A Near Proof of Weak Graph Positivity, A New Property of Regular Random Graphs

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Abstract

One deals with r-regular bipartite graphs with $2n$ vertices. In a previous paper Butera, Pernici, and the author have introduced a quantity $d(i)$, a function of the number of i-matchings, and conjectured that as n goes to infinity the fraction of graphs that satisfy $\Delta^k d(i) \geq 0$, for all k and i, approaches 1. Here Δ is the finite difference operator. This conjecture we called the 'graph positivity conjecture'. In this paper it is formally shown that for each i and k the probability that $\Delta^k d(i) \geq 0$ goes to 1 with n going to infinity. We call this weaker result the 'weak graph positivity conjecture (theorem)'. A formalism of Wanless as systematized by Pernici is central to this effort. Our result falls short of being a rigorous proof since we make a sweeping conjecture (computer tested), of which we so far have only a portion of the proof.

1 Introduction

We deal with r-regular bipartite graphs with $v = 2n$ vertices. We let m_i be the number of i-matchings. In [1], Butera, Pernici, and I introduced the quantity $d(i)$, in eq. (10) therein,

$$d(i) \equiv \ln\left(\frac{m_i}{r^i}\right) - \ln\left(\frac{\overline{m}_i}{(v-1)^i}\right) \tag{1.1}$$

where $\overline{m}_i$ is the number of i-matchings for the complete (not bipartite complete) graph on the same vertices,

$$\overline{m}_i = \frac{v!}{(v-2i)!\, i!\, 2^i} \tag{1.2}$$

We here have changed some of the notation from [1] to agree with notation in [2]. We then considered $\Delta^k d(i)$ where Δ is the finite difference operator, so

$$\Delta d(i) = d(i+1) - d(i) \tag{1.3}$$

A graph was defined to satisfy *graph positivity* if all the meaningful $\Delta^k d(i)$ were non-negative. That is

$$\Delta^k d(i) \geq 0 \tag{1.4}$$

for $k = 0, \ldots, v$ and $i = 0, \ldots, v-k$. We made the conjecture, the 'graph positivity conjecture', supported by some computer evidence,

Conjecture. *As n goes to infinity the fraction of graphs that satisfy graph positivity approaches one.*

In this paper we "formally" prove a weaker result, the 'weak graph positivity conjecture', in the same direction.

Theorem 1.1. *For each i and k, one has*

$$\mathrm{Prob}\big(\Delta^k d(i) \geq 0\big) \xrightarrow[n\to\infty]{} 1$$

The paper relies heavily on the work of Wanless, [3], and Pernici, [2], that gives a nice representation of the m_i. The restriction to bipartite graphs is mainly because this restriction is made in [1]. In this paper the bipartite nature appears used in two places. First, the number of vertices is assumed to be even, and second, in eq.(3.5) the lower limit 4 is replaced by 3 if one does not assume the graph is bipartite.

As said in the abstract our result falls short of being a rigorous proof, since we make a sweeping conjecture (computer tested) presented in Section 10. We are assembling parts of the proof and hope to have a complete proof in the not too distant future.

One should read the *Valuable Observation* at the end of Section 4 to see why the question of convergence (of series) in this paper is a piece of cake.

We hope the reader sees the beauty of some of the arguments in this paper, simple ideas building on one another.

2 Idea of the proof

Suppose we want $\mathrm{Prob}(x > y)$ to be large. We have

$$\mathrm{Prob}(x < y) = \mathrm{Prob}(e^x < e^y) \tag{2.1}$$

Set

$$(e^x - e^y) \equiv \alpha_0 \tag{2.2}$$

and

$$\mathrm{E}(e^x - e^y) \equiv \alpha \tag{2.3}$$

We will want α to be positive, and in the present work proving this positivity will be a major component. Let

$$\mathrm{E}\big((e^x - e^y)^2\big) - \alpha^2 \equiv \beta \tag{2.4}$$

Then, assuming $\alpha > 0$,

$$\mathrm{E}\big((e^x - e^y - \alpha)^2\big) \geq \alpha^2 \ \mathrm{P}(e^x - e^y < 0) \tag{2.5}$$

And so

$$\mathrm{Prob}(e^x < e^y) \leq \frac{\beta}{\alpha^2} \tag{2.6}$$

In our problem β and α will be functions of n, and we'll want probability to go to zero with n as n goes to infinity.

We turn to the object of study, and perform some simple manipulations, working from eq. (1.1)

$$\mathrm{Prob}\big(\Delta^k d(i) > 0\big) = \mathrm{Prob}\left((-1)^k \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \ln\left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}}\right) > 0\right) \tag{2.7}$$

$$= \mathrm{Prob}\left(\sum_{\ell \in \mathcal{L}^+} \binom{k}{\ell} \ln\left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}}\right) > \sum_{\ell \in \mathcal{L}^-} \binom{k}{\ell} \ln\left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}}\right)\right) \tag{2.8}$$

where $\mathcal{L}^+$ is the set of odd ℓ, $0 \le \ell \le k$, if k is odd and is the set of even ℓ, $0 \le \ell \le k$, if k is even, and $\mathcal{L}^-$ is defined vice versa.

Returning to the language of (2.1)–(2.6), we set

$$x \equiv \sum_{\ell \in \mathcal{L}^+} \binom{k}{\ell} \ln\left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}} \right) \tag{2.9}$$

$$y \equiv \sum_{\ell \in \mathcal{L}^-} \binom{k}{\ell} \ln\left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}} \right) \tag{2.10}$$

and so

$$e^x = \prod_{\ell \in \mathcal{L}^+} \left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}} \right)^{\binom{k}{\ell}} \tag{2.11}$$

$$e^y = \prod_{\ell \in \mathcal{L}^-} \left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}} \right)^{\binom{k}{\ell}} \tag{2.12}$$

Throughout, the caveat "for n large enough" is understood.

3 The Work of Wanless and Pernici

In [3] Wanless developed a formalism to compute the m_i of any regular graph. We here only give a flavor of this formalism, but present some of the consequences we will use in this paper. For each i there are defined a set of graphs $g_{i1}, g_{i2}, \ldots, g_{in(i)}$. Given a regular graph g, one computes for each j the number of subgraphs of g isomorphic to g_{ij}, call this $g \mathbin{/\!/} g_{ij}$. Then m_i for g is determined by the $n(i)$ values of $g \mathbin{/\!/} g_{ij}$. We define M_i to be the value of m_i assigned to any graph with all $n(i)$ values of $g \mathbin{/\!/} g_{ij}$ zero. Such graphs will exist only for large enough n. Initially M_i is defined only for such n. But it may be extended as a finite polynomial in $\frac{1}{n}$ to all non-zero n. $M_i = M_i(r, n)$ is an important object of study to us.

In [2] Pernici systematized the results of Wanless; we take a number of formulae from this paper. From eq. (12) and eq. (13) of [2] we write

$$M_j = \frac{n^j r^j}{j!}(1 + H_j) \tag{3.1}$$

$$H_j = \sum_{h=1}^{j-1} \frac{a_h(r,j)}{n^h} \tag{3.2}$$

a_h is a polynomial of degree at most $2h$ in j. We view M_j and H_j as formal polynomials in j amd $\frac{1}{n}$. We will sometimes need values of a_h, as given by eq. (18) and eq. (45) of [2].

We will use eq. (16) and (17) of [2]

$$[j^k n^{-h}] \ln\left(1 + H_j\right) = [j^k n^{-h}] \ln\left(1 + \sum_{s=1}^{j-1} \frac{a_s(r,j)}{n^s}\right) = 0, \qquad k \ge h+2 \tag{3.3}$$

$$[j^{h+1} n^{-h}] \ln\left(1 + H_j\right) = [j^{h+1} n^{-h}] \ln\left(1 + \sum_{s=1}^{j-1} \frac{a_s(r,j)}{n^s}\right) = \frac{1}{(h+1)h}\left(\frac{1}{r^h} - 2\right) \tag{3.4}$$

where $[j^k n^{-h}]$ in front of an expression picks out the coefficient of the $\frac{j^k}{n^h}$ term, $c(k,h)$, in an expression $\sum_{\alpha,\beta} c(\alpha,\beta)\frac{j^\alpha}{n^\beta}$. We are here working with the formal polynomials with r fixed. Eq. (3.3) and (3.4) are special cases of the conjecture of Section 10. We already have a rigorous proof of (3.3) due to Robin Chapman, [5]. We have some good ideas toward the full proof.

We set $M_0 = 1$ and $M_s = 0$ if $s < 0$. Then m_j is recovered from M_j by the formula

$$m_j = \exp\left(\sum_{s\geq 4} \frac{\varepsilon_s}{2s}(-\hat{x})^s\right) M_j \tag{3.5}$$

eq. (11) of [2]. Here

$$\hat{x} M_j = M_{j-1} \tag{3.6}$$

ε_s for a graph g is a linear function of a finite number of $g \,/\!\!/\, \ell_i$, ℓ_i a set of given graphs, the 'contributors'. The only thing we need to know is that for any given product of ε_s's, $\prod_i \varepsilon_{s(i)}$, one has that

$$\mathrm{E}\left(\prod_i \varepsilon_{s(i)}\right) \leq C \tag{3.7}$$

i.e. it is a bounded function of n. Here as everywhere in this paper the expectation is the average value of the function over all r regular bipartite graphs of order $2n$.

The result one needs to see this is that the number of s-cycles are independent Poisson random variables of finite means in the fixed r, n goes to infinity limit, [4]. One then uses the fact that the ℓ_i and g_{ij} graphs discussed above all are either single cycle or multicycle in nature.

Working from eq. (3.5) one can arrange the resultant terms arising into the following expression for m_j

$$m_j = \frac{n^j r^j}{j!}(1 + \hat{H}_j) \tag{3.8}$$

$$\hat{H}_j = \sum_{h=1}^{j-1} \frac{\hat{a_h}(r, j, \{\epsilon_i\})}{n^h} \tag{3.9}$$

m_j is a function on graphs, eq. (3.5) or eq. (3.8)-(3.9) in turn expresses m_j as a polynomial in the $\{\epsilon_i\}$, these also functions on the graphs. We will be dealing with expectations of polynomials in the $\{m_j\}$, for example eq. (4.3). We make the important observation that, for the sum in eq. (3.9) appearing in an expectation, the n dependence of the ϵ_i does not effect the formal expected asymptotic expansion by powers of $1/n$, from the discussion surrounding eq. (3.7).

Assuming as we do the conjecture of Section 10 there follows from eq. (3.1)-(3.6) and (3.8)-(3.9)

$$[j^k n^{-h}] \ln\left(1 + \hat{H}_j\right) = [j^k n^{-h}] \ln\left(1 + \sum_{s=1}^{j-1} \frac{\hat{a_s}(r, j, \{\epsilon_i\})}{n^s}\right) = 0, \qquad k \geq h+2 \tag{3.10}$$

$$[j^{h+1} n^{-h}] \ln\left(1 + \hat{H}_j\right) = [j^{h+1} n^{-h}] \ln\left(1 + \sum_{s=1}^{j-1} \frac{\hat{a_s}(r, j, \{\epsilon_i\})}{n^s}\right) = \frac{1}{(h+1)h}\left(\frac{1}{r^h} - 2\right) \tag{3.11}$$

Again we are working with formal polynomials, for fixed r and ϵ_i.

4 Some simple reorganization

We define

$$1 + K_i \equiv \frac{(v-1)^i}{r^i} \cdot \frac{(v-2i)!\, i!\, 2^i}{v!} \cdot \frac{r^i n^i}{i!} \tag{4.1}$$

using notably eq. (1.2). Then with

$$\alpha_0 = \left(\prod_{\ell \in \mathcal{L}^+} \big((1 + \hat{H_{i+\ell}})(1 + K_{i+\ell})\big)^{\binom{k}{\ell}} - \prod_{\ell \in \mathcal{L}^-} \big((1 + \hat{H_{i+\ell}})(1 + K_{i+\ell})\big)^{\binom{k}{\ell}} \right) \tag{4.2}$$

α becomes

$$\alpha = \mathrm{E}(\alpha_0) \tag{4.3}$$

Further we set

$$1 + K_i \equiv e^{G_i} \tag{4.4}$$

where

$$G_i \equiv G_{i,1} + G_{i,2} + G_{i,3} + G_{i,4} + G_{i,5} \tag{4.5}$$

$$G_{i,1} \equiv i \ \ln\Big(1 - \frac{1}{2n}\Big) \tag{4.6}$$

$$G_{i,2} \equiv (2n - 2i) \ \ln\Big(1 - \frac{i}{n}\Big) \tag{4.7}$$

$$G_{i,3} \equiv 2i \tag{4.8}$$

$$G_{i,4} \equiv \frac{1}{2} \ \ln\Big(1 - \frac{i}{n}\Big) \tag{4.9}$$

$$G_{i,5} \equiv \sum_{j \text{ odd}} c_j \left(\frac{1}{n^j} - \frac{1}{(n-i)^j} \right) \tag{4.10}$$

We have used the Stirling series to expand $\ln n!$. We also note that for example $c_1 = -\frac{1}{24}$. K_i is easily developed as a series in inverse powers of n.

Valuable Observation The convergence problem for series, except inside expectation values, is trivial, since one deals with r, j, and ϵ_i (taken as a number) fixed and n large enough. BUT, the only expectations we take are of α_0 and α_0^2 (for β). And, see (4.2) and the discussion after (3.8)-(3.9), these both are finite polynomials in the $\{\epsilon_i\}$! (So to study α_0 and α it is a good idea to expand α_0 in the formal series in powers of $\frac{1}{n}$ taking the coefficients of the terms through $\frac{1}{n^{k-1}}$ from eq.(6.2) and the rest of the terms from eq.(4.2).)

5 $k = 1$ and $k = 0$

Not only is $k = 1$ the first case, but it is different from $k \geq 2$ in some essential ways. We proceed to compute α for $k = 1$. From eq. 4.2 we have

$$\alpha_0 = \big((1 + \hat{H_{i+1}})(1 + K_{i+1}) - (1 + \hat{H_i})(1 + K_i)\big) \tag{5.1}$$

From eq. (18) and eq. (45) of [2] one gets

$$\hat{H_i} = i(i-1)\Big(-1 + \frac{1}{2r}\Big)\frac{1}{n} + \mathcal{O}\Big(\frac{1}{n^2}\Big) \tag{5.2}$$

In such asymptotic series bounds we treat the ϵ_i as constants.

Using eq. (4.4)–(4.10) one gets

$$K_i = (i^2 - i)\frac{1}{n} + \mathcal{O}\Big(\frac{1}{n^2}\Big) \tag{5.3}$$

There follows

Theorem 5.1. *For $k = 1$*

$$\alpha_0 = \frac{i}{rn} + \mathcal{O}\Big(\frac{1}{n^2}\Big) \tag{5.4}$$

One easily gets

Theorem 5.2. *For $k = 0$*

$$\alpha_0 = 1 + \mathcal{O}\Big(\frac{1}{n}\Big) \tag{5.5}$$

6 $k \geq 2$

The goal of this section is proving the following theorem.

Theorem 6.1. *For $k \geq 2$*

$$\alpha_0 = \frac{(k-2)!}{r^{k-1}n^{k-1}} + \mathcal{O}(\frac{1}{n^k}) \tag{6.1}$$

From Section 8 using the Second Identity we have

$$\alpha_0 = \Big(\Big(1 + t_+ + \frac{1}{2}t_+^2 + \cdots\Big) - \Big(1 + t_- + \frac{1}{2}t_-^2 + \cdots\Big)\Big) \tag{6.2}$$

where with

$$1 + U_i = (1 + \hat{H}_i)(1 + K_i) \tag{6.3}$$

one defines

$$t_+ = \sum_{\ell \in \mathcal{L}^+} \binom{k}{\ell}\Big(U_{i+\ell} - \frac{1}{2}(U_{i+\ell})^2 + \frac{1}{3}(U_{i+\ell})^3 \cdots\Big) \tag{6.4}$$

$$t_- = \sum_{\ell \in \mathcal{L}^-} \binom{k}{\ell}\Big(U_{i+\ell} - \frac{1}{2}(U_{i+\ell})^2 + \frac{1}{3}(U_{i+\ell})^3 \cdots\Big) \tag{6.5}$$

We treat the terms written explicitly in (6.2); the induction to higher powers of t is trivial.

$$\Big[\frac{1}{n^d}\Big](t_+ - t_-) = \Big[\frac{1}{n^d}\Big]\sum_{\ell}\binom{k}{\ell}(-1)^{k+\ell}\Big(U_{i+\ell} - \frac{1}{2}(U_{i+\ell})^2 + \frac{1}{3}(U_{i+\ell})^3 \cdots\Big) \tag{6.6}$$

$$= \begin{cases} 0 & d < k-1 \\ \dfrac{(k-2)!}{r^{k-1}} & d = k-1 \end{cases} \tag{6.7}$$

by the First Identity, Theorem 7.1. In applying $\Big[\frac{1}{n^d}\Big]$ we treat the ϵ_i as constants. Next we want to prove that the higher powers of t's make no contribution in (6.2)! This is amazing when one first sees it.

We want to show

$$\left[\frac{1}{n^d}\right](t_+^2 - t_-^2) = 0 \quad \text{for} \quad d \le k-1 \tag{6.8}$$

We proceed by looking at the powers of $\frac{1}{n}$.

$$\left[\frac{1}{n^d}\right](t_+^2 - t_-^2) = \sum_{s=1}^{d-1}\left[\left(\left[\frac{1}{n^s}\right]t_+\right)\left(\left[\frac{1}{n^{d-s}}\right]t_+\right) - \left(\left[\frac{1}{n^s}\right]t_-\right)\left(\left[\frac{1}{n^{d-s}}\right]t_-\right)\right] \tag{6.9}$$

All we need to complete a proof of (6.8) is to show

$$\left[\frac{1}{n^s}\right]t_+ = \left[\frac{1}{n^s}\right]t_- \qquad 1 \le s \le d-1 \tag{6.10}$$

But this follows from (6.6),(6.7) above. Pretty neat.

7 First Identity

For convenience we introduce

$$F_i \equiv (1+\hat{H}_i)(1+K_i) \tag{7.1}$$

Theorem 7.1 (First Identity)**.** *For all r, $i \ge 0$, and $k \ge 2$*

$$\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell+k}\left[\frac{1}{n^{k-1}}\right]\sum_{m=1}^{k-1}(-1)^{m+1}\frac{1}{m}\big(F_{(i+\ell)}-1\big)^m = \frac{(k-2)!}{r^{k-1}} \tag{7.2}$$

Theorem 7.2. *For all r, and $k \ge 2$*

$$\left[\frac{1}{n^{k-1}}\right]\ln(F_i) \tag{7.3}$$

has highest power of $i = i^k$, and this term is

$$\frac{(k-2)!}{k!}\frac{i^k}{r^{k-1}} \tag{7.4}$$

For example for $k=3$

$$\left[\frac{1}{n^2}\right]\ln(F_s) = -\frac{1}{12}\frac{s\big(3r^2s - 3r^2 - 12rs - 2s^2 + 12r + 9s - 7\big)}{r^2} \tag{7.5}$$

We now note

$$\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell+k}\ell^d = \begin{cases} 0 & d<k \\ k! & d=k \end{cases} \tag{7.6}$$

that follows from

$$\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell+k}\ell^d = \Delta^k i^d \tag{7.7}$$

which like $(\frac{d}{dx})^k x^d$ has values in (7.6). From eq. (7.6) one can deduce that Theorem 7.1 follows from Theorem 7.2, which we proceed to prove.

$$\left[\frac{1}{n^{k-1}}\right]\ln(F_i) = \left[\frac{1}{n^{k-1}}\right]\ln(1+\hat{H}_i) + \left[\frac{1}{n^{k-1}}\right]\ln(1+K_i) \tag{7.8}$$

From eqs.(3.10) (3.11) with $k \geq 2$ we see that the highest power of i in $[\frac{1}{n^{k-1}}] \ln(1+\hat{H_i})$ is k and its coefficient is

$$\frac{1}{k(k-1)}\left(\frac{1}{r^{k-1}} - 2\right) \tag{7.9}$$

To study $[\frac{1}{n^{k-1}}] \ln(1+K_i)$ we turn to equations (4.4)-(4.10). We note the highest power of i arises from the expansion of the term $G_{i,2}$, eq.(4.7), and $[\frac{1}{n^{k-1}}] \ln(1+K_i)$ has highest power i

$$\frac{2}{k(k-1)}$$

So

$$\left[\frac{1}{n^{k-1}}\right] \ln(F_i) = \frac{(k-2)!}{k!}\frac{1}{r^{k-1}} i^k \tag{7.10}$$

Quod erat demonstrandum.

8 Second Identity

We start with some simple manipulations

$$\prod_i (1+x_i)^{e_i} = e^{\sum e_i \ln(1+x_i)} = 1 + \left(\sum e_i \ln(1+x_i)\right) + \frac{1}{2!}\left(\sum e_i \ln(1+x_i)\right)^2 + \cdots \tag{8.1}$$

With the notation

$$(1+\hat{H_i})(1+K_i) \equiv 1 + U_i \tag{8.2}$$

we substitute U_i for x_i and $\binom{k}{\ell}$ for e_i in (8.1)

$$\prod_{\ell \in \mathcal{L}^+} (1+U_{i+\ell})^{\binom{k}{\ell}} = 1 + t_+ + \frac{1}{2} t_+^2 + \frac{1}{3!} t_+^3 \cdots \tag{8.3}$$

where

$$t_+ \equiv \sum_{\ell \in \mathcal{L}^+} \binom{k}{\ell} \left(U_{i+\ell} - \frac{1}{2}(U_{i+\ell})^2 + \frac{1}{3}(U_{i+\ell})^3 \cdots \right) \tag{8.4}$$

The Second Identity consists of (8.3) and (8.4) and the same expressions with $\mathcal{L}^+$, t_+ replaced by $\mathcal{L}^-$, t_-.

9 Completion

The information we need from the calculations of this paper are **Theorem 5.1**, **Theorem 5.2**, and **Theorem 6.1**. From these respectively we get:

1) For $k = 1, i \neq 0$

$$\alpha \geq \frac{c}{n}, \qquad c \quad \textit{positive} \tag{9.1}$$

$$\beta \leq \frac{c}{n^4}, \tag{9.2}$$

2) For $k = 0$

$$\alpha \geq c, \qquad c \quad \textit{positive} \tag{9.3}$$

$$\beta \leq \frac{c}{n^2}, \tag{9.4}$$

3) For $k \geq 2$

$$\alpha \geq \frac{c}{n^{k-1}}, \qquad c \quad positive \tag{9.5}$$

$$\beta \leq \frac{c}{n^{2k}}, \tag{9.6}$$

We are notationally using a c that varies from equation to equation. (We due not pursue stronger results that follow from the fact that ϵ_i is zero for $i = 1, 2, 3$ among other possible improvements.) Referring to Section 2, **Theorem 1.1** follows from fact that $\frac{\beta}{\alpha^2}$ goes to zero as n goes to infinity in each case.

10 An Awesome Conjecture

Let z_i be positive integers. We set:

$$F = \sum_{s\geq 0} \frac{a_s(r,j)}{n^s} + \sum_i c_i j(j-1)\cdots(j-z_i+1)\frac{1}{n^{z_i}r^{z_i}} \sum_{s\geq 0} \frac{a_s(r,j-z_i)}{n^s} \tag{10.1}$$

Then we conjecture:

$$[j^k n^{-h}] \ln(F) = 0, \qquad k \geq h+2 \tag{10.2}$$

$$[j^{h+1} n^{-h}] \ln(F) = \frac{1}{(h+1)h}\left(\frac{1}{r^h} - 2\right) \tag{10.3}$$

Compare eq. (10.2) - (10.3) to eq. (3.4) - (3.5).

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