Inverse problems for some systems of parabolic equations with coefficient depending on time

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Abstract: We consider the system $\vec{u}_t - a(t)M\vec{u}_{xx} = \vec{f}(x,t)$, where $0 \le x \le \pi,\ t \ge 0$, assuming that $\vec{u}(0,t) = \vec{u}_1(t),\ \vec{u}(\pi,t) = \vec{u}_2(t),\ \vec{u}(x,0) = \vec{h}(x)$, and the extra data $\vec{u}_x(0,t) = \vec{g}(t)$ are known. The coupling matrix M is a real, diagonalizable matrix for which all of the eigenvalues are positive reals. The inverse problem is: *How does one determine the unknown $a(t)$?* The function $a(t)$ is assumed positive, continuous and bounded. This problem is solved and a method to recover $a(t)$ is proposed. The method presented in this work enables us to evaluate the unknown coefficient $a(t)$ in closed form if the data (which can be chosen by experimenter) are properly chosen.

AMS Subject Classification: 35K20, 35R30
Key words: Parabolic systems, Inverse problems

1. Introduction

In the article an unknown coefficient $a(t)$ of the system of heat equations is evaluated in closed form from the properly chosen data. The data can be chosen at will by the experimenter. It is chosen in this work such that the inverse problem of finding $a(t)$ from the data is solved easily and $a(t)$ is determined precisely, in closed form. Let us consider the system

$$\begin{aligned} &\vec{u}_t - a(t)M\vec{u}_{xx} = \vec{f}(x,t) \quad \text{for } (x,t) \in [0,\pi] \times [0,\infty), \\ \text{with} \quad &\vec{u}(0,t) = \vec{u}_1(t), \qquad \vec{u}(\pi,t) = \vec{u}_2(t), \qquad \text{and} \qquad \vec{u}(x,0) = \vec{h}(x), \end{aligned} \tag{1.1}$$

where the vector functions $\{\vec{u}_1, \vec{u}_2, \vec{h}\}$ are known. The extra data are

$$\vec{u}_x(0,t) = \vec{g}(t). \tag{1.2}$$

The solution of the system of equations (1.1) is a real vector function given by

$$\vec{u}(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The regularity of $\vec{u}$ is related to the smoothness of $\{\vec{u}_1, \vec{u}_2, \vec{h}\}$. Analogously to the article [5], which was dealing with the studies of the single parabolic equation of this kind, in the present work we are not focused on the well-posedness of system (1.1). We are interested in the following inverse problem:

How to find $a(t)$ given the data?

Inverse problems for the scalar heat equation have been studied extensively (see [1], [2], [4] and the references therein). An inverse source problem for the multidimensional heat equation in which the source was assumed to be a finite sum of point sources was treated in [3]. The inverse problem there was to find the location and the intensity of these point sources from the experimental data. But there was no method for evaluating $a(t)$ explicitly, in closed form, as far as we know, except in [5]. The existence of stationary solutions of certain systems of parabolic equations was studied actively in recent years, see for example [6] and [7] and the references therein.

Let us use $\langle\ ,\ \rangle$ to denote the standard inner product on $L^2[0,\pi]$. Hence,

$$\langle G, F\rangle = \int_0^\pi G(x)F(x)dx. \tag{1.3}$$

Evidently, (1.3) induces the following norm on $L^2[0,\pi]$:

$$\|F\| = \sqrt{\int_0^\pi F^2(x)dx}.$$

We extend the inner product notation to the case when the first argument is a vector function, for which each component is an element of $L^2[0,\pi]$. In such case the result is obtained by evaluating the inner product of each component with the second argument. For instance,

$$\begin{aligned} \langle \vec{g}, F\rangle &= \Big(\langle g_1, F\rangle, \ldots, \langle g_N, F\rangle\Big)^T \\ &= \left(\int_0^\pi g_1(x)F(x)dx, \ldots, \int_0^\pi g_N(x)F(x)dx\right)^T \\ &= \int_0^\pi \vec{g}(x)F(x)dx. \end{aligned} \tag{1.4}$$

Similarly,

$$\begin{aligned} \langle \vec{u}(\cdot,t), F\rangle &= \int_0^\pi \vec{u}(x,t)F(x)dx \\ &= \left(\int_0^\pi u_1(x,t)F(x)dx, \ldots, \int_0^\pi u_N(x,t)F(x)dx\right)^T, \end{aligned}$$

giving a vector valued function of t.

Let $p_m(x) = \sqrt{\dfrac{2}{\pi}}\sin(mx)$ for $m \in \mathbb{N} = \{1, 2, \dots\}$. Then

$$p_m(0) = p_m(\pi) = 0, \quad \|p_m\| = 1 \;\text{ and }\; -\frac{d^2 p_m}{dx^2}(x) = m^2 p_m(x) \;\text{ for }\; 0 \le x \le \pi,$$

such that $\{p_m(x)\}_{m=1}^{\infty}$ is the orthonormal set of the eigenfunctions of the one dimensional negative Dirichlet Laplacian on the interval $[0, \pi]$. Let us proceed to the estimation of $a(t)$.

2. Evaluation of $a(t)$.

Let us introduce the change of variables

$$\vec{u} = \vec{v} + \vec{u}_1 + \frac{x}{\pi}(\vec{u}_2 - \vec{u}_1) = \vec{v} + \vec{r}$$

for our system (1.1). Thus, we arrive at the new system

$$\vec{v}_t - a(t)M\vec{v}_{xx} = \vec{f}(x,t) - \vec{r}_t := \vec{F}(x,t) \quad \text{for } (x,t) \in [0,\pi] \times [0,\infty), \tag{2.1}$$

with

$$\vec{v}(0,t) = \vec{0}, \qquad \vec{v}(\pi,t) = \vec{0}, \qquad \vec{v}(x,0) = \vec{h}(x) - \vec{r}(x,0) := \vec{H}(x).$$

We assume that the coupling matrix M in our system (1.1) is real, constant in space and time, $N \times N$ with $N \ge 2$, diagonalizable and its eigenvalues $\{d_k\}_{k=1}^{N}$ are positive reals. Hence, it follows that there exists an invertible real matrix P such that

$$PMP^{-1} = D = \text{diag}\,(d_1, \dots, d_N),$$

and

$$M = P^{-1}DP. \tag{2.2}$$

By virtue of (2.2), multiplying the system of equations (2.1) on the left by P yields

$$P\vec{v}_t - a(t)DP\vec{v}_{xx} = P\vec{F}(x,t). \tag{2.3}$$

We introduce new vector functions:

$$\tilde{v}(x,t) := P\vec{v}(x,t), \qquad \tilde{H}(x) := P\vec{H}(x) \;\text{ and }\; \tilde{F}(x,t) := P\vec{F}(x,t).$$

This enables us to write our system in terms of the new variables:

$$\begin{gathered} \frac{\partial \tilde{v}}{\partial t} - a(t)D\frac{\partial^2 \tilde{v}}{\partial x^2} = \tilde{F}(x,t), \\ \text{with} \quad \tilde{v}(0,t) = \vec{0}, \qquad \tilde{v}(\pi,t) = \vec{0} \quad \text{and} \quad \tilde{v}(x,0) = \tilde{H}(x). \end{gathered} \tag{2.4}$$

The reason that we have done this is that (2.4) consists of N fully decoupled scalar equations, allowing for solutions to be more easily obtained.

For $m \in \mathbb{N} = \{1, 2, \dots\}$ let

$$\tilde{v}_m(t) := \langle \tilde{v}(\cdot, t), p_m \rangle, \qquad \tilde{F}_m(t) := \langle \tilde{F}(\cdot, t), p_m \rangle \quad \text{and} \quad \tilde{H}_m := \langle \tilde{H}, p_m \rangle \in \mathbb{R}^N,$$

with the inner product defined in (1.4). Evidently,

$$\tilde{F}(x,t) = \sum_{m=1}^{\infty} \tilde{F}_m(t) p_m(x), \qquad \tilde{H}(x) = \sum_{m=1}^{\infty} \tilde{H}_m p_m(x). \tag{2.5}$$

Similarly,

$$\vec{F}(x,t) = \sum_{m=1}^{\infty} F_m(t) p_m(x), \qquad \vec{H}(x) = \sum_{m=1}^{\infty} \vec{H}_m p_m(x), \tag{2.6}$$

where

$$\vec{F}_m(t) := \langle \vec{F}(\cdot, t), p_m \rangle \quad \text{and} \quad \vec{H}_m := \langle \vec{H}, p_m \rangle \in \mathbb{R}^N. \tag{2.7}$$

Also,

$$\vec{f}_m(t) := \langle \vec{f}(\cdot, t), p_m \rangle \quad \text{and} \quad \vec{h}_m := \langle \vec{h}, p_m \rangle \in \mathbb{R}^N. \tag{2.8}$$

Let us seek the solution to (2.4) in the form

$$\tilde{v}(x,t) = \sum_{m=1}^{\infty} \tilde{v}_m(t) p_m(x) = \sum_{m=1}^{\infty} \Big\langle \tilde{v}(\cdot, t), p_m \Big\rangle p_m(x). \tag{2.9}$$

It is a standard result that such a solution exists. Taking the inner product of p_m with each side of the system of partial differential equations in (2.4) gives us

$$\left\langle \frac{\partial \tilde{v}}{\partial t} - a(t) D \frac{\partial^2 \tilde{v}}{\partial x^2}, p_m(x) \right\rangle = \langle \tilde{F}(x,t), p_m(x) \rangle. \tag{2.10}$$

Since the negative second derivative operator acting on square integrable functions on the $[0, \pi]$ interval with Dirichlet boundary conditions is self-adjoint, we obtain

$$\frac{d\tilde{v}_m(t)}{dt} + a(t) m^2 D \tilde{v}_m(t) = \tilde{F}_m(t), \tag{2.11}$$

for $m \in \mathbb{N}$. System (2.11) decouples into N scalar linear equations of the form $y' + Ky = b(t)$, which can be easily solved. The initial condition for (2.11) is

$$\tilde{v}_m(0) = \langle \tilde{v}(\cdot, 0), p_m \rangle = \langle \tilde{H}, p_m \rangle = \tilde{H}_m, \qquad m \in \mathbb{N}. \tag{2.12}$$

Let us assume that the function $a(t)$ is positive, continuous and bounded, such that $0 < a_0 \le a(t) \le a_1$ and introduce

$$A(t) := \int_0^t a(s) ds. \tag{2.13}$$

Clearly, we have

$$a_0 t \le A(t) \le a_1 t, \qquad lim_{t\to\infty} A(t) = \infty.$$

Also,

$$\frac{d}{dt}\left(e^{Dm^2A(t)}\right) = e^{Dm^2A(t)} Dm^2 a(t), \qquad m \in \mathbb{N}.$$

From (2.11) we derive that

$$\tilde{v}_m(t) = \mathrm{e}^{-Dm^2A(t)}\tilde{H}_m + \mathrm{e}^{-Dm^2A(t)}\int_0^t \mathrm{e}^{Dm^2A(s)}\tilde{F}_m(s)ds. \tag{2.14}$$

Therefore,

$$\tilde{v}(x,t) = \sum_{m=1}^{\infty}\left[\mathrm{e}^{-Dm^2A(t)}\tilde{H}_m + \mathrm{e}^{-Dm^2A(t)}\int_0^t \mathrm{e}^{Dm^2A(s)}\tilde{F}_m(s)ds\right]p_m(x),$$

which yields

$$\vec{v}(x,t) = P^{-1}\tilde{v}(x,t) =$$

$$= \sum_{m=1}^{\infty}\left[P^{-1}\mathrm{e}^{-Dm^2A(t)}\tilde{H}_m + P^{-1}\mathrm{e}^{-Dm^2A(t)}\int_0^t \mathrm{e}^{Dm^2A(s)}\tilde{F}_m(s)ds\right]p_m(x),$$

such that $\vec{u}(x,t) = \vec{r}(x,t) +$

$$= \sum_{m=1}^{\infty}\left[P^{-1}\mathrm{e}^{-Dm^2A(t)}\tilde{H}_m + P^{-1}\mathrm{e}^{-Dm^2A(t)}\int_0^t \mathrm{e}^{Dm^2A(s)}\tilde{F}_m(s)ds\right]p_m(x). \tag{2.15}$$

Clearly, we have

$$P^{-1}\mathrm{e}^{-Dm^2A(t)}P = \mathrm{e}^{-Mm^2A(t)},$$

which implies that

$$P^{-1}\mathrm{e}^{-Dm^2A(t)}\tilde{H}_m = \mathrm{e}^{-Mm^2A(t)}\vec{H}_m, \qquad m \in \mathbb{N}. \tag{2.16}$$

Similarly,

$$P^{-1}\mathrm{e}^{-Dm^2(A(t)-A(s))}P = \mathrm{e}^{-Mm^2(A(t)-A(s))}, \qquad m \in \mathbb{N},$$

such that

$$P^{-1}\mathrm{e}^{-Dm^2(A(t)-A(s))}\tilde{F}_m(s) = \mathrm{e}^{-Mm^2(A(t)-A(s))}\vec{F}_m(s), \qquad m \in \mathbb{N}. \tag{2.17}$$

Hence, by means of (2.15) along with identities (2.16) and (2.17) we arrive at $\vec{u}(x,t) = \vec{r}(x,t) +$

$$+\sum_{m=1}^{\infty}\left[\mathrm{e}^{-Mm^2A(t)}\vec{H}_m + \int_0^t \mathrm{e}^{-Mm^2(A(t)-A(s))}\vec{F}_m(s)ds\right]p_m(x). \tag{2.18}$$

We use extra data (1.2), such that $\vec{u}_x(0,t) = \dfrac{1}{\pi}(\vec{u}_2(t) - \vec{u}_1(t)) +$

$$+\sum_{m=1}^{\infty}\left[\mathrm{e}^{-Mm^2A(t)}\vec{H}_m + \int_0^t \mathrm{e}^{-Mm^2(A(t)-A(s))}\vec{F}_m(s)ds\right]\sqrt{\frac{2}{\pi}}m = \vec{g}(t) \qquad (2.19)$$

and denote

$$\vec{g}_1(t) := \vec{g}(t) - \frac{1}{\pi}(\vec{u}_2(t) - \vec{u}_1(t)).$$

Let us introduce the vector functions

$$\vec{Q}_0(z) := \sum_{m=1}^{\infty}\mathrm{e}^{-Mm^2z}\vec{H}_m\sqrt{\frac{2}{\pi}}m \qquad (2.20)$$

and

$$\vec{Q}(z,s) := \sum_{m=1}^{\infty}\mathrm{e}^{-Mm^2z}\vec{F}_m(s)\sqrt{\frac{2}{\pi}}m. \qquad (2.21)$$

Then we arrive at the system of equation for $A(t)$, namely

$$\vec{Q}_0(A(t)) + \int_0^t \vec{Q}(A(t)-A(s),s)ds = \vec{g}_1(t). \qquad (2.22)$$

If we manage to find $A(t)$, then

$$a(t) = \frac{dA(t)}{dt}. \qquad (2.23)$$

Let us consider a relatively simple situation, when $a(t)$ can be found in closed form. Let

$$\vec{u}_1(t) = \vec{u}_2(t) = \vec{0}. \qquad (2.24)$$

Then

$$\vec{g}_1(t) = \vec{g}(t), \qquad \vec{r} = \vec{0}, \qquad \vec{F} = \vec{f}, \qquad \vec{H} = \vec{h}. \qquad (2.25)$$

Let $\vec{w}_1$ be the eigenvector of matrix M corresponding to the eigenvalue d_1, namely

$$M\vec{w}_1 = d_1\vec{w}_1. \qquad (2.26)$$

We also assume that

$$\vec{g}(t) = g(t)\vec{w}_1, \qquad \vec{f}_m = \vec{h}_m = \vec{0}, \qquad m > 1 \qquad (2.27)$$

and when $m = 1$

$$\vec{f}_1 = \vec{h}_1 = \sqrt{\frac{\pi}{2}}\vec{w}_1. \qquad (2.28)$$

Then by means of (2.8), (2.7) and (2.25) we have the analogous identities for $\vec{F}_m$ and $\vec{H}_m$ with $m \in \mathbb{N}$. Therefore, by virtue of (2.22) we arrive at

$$\mathrm{e}^{-MA(t)}\vec{w}_1 + \int_0^t \mathrm{e}^{-M(A(t)-A(s))}\vec{w}_1 ds = g(t)\vec{w}_1.$$

Evidently,

$$\mathrm{e}^{-MA(t)}\vec{w}_1 = \mathrm{e}^{-d_1 A(t)}\vec{w}_1, \qquad \mathrm{e}^{-M(A(t)-A(s))}\vec{w}_1 = \mathrm{e}^{-d_1(A(t)-A(s))}\vec{w}_1,$$

which yields

$$\mathrm{e}^{-d_1 A(t)}\left(1 + \int_0^t \mathrm{e}^{d_1 A(s)} ds\right) = g(t). \tag{2.29}$$

Let $\phi(t) := \displaystyle\int_0^t \mathrm{e}^{d_1 A(s)} ds$, such that

$$\phi'(t) = \mathrm{e}^{d_1 A(t)}, \qquad \phi''(t) = d_1 a(t)\phi'(t). \tag{2.30}$$

For ϕ we easily obtain a first order differential equation, using (2.29), namely

$$\phi'(t) = \frac{\phi(t)}{g(t)} + \frac{1}{g(t)}, \qquad \phi(0) = 0. \tag{2.31}$$

Apparently, the solution of equation (2.31) is given by

$$\phi(t) = \int_0^t \frac{1}{g(s)} \mathrm{e}^{\int_s^t \frac{1}{g(\tau)} d\tau} ds. \tag{2.32}$$

Clearly, (2.30) yields

$$a(t) = \frac{\phi''(t)}{d_1 \phi'(t)}, \tag{2.33}$$

which enables us to find $a(t)$ in closed form via $\phi(t)$ when special data (2.24), (2.27) and (2.28) is chosen. ∎

Acknowledgement

Stimulating discussions with A.G. Ramm are gratefully acknowledged. The work was partially supported by the NSERC Discovery grant.

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