On the solvability of some systems of integro-differential equations with drift

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Abstract. We establish the existence in the sense of sequences of solutions for certain systems of integro-differential equations involving the drift terms in the appropriate H^2 spaces by means of the fixed point technique when the elliptic problems contain second order differential operators with and without Fredholm property. It is proven that, under the reasonable technical conditions, the convergence in L^1 of the integral kernels implies the existence and convergence in H^2 of the solutions. We emphasize that the study of the system case is more difficult than of the scalar case and requires to overcome more cumbersome technicalities.

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1 Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. As a consequence, the equation $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. These properties of Fredholm operators are widely used in many methods of linear and nonlinear analysis.

Elliptic problems in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Lopatinskii conditions are fulfilled (see e.g. [1], [7], [11], [13]). This is the main result of the theory of linear elliptic equations. In the case of unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For instance, Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$

does not satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic equations in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions stated above, limiting operators are invertible (see [14]). In some trivial cases, limiting operators can be explicitly constructed. For instance, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinity,

$$a_{\pm} = \lim_{x \to \pm\infty} a(x), \quad b_{\pm} = \lim_{x \to \pm\infty} b(x), \quad c_{\pm} = \lim_{x \to \pm\infty} c(x),$$

the limiting operators are given by:

$$L_{\pm}u = a_{\pm}u'' + b_{\pm}u' + c_{\pm}u.$$

Since the coefficients are constants, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ fails to satisfy the Fredholm property, can be explicitly found by virtue of the Fourier transform:

$$\lambda_{\pm}(\xi) = -a_{\pm}\xi^2 + b_{\pm}i\xi + c_{\pm}, \quad \xi \in \mathbb{R}.$$

Invertibility of limiting operators is equivalent to the condition that the essential spectrum does not contain the origin.

In the case of general elliptic problems, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, these conditions may not be explicitly written.

In the case of non-Fredholm operators the usual solvability conditions may not be applicable and solvability conditions are, in general, unknown. There are some classes of operators for which solvability relations are derived. Let us illustrate them with the following example. Consider the problem

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where a is a positive constant. Such operator L coincides with its limiting operators. The homogeneous equation has a nonzero bounded solution. Hence the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can apply the Fourier transform and find the solution explicitly. Solvability conditions can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exist a solution of this equation in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [21]). Here and further down S_r^d stands for the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Hence, though the operator does not satisfy the Fredholm property, solvability relations are formulated in similarly. However, this similarity is only formal because the range of the operator is not closed.

In the case of the operator with a potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

Fourier transform is not directly applicable. Nevertheless, solvability relations in $\mathbb{R}^3$ can be derived by a rather sophisticated application of the theory of self-adjoint operators (see [19]). As before, solvability conditions are formulated in terms of orthogonality to solutions of the homogeneous adjoint equation. There are several other examples of linear elliptic non Fredholm operators for which solvability conditions can be derived (see [14], [15], [19], [20], [21]).

Solvability relations play a significant role in the analysis of nonlinear elliptic problems. In the case of non-Fredholm operators, in spite of some progress in understanding of linear equations, there exist only few examples where nonlinear non-Fredholm operators are analyzed (see [6], [16], [21], [22]).

In the present work we consider another class of stationary nonlinear systems of equations, for which the Fredholm property may not be satisfied:

$$\frac{d^2u_k}{dx^2} + b_k\frac{du_k}{dx} + a_k u_k + \int_\Omega G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy = 0, \tag{1.2}$$

with $a_k \geq 0,\ b_k \in \mathbb{R},\ b_k \neq 0,\ 1 \leq k \leq N,\ N \geq 2$ and $x \in \Omega$. Here and further down the vector function

$$u := (u_1, u_2, ..., u_N)^T \in \mathbb{R}^N. \tag{1.3}$$

For the simplicity of presentation we restrict ourselves to the one dimensional case (the multidimensional case will be treated in our forthcoming paper). Thus, Ω is a domain on the real line. Work [8] deals with a single equation anologous to system (1.2) above. In population dynamics the integro-differential equations describe models with intra-specific competition and nonlocal consumption of resources (see e.g. [2], [3]). The studies of the systems of integro- differential equations are of interest to us in the context of the complicated biological systems, where $u_k(x,t),\ k = 1, ..., N$ denote the cell densities for various groups of cells in the organism. We use the explicit form of the solvability conditions and study the existence of solutions of such nonlinear systems. We would like especially to emphasize that the solutions of the integro-differential equations with the drift term are relevant to the understanding of the emergence and propagation of patterns in the theory of speciation (see [17]). The solvability of the linear equation involving the Laplace operator with the drift term was treated in [20], see also [4]. In the case of the vanishing drift terms, namely when $b_k = 0,\ 1 \leq k \leq N$, the system analogous to (1.2) was treated in [16] and [22]. Weak solutions of the Dirichlet and Neumann problems with drift were considered in [12].

2 Formulation of the results

Our technical assumptions are analogous to the ones of the [8], adapted to the work with vector functions. It is also more complicated to work in the Sobolev spaces for vector functions, especially in the problem on the finite interval with periodic boundary conditions when the constraints are imposed on some of the components. The nonlinear parts of system (1.2) will satisfy the following regularity conditions.

Assumption 1. *Let $1 \leq k \leq N$. Functions $F_k(u, x) : \mathbb{R}^N \times \Omega \to \mathbb{R}$ are satisfying the Caratheodory condition (see [10]), such that*

$$\sqrt{\sum_{k=1}^{N} F_k^2(u, x)} \leq K|u|_{\mathbb{R}^N} + h(x) \quad \text{for} \quad u \in \mathbb{R}^N, \ x \in \Omega \tag{2.1}$$

with a constant $K > 0$ and $h(x) : \Omega \to \mathbb{R}^+$, $h(x) \in L^2(\Omega)$. Moreover, they are Lipschitz continuous functions, such that for any $u^{(1),(2)} \in \mathbb{R}^N$, $x \in \Omega$:

$$\sqrt{\sum_{k=1}^{N} (F_k(u^{(1)}, x) - F_k(u^{(2)}, x))^2} \leq L|u^{(1)} - u^{(2)}|_{\mathbb{R}^N}, \tag{2.2}$$

with a constant $L > 0$.

Here and below the norm of a vector function given by (1.3) is

$$|u|_{\mathbb{R}^N} := \sqrt{\sum_{k=1}^{N} u_k^2}.$$

Note that the solvability of a local elliptic problem in a bounded domain in $\mathbb{R}^N$ was studied in [5], where the nonlinear function was allowed to have a sublinear growth. For the purpose of the study of the existence of solutions of (1.2), we introduce the auxiliary system of equations with $1 \leq k \leq N$ as

$$-\frac{d^2 u_k}{dx^2} - b_k \frac{du_k}{dx} - a_k u_k = \int_{\Omega} G_k(x - y) F_k(v_1(y), v_2(y), ..., v_N(y), y) dy. \tag{2.3}$$

We denote $(f_1(x), f_2(x))_{L^2(\Omega)} := \int_{\Omega} f_1(x) \bar{f}_2(x) dx$, with a slight abuse of notations when these functions are not square integrable, like for example those involved in orthogonality condition (5.5) below. In the first part of the article we consider the case of the whole real line, $\Omega = \mathbb{R}$, such that the appropriate Sobolev space is equipped with the norm

$$\|\phi\|_{H^2(\mathbb{R})}^2 := \|\phi\|_{L^2(\mathbb{R})}^2 + \left\| \frac{d^2 \phi}{dx^2} \right\|_{L^2(\mathbb{R})}^2. \tag{2.4}$$

For a vector function given by (1.3), we have

$$\|u\|^2_{H^2(\mathbb{R},\mathbb{R}^N)} := \sum_{k=1}^N \|u_k\|^2_{H^2(\mathbb{R})} = \sum_{k=1}^N \left\{ \|u_k\|^2_{L^2(\mathbb{R})} + \left\| \frac{d^2 u_k}{dx^2} \right\|^2_{L^2(\mathbb{R})} \right\}. \tag{2.5}$$

Let us also use the norm

$$\|u\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} := \sum_{k=1}^N \|u_k\|^2_{L^2(\mathbb{R})}.$$

Due to Assumption 1 above, we are not allowed to consider the higher powers of the nonlinearities, than the first one, which is restrictive from the point of view of the applications. But this guarantees that our nonlinear vector function is a bounded and continuous map from $L^2(\Omega, \mathbb{R}^N)$ to $L^2(\Omega, \mathbb{R}^N)$. The main issue for the problem above is that in the absence of the drift terms we were dealing with the self-adjoint, non Fredholm operators

$$-\frac{d^2}{dx^2} - a_k : H^2(\mathbb{R}) \to L^2(\mathbb{R}), \ a_k \geq 0,$$

which was the obstacle to solve our system. The similar situations but in linear problems, both self- adjoint and non self-adjoint involving non Fredholm differential operators have been treated extensively in recent years (see [14], [15], [19], [20], [21]). However, the situation is different when the constants in the drift terms $b_k \neq 0$. For $1 \leq k \leq N$, the operators

$$L_{a,\ b,\ k} := -\frac{d^2}{dx^2} - b_k \frac{d}{dx} - a_k : \quad H^2(\mathbb{R}) \to L^2(\mathbb{R}) \tag{2.6}$$

with $a_k \geq 0$ and $b_k \in \mathbb{R}, \ b_k \neq 0$ involved in the left side of system (2.3) are non-selfadjoint. By virtue of the standard Fourier transform, it can be easily verified that the essential spectra of the operators $L_{a,\ b,\ k}$ are given by

$$\lambda_{a,\ b,\ k}(p) = p^2 - a_k - ib_k p, \quad p \in \mathbb{R}.$$

Evidently, when $a_k > 0$ the operators $L_{a,\ b,\ k}$ are Fredholm, since their essential spectra stay away from the origin. But for $a_k = 0$ our operators $L_{a,\ b,\ k}$ do not satisfy the Fredholm property since the origin belongs to their essential spectra. We manage to prove that under the reasonable technical assumptions system (2.3) defines a map $T_{a,b} : H^2(\mathbb{R}, \mathbb{R}^N) \to H^2(\mathbb{R}, \mathbb{R}^N)$, which is a strict contraction.

Theorem 1. *Let $\Omega = \mathbb{R}, \ N \geq 2, \ 1 \leq l \leq N-1, \ 1 \leq k \leq N, \ b_k \in \mathbb{R}, \ b_k \neq 0$ and $G_k(x) : \mathbb{R} \to \mathbb{R}, \ G_k(x) \in L^1(\mathbb{R})$ and Assumption 1 holds.*

I) Let $a_k > 0$ for $1 \leq k \leq l$.

II) Let $a_k = 0$ for $l+1 \leq k \leq N$, in addition $xG_k(x) \in L^1(\mathbb{R})$, orthogonality relations (5.5) hold and $2\sqrt{\pi} N_{a,\ b} L < 1$ with $N_{a,\ b}$ defined by (5.4) below. Then the map $v \mapsto T_{a,b} v = u$ on $H^2(\mathbb{R}, \mathbb{R}^N)$ defined by system (2.3) has a unique fixed point $v^{(a,b)}$, which is the only solution of the system of equations (1.2) in $H^2(\mathbb{R}, \mathbb{R}^N)$.

The fixed point $v^{(a,b)}$ is nontrivial provided that for some $1 \le k \le N$ the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0,x)} \cap supp\widehat{G_k}$ is a set of nonzero Lebesgue measure in $\mathbb{R}$.

Note that in the case I) of the theorem above, when $a_k > 0$, as distinct from part I) of Assumption 2 of [22], the orthogonality relations are not needed. Related to system (1.2) on the real line, we consider the sequence of approximate systems of equations with $m \in \mathbb{N},\ 1 \le k \le N,\ a_k \ge 0,\ b_k \in \mathbb{R},\ b_k \ne 0$, namely

$$\frac{d^2 u_k^{(m)}}{dx^2} + b_k \frac{du_k^{(m)}}{dx} + a_k u_k^{(m)} + \int_{-\infty}^{\infty} G_{k,m}(x-y) F_k(u_1^{(m)}(y), u_2^{(m)}(y), ..., u_N^{(m)}(y), y) dy = 0. \quad (2.7)$$

Each sequence of kernels $\{G_{k,m}(x)\}_{m=1}^{\infty}$ converges to $G_k(x)$ as $m \to \infty$ in the appropriate function spaces discussed further down. We will prove that, under the certain technical assumptions, each of systems (2.7) admits a unique solution $u^{(m)}(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$, limiting system (1.2) has a unique solution $u(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$, and $u^{(m)}(x) \to u(x)$ in $H^2(\mathbb{R}, \mathbb{R}^N)$ as $m \to \infty$, which is the so-called *existence of solutions in the sense of sequences*. In this case, the solvability conditions can be formulated for the iterated kernels $G_{k,m}$. They imply the convergence of the kernels in terms of the Fourier transforms (see the Appendix) and, as a consequence, the convergence of the solutions (Theorems 2, 4). Similar ideas in the sense of standard Schrödinger type operators were exploited in [18]. Our second main statement is as follows.

Theorem 2. *Let $\Omega = \mathbb{R},\ N \ge 2,\ 1 \le l \le N-1,\ 1 \le k \le N,\ b_k \in \mathbb{R},\ b_k \ne 0,\ m \in \mathbb{N},\ G_{k,m}(x) : \mathbb{R} \to \mathbb{R},\ G_{k,m}(x) \in L^1(\mathbb{R})$ are such that $G_{k,m}(x) \to G_k(x)$ in $L^1(\mathbb{R})$ as $m \to \infty$. Let Assumption 1 hold.*

I) Let $a_k > 0$ for $1 \le k \le l$.

II) Let $a_k = 0$ for $l+1 \le k \le N$. Assume that $xG_{k,m}(x) \in L^1(\mathbb{R}),\ xG_{k,m}(x) \to xG_k(x)$ in $L^1(\mathbb{R})$ as $m \to \infty$, orthogonality relations (5.11) hold along with upper bound (5.12). Then each system (2.7) admits a unique solution $u^{(m)}(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$, and limiting problem (1.2) possesses a unique solution $u(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$, such that $u^{(m)}(x) \to u(x)$ in $H^2(\mathbb{R}, \mathbb{R}^N)$ as $m \to \infty$.

The unique solution $u^{(m)}(x)$ of each system (2.7) is nontrivial provided that for some $1 \le k \le N$ the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0,x)} \cap supp\widehat{G_{k,m}}$ is a set of nonzero Lebesgue measure in $\mathbb{R}$. Analogously, the unique solution $u(x)$ of limiting system (1.2) does not vanish identically if $supp\widehat{F_k(0,x)} \cap supp\widehat{G_k}$ is a set of nonzero Lebesgue measure in $\mathbb{R}$ for a certain $1 \le k \le N$.

In the second part of the work we consider the analogous system on the finite interval with periodic boundary conditions, i.e. $\Omega = I := [0,\ 2\pi]$ and the appropriate functional space is

$$H^2(I) = \{v(x) : I \to \mathbb{R} \mid v(x), v''(x) \in L^2(I), \quad v(0) = v(2\pi), \quad v'(0) = v'(2\pi)\},$$

aiming at $u_k(x) \in H^2(I),\ 1 \le k \le l$. For the technical purposes, we introduce the following auxiliary constrained subspace

$$H_0^2(I) = \{v(x) \in H^2(I) \mid (v(x), 1)_{L^2(I)} = 0\}. \tag{2.8}$$

The aim is to have $u_k(x) \in H_0^2(I),\ l+1 \le k \le N$. The constrained subspace (2.8) is a Hilbert space as well (see e.g. Chapter 2.1 of [9]). The resulting space used for proving the existence in the sense of sequences of solutions $u(x) : I \to \mathbb{R}^N$ of system (1.2) will be the direct sum of the spaces given above, namely

$$H_c^2(I, \mathbb{R}^N) := \oplus_{k=1}^{l} H^2(I) \oplus_{k=l+1}^{N} H_0^2(I).$$

The corresponding Sobolev norm is given by

$$\|u\|^2_{H_c^2(I,\mathbb{R}^N)} := \sum_{k=1}^{N} \{\|u_k\|^2_{L^2(I)} + \|u_k''\|^2_{L^2(I)}\},$$

where $u(x) : I \to \mathbb{R}^N$. Another useful norm is given by

$$\|u\|^2_{L^2(I,\mathbb{R}^N)} = \sum_{k=1}^{N} \|u_k\|^2_{L^2(I)}.$$

Let us prove that system (2.3) in this situation defines a map $\tau_{a,b} : H_c^2(I, \mathbb{R}^N) \to H_c^2(I, \mathbb{R}^N)$, which will be a strict contraction under the given technical assumptions.

Theorem 3. *Let $\Omega = I,\ N \ge 2,\ 1 \le l \le N-1,\ 1 \le k \le N,\ b_k \in \mathbb{R},\ b_k \ne 0$ and $G_k(x) : I \to \mathbb{R},\ G_k(x) \in L^\infty(I),\ G_k(0) = G_k(2\pi),\ F_k(u, 0) = F_k(u, 2\pi)$ for $u \in \mathbb{R}^N$ and Assumption 1 holds.*

I) Let $a_k > 0$ for $1 \le k \le l$.

II) Let $a_k = 0$ for $l+1 \le k \le N$, orthogonality conditions (5.27) hold and $2\sqrt{\pi}\mathcal{N}_{a,\ b}L < 1$, where $\mathcal{N}_{a,\ b}$ is defined by (5.26). Then the map $\tau_{a,b}v = u$ on $H_c^2(I, \mathbb{R}^N)$ defined by system (2.3) possesses a unique fixed point $v^{(a,b)}$, the only solution of the system of equations (1.2) in $H_c^2(I, \mathbb{R}^N)$.

The fixed point $v^{(a,b)}$ is nontrivial provided that the Fourier coefficients $G_{k,n}F_k(0, x)_n \ne 0$ for a certain $1 \le k \le N$ and for some $n \in \mathbb{Z}$.

Remark 1. *We use the constrained subspace $H_0^2(I)$ involved in the direct sum of spaces $H_c^2(I, \mathbb{R}^N)$, such that the Fredholm operators $-\dfrac{d^2}{dx^2} - b_k\dfrac{d}{dx} : H_0^2(I) \to L^2(I)$, have the trivial kernels.*

To study the existence in the sense of sequences of solutions for our integro- differential system on the interval I, we consider the sequence of iterated systems of equations, analogously to the whole real line case with $m \in \mathbb{N},\ 1 \le k \le N,\ a_k \ge 0,\ b_k \in \mathbb{R},\ b_k \ne 0$, such that

$$\frac{d^2 u_k^{(m)}}{dx^2} + b_k \frac{du_k^{(m)}}{dx} + a_k u_k^{(m)} + \int_0^{2\pi} G_{k,m}(x-y) F_k(u_1^{(m)}(y), u_2^{(m)}(y), ..., u_N^{(m)}(y), y) dy = 0. \quad (2.9)$$

Our final main proposition is as follows.

Theorem 4. *Let $\Omega = I,\ N \ge 2,\ 1 \le l \le N-1,\ 1 \le k \le N,\ b_k \in \mathbb{R},\ b_k \ne 0,\ m \in \mathbb{N},\ G_{k,m}(x) : I \to \mathbb{R},\ G_{k,m}(0) = G_{k,m}(2\pi),\ G_{k,m}(x) \in L^\infty(I)$ are such that $G_{k,m}(x) \to G_k(x)$ in $L^\infty(I)$ as $m \to \infty,\ F_k(u, 0) = F_k(u, 2\pi)$ for $u \in \mathbb{R}^N$. Let Assumption 1 hold.*

I) Let $a_k > 0$ for $1 \le k \le l$.

II) Let $a_k = 0$ for $l+1 \le k \le N$. Assume that orthogonality conditions (5.33) hold along with estimate from above (5.34) Then each system (2.9) has a unique solution $u^{(m)}(x) \in H_c^2(I, \mathbb{R}^N)$ and the limiting system of equations (1.2) admits a unique solution $u(x) \in H_c^2(I, \mathbb{R}^N)$, such that $u^{(m)}(x) \to u(x)$ in $H_c^2(I, \mathbb{R}^N)$ as $m \to \infty$.

The unique solution $u^{(m)}(x)$ of each system (2.9) is nontrivial provided that the Fourier coefficients $G_{k,m,n} F_k(0, x)_n \ne 0$ for a certain $1 \le k \le N$ and some $n \in \mathbb{Z}$. Analogously, the unique solution $u(x)$ of the limiting system of equations (1.2) does not vanish identically if $G_{k,n} F_k(0, x)_n \ne 0$ for some $1 \le k \le N$ and a certain $n \in \mathbb{Z}$.

Remark 2. *Note that in the article we work with real valued vector functions by virtue of the assumptions on $F_k(u, x)$, $G_{k,m}(x)$ and $G_k(x)$ involved in the nonlocal terms of the iterated and limiting systems of equations discussed above.*

Remark 3. *The importance of Theorems 2 and 4 above is the continuous dependence of solutions with respect to the integral kernels.*

3 The Whole Real Line Case

Proof of Theorem 1. Let us first suppose that in the case of $\Omega = \mathbb{R}$ for some $v \in H^2(\mathbb{R}, \mathbb{R}^N)$ there exist two solutions $u^{(1),(2)} \in H^2(\mathbb{R}, \mathbb{R}^N)$ of system (2.3). Then their difference $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$ will be a solution of the homogeneous system of equations

$$-\frac{d^2 w_k}{dx^2} - b_k \frac{dw_k}{dx} - a_k w_k = 0, \quad 1 \le k \le N.$$

Since the operator $L_{a,\ b,\ k}$ defined in (2.6) acting on the whole real line does not possess any nontrivial square integrable zero modes, $w(x) = 0$ on $\mathbb{R}$.

We choose arbitrarily $v(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$. Let us apply the standard Fourier transform (5.1) to both sides of (2.3). This yields

$$\widehat{u_k}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k}(p)}{p^2 - a_k - ib_k p}, \quad 1 \le k \le N, \tag{3.1}$$

where $\widehat{f_k}(p)$ denotes the Fourier image of $F_k(v(x), x)$. Evidently, for $1 \le k \le N$, we have the estimates from above

$$|\widehat{u_k}(p)| \le \sqrt{2\pi} N_{a,\ b,\ k} |\widehat{f_k}(p)| \quad and \quad |p^2\widehat{u_k}(p)| \le \sqrt{2\pi} N_{a,\ b,\ k} |\widehat{f_k}(p)|,$$

where $N_{a,\ b,\ k} < \infty$ by means of Lemma 5 of the Appendix without any orthogonality relations for $a_k > 0$ and under orthogonality condition (5.5) when $a_k = 0$. This enables us to estimate the norm

$$\|u\|^2_{H^2(\mathbb{R},\mathbb{R}^N)} = \sum_{k=1}^N \{\|\widehat{u_k}(p)\|^2_{L^2(\mathbb{R})} + \|p^2\widehat{u_k}(p)\|^2_{L^2(\mathbb{R})}\} \le \sum_{k=1}^N 4\pi N^2_{a,\ b,\ k}\|F_k(v(x), x)\|^2_{L^2(\mathbb{R})},$$

which is finite due to (2.1) of Assumption 1 because $|v(x)|_{\mathbb{R}^N} \in L^2(\mathbb{R})$. Hence, for an arbitrary $v(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$ there exists a unique solution $u(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$ of system (2.3), such that its Fourier image is given by (3.1) and the map $T_{a,b} : H^2(\mathbb{R}, \mathbb{R}^N) \to H^2(\mathbb{R}, \mathbb{R}^N)$ is well defined. This allows us to choose arbitrarily $v^{(1),(2)}(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$, such that their images $u^{(1),(2)} = T_{a,b}v^{(1),(2)} \in H^2(\mathbb{R}, \mathbb{R}^N)$. Clearly,

$$\widehat{u_k^{(1)}}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k^{(1)}}(p)}{p^2 - a_k - ib_k p}, \quad \widehat{u_k^{(2)}}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{f_k^{(2)}}(p)}{p^2 - a_k - ib_k p}, \quad 1 \le k \le N,$$

where $\widehat{f_k^{(1)}}(p)$ and $\widehat{f_k^{(2)}}(p)$ stand for the Fourier transforms of $F_k(v^{(1)}(x), x)$ and $F_k(v^{(2)}(x), x)$ respectively. Thus, for $1 \le k \le N$, we easily obtain

$$\left|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\right| \le \sqrt{2\pi} N_{a,\ b,\ k}\left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|,$$

$$\left|p^2\widehat{u_k^{(1)}}(p) - p^2\widehat{u_k^{(2)}}(p)\right| \le \sqrt{2\pi} N_{a,\ b,\ k}\left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|.$$

For the appropriate norms of vector functions we arrive at

$$\|u^{(1)} - u^{(2)}\|^2_{H^2(\mathbb{R},\mathbb{R}^N)} \le 4\pi N^2_{a,\ b} \sum_{k=1}^N \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(\mathbb{R})}.$$

Note that $v_k^{(1),(2)}(x) \in H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$ via the Sobolev embedding. Condition (2.2) yields

$$\sum_{k=1}^N \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(\mathbb{R})} \le L^2\|v^{(1)} - v^{(2)}\|^2_{L^2(\mathbb{R},\mathbb{R}^N)}.$$

Therefore,

$$\|T_{a,b}v^{(1)} - T_{a,b}v^{(2)}\|_{H^2(\mathbb{R},\mathbb{R}^N)} \le 2\sqrt{\pi}N_{a,\ b}L\|v^{(1)} - v^{(2)}\|_{H^2(\mathbb{R},\mathbb{R}^N)}$$

and the constant in the right side of this inequality is less than one as assumed. Hence, by means of the Fixed Point Theorem, there exists a unique vector function $v^{(a,b)} \in H^2(\mathbb{R},\mathbb{R}^N)$ with the property $T_{a,b}v^{(a,b)} = v^{(a,b)}$, which is the only solution of system (1.2) in $H^2(\mathbb{R},\mathbb{R}^N)$. Suppose $v^{(a,b)}(x) = 0$ identically on the real line. This will contradict to our assumption that for a certain $1 \le k \le N$, the Fourier transforms of $G_k(x)$ and $F_k(0,x)$ do not vanish on a set of nonzero Lebesgue measure in $\mathbb{R}$. ∎

Let us turn our attention to showing the existence in the sense of sequences of the solutions for our system of integro-differential equations on the real line.

Proof of Theorem 2. By virtue of the result of Theorem 1 above, each system (2.7) has a unique solution $u^{(m)}(x) \in H^2(\mathbb{R},\mathbb{R}^N),\ m \in \mathbb{N}$. Limiting system (1.2) possesses a unique solution $u(x) \in H^2(\mathbb{R},\mathbb{R}^N)$ by means of Lemma 6 below along with Theorem 1. Let us apply the standard Fourier transform (5.1) to both sides of (1.2) and (2.7), which yields for $1 \le k \le N,\ m \in \mathbb{N}$

$$\widehat{u_k}(p) = \sqrt{2\pi}\frac{\widehat{G_k}(p)\widehat{\varphi_k}(p)}{p^2 - a_k - ib_kp}, \quad \widehat{u_k^{(m)}}(p) = \sqrt{2\pi}\frac{\widehat{G_{k,m}}(p)\widehat{\varphi_{k,m}}(p)}{p^2 - a_k - ib_kp}, \tag{3.2}$$

where $\widehat{\varphi_k}(p)$ and $\widehat{\varphi_{k,m}}(p)$ denote the Fourier images of $F_k(u(x),x)$ and $F_k(u^{(m)}(x),x)$ respectively. Apparently,

$$\left|\widehat{u_k^{(m)}}(p) - \widehat{u_k}(p)\right| \le \sqrt{2\pi}\left\|\frac{\widehat{G_{k,m}}(p)}{p^2 - a_k - ib_kp} - \frac{\widehat{G_k}(p)}{p^2 - a_k - ib_kp}\right\|_{L^\infty(\mathbb{R})}|\widehat{\varphi_k}(p)| +$$

$$+\sqrt{2\pi}\left\|\frac{\widehat{G_{k,m}}(p)}{p^2 - a_k - ib_kp}\right\|_{L^\infty(\mathbb{R})}|\widehat{\varphi_{k,m}}(p) - \widehat{\varphi_k}(p)|.$$

Therefore,

$$\|u_k^{(m)} - u_k\|_{L^2(\mathbb{R})} \le \sqrt{2\pi}\left\|\frac{\widehat{G_{k,m}}(p)}{p^2 - a_k - ib_kp} - \frac{\widehat{G_k}(p)}{p^2 - a_k - ib_kp}\right\|_{L^\infty(\mathbb{R})}\|F_k(u(x),x)\|_{L^2(\mathbb{R})} +$$

$$+\sqrt{2\pi}\left\|\frac{\widehat{G_{k,m}}(p)}{p^2 - a_k - ib_kp}\right\|_{L^\infty(\mathbb{R})}\|F_k(u^{(m)}(x),x) - F_k(u(x),x)\|_{L^2(\mathbb{R})}.$$

By virtue of inequality (2.2) of Assumption 1, we derive

$$\sqrt{\sum_{k=1}^{N}\|F_k(u^{(m)}(x),x) - F_k(u(x),x)\|^2_{L^2(\mathbb{R})}} \le L\|u^{(m)}(x) - u(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)}. \tag{3.3}$$

Clearly, $u_k^{(m)}(x), \ u_k(x) \in H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$ for $1 \leq k \leq N, \ m \in \mathbb{N}$ due to the Sobolev embedding. Therefore, we arrive at

$$\|u^{(m)}(x)-u(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \leq 4\pi \sum_{k=1}^N \left\| \frac{\widehat{G_{k,m}}(p)}{p^2 - a_k - ib_k p} - \frac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p} \right\|^2_{L^\infty(\mathbb{R})} \|F_k(u(x), x)\|^2_{L^2(\mathbb{R})} +$$

$$+4\pi \left[N_{a,\ b}^{(m)} \right]^2 L^2 \|u^{(m)}(x) - u(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)},$$

such that $\|u^{(m)}(x) - u(x)\|^2_{L^2(\mathbb{R},\mathbb{R}^N)} \leq$

$$\leq \frac{4\pi}{\varepsilon(2-\varepsilon)} \sum_{k=1}^N \left\| \frac{\widehat{G_{k,m}}(p)}{p^2 - a_k - ib_k p} - \frac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p} \right\|^2_{L^\infty(\mathbb{R})} \|F_k(u(x), x)\|^2_{L^2(\mathbb{R})}.$$

By virtue of inequality (2.1) of Assumption 1, we have $F_k(u(x), x) \in L^2(\mathbb{R}), \ 1 \leq k \leq N$ for $u(x) \in H^2(\mathbb{R}, \mathbb{R}^N)$. This implies that

$$u^{(m)}(x) \to u(x), \quad m \to \infty \tag{3.4}$$

in $L^2(\mathbb{R}, \mathbb{R}^N)$ via the result of Lemma 6 of the Appendix. Evidently, for $1 \leq k \leq N, \ m \in \mathbb{N}$, we have

$$p^2 \widehat{u_k}(p) = \sqrt{2\pi} \frac{p^2 \widehat{G_k}(p) \widehat{\varphi_k}(p)}{p^2 - a_k - ib_k p}, \quad p^2 \widehat{u_k^{(m)}}(p) = \sqrt{2\pi} \frac{p^2 \widehat{G_{k,m}}(p) \widehat{\varphi_{k,m}}(p)}{p^2 - a_k - ib_k p}.$$

Therefore,

$$\left| p^2 \widehat{u_k^{(m)}}(p) - p^2 \widehat{u_k}(p) \right| \leq \sqrt{2\pi} \left\| \frac{p^2 \widehat{G_{k,m}}(p)}{p^2 - a_k - ib_k p} - \frac{p^2 \widehat{G_k}(p)}{p^2 - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})} |\widehat{\varphi_k}(p)| +$$

$$+\sqrt{2\pi} \left\| \frac{p^2 \widehat{G_{k,m}}(p)}{p^2 - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})} |\widehat{\varphi_{k,m}}(p) - \widehat{\varphi_k}(p)|.$$

By virtue of (3.3), for $1 \leq k \leq N, \ m \in \mathbb{N}$, we arrive at

$$\left\| \frac{d^2 u_k^{(m)}}{dx^2} - \frac{d^2 u_k}{dx^2} \right\|_{L^2(\mathbb{R})} \leq \sqrt{2\pi} \left\| \frac{p^2 \widehat{G_{k,m}}(p)}{p^2 - a_k - ib_k p} - \frac{p^2 \widehat{G_k}(p)}{p^2 - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})} \|F_k(u(x), x)\|_{L^2(\mathbb{R})} +$$

$$+\sqrt{2\pi} \left\| \frac{p^2 \widehat{G_{k,m}}(p)}{p^2 - a_k - ib_k p} \right\|_{L^\infty(\mathbb{R})} L \|u^{(m)}(x) - u(x)\|_{L^2(\mathbb{R},\mathbb{R}^N)}.$$

By means of the result of Lemma 6 of the Appendix along with (3.4), we derive that $\dfrac{d^2 u^{(m)}}{dx^2} \to \dfrac{d^2 u}{dx^2}$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ as $m \to \infty$. Definition (2.5) of the norm gives us that $u^{(m)}(x) \to u(x)$ in $H^2(\mathbb{R}, \mathbb{R}^N)$ as $m \to \infty$.

Let us assume that the solution $u^{(m)}(x)$ of system (2.7) studied above vanishes on the real line for a certain $m \in \mathbb{N}$. This will contradict to our assumption that the Fourier images of $G_{k,m}(x)$ and $F_k(0, x)$ are nontrivial on a set of nonzero Lebesgue measure in $\mathbb{R}$. The analogous argument holds for the solution $u(x)$ of limiting system (1.2). ∎

4 The Problem on the Finite Interval

Proof of Theorem 3. Apparently, each operator involved in the left side of system (2.3)

$$\mathcal{L}_{a,\ b,\ k} := -\frac{d^2}{dx^2} - b_k \frac{d}{dx} - a_k : \quad H^2(I) \to L^2(I), \tag{4.1}$$

where $1 \le k \le N,\ a_k > 0,\ b_k \in \mathbb{R},\ b_k \neq 0$ is Fredholm, non-selfadjoint, its set of eigenvalues is given by

$$\lambda_{a,\ b,\ k}(n) = n^2 - a_k - i b_k n, \quad n \in \mathbb{Z} \tag{4.2}$$

and its eigenfunctions are the standard Fourier harmonics $\frac{e^{inx}}{\sqrt{2\pi}},\ n \in \mathbb{Z}$. When $a_k = 0$, we will use the similar ideas in the constrained subspace (2.8) instead of $H^2(I)$. Evidently, the eigenvalues of each operator $\mathcal{L}_{a,\ b,\ k}$ are simple, as distinct from the analogous situation without the drift term, when the eigenvalues corresponding to $n \neq 0$ have the multiplicity of two (see [16]).

Let us first suppose that for some $v(x) \in H_c^2(I, \mathbb{R}^N)$ there exist two solutions $u^{(1),(2)}(x) \in H_c^2(I, \mathbb{R}^N)$ of system (2.3) with $\Omega = I$. Then the vector function $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H_c^2(I, \mathbb{R}^N)$ will be a solution of the homogeneous system of equations

$$-\frac{d^2 w_k}{dx^2} - b_k \frac{dw_k}{dx} - a_k w_k = 0, \quad 1 \le k \le N.$$

But the operator $\mathcal{L}_{a,\ b,\ k} :\ H^2(I) \to L^2(I),\ a_k > 0$ discussed above does not possess nontrivial zero modes. Hence, $w(x)$ vanishes in I.

We choose arbitrarily $v(x) \in H_c^2(I, \mathbb{R}^N)$ and apply the Fourier transform (5.23) to system (2.3) considered on the interval I. This yields

$$u_{k,n} = \sqrt{2\pi} \frac{G_{k,n} f_{k,n}}{n^2 - a_k - i b_k n}, \quad n^2 u_{k,n} = \sqrt{2\pi} \frac{n^2 G_{k,n} f_{k,n}}{n^2 - a_k - i b_k n}, \quad 1 \le k \le N \qquad n \in \mathbb{Z}, \tag{4.3}$$

with $f_{k,n} := F_k(v(x), x)_n$. Thus, we obtain

$$|u_{k,n}| \le \sqrt{2\pi} \mathcal{N}_{a,\ b,\ k} |f_{k,n}|, \quad |n^2 u_{k,n}| \le \sqrt{2\pi} \mathcal{N}_{a,\ b,\ k} |f_{k,n}|,$$

where $\mathcal{N}_{a,\ b,\ k} < \infty$ under the given auxiliary assumptions by virtue of Lemma 7 of the Appendix. Hence,

$$\|u\|^2_{H_c^2(I,\mathbb{R}^N)} = \sum_{k=1}^N \left[\sum_{n=-\infty}^{\infty} |u_{k,n}|^2 + \sum_{n=-\infty}^{\infty} |n^2 u_{k,n}|^2 \right] \le 4\pi \mathcal{N}^2_{a,\ b} \sum_{k=1}^N \|F_k(v(x), x)\|^2_{L^2(I)} < \infty$$

due to (2.1) of Assumption 1 for $|v(x)|_{\mathbb{R}^N} \in L^2(I)$. Thus, for any $v(x) \in H_c^2(I, \mathbb{R}^N)$ there exists a unique $u(x) \in H_c^2(I, \mathbb{R}^N)$ solving system (2.3) with its Fourier transform given by (4.3) and the map $\tau_{a,b} : H_c^2(I, \mathbb{R}^N) \to H_c^2(I, \mathbb{R}^N)$ is well defined.

Let us consider arbitrary $v^{(1),(2)}(x) \in H_c^2(I, \mathbb{R}^N)$ with their images under the map discussed above $u^{(1),(2)} = \tau_{a,b} v^{(1),(2)} \in H_c^2(I, \mathbb{R}^N)$. By applying the Fourier transform (5.23), we easily derive

$$u_{k,n}^{(1)} = \sqrt{2\pi} \frac{G_{k,n} f_{k,n}^{(1)}}{n^2 - a_k - i b_k n}, \quad u_{k,n}^{(2)} = \sqrt{2\pi} \frac{G_{k,n} f_{k,n}^{(2)}}{n^2 - a_k - i b_k n}, \quad 1 \le k \le N, \quad n \in \mathbb{Z},$$

with $f_{k,n}^{(j)} := F_k(v^{(j)}(x), x)_n, \ j = 1, 2$. Hence,

$$|u_{k,n}^{(1)} - u_{k,n}^{(2)}| \le \sqrt{2\pi} \mathcal{N}_{a,\ b,\ k} |f_{k,n}^{(1)} - f_{k,n}^{(2)}|, \quad |n^2(u_{k,n}^{(1)} - u_{k,n}^{(2)})| \le \sqrt{2\pi} \mathcal{N}_{a,\ b,\ k} |f_{k,n}^{(1)} - f_{k,n}^{(2)}|.$$

Therefore,

$$\|u^{(1)} - u^{(2)}\|_{H_c^2(I, \mathbb{R}^N)}^2 = \sum_{k=1}^N \left[\sum_{n=-\infty}^{\infty} |u_{k,n}^{(1)} - u_{k,n}^{(2)}|^2 + \sum_{n=-\infty}^{\infty} |n^2(u_{k,n}^{(1)} - u_{k,n}^{(2)})|^2 \right] \le$$

$$\le 4\pi \mathcal{N}_{a,\ b}^2 \sum_{k=1}^N \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|_{L^2(I)}^2.$$

Clearly, $v_k^{(1),(2)}(x) \in H^2(I) \subset L^\infty(I), \ 1 \le k \le N$ due to the Sobolev embedding. By virtue of (2.2) we easily arrive at

$$\|\tau_{a,b} v^{(1)} - \tau_{a,b} v^{(2)}\|_{H_c^2(I, \mathbb{R}^N)} \le 2\sqrt{\pi} \mathcal{N}_{a,\ b} L \|v^{(1)} - v^{(2)}\|_{H_c^2(I, \mathbb{R}^N)},$$

with the constant in the right side of this estimate less than one as assumed. Thus, the Fixed Point Theorem yields the existence and uniqueness of a vector function $v^{(a,b)} \in H_c^2(I, \mathbb{R}^N)$, satisfying $\tau_{a,b} v^{(a,b)} = v^{(a,b)}$, which is the only solution of system (1.2) in $H_c^2(I, \mathbb{R}^N)$. Suppose $v^{(a,b)}(x) = 0$ identically in I. This gives us the contradiction to the assumption that $G_{k,n} F_k(0, x)_n \neq 0$ for some $1 \le k \le N$ and a certain $n \in \mathbb{Z}$. ■

We proceed to establishing the final main statement of the article.

Proof of Theorem 4. Apparently, the limiting kernels $G_k(x), \ 1 \le k \le N$ are also periodic on the interval I (see the argument of Lemma 8 of the Appendix). Each system (2.9) has a unique solution $u^{(m)}(x), \ m \in \mathbb{N}$ belonging to $H_c^2(I, \mathbb{R}^N)$ by virtue of Theorem 3 above. The limiting system of equations (1.2) admits a unique solution $u(x)$, which belongs to $H_c^2(I, \mathbb{R}^N)$ by means of Lemma 8 of the Appendix along with Theorem 3.

Let us apply Fourier transform (5.23) to both sides of systems (1.2) and (2.9). This yields

$$u_{k,n} = \sqrt{2\pi} \frac{G_{k,n} \varphi_{k,n}}{n^2 - a_k - i b_k n}, \quad u_{k,n}^{(m)} = \sqrt{2\pi} \frac{G_{k,m,n} \varphi_{k,n}^{(m)}}{n^2 - a_k - i b_k n}, \tag{4.4}$$

where $1 \leq k \leq N$, $n \in \mathbb{Z}$, $m \in \mathbb{N}$ with $\varphi_{k,n}$ and $\varphi_{k,n}^{(m)}$ denoting the Fourier images of $F_k(u(x), x)$ and $F_k(u^{(m)}(x), x)$ respectively under transform (5.23). We easily obtain the upper bound

$$|u_{k,n}^{(m)} - u_{k,n}| \leq \sqrt{2\pi}\left\|\frac{G_{k,m,n}}{n^2 - a_k - ib_k n} - \frac{G_{k,n}}{n^2 - a_k - ib_k n}\right\|_{l^\infty}|\varphi_{k,n}| +$$

$$+\sqrt{2\pi}\left\|\frac{G_{k,m,n}}{n^2 - a_k - ib_k n}\right\|_{l^\infty}|\varphi_{k,n}^{(m)} - \varphi_{k,n}|.$$

Hence,

$$\|u_k^{(m)} - u_k\|_{L^2(I)} \leq \sqrt{2\pi}\left\|\frac{G_{k,m,n}}{n^2 - a_k - ib_k n} - \frac{G_{k,n}}{n^2 - a_k - ib_k n}\right\|_{l^\infty}\|F_k(u(x), x)\|_{L^2(I)} +$$

$$+\sqrt{2\pi}\left\|\frac{G_{k,m,n}}{n^2 - a_k - ib_k n}\right\|_{l^\infty}\|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|_{L^2(I)}.$$

By virtue of bound (2.2) of Assumption 1, we arrive at

$$\sqrt{\sum_{k=1}^{N}\|F_k(u^{(m)}(x), x) - F_k(u(x), x)\|_{L^2(I)}^2} \leq L\|u^{(m)}(x) - u(x)\|_{L^2(I, \mathbb{R}^N)}. \tag{4.5}$$

Note that $u_k^{(m)}(x), u_k(x) \in H^2(I) \subset L^\infty(I)$ via the Sobolev embedding. Evidently,

$$\|u^{(m)}(x) - u(x)\|_{L^2(I,\mathbb{R}^N)}^2 \leq 4\pi\sum_{k=1}^{N}\left\|\frac{G_{k,m,n}}{n^2 - a_k - ib_k n} - \frac{G_{k,n}}{n^2 - a_k - ib_k n}\right\|_{l^\infty}^2\|F_k(u(x), x)\|_{L^2(I)}^2 +$$

$$+4\pi\left[\mathcal{N}_{a,b}^{(m)}\right]^2 L^2\|u^{(m)}(x) - u(x)\|_{L^2(I,\mathbb{R}^N)}^2.$$

Thus, we derive $\|u^{(m)}(x) - u(x)\|_{L^2(I,\mathbb{R}^N)}^2 \leq$

$$\leq \frac{4\pi}{\varepsilon(2-\varepsilon)}\sum_{k=1}^{N}\left\|\frac{G_{k,m,n}}{n^2 - a_k - ib_k n} - \frac{G_{k,n}}{n^2 - a_k - ib_k n}\right\|_{l^\infty}^2\|F_k(u(x), x)\|_{L^2(I)}^2.$$

Clearly, $F_k(u(x), x) \in L^2(I)$, $1 \leq k \leq N$ for $u(x) \in H_c^2(I, \mathbb{R}^N)$ by means of bound (2.1) of Assumption 1. Lemma 8 below implies that

$$u^{(m)}(x) \to u(x), \quad m \to \infty \tag{4.6}$$

in $L^2(I, \mathbb{R}^N)$. Obviously,

$$|n^2 u_{k,n}^{(m)} - n^2 u_{k,n}| \leq \sqrt{2\pi}\left\|\frac{n^2 G_{k,m,n}}{n^2 - a_k - ib_k n} - \frac{n^2 G_{k,n}}{n^2 - a_k - ib_k n}\right\|_{l^\infty}|\varphi_{k,n}| +$$

$$+\sqrt{2\pi}\Big\|\frac{n^2 G_{k,m,n}}{n^2-a_k-ib_kn}\Big\|_{l^\infty}|\varphi_{k,n}^{(m)}-\varphi_{k,n}|.$$

By means of (4.5), we obtain

$$\Big\|\frac{d^2u_k^{(m)}}{dx^2}-\frac{d^2u_k}{dx^2}\Big\|_{L^2(I)}\le\sqrt{2\pi}\Big\|\frac{n^2G_{k,m,n}}{n^2-a_k-ib_kn}-\frac{n^2G_{k,n}}{n^2-a_k-ib_kn}\Big\|_{l^\infty}\|F_k(u(x),x)\|_{L^2(I)}+$$

$$+\sqrt{2\pi}\Big\|\frac{n^2G_{k,m,n}}{n^2-a_k-ib_kn}\Big\|_{l^\infty}L\|u^{(m)}-u\|_{L^2(I,\mathbb{R}^N)}.$$

Lemma 8 of the Appendix along with (4.6) give us $\dfrac{d^2u^{(m)}}{dx^2}\to\dfrac{d^2u}{dx^2}$ as $m\to\infty$ in $L^2(I,\mathbb{R}^N)$. Hence, $u^{(m)}(x)\to u(x)$ in the $H_c^2(I,\mathbb{R}^N)$ norm as $m\to\infty$.

Suppose that $u^{(m)}(x)=0$ identically in the interval I for some $m\in\mathbb{N}$. This gives us a contradiction to the assumption that $G_{k,m,n}F_k(0,x)_n\neq0$ for some $1\le k\le N$ and a certain $n\in\mathbb{Z}$. The analogous argument holds for the solution $u(x)$ of the limiting system of equations (1.2). ∎

5 Appendix

Let $G_k(x)$ be a function, $G_k(x):\mathbb{R}\to\mathbb{R}$, for which we denote its standard Fourier transform using the hat symbol as

$$\widehat{G_k}(p):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}G_k(x)e^{-ipx}dx,\quad p\in\mathbb{R},\tag{5.1}$$

such that

$$\|\widehat{G_k}(p)\|_{L^\infty(\mathbb{R})}\le\frac{1}{\sqrt{2\pi}}\|G_k\|_{L^1(\mathbb{R})}\tag{5.2}$$

and $G_k(x)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}\widehat{G_k}(q)e^{iqx}dq,\ x\in\mathbb{R}$. For the technical purposes we define the auxiliary quantities

$$N_{a,\ b,\ k}:=max\Big\{\Big\|\frac{\widehat{G_k}(p)}{p^2-a_k-ib_kp}\Big\|_{L^\infty(\mathbb{R})},\quad\Big\|\frac{p^2\widehat{G_k}(p)}{p^2-a_k-ib_kp}\Big\|_{L^\infty(\mathbb{R})}\Big\},\tag{5.3}$$

with $a_k\ge0,\ b_k\in\mathbb{R},\ b_k\neq0,\ 1\le k\le N,\ N\ge2$. Let $N_{0,\ b,\ k}$ stand for (5.3) when a_k vanishes. Under the conditions of Lemma 5 below, quantities (5.3) will be finite. This will enable us to define

$$N_{a,\ b}:=max_{1\le k\le N}N_{a,\ b,\ k}<\infty.\tag{5.4}$$

The technical lemmas below are the adaptations of the ones established in [8] for the studies of the single integro-differential equation with drift, analogous to system (1.2). We provide them for the convenience of the readers.

Lemma 5. *Let $N \geq 2,\ 1 \leq k \leq N,\ b_k \in \mathbb{R},\ b_k \neq 0$ and $G_k(x) : \mathbb{R} \to \mathbb{R},\ G_k(x) \in L^1(\mathbb{R})$ and $1 \leq l \leq N-1$.*

a) Let $a_k > 0$ for $1 \leq k \leq l$. Then $N_{a,\ b,\ k} < \infty$.

b) Let $a_k = 0$ for $l+1 \leq k \leq N$ and in addition $xG_k(x) \in L^1(\mathbb{R})$. Then $N_{0,\ b,\ k} < \infty$ if and only if

$$(G_k(x), 1)_{L^2(\mathbb{R})} = 0 \tag{5.5}$$

holds.

Proof. First of all, let us observe that in both cases a) and b) of our lemma the boundedness of $\dfrac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p}$ yields the boundedness of $\dfrac{p^2\widehat{G_k}(p)}{p^2 - a_k - ib_k p}$. Indeed, we can write $\dfrac{p^2\widehat{G_k}(p)}{p^2 - a_k - ib_k p}$ as the following sum

$$\widehat{G_k}(p) + a_k \frac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p} + ib_k \frac{p\widehat{G_k}(p)}{p^2 - a_k - ib_k p}. \tag{5.6}$$

Evidently, the first term in (5.6) is bounded by means of (5.2) since $G_k(x) \in L^1(\mathbb{R})$ as assumed. The third term in (5.6) can be estimated from above in the absolute value via (5.2) as

$$\frac{|b_k||p||\widehat{G_k}(p)|}{\sqrt{(p^2 - a_k)^2 + b_k^2 p^2}} \leq \frac{1}{\sqrt{2\pi}} \|G_k(x)\|_{L^1(\mathbb{R})} < \infty.$$

Thus, $\dfrac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p} \in L^\infty(\mathbb{R})$ implies that $\dfrac{p^2\widehat{G_k}(p)}{p^2 - a_k - ib_k p} \in L^\infty(\mathbb{R})$. To obtain the result of the part a) of the lemma, we need to estimate

$$\frac{|\widehat{G_k}(p)|}{\sqrt{(p^2 - a_k)^2 + b_k^2 p^2}}. \tag{5.7}$$

Apparently, the numerator of (5.7) can be bounded from above by virtue of (5.2) and the denominator in (5.7) can be easily estimated below by a finite, positive constant, such that

$$\frac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p} \leq C\|G_k(x)\|_{L^1(\mathbb{R})} < \infty.$$

Here and further down C will denote a finite, positive constant. This yields that under the given conditions, when $a_k > 0$ we have $N_{a,\ b,\ k} < \infty$. In the case of $a_k = 0$, we express

$$\widehat{G_k}(p) = \widehat{G_k}(0) + \int_0^p \frac{d\widehat{G_k}(s)}{ds} ds,$$

such that

$$\frac{\widehat{G_k}(p)}{p^2 - ib_k p} = \frac{\widehat{G_k}(0)}{p(p - ib_k)} + \frac{\int_0^p \frac{d\widehat{G_k}(s)}{ds} ds}{p(p - ib_k)}. \tag{5.8}$$

By means of definition (5.1) of the standard Fourier transform, we easily estimate

$$\left|\frac{d\widehat{G_k}(p)}{dp}\right| \leq \frac{1}{\sqrt{2\pi}}\|xG_k(x)\|_{L^1(\mathbb{R})}.$$

Hence, we derive

$$\left|\frac{\int_0^p \frac{d\widehat{G_k}(s)}{ds}ds}{p(p-ib_k)}\right| \leq \frac{\|xG_k(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|b_k|} < \infty$$

via the one of our assumptions. Therefore, the expression in the left side of (5.8) is bounded if and only if $\widehat{G_k}(0) = 0$, which equivalent to orthogonality condition (5.5). ■

For the purpose of the study of systems (2.7), we define the following auxiliary expressions

$$N_{a,\ b,\ k}^{(m)} := max\left\{\left\|\frac{\widehat{G_{k,m}}(p)}{p^2 - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})},\quad \left\|\frac{p^2\widehat{G_{k,m}}(p)}{p^2 - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})}\right\} \tag{5.9}$$

with $a_k \geq 0,\ b_k \in \mathbb{R},\ b_k \neq 0$ and $m \in \mathbb{N}$. Under the assumptions of Lemma 6 below, expressions (5.9) will be finite. This will allow us to define

$$N_{a,\ b}^{(m)} := max_{1\leq k\leq N} N_{a,\ b,\ k}^{(m)} < \infty, \tag{5.10}$$

where $m \in \mathbb{N}$. We have the following technical statement.

Lemma 6. *Let $N \geq 2,\ 1 \leq k \leq N,\ b_k \in \mathbb{R},\ b_k \neq 0$ and $G_{k,m}(x) : \mathbb{R} \to \mathbb{R},\ G_{k,m}(x) \in L^1(\mathbb{R})$, such that $G_{k,m}(x) \to G_k(x)$ in $L^1(\mathbb{R})$ as $m \to \infty$ and $1 \leq l \leq N-1$.*

a) Let $a_k > 0$ for $1 \leq k \leq l$.

b) Let $a_k = 0$ for $l+1 \leq k \leq N$ and additionally $xG_{k,m}(x) \in L^1(\mathbb{R})$, such that $xG_{k,m}(x) \to xG_k(x)$ in $L^1(\mathbb{R})$ as $m \to \infty$ and

$$(G_{k,m}(x), 1)_{L^2(\mathbb{R})} = 0, \quad m \in \mathbb{N}. \tag{5.11}$$

We also assume that

$$2\sqrt{\pi}N_{a,\ b}^{(m)}L \leq 1 - \varepsilon \tag{5.12}$$

holds for all $m \in \mathbb{N}$ as well with some fixed $0 < \varepsilon < 1$. Then, for all $1 \leq k \leq N$, we have

$$\frac{\widehat{G_{k,m}}(p)}{p^2 - a_k - ib_k p} \to \frac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p}, \quad m \to \infty, \tag{5.13}$$

$$\frac{p^2\widehat{G_{k,m}}(p)}{p^2 - a_k - ib_k p} \to \frac{p^2\widehat{G_k}(p)}{p^2 - a_k - ib_k p}, \quad m \to \infty \tag{5.14}$$

in $L^\infty(\mathbb{R})$, such that

$$\left\|\frac{\widehat{G_{k,m}}(p)}{p^2 - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})} \to \left\|\frac{\widehat{G_k}(p)}{p^2 - a_k - ib_k p}\right\|_{L^\infty(\mathbb{R})}, \quad m \to \infty, \tag{5.15}$$

$$\left\| \frac{p^2 \widehat{G_{k,m}}(p)}{p^2 - a_k - i b_k p} \right\|_{L^\infty(\mathbb{R})} \to \left\| \frac{p^2 \widehat{G_k}(p)}{p^2 - a_k - i b_k p} \right\|_{L^\infty(\mathbb{R})}, \quad m \to \infty. \tag{5.16}$$

Moreover,

$$2\sqrt{\pi} N_{a,\ b} L \le 1 - \varepsilon. \tag{5.17}$$

Proof. Obviously, for all $1 \le k \le N$, we have

$$\|\widehat{G_{k,m}}(p) - \widehat{G_k}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}} \|G_{k,m}(x) - G_k(x)\|_{L^1(\mathbb{R})} \to 0, \quad m \to \infty, \tag{5.18}$$

as assumed. Let us prove that (5.13) implies (5.14). Indeed, $\dfrac{p^2[\widehat{G_{k,m}}(p) - \widehat{G_k}(p)]}{p^2 - a_k - i b_k p}$ can be written as the sum

$$[\widehat{G_{k,m}}(p) - \widehat{G_k}(p)] + a_k \left[\frac{\widehat{G_{k,m}}(p)}{p^2 - a_k - i b_k p} - \frac{\widehat{G_k}(p)}{p^2 - a_k - i b_k p} \right] + i b_k p \frac{[\widehat{G_{k,m}}(p) - \widehat{G_k}(p)]}{p^2 - a_k - i b_k p}. \tag{5.19}$$

The first term in (5.19) tends to zero as $m \to \infty$ in the $L^\infty(\mathbb{R})$ norm by virtue of (5.18). The third term in (5.19) can be bounded from above in the absolute value as

$$|b_k| \frac{|p| |\widehat{G_{k,m}}(p) - \widehat{G_k}(p)|}{\sqrt{(p^2 - a_k)^2 + b_k^2 p^2}} \le \|\widehat{G_{k,m}}(p) - \widehat{G_k}(p)\|_{L^\infty(\mathbb{R})},$$

hence it converges to zero as $m \to \infty$ in the $L^\infty(\mathbb{R})$ norm via (5.18) as well. Therefore, the statement of (5.13) yields (5.14). Evidently, (5.15) and (5.16) will follow from the statements of (5.13) and (5.14) respectively by virtue of the triangle inequality.

First we prove (5.13) in the case a) when $a_k > 0$. Then one needs to estimate

$$\frac{|\widehat{G_{k,m}}(p) - \widehat{G_k}(p)|}{\sqrt{(p^2 - a_k)^2 + b_k^2 p^2}}. \tag{5.20}$$

Apparently, the denominator in fraction (5.20) can be bounded from below by a positive constant and the numerator in (5.20) can be estimated from above by means of (5.18). This gives us the result of (5.13) when the constant a_k is positive.

Then let us turn our attention to establishing (5.13) in the case b) when $a_k = 0$. Hence, we have orthogonality relations (5.11). Let us prove that the analogous statement will hold in the limit. Indeed,

$$|(G_k(x), 1)_{L^2(\mathbb{R})}| = |(G_k(x) - G_{k,m}(x), 1)_{L^2(\mathbb{R})}| \le \|G_{k,m}(x) - G_k(x)\|_{L^1(\mathbb{R})} \to 0$$

as $m \to \infty$ due to the one of our assumptions. Therefore,

$$(G_k(x), 1)_{L^2(\mathbb{R})} = 0, \quad 1 \le k \le N. \tag{5.21}$$

We express

$$\widehat{G_k}(p) = \widehat{G_k}(0) + \int_0^p \frac{d\widehat{G_k}(s)}{ds} ds, \quad \widehat{G_{k,m}}(p) = \widehat{G_{k,m}}(0) + \int_0^p \frac{d\widehat{G_{k,m}}(s)}{ds} ds,$$

where $1 \leq k \leq N,\ m \in \mathbb{N}$. By means of (5.21) and (5.11), we obtain

$$\widehat{G_k}(0) = 0, \quad \widehat{G_{k,m}}(0) = 0, \quad 1 \leq k \leq N, \quad m \in \mathbb{N}.$$

Thus,

$$\left| \frac{\widehat{G_{k,m}}(p)}{p^2 - ib_k p} - \frac{\widehat{G_k}(p)}{p^2 - ib_k p} \right| = \left| \frac{\int_0^p \left[\frac{d\widehat{G_{k,m}}(s)}{ds} - \frac{d\widehat{G_k}(s)}{ds} \right] ds}{p(p - ib_k)} \right|. \tag{5.22}$$

From the definition of the standard Fourier transform (5.1) we easily derive

$$\left| \frac{d\widehat{G_{k,m}}(p)}{dp} - \frac{d\widehat{G_k}(p)}{dp} \right| \leq \frac{1}{\sqrt{2\pi}} \|xG_{k,m}(x) - xG_k(x)\|_{L^1(\mathbb{R})}.$$

This enables us to obtain the upper bound on the right side of (5.22) by

$$\frac{\|xG_{k,m}(x) - xG_k(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|b_k|} \to 0, \quad m \to \infty,$$

as assumed, which proves (5.13) when $a_k = 0$. Apparently, under our conditions

$$N_{a,\ b,\ k} < \infty, \quad N_{a,\ b,\ k}^{(m)} < \infty, \quad m \in \mathbb{N}, \quad 1 \leq k \leq N, \quad a_k \geq 0, \quad b_k \in \mathbb{R}, \quad b_k \neq 0$$

by virtue of the result of Lemma 5 above. We have bounds (5.12). A trivial limiting argument using (5.15) and (5.16) yields (5.17). ■

Let the function $G_k(x) : I \to \mathbb{R}, \quad G_k(0) = G_k(2\pi)$ and its Fourier transform on the finite interval is defined as

$$G_{k,n} := \int_0^{2\pi} G_k(x) \frac{e^{-inx}}{\sqrt{2\pi}} dx, \quad n \in \mathbb{Z} \tag{5.23}$$

and $G_k(x) = \sum_{n=-\infty}^{\infty} G_{k,n} \frac{e^{inx}}{\sqrt{2\pi}}$. Evidently, we have the estimate from above

$$\|G_{k,n}\|_{l^\infty} \leq \frac{1}{\sqrt{2\pi}} \|G_k(x)\|_{L^1(I)}. \tag{5.24}$$

Similarly to the whole real line case, we will use

$$\mathcal{N}_{a,\ b,\ k} := max \left\{ \left\| \frac{G_{k,n}}{n^2 - a_k - ib_k n} \right\|_{l^\infty}, \quad \left\| \frac{n^2 G_{k,n}}{n^2 - a_k - ib_k n} \right\|_{l^\infty} \right\} \tag{5.25}$$

for $a_k \geq 0,\ b_k \in \mathbb{R},\ b_k \neq 0,\ 1 \leq k \leq N,\ N \geq 2$. Let $\mathcal{N}_{0,\ b,\ k}$ stand for (5.25) when $a_k = 0$. Under the conditions of Lemma 7 below, expressions (5.25) will be finite. This will allow us to define

$$\mathcal{N}_{a,\ b} := max_{1 \leq k \leq N} \mathcal{N}_{a,\ b,\ k} < \infty. \tag{5.26}$$

We have the following trivial statement.

Lemma 7. *Let $N \geq 2,\ 1 \leq k \leq N,\ b_k \in \mathbb{R},\ b_k \neq 0$ and $G_k(x) : I \to \mathbb{R},\ G_k(x) \in L^\infty(I),\ G_k(0) = G_k(2\pi)$ and $1 \leq l \leq N-1$.*

a) Let $a_k > 0$ for $1 \leq k \leq l$. Then $\mathcal{N}_{a,\ b,\ k} < \infty$.

b) If $a_k = 0$ for $l+1 \leq k \leq N$. Then $\mathcal{N}_{0,\ b,\ k} < \infty$ if and only if

$$(G_k(x), 1)_{L^2(I)} = 0. \tag{5.27}$$

Proof. Clearly, in both cases a) and b) of our lemma the boundedness of $\dfrac{G_{k,n}}{n^2 - a_k - ib_k n}$ implies the boundedness of $\dfrac{n^2 G_{k,n}}{n^2 - a_k - ib_k n}$. Indeed, $\dfrac{n^2 G_{k,n}}{n^2 - a_k - ib_k n}$ can be easily expressed as

$$G_{k,n} + a\frac{G_{k,n}}{n^2 - a_k - ib_k n} + ib_k \frac{nG_{k,n}}{n^2 - a_k - ib_k n}. \tag{5.28}$$

Obviously, the first term in (5.28) can be trivially estimated from above via (5.24) for $G_k(x) \in L^\infty(I) \subset L^1(I)$. The third term in (5.28) can be bounded from above by means of (5.24) as well, namely

$$|b_k| \frac{|n||G_{k,n}|}{\sqrt{(n^2 - a_k)^2 + b_k^2 n^2}} \leq |G_{k,n}| \leq \frac{1}{\sqrt{2\pi}} \|G_k(x)\|_{L^1(I)} < \infty.$$

Hence, $\dfrac{G_{k,n}}{n^2 - a_k - ib_k n} \in l^\infty$ yields $\dfrac{n^2 G_{k,n}}{n^2 - a_k - ib_k n} \in l^\infty$. To establish the statement of the part a) of the lemma, we need to consider

$$\frac{|G_{k,n}|}{\sqrt{(n^2 - a_k)^2 + b_k^2 n^2}}. \tag{5.29}$$

Apparently, the denominator in (5.29) can be estimated from below by a positive constant and the numerator in (5.29) can be trivially treated by virtue of (5.24). Thus, $\mathcal{N}_{a,\ b,\ k} < \infty$ when $a_k > 0$. To prove the result of the part b), we observe that

$$\left| \frac{G_{k,n}}{n(n - ib_k)} \right| \tag{5.30}$$

is bounded if and only if $G_{k,0} = 0$, which is equivalent to orthogonality relation (5.27). In this case (5.30) can be easily bounded from above by

$$\frac{1}{\sqrt{2\pi}} \frac{\|G_k(x)\|_{L^1(I)}}{\sqrt{n^2 + b_k^2}} \leq \frac{1}{\sqrt{2\pi}} \frac{\|G_k(x)\|_{L^1(I)}}{|b_k|} < \infty$$

by means of (5.24) and the one of our assumptions. ■

In order to treat systems (2.9), we define for the technical purposes

$$\mathcal{N}_{a,\ b,\ k}^{(m)} := max\left\{ \left\| \frac{G_{k,m,n}}{n^2 - a_k - ib_k n} \right\|_{l^\infty}, \quad \left\| \frac{n^2 G_{k,m,n}}{n^2 - a_k - ib_k n} \right\|_{l^\infty} \right\}, \tag{5.31}$$

where $a_k \geq 0$, $b_k \in \mathbb{R}$, $b_k \neq 0$ and $m \in \mathbb{N}$. Under the conditions of Lemma 8 below, quantities (5.31) will be finite. This will enable us to define

$$\mathcal{N}_{a,\ b}^{(m)} := max_{1 \leq k \leq N} \mathcal{N}_{a,\ b,\ k}^{(m)}, \tag{5.32}$$

with $m \in \mathbb{N}$. Our final auxiliary statement is as follows.

Lemma 8. *Let $N \geq 2$, $1 \leq k \leq N$, $b_k \in \mathbb{R}$, $b_k \neq 0$ and $G_{k,m}(x) : I \to \mathbb{R}$, $G_{k,m}(0) = G_{k,m}(2\pi)$, $G_{k,m}(x) \in L^\infty(I)$, such that $G_{k,m}(x) \to G_k(x)$ in $L^\infty(I)$ as $m \to \infty$ and $1 \leq l \leq N-1$.*

a) Let $a_k > 0$ for $1 \leq k \leq l$.

b) Let $a_k = 0$ for $l+1 \leq k \leq N$ and additionally

$$(G_{k,m}(x), 1)_{L^2(I)} = 0, \quad m \in \mathbb{N}. \tag{5.33}$$

Let us also assume that

$$2\sqrt{\pi}\mathcal{N}_{a,\ b}^{(m)} L \leq 1 - \varepsilon \tag{5.34}$$

holds for all $m \in \mathbb{N}$ as well with a certain fixed $0 < \varepsilon < 1$. Then, for all $1 \leq k \leq N$, we have

$$\frac{G_{k,m,n}}{n^2 - a_k - ib_k n} \to \frac{G_{k,n}}{n^2 - a_k - ib_k n}, \quad m \to \infty, \tag{5.35}$$

$$\frac{n^2 G_{k,m,n}}{n^2 - a_k - ib_k n} \to \frac{n^2 G_{k,n}}{n^2 - a_k - ib_k n}, \quad m \to \infty \tag{5.36}$$

in l^∞, such that

$$\left\| \frac{G_{k,m,n}}{n^2 - a_k - ib_k n} \right\|_{l^\infty} \to \left\| \frac{G_{k,n}}{n^2 - a_k - ib_k n} \right\|_{l^\infty}, \quad m \to \infty, \tag{5.37}$$

$$\left\| \frac{n^2 G_{k,m,n}}{n^2 - a_k - ib_k n} \right\|_{l^\infty} \to \left\| \frac{n^2 G_{k,n}}{n^2 - a_k - ib_k n} \right\|_{l^\infty}, \quad m \to \infty. \tag{5.38}$$

Moreover,

$$2\sqrt{\pi}\mathcal{N}_{a,\ b}L \le 1-\varepsilon. \tag{5.39}$$

Proof. Evidently, under the given conditions, the limiting kernel functions $G_k(x),\ 1 \le k \le N$ are periodic as well. Indeed, we derive

$$|G_k(0)-G_k(2\pi)| \le |G_k(0)-G_{k,m}(0)|+|G_{k,m}(2\pi)-G_k(2\pi)| \le 2\|G_{k,m}(x)-G_k(x)\|_{L^\infty(I)} \to 0$$

when $m \to \infty$ due to the one of our assumptions. Hence, $G_k(0) = G_k(2\pi)$ with $1 \le k \le N$ holds. Apparently,

$$\|G_{k,m,n}-G_{k,n}\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}}\|G_{k,m}-G_k\|_{L^1(I)} \le \sqrt{2\pi}\|G_{k,m}-G_k\|_{L^\infty(I)} \to 0, \quad m \to \infty \tag{5.40}$$

as assumed. We observe that the statements of (5.35) and (5.36) yield (5.37) and (5.38) respectively by means of the triangle inequality. Let us show that (5.35) yields (5.36). For this purpose, we write $\frac{n^2[G_{k,m,n}-G_{k,n}]}{n^2-a_k-ib_kn}$ as

$$[G_{k,m,n}-G_{k,n}]+a_k\frac{G_{k,m,n}-G_{k,n}}{n^2-a_k-ib_kn}+ib_k\frac{n[G_{k,m,n}-G_{k,n}]}{n^2-a_k-ib_kn}. \tag{5.41}$$

The first term in (5.41) converges to zero in the l^∞ norm as $m \to \infty$ due to estimate (5.40). The third term in (5.41) can be estimated from above in the absolute value as

$$\frac{|b_k||n||G_{k,m,n}-G_{k,n}|}{\sqrt{(n^2-a_k)^2+b_k^2n^2}} \le \|G_{k,m,n}-G_{k,n}\|_{l^\infty}.$$

Therefore, it tends to zero as $m \to \infty$ in the l^∞ norm via (5.40) as well. This proves that (5.35) implies (5.36).

First we prove (5.35) in the case a) when $a_k > 0$. Let us consider the expression

$$\frac{|G_{k,m,n}-G_{k,n}|}{\sqrt{(n^2-a_k)^2+b_k^2n^2}}. \tag{5.42}$$

Evidently, the denominator of (5.42) can be estimated from below by a positive constant and the numerator bounded from above by virtue of (5.40). This implies (5.35) for $a_k > 0$.

Then we turn our attention to establishing (5.35) in the case b) when $a_k = 0$. By virtue of the one of our assumptions, we have orthogonality conditions (5.33). Let us show that the analogous relations hold in the limit. Indeed,

$$|(G_k(x),1)_{L^2(I)}| = |(G_k(x)-G_{k,m}(x),1)_{L^2(I)}| \le 2\pi\|G_{k,m}(x)-G_k(x)\|_{L^\infty(I)} \to 0, \quad m \to \infty$$

as assumed. Hence,

$$(G_k(x),1)_{L^2(I)} = 0, \quad 1 \le k \le N,$$

which is equivalent to $G_{k,0}=0$. Clearly, $G_{k,m,0}=0,\ 1\leq k\leq N,\ m\in\mathbb{N}$ due to orthogonality relations (5.33). Then by virtue of (5.40), we arrive at

$$\left|\frac{G_{k,m,n}-G_{k,n}}{n(n-ib_k)}\right|\leq\frac{\sqrt{2\pi}\|G_{k,m}(x)-G_k(x)\|_{L^\infty(I)}}{|b_k|}.$$

The norm in the right side of this estimate converges to zero as $m\to\infty$. Thus, (5.35) holds when $a_k=0$ as well. Apparently, under the given conditions

$$\mathcal{N}_{a,\ b,\ k}<\infty,\quad \mathcal{N}^{(m)}_{a,\ b,\ k}<\infty,\qquad m\in\mathbb{N},\quad 1\leq k\leq N,\quad a_k\geq 0,\quad b_k\in\mathbb{R},\quad b_k\neq 0$$

by means of the result of Lemma 7 above. We have inequality (5.34). A trivial limiting argument using (5.37) and (5.38) yields (5.39). ■

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