

A GELFAND-LEVITAN TRACE FORMULA FOR GENERIC QUANTUM GRAPHS

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ABSTRACT. We formulate and prove a Gelfand-Levitan trace formula for general quantum graphs with arbitrary edge lengths and coupling conditions which cover all self-adjoint operators on quantum graphs, except for a set of measure zero. The formula is reminiscent of the original Gelfand-Levitan result on the segment with Neumann boundary conditions.

1. INTRODUCTION

Given a Schrödinger operator with a potential q on a line segment of length π with Neumann boundary conditions, let us denote the corresponding eigenvalues by $\lambda_n(q)$. In [GL53] Gelfand and Levitan found and proved a formula for the sum of the differences between $\lambda_n(q)$ and the eigenvalues of the null potential $\lambda_n(0)$, namely,

$$(1) \quad \sum_{n=1}^{\infty} \left[\lambda_n(q) - \lambda_n(0) - \frac{1}{\pi} \int_0^{\pi} q(x) \, dx \right] = \frac{1}{4} [q(\pi) + q(0)] - \frac{1}{2\pi} \int_0^{\pi} q(x) \, dx,$$

under certain regularity conditions – see also [Dik53, HK60]. Since then, regularised trace formulas of this type have been present in the literature more or less continuously and were extended to many different settings and forms, including more general operators and potentials – see [SP06] for a review of the topic, including some historical notes. Also, and as was pointed out in [Bar74], there is a relation between the trace formula (1) and the short-time asymptotic expansion of the trace of the heat kernel.

Of interest to us here are the extensions to the case of quantum graphs, where this type of result may be traced back to a paper by Roth in 1983 [Rot83], with further developments in several directions such as those in [BER15, BK13, FK16, Nic87]. Quantum graphs have also received much attention in the literature within the past 30 years and, in particular, there have been several attempts at generalizing Gelfand and Levitan’s result to this setting. So far, the results obtained have been restricted to specific graphs and include, for instance, the case of equilateral graphs for which Carlson proved a formula involving integrals of the potential and the eigenfunctions [Car12], and the work of C-F. Yang and J.-X. Yang for equilateral star graphs with different boundary conditions and coupling at the central node, which are closer in form to (1) [YY07, Yan13].

A main difficulty with extending (1) to graphs with a general topology and arbitrary edge lengths is that there will then exist eigenvalue sequences with different asymptotic behaviours, making the regularisation of the trace by associating the different eigenvalues of the problem with a potential to those with the null potential a delicate issue. The purpose of the present paper is to provide an answer to this question in this general setting. We thus consider graphs with arbitrary edge lengths and topology, while the coupling is generic in the following sense. The whole class of coupling conditions defining a self-adjoint operator is allowed, with the exception of a set of measure zero

corresponding to a particular eigenvalue in the coupling matrix. This exception leaves out some important coupling conditions such as Dirichlet, standard or δ -coupling, but it does include Robin, Neumann or δ' -coupling (the last one with the exception of the case when the coupling parameter is zero). On the other hand, notice also that the trace formula is expected to be slightly different in the cases which are left out by the present approach, as this is already the case for one single interval.

A key point in our approach is that, unlike in [YY07, Yan13], for instance, we do not subtract from the eigenvalue of the Hamiltonian with the potential a particular known value – in (1), $\lambda_n(0)$ is in fact $k^2 -$, but for elegance of the result, we find the formula for the difference between the eigenvalues of the Hamiltonian with the potential and those for when this potential is zero. This is, in fact, what allows us to assign a correspondence between the different eigenvalues in such a way as to make the involved series convergent, while not making the corresponding formula cumbersome. If, for instance, one considers the first terms in the asymptotics of the different sequences, then the formula will include terms related to the coupling matrix, for instance.

The discussion above also implies that a second ingredient which is necessary to obtain for a formula of this type to work is the asymptotic behaviour of the different sequences up to an order such that we can both group eigenvalues according to their asymptotic behaviour and ensure convergence of the series involved. As far as we are aware, previous results along these lines for general graphs only considered remainders of order zero [Nic87], while for our purposes we need to go up to the term with remainder of order n^{-2} .

The separation of the full spectrum into different sequences may be done in several different ways, so we now briefly explain our procedure. It is clear that, with the end in view, the basis for this separation has to be the asymptotic behaviour of the spectrum. We first note that the leading term of the secular equation is $\prod_{i=1}^d (-k \sin(k\ell_i))$, where d is the number of the edges of the graph, k is the square root of the energy and ℓ_i are the edge lengths. In Section 4 we prove that the square roots of eigenvalues are close to the zeros of the given product and that they can be grouped in sets of at most d eigenvalues and d zeros. We thus partition the spectrum into d subsequences of eigenvalues in the following way. Denote the sequence of all eigenvalues in increasing order by $\{\lambda_n\}_{n=1}^\infty$ and let the sequence $\{\mu_n\}_{n=1}^\infty$ correspond to the non-negative zeros of the above product, also arranged in increasing order, with the first d entries being 0. We now pair λ_n with μ_n and define the subsequences $\{\lambda_{in}\}_{n=0}^\infty$ as subsequences of $\{\lambda_n\}_{n=1}^\infty$ which are paired with those zeros of $\prod_{i=1}^d (-k \sin(k\ell_i))$ which are zeros of $\sin(k\ell_i)$ for a given i (the first entry of this sequence λ_{i0} is paired with 0).

We may now formulate the main result of the paper.

Theorem A. *We assume a quantum graph with d edges with arbitrary lengths ℓ_i , $i = 1, \dots, d$, and associated coupling matrix U not having -1 in its spectrum. Then, denoting the eigenvalues of the Hamiltonian with a potential q and with the zero potential by $\lambda_{in}(q)$ and $\lambda_{in}(0)$, respectively, in the way described above, and the component of the potential on the i -th edge by $q_i \in W^{1,1}((0, \ell_i))$, the following trace formula holds*

$$\sum_{i=1}^d \sum_{n=0}^{\infty} \left[\lambda_{in}(q) - \lambda_{in}(0) - \frac{1}{\ell_i} \int_0^{\ell_i} q_i(x) dx \right] = \sum_{i=1}^d \left\{ \frac{1}{4} [q_i(\ell_i) + q_i(0)] - \frac{1}{2\ell_i} \int_0^{\ell_i} q_i(x) dx \right\}.$$

The paper is structured as follows. In the next section we describe the model of quantum graphs, and in Section 3 the secular equation is found and some preparatory calculations for Section 4 are

performed. In Section 4 we give the proofs of the main results leading to the proof of Theorem A. Several technical results used throughout the paper are given in the Appendix A.

2. DESCRIPTION OF THE MODEL

We briefly introduce the model of quantum graphs; for more details we refer the reader to [BK13]. Let us consider a metric graph Γ consisting of the set of vertices $v \in \mathcal{V}$, which are connected by the set of d finite edges $e_j \in \mathcal{E}$. The number d is finite and the lengths of the edges are $\ell_j \in (0, \infty)$. We equip the graph Γ with a self-adjoint operator

$$\mathcal{H} = -\frac{d^2}{dx^2} + q_j(x), \quad x \in e_j$$

with the real potentials $q_j \in W^{1,1}(e_j)$. The domain of \mathcal{H} consists of functions with the edge components in the Sobolev spaces $W^{2,2}(e_j)$ and satisfying the coupling conditions

$$(U_v - I)\Psi_v + i(U_v + I)\Psi'_v = 0$$

at the vertices. Here U_v is a $d_v \times d_v$ unitary matrix (d_v is the degree of the vertex v), Ψ_v is the vector of the limiting values of functions at the vertex v from its incident edge and, similarly, Ψ'_v is the vector of the derivatives outgoing from v ; I is the $d_v \times d_v$ identity matrix. Throughout the paper, we will assume that $-1 \notin \sigma(U_v)$.

With the use of the flower-like model (see [Kuc08, EL10]), where all the vertices are joined into one and the topology of the graph is described by the larger $2d \times 2d$ coupling matrix U , one may write the coupling condition as

$$(U - I)\Psi + i(U + I)\Psi' = 0.$$

Here, with a small abuse of notation, I refers now to the $2d \times 2d$ identity matrix, Ψ is the vector with the limiting values of functions defined on each edge, as the vertex is approached from either end of the edge, and Ψ' is the vector of limits of the corresponding outgoing derivatives. We assume that the first entry of the vector Ψ is the functional value at the beginning of the first edge, the second entry is the functional value at the end of the first edge, the third entry is the functional value at the beginning of the second edge, and so on, and similarly for Ψ' . Using the fact that -1 is not in $\sigma(U)$, we may write

$$(2) \quad H\Psi + \Psi' = 0,$$

where $H = -i(U + I)^{-1}(U - I)$ is a Hermitian $2d \times 2d$ matrix. We denote the entries of the matrix H in the following way

$$(3) \quad \begin{aligned} H_{2i-1,2i-1} &=: H_{11i}, & H_{2i-1,2i} &=: H_{12i}, & H_{2i,2i} &=: H_{22i}, \\ H_{2i-1,2j-1} &=: H_{11ij}, & H_{2i-1,2j} &=: H_{12ij}, \\ H_{2i,2j-1} &=: H_{21ij}, & H_{2i,2j} &=: H_{22ij}, & i < j. \end{aligned}$$

3. THE SECULAR EQUATION

In this section, we will construct the secular equation. On each edge, we assume two independent solutions of the eigenvalue problem with the eigenvalue k^2

$$\begin{aligned} c_j(x, k) &= \cos(kx) + \int_0^x \frac{\sin(k(x-t))}{k} q_j(t) c_j(t, k) dt, \\ s_j(x, k) &= \frac{\sin(kx)}{k} + \int_0^x \frac{\sin(k(x-t))}{k} q_j(t) s_j(t, k) dt. \end{aligned}$$

One can easily check that both functions satisfy $\mathcal{H}c_j(x, k) = k^2 c_j(x, k)$ and $\mathcal{H}s_j(x, k) = k^2 s_j(x, k)$.

The following lemma (the asymptotic expansion follows the idea of [Yur00]) is proven in Appendix A.

Lemma 3.1. *The functions c_j and s_j defined above satisfy*

$$c_j(x, k) = \cos(kx) + \frac{\sin(kx)}{k} \frac{1}{2} \int_0^x q_j(t) dt + \\ + \frac{\cos(kx)}{k^2} \left[\frac{1}{4}(q_j(x) - q_j(0)) - \frac{1}{8} \left(\int_0^x q_j(t) dt \right)^2 \right] + o\left(\frac{e^{|\operatorname{Im} k|x}}{k^2}\right),$$

$$c'_j(x, k) = -k \sin(kx) + \cos(kx) \frac{1}{2} \int_0^x q_j(t) dt + \\ + \frac{\sin(kx)}{k} \left[\frac{1}{4}(q_j(x) + q_j(0)) + \frac{1}{8} \left(\int_0^x q_j(t) dt \right)^2 \right] + o\left(\frac{e^{|\operatorname{Im} k|x}}{k}\right),$$

$$s_j(x, k) = \frac{\sin(kx)}{k} - \frac{\cos(kx)}{k^2} \frac{1}{2} \int_0^x q_j(t) dt + \\ + \frac{\sin(kx)}{k^3} \left[\frac{1}{4}(q_j(x) + q_j(0)) - \frac{1}{8} \left(\int_0^x q_j(t) dt \right)^2 \right] + o\left(\frac{e^{|\operatorname{Im} k|x}}{k^3}\right),$$

$$s'_j(x, k) = \cos(kx) + \frac{\sin(kx)}{k} \frac{1}{2} \int_0^x q_j(t) dt - \\ - \frac{\cos(kx)}{k^2} \left[\frac{1}{4}(q_j(x) - q_j(0)) + \frac{1}{8} \left(\int_0^x q_j(t) dt \right)^2 \right] + o\left(\frac{e^{|\operatorname{Im} k|x}}{k^2}\right).$$

We will now transform equation (2) into a form that is more appropriate for our purposes. Writing

$$a_j := \frac{1}{2} \int_0^{\ell_j} q_j(t) dt, \quad b_j := \frac{q_j(\ell_j) + q_j(0)}{4} + \frac{1}{8} \left(\int_0^{\ell_j} q_j(t) dt \right)^2,$$

and using the expression of the components of the eigenfunction as the linear combination

$$f_j(x) = A_j c_j(x, k) + B_j s_j(x, k),$$

and the corresponding conditions $c_j(0, k) = 1$, $c'_j(0, k) = 0$, $s_j(0, k) = 0$, $s'_j(0, k) = 1$, it is possible to rewrite equation (2) as

$$[HM_1(k) + M_2(k)](A_1, B_1, A_2, B_2, \dots, A_d, B_d)^T = 0.$$

Here the matrices M_1 and M_2 are given by

$$\begin{aligned}
 M_1(k) &= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ c_1(\ell_1, k) & s_1(\ell_1, k) & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & c_2(\ell_2, k) & s_2(\ell_2, k) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \cos(k\ell_1) + a_1 \frac{\sin(k\ell_1)}{k} & \frac{\sin(k\ell_1)}{k} - a_1 \frac{\cos(k\ell_1)}{k^2} & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & \cos(k\ell_2) + a_2 \frac{\sin(k\ell_2)}{k} & \frac{\sin(k\ell_2)}{k} - a_2 \frac{\cos(k\ell_2)}{k^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \\
 &\quad + o\left(\frac{e^{|\operatorname{Im} k| \max \ell_j}}{k}\right),
 \end{aligned}$$

and

$$\begin{aligned}
 M_2(k) &= \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ -c'_1(\ell_1, k) & -s'_1(\ell_1, k) & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & -c'_2(\ell_2, k) & -s'_2(\ell_2, k) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ k \sin(k\ell_1) - a_1 \cos(k\ell_1) - b_1 \frac{\sin(k\ell_1)}{k} & -\cos(k\ell_1) - a_1 \frac{\sin(k\ell_1)}{k} & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & k \sin(k\ell_2) - a_2 \cos(k\ell_2) - b_2 \frac{\sin(k\ell_2)}{k} & -\cos(k\ell_2) - a_2 \frac{\sin(k\ell_2)}{k} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \\
 &\quad + o\left(\frac{e^{|\operatorname{Im} k| \max \ell_j}}{k}\right),
 \end{aligned}$$

Hence the secular equation can be obtained as $\det [HM_1(k) + M_2(k)] = 0$. From this we obtain after a straightforward, but a bit tedious computation

$$\begin{aligned}
0 = \varphi(k) &= \prod_{i=1}^d (-k \sin(k\ell_i)) + \sum_{i=1}^d \left(\prod_{\substack{j=1 \\ j \neq i}}^d (-k \sin(k\ell_j)) \right) [\cos(k\ell_i)(a_i - \text{Tr } H_i) - 2\text{Re } H_{12i}] + \\
&+ \sum_{\substack{i,j=1 \\ i < j}}^d \left(\prod_{\substack{o=1 \\ o \neq i,j}}^d (-k \sin(k\ell_o)) \right) \left\{ \frac{\sin(k\ell_i) \sin(k\ell_j)}{d-1} (a_i \text{Tr } H_i + a_j \text{Tr } H_j - b_i - b_j - \right. \\
&\quad - \det H_i - \det H_j) + \cos(k\ell_i) \cos(k\ell_j) [a_i a_j - a_i \text{Tr } H_j - a_j \text{Tr } H_i - \\
&\quad - (|H_{11ij}|^2 + |H_{12ij}|^2 + |H_{21ij}|^2 + |H_{22ij}|^2) + \text{Tr } H_i \text{Tr } H_j] + \\
&\quad + \cos(k\ell_i) [2(\text{Tr } H_i - a_i) \text{Re } H_{12j} - 2\text{Re}(H_{11ij} \bar{H}_{12ij} + H_{22ij} \bar{H}_{21ij})] + \\
&\quad + \cos(k\ell_j) [2(\text{Tr } H_j - a_j) \text{Re } H_{12i} - 2\text{Re}(H_{11ij} \bar{H}_{21ij} + H_{22ij} \bar{H}_{12ij})] + \\
&\quad \left. + 4\text{Re } H_{12i} \text{Re } H_{12j} - 2\text{Re}(H_{12ij} \bar{H}_{21ij} + H_{11ij} \bar{H}_{22ij}) \right\} + o\left(k^{d-2} e^{|\text{Im } k| \sum_{i=1}^d \ell_i}\right).
\end{aligned}$$

where $H_i = \begin{pmatrix} H_{11i} & H_{12i} \\ \bar{H}_{12i} & H_{22i} \end{pmatrix}$ – see (3) for the definition of the entries of the matrix H .

Dividing the above formula by $\prod_{i=1}^d (-k \sin(k\ell_i))$, we write the residual term as $o\left(\frac{1}{k^2}\right)$. Although this does not hold close to the zeros of $\sin(k\ell_i)$, it does hold on the contours Γ_N and C_p defined below, thus allowing us to compute the integrals on these contours. Using Lemma A.1 we find that

$$\begin{aligned}
\frac{\varphi(k)}{\prod_{i=1}^d (-k \sin(k\ell_i))} &= 1 + \frac{1}{k} \sum_{i=1}^d \left[\cot(k\ell_i) (\text{Tr } H_i - a_i) + \frac{2\text{Re } H_{12i}}{\sin(k\ell_i)} \right] + \\
&+ \frac{1}{k^2} \left\{ \sum_{i=1}^d (a_i \text{Tr } H_i - b_i - \det H_i) + \sum_{\substack{i,j=1 \\ i < j}}^d \cot(k\ell_i) \cot(k\ell_j) [a_i a_j - a_i \text{Tr } H_j \right. \\
&\quad - a_j \text{Tr } H_i - (|H_{11ij}|^2 + |H_{12ij}|^2 + |H_{21ij}|^2 + |H_{22ij}|^2) + \text{Tr } H_i \text{Tr } H_j] + \\
&\quad + \frac{\cot(k\ell_i)}{\sin(k\ell_j)} [2\text{Tr } H_i \text{Re } H_{12j} - 2a_i \text{Re } H_{12j} - 2\text{Re}(H_{11ij} \bar{H}_{12ij} + H_{22ij} \bar{H}_{21ij})] + \\
&\quad + \frac{\cot(k\ell_j)}{\sin(k\ell_i)} [2\text{Tr } H_j \text{Re } H_{12i} - 2a_j \text{Re } H_{12i} - 2\text{Re}(H_{11ij} \bar{H}_{21ij} + H_{22ij} \bar{H}_{12ij})] + \\
&\quad \left. \frac{1}{\sin(k\ell_i) \sin(k\ell_j)} [4\text{Re } H_{12i} \text{Re } H_{12j} - 2\text{Re}(H_{12ij} \bar{H}_{21ij} + H_{11ij} \bar{H}_{22ij})] \right\} + o\left(\frac{1}{k^2}\right).
\end{aligned}$$

Using the Taylor expansion for the logarithm around one we obtain

$$\begin{aligned}
(4) \quad \ln \frac{\varphi(k)}{\prod_{i=1}^d (-k \sin(k\ell_i))} &= \frac{1}{k} \sum_{i=1}^d \left[\cot(k\ell_i)(\operatorname{Tr} H_i - a_i) + \frac{2\operatorname{Re} H_{12i}}{\sin(k\ell_i)} \right] + \\
&\frac{1}{k^2} \sum_{\substack{i,j=1 \\ i < j}}^d \left\{ \cot(k\ell_i) \cot(k\ell_j) [-(|H_{11ij}|^2 + |H_{12ij}|^2 + |H_{21ij}|^2 + |H_{22ij}|^2)] + \right. \\
&\quad + \frac{\cot(k\ell_i)}{\sin(k\ell_j)} [-2\operatorname{Re}(H_{11ij}\bar{H}_{12ij} + H_{22ij}\bar{H}_{21ij})] + \\
&\quad + \frac{\cot(k\ell_j)}{\sin(k\ell_i)} [-2\operatorname{Re}(H_{11ij}\bar{H}_{21ij} + H_{22ij}\bar{H}_{12ij})] + \\
&\quad \left. \frac{1}{\sin(k\ell_i) \sin(k\ell_j)} [-2\operatorname{Re}(H_{12ij}\bar{H}_{21ij} + H_{11ij}\bar{H}_{22ij})] \right\} \\
&+ \frac{1}{k^2} \sum_{i=1}^d \left\{ \cot^2(k\ell_i) \left[-\frac{1}{2}(\operatorname{Tr} H_i - a_i)^2 \right] + (a_i \operatorname{Tr} H_i - b_i - \det H_i) \right. \\
&\quad \left. + \frac{\cot(k\ell_i)}{\sin(k\ell_i)} [-2(\operatorname{Tr} H_i - a_i)\operatorname{Re} H_{12i}] - 2\frac{(\operatorname{Re} H_{12i})^2}{\sin^2(k\ell_i)} \right\} + o\left(\frac{1}{k^2}\right).
\end{aligned}$$

Writing $\varphi_0(k)$ for the function in the secular equation when $q_j(x) = 0$, $j = 1, \dots, d$ we obtain in a similar way

$$\begin{aligned}
(5) \quad \ln \frac{\varphi(k)}{\varphi_0(k)} &= -\frac{1}{k} \sum_{i=1}^d \cot(k\ell_i) a_i + \frac{1}{k^2} \sum_{i=1}^d \left[\frac{1}{\sin^2(k\ell_i)} a_i \operatorname{Tr} H_i - b_i \right. \\
&\quad \left. - \frac{1}{2} \cot^2(k\ell_i) a_i^2 + \frac{\cot(k\ell_i)}{\sin(k\ell_i)} 2a_i \operatorname{Re} H_{12i} \right] + o\left(\frac{1}{k^2}\right).
\end{aligned}$$

4. PROOF OF THE MAIN RESULT

Let us define the counter-clockwise contour Γ_N as a square with vertices $N - iN$, $N + iN$, $-N + iN$, $-N - iN$. Then, using the symmetric version of Rouché's theorem, we can prove the following theorem relating the number of zeros of $\prod_{i=1}^d (-k \sin(k\ell_i))$ and zeros of $\varphi(k)$ (the proof is given in the Appendix).

Theorem 4.1. *For all $\varepsilon > 0$ there exists $K > 0$ so that for all $N > K$ and $N \notin \cup_{i=1}^d \cup_{n \in \mathbb{N}_0} \left(\frac{n\pi}{\ell_i} - \frac{\varepsilon}{\ell_i}, \frac{n\pi}{\ell_i} + \frac{\varepsilon}{\ell_i} \right)$ the functions $\prod_{i=1}^d (-k \sin(k\ell_i))$ and $\varphi(k)$ have the same number of zeros inside the contour Γ_N .*

Let us denote the sequence of all eigenvalues of the operator \mathcal{H} arranged by ascending order by $\{\lambda_n\}_{n=1}^\infty$. We denote by $\{\mu_n\}_{n=1}^\infty$ the sequence in which the first d elements are 0 and all subsequent elements are positive zeros of $\prod_{i=1}^d \sin(k\ell_i)$ arranged in increasing order. We pair λ_n with μ_n . In view of Theorem 4.1, $k_n := \sqrt{\lambda_n}$ with $\operatorname{Re} k_n \geq 0$ is "close to" μ_n , as we will see in the following lemma. We will denote the sequence of eigenvalues corresponding to the zeros of $\sin(k\ell_i)$ by $\{\lambda_{in}\}_{n=0}^\infty$, where λ_{i0} corresponds to 0 and the remaining values to positive zeros of $\sin(k\ell_i)$.

Lemma 4.2. *It is possible to choose $\varepsilon > 0$ and $K > 0$ such that there exists a strictly increasing sequence $\{N_p\}_{p=1}^\infty$ with $K < N_1$ and satisfying*

$$N_p \notin \bigcup_{i=1}^d \bigcup_{n \in \mathbb{N}_0} \left(\frac{n\pi}{\ell_i} - \frac{\varepsilon}{\ell_i}, \frac{n\pi}{\ell_i} + \frac{\varepsilon}{\ell_i} \right)$$

and

$$\lim_{p \rightarrow \infty} N_p = +\infty.$$

and there are at most d eigenvalues $\lambda = k^2$ of \mathcal{H} with $N_p \leq k \leq N_{p+1}$, for all $p \in \mathbb{N}$. Furthermore, all these eigenvalues belong to different sequences λ_{in} and there are at most d zeros μ of $\prod_{i=1}^d \sin(k\ell_i)$ with $N_p \leq \mu \leq N_{p+1}$, $\forall p \in \mathbb{N}$. The number of eigenvalues and zeros with this property is the same.

Proof. We choose

$$\varepsilon < \frac{\pi}{4 \max_j \ell_j \sum_{i=1}^d \frac{1}{\ell_i}}.$$

The width of each interval $\left(\frac{n\pi}{\ell_i} - \frac{\varepsilon}{\ell_i}, \frac{n\pi}{\ell_i} + \frac{\varepsilon}{\ell_i} \right)$ is $\frac{2\varepsilon}{\ell_i}$ and so the sum of the lengths of these “forbidden” intervals for all sequences is $2\varepsilon \sum_{i=1}^d \frac{1}{\ell_i}$. We choose ε sufficiently small to ensure that this expression is smaller than $\frac{\pi}{2 \max_j \ell_j}$ (half of the smallest distance between two neighbouring zeros of the sine function from the given sequence). Hence the “forbidden intervals” do not cover the whole interval between two neighbouring zeros of a given sine function, and it is possible to choose a contour in Theorem 4.1 between them and obtain that the number of zeros of the sine and the eigenvalues in that contour is the same. \square

Now we choose for contours C_p the rectangles with vertices $N_{p+1} - iN_{p+1}$, $N_{p+1} + iN_{p+1}$, $N_p + iN_{p+1}$ and $N_p - iN_{p+1}$, traversed counter-clockwise. Inside the contour there is the same number of square roots of eigenvalues of \mathcal{H} and zeros of $\prod_{i=1}^d k \sin(k\ell_i)$ and this number is at most d . Let us first consider the case when there is only one square root of eigenvalue and one zero inside C_p .

Theorem 4.3. *Let us assume that inside the contour C_p there are the points $\frac{n\pi}{\ell_i}$ and $k_{in} = \sqrt{\lambda_{in}}$ for a given i . Then $\lambda_{in} = k_{in}^2$ behaves asymptotically as*

$$\begin{aligned} \lambda_{in} &= \left(\frac{n\pi}{\ell_i}\right)^2 + \frac{2}{\ell_i} [a_i - \text{Tr } H_i - (-1)^n 2\text{Re } H_{12i}] + \\ &+ \frac{2}{n\pi} \sum_{\substack{j=1 \\ j \neq i}}^d \left[\cot \frac{n\pi\ell_j}{\ell_i} (|H_{11ij}|^2 + |H_{12ij}|^2 + |H_{21ij}|^2 + |H_{22ij}|^2) + \right. \\ &+ \frac{1}{\sin \frac{n\pi\ell_j}{\ell_i}} 2\text{Re} (H_{11ij} \bar{H}_{12ij} + H_{22ij} \bar{H}_{21ij}) + \\ &\frac{(-1)^n}{\sin \frac{n\pi\ell_j}{\ell_i}} 2\text{Re} (H_{12ij} \bar{H}_{21ij} + H_{11ij} \bar{H}_{22ij}) + \\ &\left. (-1)^n \cot \frac{n\pi\ell_j}{\ell_i} 2\text{Re} (H_{11ij} \bar{H}_{21ij} + H_{22ij} \bar{H}_{12ij}) \right] + \mathcal{O}\left(\frac{1}{n^2}\right). \end{aligned}$$

Proof. We use the integral

$$k_{in}^2 - \left(\frac{n\pi}{\ell_i}\right)^2 = -\frac{1}{2\pi i} \oint_{C_p} \ln \frac{\varphi(k)}{\prod_{j=1}^d (-k \sin(k\ell_j))} 2k \, dk.$$

A straightforward computation using equation (4) and Lemma A.3 leads to the result. \square

When the number of square roots of the eigenvalues (and zeros of the product) is larger than one, we sum over the eigenvalues.

Theorem 4.4. *Let us assume that inside the contour C_p there are the points $\frac{n_i\pi}{\ell_i}$ and $k_{in} = \sqrt{\lambda_{in_i}}$ for i from the index set I . Then $\sum_{i \in I} \lambda_{in_i}$ behaves asymptotically as*

$$\begin{aligned} \sum_{i \in I} \lambda_{in_i} &= \sum_{i \in I} \left(\frac{n_i\pi}{\ell_i}\right)^2 + \sum_{i \in I} \frac{2}{\ell_i} [a_i - \text{Tr } H_i - (-1)^{n_i} 2\text{Re } H_{12i}] + \\ &+ \sum_{i \in I} \frac{2}{n_i\pi} \sum_{\substack{j=1 \\ n_i\ell_j \neq n_j\ell_i}}^d \left[\cot \frac{n_i\pi\ell_j}{\ell_i} (|H_{11ij}|^2 + |H_{12ij}|^2 + |H_{21ij}|^2 + |H_{22ij}|^2) + \right. \\ &+ \frac{1}{\sin \frac{n_i\pi\ell_j}{\ell_i}} 2\text{Re} (H_{11ij} \bar{H}_{12ij} + H_{22ij} \bar{H}_{21ij}) + \\ &\frac{(-1)^{n_i}}{\sin \frac{n_i\pi\ell_j}{\ell_i}} 2\text{Re} (H_{12ij} \bar{H}_{21ij} + H_{11ij} \bar{H}_{22ij}) + \\ &\left. (-1)^{n_i} \cot \frac{n_i\pi\ell_j}{\ell_i} 2\text{Re} (H_{11ij} \bar{H}_{21ij} + H_{22ij} \bar{H}_{12ij}) \right] + \mathcal{O}\left(\max_{i \in I} \frac{1}{n_i^2}\right). \end{aligned}$$

Proof. Again, we obtain a similar integral as in the previous lemma

$$\sum_{i \in I} \left[k_{in_i}^2 - \left(\frac{n_i\pi}{\ell_i}\right)^2 \right] = -\frac{1}{2\pi i} \oint_{C_p} \ln \frac{\varphi(k)}{\prod_{j=1}^d (-k \sin(k\ell_j))} 2k \, dk.$$

If there are no common zeros of the different sine functions, we may apply the same argument as in the previous lemma and obtain the sum of the right-hand side of the previous theorem. If there is a multiple zero of a sine function (i.e. $n_i \ell_j = n_j \ell_i$ for any i, j so that $\frac{n_i \pi}{\ell_i}$ lies inside the contour C_p), we may apply Lemma A.3 g), i), and k) to show that the contribution of this zero to the third term on the *rhs* is of order $O\left(\frac{1}{n_i^2}\right)$. \square

Combining the previous two theorems together yields the following corollary

Corollary 4.5. *The sum*

$$\sum_{i=1}^d \sum_{n=0}^{\infty} \left[\lambda_{in}(q) - \lambda_{in}(0) - \frac{2a_i}{\ell_i} \right]$$

is absolutely convergent, where $\lambda_{in}(q)$ and $\lambda_{in}(0)$ denote the eigenvalues for the potential q and for the null potential, respectively.

Proof. Subtracting the right-hand side of the formulæ in Theorems 4.3 and 4.4 one obtains the terms of the sum. (Note that the term by $\frac{1}{n}$ depends only on the matrix H and not on the potential.) Hence the sum $\sum_{i \in I} \lambda_{in}(q) - \lambda_{in}(0) - \frac{2a_i}{\ell_i}$ is of order $O\left(\max_{i \in I} \frac{1}{n_i^2}\right)$ and the sum of these sums is absolutely convergent. \square

Finally, we can prove the main result.

Proof of Theorem A. We integrate around the contours Γ_N in the “allowed regions” with N going to infinity. For sufficiently large N , there are $d + \sum_{i=1}^d \lfloor \frac{N\ell_i}{\pi} \rfloor$ eigenvalues of \mathcal{H} with square roots smaller than N (here $\lfloor \cdot \rfloor$ denotes the floor function, that is, the largest integer not larger than its argument). The number of k_n with the same property in the k -plane is double. We obtain

$$(6) \quad 2 \sum_{i=1}^d \sum_{n=0}^{\lfloor \frac{N\ell_i}{\pi} \rfloor} [\lambda_{in}(q) - \lambda_{in}(0)] = -\frac{1}{2\pi i} \oint_{\Gamma_N} \ln \frac{\varphi(k)}{\varphi_0(k)} 2k \, dk.$$

We can evaluate the integral with the use of equation (5) and Lemma A.3, we find after dividing the equation by 2

$$(7) \quad \sum_{i=1}^d \sum_{n=0}^{\lfloor \frac{N\ell_i}{\pi} \rfloor} [\lambda_{in}(q) - \lambda_{in}(0)] = \sum_{i=1}^d \frac{a_i}{\ell_i} \left(1 + 2 \left\lfloor \frac{N\ell_i}{\pi} \right\rfloor \right) + \sum_{i=1}^d \left(b_i - \frac{1}{2} a_i^2 \right) + O\left(\frac{1}{N}\right).$$

We have used the sums

$$\sum_{n=1}^M \frac{1}{n^2} = \frac{\pi^2}{6} + O\left(\frac{1}{M}\right), \quad \sum_{n=1}^M \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} + O\left(\frac{1}{M}\right).$$

Subtracting $\sum_{i=1}^d \frac{2a_i}{\ell_i} \left(1 + \left\lfloor \frac{N\ell_i}{\pi} \right\rfloor \right)$ from both sides of (7), using

$$b_i - \frac{1}{2} a_i^2 = \frac{1}{4} [q_i(\ell_i) + q_i(0)],$$

and sending N to infinity we find the sought result. The contribution of the term $o\left(\frac{1}{k^2}\right)$ in (6) resulting from the logarithm expansion (5) goes to zero as $N \rightarrow \infty$, because the length of the contour is of order N and the value of the function on it is $o\left(\frac{1}{N^2}\right) \times N$. \square

APPENDIX A. SOME AUXILIARY RESULTS

First, we prove Lemma 3.1 – this is already partially given in [Yur00].

Proof of Lemma 3.1. For the sake of simplicity we omit the subscript j . Repeatedly substituting c_j into its defining formula we get

$$\begin{aligned} c(x, k) &= \cos(kx) + \int_0^x \frac{\sin(k(x-t))}{k} \cos(kt) q(t) dt + \\ &\quad + \int_0^x \frac{\sin(k(x-t))}{k} q(t) \int_0^t \frac{\sin(k(t-s))}{k} q(s) \cos(ks) ds dt + o\left(\frac{e^{|\operatorname{Im} k|x}}{k^2}\right). \end{aligned}$$

Then we use the formula

$$\sin(\alpha - \beta) \cos \beta = \frac{1}{2} [\sin \alpha + \sin(\alpha - 2\beta)]$$

to obtain

$$\begin{aligned} c(x, k) &= \cos(kx) + \frac{1}{2k} \int_0^x [\sin(kx) + \sin(k(x-2t))] q(t) dt + \\ &\quad + \int_0^x \int_0^t \frac{\sin(k(x-t))}{k} q(t) \frac{1}{2k} [\sin(kt) + \sin(k(t-2s))] q(s) ds dt + o\left(\frac{e^{|\operatorname{Im} k|x}}{k^2}\right). \end{aligned}$$

Finally, using integration by parts we have

$$\begin{aligned} \int_0^x \sin(k(x-2t)) q(t) dt &= \int_0^x q(t) \frac{\partial}{\partial t} \frac{\cos(k(x-2t))}{2k} dt = \\ &= \frac{1}{2k} [q(x) - q(0)] \cos(kx) - \frac{1}{2k} \int_0^x \cos(k(x-2t)) \frac{\partial q(t)}{\partial t} dt = \\ &= \frac{1}{2k} [q(x) - q(0)] \cos(kx) + o\left(\frac{e^{|\operatorname{Im} k|x}}{k}\right), \end{aligned}$$

where we have used the fact that $q \in W^{1,1}(e)$. Using this we can write

$$\begin{aligned} c(x, k) &= \cos(kx) + \frac{\sin(kx)}{k} \frac{1}{2} \int_0^x q(t) dt + \\ &\quad + \frac{1}{4k^2} [q(x) - q(0)] \cos(kx) + \frac{1}{4k^2} \int_0^x \int_0^t q(t) q(s) [\cos(k(x-2t)) - \cos(kx)] ds dt + \\ &\quad + \frac{1}{2k^2} \int_0^x q(t) \sin(k(x-t)) \int_0^t \sin(k(t-2s)) q(s) ds dt + o\left(\frac{e^{|\operatorname{Im} k|x}}{k^2}\right). \end{aligned}$$

By similar arguments as before (with the use of integration by parts) the term in the last line and the term $\frac{1}{4k^2} \int_0^x \int_0^t q(t) q(s) [\cos(k(x-2t)) - \cos(kx)] ds dt$ are of order $o\left(\frac{e^{|\operatorname{Im} k|x}}{k^2}\right)$. Finally, since

$$\frac{\cos(kx)}{4k^2} \int_0^x q(t) \int_0^t q(s) ds dt = \frac{\cos(kx)}{8k^2} \int_0^x \int_0^x q(t) q(s) ds dt = \frac{\cos(kx)}{8k^2} \left(\int_0^x q(t) dt \right)^2,$$

we obtain the formula for $c(x, k)$.

The formulæ for the function $s(x, k)$ and the corresponding derivatives can be derived in a similar way. For c' we have

$$\begin{aligned} c'(x, k) &= -k \sin(kx) + \int_0^x \cos(k(x-t))q(t) \cos(kt) dt + \\ &+ \int_0^x \cos(k(x-t))q(t) \int_0^t \frac{\sin(k(t-s))}{k} q(s) \cos(ks) ds dt + o\left(\frac{e^{|\operatorname{Im} k|x}}{k}\right) = \\ &= -k \sin(kx) + \int_0^x q(t) \frac{1}{2} [\cos(kx) + \cos(k(x-2t))] dt + \\ &+ \int_0^x \cos(k(x-t))q(t) \int_0^t \frac{1}{2k} q(s) [\sin(kt) + \sin(k(t-2s))] ds dt + o\left(\frac{e^{|\operatorname{Im} k|x}}{k}\right) \end{aligned}$$

For the different particular terms we get

$$\begin{aligned} \frac{1}{2} \int_0^x q(t) \cos(k(x-2t)) dt &= \frac{1}{2} \int_0^x q(t) \frac{\partial \sin(k(x-2t))}{\partial t} \left(-\frac{1}{2k}\right) dt = \\ &= \frac{1}{4k} \int_0^x \frac{\partial q(t)}{\partial t} \sin(k(x-2t)) dt + \frac{1}{4k} [q(x) + q(0)] \sin(kx) = \\ &= \frac{1}{4k} [q(x) + q(0)] \sin(kx) + o\left(\frac{e^{|\operatorname{Im} k|x}}{k}\right). \end{aligned}$$

$$\begin{aligned} \int_0^x \cos(k(x-t))q(t) \int_0^t \frac{1}{2k} q(s) \sin(kt) ds dt &= \\ &= \frac{1}{4k} \int_0^x q(t) [\sin(kx) - \sin(k(x-2t))] \int_0^t q(s) dt = \\ &= \frac{\sin(kx)}{8k} \left(\int_0^x q(t) dt\right)^2 + o\left(\frac{e^{|\operatorname{Im} k|x}}{k}\right). \end{aligned}$$

$$\int_0^x \cos(k(x-t))q(t) \int_0^t \frac{1}{2k} q(s) \sin(k(t-2s)) ds dt = o\left(\frac{e^{|\operatorname{Im} k|x}}{k}\right).$$

We also briefly show the derivation of formulæ for s and s' .

$$\begin{aligned} s(x, k) &= \frac{\sin(kx)}{k} + \int_0^x \frac{\sin(k(x-t))}{k^2} q(t) \sin(kt) dt + \\ &+ \int_0^x \frac{\sin(k(x-t))}{k^3} q(t) \int_0^t \sin(k(t-s))q(s) \sin(ks) ds dt + o\left(\frac{e^{|\operatorname{Im} k|x}}{k^3}\right) = \\ &= \frac{\sin(kx)}{k} - \frac{\cos(kx)}{k^2} \frac{1}{2} \int_0^x q(t) dt + \frac{1}{4k^3} \sin(kx) [q(x) + q(0)] - \\ &- \frac{1}{4k^3} \int_0^x [\sin(kx) + \sin(k(x-2t))] q(t) \int_0^t q(s) ds dt + \\ &+ \frac{1}{2k^3} \int_0^x \sin(k(x-t))q(t) \int_0^t \cos(k(t-2s))q(s) ds dt + o\left(\frac{e^{|\operatorname{Im} k|x}}{k^3}\right). \end{aligned}$$

$$\begin{aligned}
 s'(x, k) &= \cos(kx) + \frac{1}{k} \int_0^x \cos(k(x-t)) \sin(kt) q(t) dt + \\
 &+ \frac{1}{k^2} \int_0^x \cos(k(x-t)) q(t) \int_0^t \sin(k(t-s)) \sin(ks) q(s) ds dt + o\left(\frac{e^{|\operatorname{Im} k|x}}{k^2}\right) = \\
 &= \cos(kx) + \frac{1}{2k} \int_0^x [\sin(kx) - \sin(k(x-2t))] q(t) dt + \\
 &+ \frac{1}{2k^2} \int_0^x \cos(k(x-t)) q(t) \int_0^t [\cos(k(t-2s)) - \cos(kt)] q(s) ds dt = \\
 &+ o\left(\frac{e^{|\operatorname{Im} k|x}}{k^2}\right) = \cos(kx) + \frac{\sin(kx)}{k} \frac{1}{2} \int_0^x q(t) dt - \frac{\cos(kx)}{4k^2} [q(x) - q(0)] - \\
 &- \frac{1}{4k^2} \int_0^x \int_0^t \cos(k(x-2t)) q(t) q(s) ds dt - \\
 &- \frac{1}{4k^2} \int_0^x \cos(kx) q(t) \int_0^t q(s) ds dt + o\left(\frac{e^{|\operatorname{Im} k|x}}{k^2}\right).
 \end{aligned}$$

□

Lemma A.1. *On the contour Γ_N defined in Section 4 with large enough N satisfying*

$$N \notin \cup_{n \in \mathbb{Z}} \left(\frac{n\pi}{\ell_j} - \frac{\varepsilon}{\ell_j}, \frac{n\pi}{\ell_j} + \frac{\varepsilon}{\ell_j} \right),$$

it holds

$$\frac{e^{|\operatorname{Im} k|\ell_j}}{|\sin(k\ell_j)|} \leq K_\varepsilon,$$

where the constant K_ε depends only on ε .

Proof. The proof will be similar to the proof of [Yan14, Lemma 2.4]. We will first prove the inequality for the right edge of the square Γ_N , i.e. for $k = N + i\tau$, $\tau \in (-N, N)$. We know that there exist such $C_\varepsilon > 0$ that $|\sin(k\ell_j)| > C_\varepsilon$. We have

$$\begin{aligned}
 |\sin(k\ell_j)| &= |\sin(N\ell_j) \cos(i\tau\ell_j) + \cos(N\ell_j) \sin(i\tau\ell_j)| = \\
 &= \frac{1}{2} \sqrt{|\sin(N\ell_j)(e^{-\tau\ell_j} + e^{\tau\ell_j})|^2 + |\cos(N\ell_j)(e^{-\tau\ell_j} - e^{\tau\ell_j})|^2} \geq \\
 &\geq \frac{1}{2} |\sin(N\ell_j)| |e^{-\tau\ell_j} + e^{\tau\ell_j}| > \frac{1}{2} C_\varepsilon e^{|\tau|\ell_j}
 \end{aligned}$$

and hence

$$\frac{e^{|\operatorname{Im} k|\ell_j}}{|\sin(k\ell_j)|} \leq \frac{2e^{|\operatorname{Im} k|\ell_j}}{C_\varepsilon e^{|\operatorname{Im} k|\ell_j}} = \frac{2}{C_\varepsilon}.$$

For the upper edge of the square $k = \sigma + iN$, $\sigma \in (-N, N)$ we have for sufficiently large N

$$|\sin(k\ell_j)| = \frac{1}{2} |e^{-N\ell_j + i\sigma\ell_j} - e^{N\ell_j - i\sigma\ell_j}| \geq \frac{1}{2} (e^{N\ell_j} - e^{-N\ell_j})$$

and hence for N large enough

$$\frac{e^{|\operatorname{Im} k|\ell_j}}{|\sin(k\ell_j)|} \leq \frac{2e^{N\ell_j}}{e^{N\ell_j} - e^{-N\ell_j}} \leq 4.$$

We have chosen N such that $e^{-2N\ell_j} < \frac{1}{2}$. The proof for the other edges of the square Γ_N is similar. \square

For the sake of completeness we present the symmetric version of Rouché's theorem (for the proof see e.g. [Est62, p. 156] or [Bur79, p. 265]).

Theorem A.2. *Let f and g be holomorphic functions in the bounded subset V of \mathbb{C} and continuous at its closure \bar{V} . Let us assume that on the boundary ∂V of V the following relation holds*

$$|f - g| < |f| + |g|.$$

Then functions f and g have the same (finite) number of zeros in V .

Now we can proceed with the proof of Theorem 4.1.

Proof of Theorem 4.1. Since we assume that $N \notin \cup_{n \in \mathbb{N}_0} \left(\frac{n\pi}{\ell_i} - \frac{\varepsilon}{\ell_i}, \frac{n\pi}{\ell_i} + \frac{\varepsilon}{\ell_i} \right)$ for each i , we have $|\sin(N\ell_i)| > C_\varepsilon > 0$ with C_ε depending only on ε . We use the Rouché's theorem with $f = \varphi(k)$ and $g = \prod_{i=1}^d (-k \sin(k\ell_i))$.

On the contour Γ_N we have

$$\begin{aligned} |f| + |g| &= 2|k|^d \prod_{i=1}^d |\sin(k\ell_i)| + O(|k|^{d-1} e^{|\operatorname{Im} k| \sum_{i=1}^d \ell_i}), \\ |f - g| &\leq O(|k|^{d-1} e^{|\operatorname{Im} k| \sum_{i=1}^d \ell_i}). \end{aligned}$$

Using Lemma A.1 we obtain

$$|f| + |g| - |f - g| = |k|^d \prod_{i=1}^d |\sin(k\ell_i)| \left(2 + \frac{1}{|k|} O \left(\frac{e^{|\operatorname{Im} k| \sum_{j=1}^d \ell_j}}{\prod_{o=1}^d |\sin(k\ell_o)|} \right) \right) > |k|^d C_\varepsilon^d > 0$$

for $|k|$ large enough and hence the inequality in Theorem A.2 is satisfied which completes the proof. \square

Lemma A.3. *Let us assume a counterclockwise contour γ_n which encircles $\frac{n\pi}{\ell_i}$ once and does not encircle any other zeros of $\sin(k\ell_i)$. Then*

- a) $\frac{1}{2\pi i} \oint_{\gamma_n} \cot(k\ell_i) dk = \frac{1}{\ell_i}, n \in \mathbb{Z},$
- b) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{1}{\sin(k\ell_i)} dk = \frac{(-1)^n}{\ell_i}, n \in \mathbb{Z},$
- c) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{1}{k} \cot(k\ell_i) dk = \frac{1}{n\pi}, n \in \mathbb{Z} \setminus \{0\},$
- d) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{1}{k} \cot(k\ell_i) dk = 0, n = 0,$
- e) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{1}{k \sin(k\ell_i)} dk = \frac{(-1)^n}{n\pi}, n \in \mathbb{Z} \setminus \{0\},$
- f) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{1}{k \sin(k\ell_i)} dk = 0, n = 0,$

- g) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{1}{k} \cot^2(k\ell_i) dk = -\frac{1}{n^2\pi^2}, n \in \mathbb{Z} \setminus \{0\},$
- h) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{1}{k} \cot^2(k\ell_i) dk = -\frac{2}{3}, n = 0,$
- i) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{\cot(k\ell_i)}{k \sin(k\ell_i)} dk = -\frac{(-1)^n}{n^2\pi^2}, n \in \mathbb{Z} \setminus \{0\},$
- j) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{\cot(k\ell_i)}{k \sin(k\ell_i)} dk = -\frac{1}{6}, n = 0,$
- k) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{1}{k \sin^2(k\ell_i)} dk = -\frac{1}{n^2\pi^2}, n \in \mathbb{Z} \setminus \{0\},$
- l) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{1}{k \sin^2(k\ell_i)} dk = \frac{1}{3}, n = 0,$
- m) $\frac{1}{2\pi i} \oint_{\gamma_n} \frac{1}{k} dk = 1, n = 0.$

Proof. The lemma can be proven by standard techniques of complex analysis, i.e. the residue theorem, see e.g. [Bur79]. \square

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