with a quantum mechanical treatment a useful reference is [15, Smolyanov et alia].

5 Dedication and Acknowledgement

This paper is dedicated to Jane, AT's late wife, who enjoyed poetry and art and was fascinated by the Golden Ratio, but she feared algebra and analysis not realising their importance in revealing truth and beauty. Herein, one of our most striking results is the emergence of the tiny crescent shaped island near the origin O of the (Z, W) plane, forming parts of the curve $\Delta = 0$, where the physics changes dramatically for the tiniest changes in the constants of the motion, Z and W. We call this singularity "Jane's Little Island" or "Ynys Fechan Siân" in Welsh (YFS). Since this corresponds to small B it should be physically observable.

It is a pleasure for AT to thank GRS who taught him to love mathematics, especially classical analysis even when it was relatively unfashionable, and to thank JCT who inspired his love of mathematical physics, without both of whom this paper could not have been written. Further, it is a pleasure to thank ADN for helpful conversations and IMD for his constant support over the last 40 years; one learns more from one's students than anyone else. Finally, a word about the passing of Sir Michael Francis Atiyah – an inspiration and hero to all of us who love the truth and beauty of mathematics. He was unique and we all miss him greatly. He was a powerful friend and ally of Swansea University.

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