Macroscopic Long-Range Dynamics of Fermions and Quantum Spins on the Lattice - An Introduction

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Abstract

The aim of the current paper is to present, in a concise way, our recent, very general, mathematically rigorous studies [1, 2] on the dynamical properties of fermions and quantum-spin systems with long-range, or mean-field, interactions. In particular, they show that long-range dynamics in infinite volume are equivalent to intricate combinations of classical and quantum short-range dynamics, opening new theoretical perspectives, as explained in [3]. This phenomenon is a direct consequence of the highly non-local character of long-range, or mean-field, interactions. Note that [1–3] are altogether about 200 pages long. Therefore, as a simple example allowing to emphasize the key points of [1–3], we consider here the strong-coupling BCS-Hubbard model. The dynamical properties of this model are technically easy to study, albeit non-trivial, and this example is thus very pedagogical. From the physical point of view, this model is also interesting because it highlights the possible thermodynamic impact of the (screened) Coulomb repulsion on (s-wave) superconductivity, in the strong-coupling approximation. Its behavior at thermodynamical equilibrium was rigorously known, but not its infinite-volume dynamics.

Keywords: superconductivity, mean-field, long-range, BCS, quantum dynamics.

Contents

1 Introduction

More than seventy years ago, Bogoliubov proposes an ansatz, widely known as the Bogoliubov approximation, which corresponds to replace, in many-boson Hamiltonians, the annihilation and creation operators of zero-impulsion particles with complex numbers to be determined *self-consistently*. See [4, Section 1.1] for more details. However, even nowadays, the mathematical validity of this approximation with respect to the primordial dynamics of (stable) many-boson Hamiltonians with usual two-body interactions is an open problem.

In the context of many-fermion systems, ten years after Bogoliubov's ansatz, a similar approximation is used in the BCS theory of (conventional) superconductivity, as explained by Bogliubov in 1958 [5] and Haag in 1962 [6]. In 1966, this approximation is shown [7] to be exact at the level of the thermodynamic pressure for fermion systems that are similar to the BCS model. See also [8, 9]. The validity of the approximation with respect to the primordial dynamics was an open question that Thirring and Wehrl [10, 11] solve in 1967 for an exactly solvable permutation-invariant fermion model. An attempt to generalize Thirring and Wehrl's results to a general class of fermionic models, including the BCS theory, has been done in 1978 [12], but at the cost of technical assumptions that are difficult to verify in practice.

In 1973, Hepp and Lieb [13] made explicit, for the first time, the existence of Poisson brackets in some (commutative) algebra of functions, related to the classical effective dynamics. This is done for a permutation-invariant quantum-spin system with mean-field interactions. This research direction has been strongly developed by many authors until 1992, see [14–33]. All these papers study dynamical properties of *permutation-invariant* quantum-spin systems with mean-field interactions.

Thereafter, the mathematical research activity on this subject considerably decreases until the early 2000s when emerges, within the mathematical physics community, a new interest in such quantum systems, partially because of new experiments like those on ultracold atoms (via laser and evaporative coolings). See, for instance, the paper [34] on mean-field dynamics, published in the year 2000. There is also an important research activity on the mathematical foundation of the Gross-Pitaevskii[1] (GP) or Hartree theories, starting after 1998. For more details on the GP theory and mean-field dynamics for indistinguishable particles (bosons), see [35–39] and references therein. In which concern lattice-fermion or quantum-spin systems with long-range, or mean-field, interactions at equilibrium, see, e.g., [40–43]. Concerning the dynamics of fermion systems in the continuum with mean-field interactions, see [44–53], as well as [36, Sections 6 and 7]. Such mean-field problems are even related to other academic disciplines, like mathematical economics, via the so-called mean-field game theory [54] developed from 2006 by Lasry and Lions. Mean-field theory in its extended sense is, in fact,

[1]The so-called GP limit is not really a mean-field limit, but it looks similar.

a major research field of mathematics, even nowadays, and is still studied in physics, see, e.g., [55] and references therein.

The current paper belongs to this research field, since it is an application of our recent studies [1,2] on the dynamical properties of fermion and quantum-spin systems with long-range, or mean-field, interactions. See also [3], on which [1,2] are conceptually based.

By long-range interactions for fermion systems, we mean for instance something like the BCS interaction, which, written in the x-space, is of the form

$$-\frac{1}{|\Lambda_L|}\sum_{x,y\in\Lambda_L} a_{x,\uparrow}^{*}a_{x,\downarrow}^{*}a_{y,\downarrow}a_{y,\uparrow}\ , \tag{1}$$

where $a_{x,\mathrm{s}}^{*}$ (resp. $a_{x,\mathrm{s}}$) creates (resp. annihilates) a fermion with spin $\mathrm{s}\in\{\uparrow,\downarrow\}$ at lattice position x in a cubic box $\Lambda_L \doteq \{\mathbb{Z}\cap[-L,L]\}^d$ (d-dimensional crystal). For more details, see Section 2.1, in particular discussions until Equation (6). (1) explicitly shows the long-range character of the interaction. It is a mean-field interaction since

$$\frac{1}{|\Lambda_L|}\sum_{x,y\in\Lambda_L} a_{x,\uparrow}^{*}a_{x,\downarrow}^{*}a_{y,\downarrow}a_{y,\uparrow}=\sum_{y\in\Lambda_L}\left(\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}a_{x,\uparrow}^{*}a_{x,\downarrow}^{*}\right)a_{y,\downarrow}a_{y,\uparrow}\ .$$

This is an important, albeit elementary, example of the far more general case studied in [1,2]. Indeed, the results obtained in [1, 2] are far beyond previous ones because the permutation-invariance of lattice-fermion or quantum-spin systems is *not* required:

- The short-range part of the corresponding Hamiltonian is very general since only a sufficiently strong polynomial decay of its interactions and a translation invariance are necessary.
- The long-range part is also very general, being an infinite sum (over n) of mean-field terms of order $n\in\mathbb{N}$. In fact, even for permutation-invariant systems, the class of long-range, or mean-field, interactions we are able to handle is much larger than what was previously studied.
- The initial state is only required to be periodic. By [1, Proposition 2.2], observe that the set of all such initial states is dense within the set of all even states, the physically relevant ones.

Long-range, or mean-field, effective models are essential in condensed matter physics to study, from microscopic considerations, macroscopic phenomena like superconductivity. What's more, they are possibly not merely effective interactions. We discuss this issue in more detail in Section 3.

Observe also that, following Bóna [56], [1–3] show that classical mechanics does not only appear in the limit $\hbar\to 0$, as explained for instance in [55, 57]. In fact, Bóna's major conceptual contribution [56, Section 1.1-a] is to highlight the emergence of classical mechanics without necessarily the disappearance of the quantum world. In [3], a more general approach is proposed, leading to a new mathematical framework in which the classical and quantum worlds are entangled. Such a feature is demonstrated in [1,2] for the dynamics of macroscopic lattice Fermi systems with long-range interactions, as shortly explained in Section 4.3.4.

The aim of the current paper is to present, in a concise way, central results of [1,2]. These articles are indeed quite long and very technical and Section 2 is a pedagogical explanation of their key points, by using the strong-coupling BCS-Hubbard model as a paradigm. This model is technically easy, albeit non-trivial, and it is rigorously studied at equilibrium in [41] in order to understand the possible thermodynamic impact of the Coulomb repulsion on (s-wave) superconductivity, in the strong-coupling approximation. The origin of this model and all its dynamical properties are explained in Section 2. Without the Hubbard part of the interaction, one gets the usual strong-coupling

BCS model and the results of [21, Section A] are recovered. Section 4 explains the general results of [1, 2] in simple terms. In particular, we formulate them in Section 4.4 in the special context of permutation-invariant models, making the link with previous results on permutation-invariant quantum-spin systems.

Here, we focus on lattice Fermi systems which are, from a technical point of view, slightly more difficult than quantum-spin systems, because of the non-commutativity of fermionic creation and annihilation operators on different lattice sites. However, all the results presented here or in [1,2] can be applied to quantum-spin systems via obvious modifications.

2 The Strong-Coupling BCS-Hubbard Model

2.1 Presentation of the Model

The most general form of a translation invariant model for fermions (spin set S) with two-body interactions in a cubic box $\Lambda_L \doteq \{\mathbb{Z} \cap [-L, L]\}^d$ (d-dimensional crystal) of volume $|\Lambda_L|$, $L \in \mathbb{N}_0$ is given in momentum space by

$$\begin{aligned} \mathrm{H}_L^{Full} \quad = \quad & \sum_{k \in \Lambda_L^*,\ \mathrm{s} \in \mathrm{S}} (\varepsilon_k - \mu)\, \tilde{a}_k^* \tilde{a}_k \\ & + \frac{1}{|\Lambda_L|} \sum_{\substack{k,k',q \in \Lambda_L^* \\ \mathrm{s}_1,\mathrm{s}_2,\mathrm{s}_3,\mathrm{s}_4 \in \mathrm{S}}} g_{\mathrm{s}_1,\mathrm{s}_2,\mathrm{s}_3,\mathrm{s}_4}(k, k', q)\, \tilde{a}_{k+q,\mathrm{s}_1}^* \tilde{a}_{k'-q,\mathrm{s}_2}^* \tilde{a}_{k',\mathrm{s}_3} \tilde{a}_{k,\mathrm{s}_4}\ . \end{aligned} \tag{2}$$

See [58, Eq. (2.1)]. Here, Λ_L^* is the reciprocal lattice of quasi-momenta (periodic boundary conditions) associated with Λ_L and the operator $\tilde{a}_{k,\mathrm{s}}^*$ (respectively $\tilde{a}_{k,\mathrm{s}}$) creates (respectively annihilates) a fermion with spin $\mathrm{s} \in \mathrm{S}$ and (quasi-) momentum $k \in \Lambda_L^*$. The function ε_k represents the kinetic energy of a fermion with (quasi-) momentum k and the real number μ is the chemical potential. The last term of (2) corresponds to a translation-invariant two-body interaction written in the momentum space.

One important example of a lattice-fermion system with long-range interactions is given in the scope of the celebrated BCS theory – proposed in the late 1950s (1957) to explain conventional type I superconductors. The lattice version of this theory is obtained from (2) by taking $\mathrm{S} \doteq \{\uparrow, \downarrow\}$ and imposing

$$g_{\mathrm{s}_1,\mathrm{s}_2,\mathrm{s}_3,\mathrm{s}_4}(k, k', q) = \delta_{k,-k'} \delta_{\mathrm{s}_1,\uparrow} \delta_{\mathrm{s}_2,\downarrow} \delta_{\mathrm{s}_3,\downarrow} \delta_{\mathrm{s}_4,\uparrow} f(k, -k, q)$$

for some function f: It corresponds to the so-called (reduced) BCS Hamiltonian

$$\mathrm{H}_L^{BCS} \doteq \sum_{k \in \Lambda_L^*} (\varepsilon_k - \mu) \left(\tilde{a}_{k,\uparrow}^* \tilde{a}_{k,\uparrow} + \tilde{a}_{k,\downarrow}^* \tilde{a}_{k,\downarrow} \right) - \frac{1}{|\Lambda_L|} \sum_{k,q \in \Lambda_L^*} \gamma_{k,q} \tilde{a}_{k,\uparrow}^* \tilde{a}_{-k,\downarrow}^* \tilde{a}_{-q,\downarrow} \tilde{a}_{q,\uparrow}\ , \tag{3}$$

where $\gamma_{k,q}$ is a positive[2] function. Because of the term $\delta_{k,-k'}$, the interaction of this model has a long-range character, in position space. The choice $\gamma_{k,q} = \gamma > 0$ in (3) is a very interesting simplification since, even when $\varepsilon_k = 0$, the BCS Hamiltonian qualitatively displays most of basic properties of real conventional type I superconductors. See, e.g. [59, Chapter VII, Section 4]. The case $\varepsilon_k = 0$ is known as the strong coupling limit of the BCS model. The dynamical properties of the BCS Hamiltonian H_L^{BCS} with $\gamma_{k,q} = \gamma > 0$ can be *explicitly* computed from results of [1, 2], but we prefer here to consider another BCS-type model including the Hubbard interaction, as a richer example.

An important phenomenon not taken into account in the BCS theory is the Coulomb interaction between electrons or holes, which can imply strong correlations. This problem was of course already

[2]The positivity of $\gamma_{k,q}$ imposes constraints on the choice of the function f.

addressed in theoretical physics right after the emergence of the Fröhlich model and the BCS theory, see, e.g., [60]. This is a drawback of the conventional BCS theory since the Coulomb interaction can be very important, for instance in cuprate superconductivity: In all cuprates, there is undeniable experimental evidence of strong on-site Coulomb repulsions, leading to the universally observed Mott transition at zero doping [61, 62]. Recall that this phase is characterized by a periodic distribution of fermions (electrons or holes) with exactly one particle per lattice site. Doping copper oxides with holes or electrons can prevent from this situation. In this case, at sufficiently small temperatures, a superconducting phase is achieved, as first discovered in 1986 for the copper oxide perovskite $\mathrm{La}_{2-x}\mathrm{Ba}_x\mathrm{CuO}_4$ [63].

However, even after three decades of theoretical studies, including the metaphoric string theory approach to condensed matter via the AdS/CFT duality, and in spite of many significant advances, there is still no widely accepted explanation of the microscopic origin of cuprate, or more generally high-T_c, superconductivity. A large amount of numerical and experimental data is available, but no particular pairing mechanism (through, for instance, antiferromagnetic spin fluctuations, phonons, etc.) has been firmly established [64, Section 7.6]. In fact, the debate seems to be strongly polarized between those using a purely electronic/magnetic microscopic mechanism and those using electron-phonon mechanisms.

It is not the subject of this paper to discuss further theories for high-T_c superconductivity, as we recently did in [65]. In fact, even if many theoretical approaches have been successful in explaining various physical properties of superconductors, only a few mathematically rigorous results related to a microscopic description of such a material via a quantum many-body problem are available. We present below a model, named here the strong-coupling BCS-Hubbard Hamiltonian, which is rigorously studied at equilibrium in [41] in order to understand the possible thermodynamic impact of the Coulomb repulsion on (s-wave) superconductivity. An interesting mathematical outcome of [41] on the strong-coupling BCS-Hubbard Hamiltonian is the existence of a superconductor-Mott insulator phase transition, like in cuprates which must be doped to become superconductors.

The results of [41] are based on an *exact* study of the phase diagram of the strong-coupling BCS-Hubbard model defined, in a cubic box $\Lambda_L \doteq \{\mathbb{Z} \cap [-L, L]\}^d$ ($d \in \mathbb{N}$) of volume $|\Lambda_L|$ for $L \in \mathbb{N}_0$, by the Hamiltonian

$$\begin{aligned} \mathrm{H}_L \doteq & -\mu \sum_{x \in \Lambda_L} (n_{x,\uparrow} + n_{x,\downarrow}) - h \sum_{x \in \Lambda_L} (n_{x,\uparrow} - n_{x,\downarrow}) \\ & + 2\lambda \sum_{x \in \Lambda_L} n_{x,\uparrow} n_{x,\downarrow} - \frac{\gamma}{|\Lambda_L|} \sum_{x,y \in \Lambda_L} a^*_{x,\uparrow} a^*_{x,\downarrow} a_{y,\downarrow} a_{y,\uparrow} \end{aligned} \tag{4}$$

for real parameters $\mu, h \in \mathbb{R}$ and $\lambda, \gamma \geq 0$. The operator $a^*_{x,\mathrm{s}}$ (resp. $a_{x,\mathrm{s}}$) creates (resp. annihilates) a fermion with spin $\mathrm{s} \in \{\uparrow, \downarrow\}$ at lattice position $x \in \mathbb{Z}^d$, $d = 1, 2, 3, ...$, whereas $n_{x,\mathrm{s}} \doteq a^*_{x,\mathrm{s}} a_{x,\mathrm{s}}$ is the particle number operator at position x and spin s. They are linear operators acting on the fermion Fock space $\mathcal{F}_{\Lambda_L}$, where

$$\mathcal{F}_\Lambda \doteq \bigwedge \mathbb{C}^{\Lambda \times \{\uparrow,\downarrow\}} \equiv \mathbb{C}^{2^{\Lambda \times \{\uparrow,\downarrow\}}} \tag{5}$$

for any $\Lambda \subseteq \mathbb{Z}^d$ and $d \in \mathbb{N}$. The first term of the right-hand side of (4) represents the strong-coupling limit of the kinetic energy, also called "atomic limit" in the context of the Hubbard model, see, e.g., [66, 67]. The second term corresponds to the interaction between spins and the external magnetic field h. The on-site interaction with positive coupling constant $\lambda \geq 0$ represents the (screened) Coulomb repulsion as in the celebrated Hubbard model. The last term is the BCS interaction written in the x-space since

$$\frac{\gamma}{|\Lambda_L|} \sum_{x,y \in \Lambda_L} a^*_{x,\uparrow} a^*_{x,\downarrow} a_{y,\downarrow} a_{y,\uparrow} = \frac{\gamma}{|\Lambda_L|} \sum_{k,q \in \Lambda^*_L} \tilde{a}^*_{k,\uparrow} \tilde{a}^*_{-k,\downarrow} \tilde{a}_{q,\downarrow} \tilde{a}_{-q,\uparrow} \, . \tag{6}$$

See (3) with $\gamma_{k,q} = \gamma > 0$. This homogeneous BCS interaction should be seen as a long-range effective interaction, the precise mediators of which are not relevant, i.e., they could be phonons, as in conventional type I superconductors, or anything else.

2.2 Approximating Hamiltonians

The thermodynamic impact of the Coulomb repulsion on s-wave superconductors is analyzed in [41], via a rigorous study of equilibrium and ground states of the strong-coupling BCS-Hubbard Hamiltonian: An Hamiltonian like H_L defines in the thermodynamic limit $L \to \infty$ a free-energy density functional on a suitable set of states of the CAR algebra $\mathcal{U}$ of the lattice $\mathbb{Z}^d$. See [41, Section 6.2] for more details. Minimizers ω of the free-energy density are called equilibrium states of the model and, for any $L \in \mathbb{N}_0$, the Gibbs states $\omega^{(L)}$, defined on the algebra $\mathcal{B}(\mathcal{F}_{\Lambda_L})$ of linear operators acting on the fermion Fock space $\mathcal{F}_{\Lambda_L}$ (5) by

$$\omega^{(L)}(A) \doteq \mathrm{Trace}_{\mathcal{F}_{\Lambda_L}}\left(A\frac{e^{-\beta \mathrm{H}_L}}{\mathrm{Trace}_{\mathcal{F}_{\Lambda_L}}\left(e^{-\beta \mathrm{H}_L}\right)}\right), \qquad A \in \mathcal{B}(\mathcal{F}_{\Lambda_L}), \tag{7}$$

at inverse temperature $\beta > 0$, converges[3] in the thermodynamic limit $L \to \infty$ to a well-defined equilibrium state. The important point in such an analysis is the study of a variational problem over complex numbers: By the so-called approximating Hamiltonian method [8,9,68] one uses an approximation of the Hamiltonian, which is, in the case of the strong-coupling BCS-Hubbard Hamiltonian, equal to the c-dependent Hamiltonian

$$\begin{aligned}\mathrm{H}_L(c) \doteq\; & -\mu \sum_{x\in\Lambda_L}(n_{x,\uparrow}+n_{x,\downarrow}) - h\sum_{x\in\Lambda_L}(n_{x,\uparrow}-n_{x,\downarrow}) + 2\lambda\sum_{x\in\Lambda_L} n_{x,\uparrow}n_{x,\downarrow} \\ & -\gamma\sum_{x\in\Lambda_L}\left(c a^*_{x,\uparrow}a^*_{x,\downarrow} + \bar{c} a_{x,\downarrow}a_{x,\uparrow}\right),\end{aligned} \tag{8}$$

with $c \in \mathbb{C}$, see also [10,11]. The main advantage of using this c-dependent Hamiltonian, in comparison with H_L, is the fact that it is a sum of shifts of the same on-site operator. For an appropriate choice of (order) parameter $c \in \mathbb{C}$, it leads to the exact thermodynamics of the strong-coupling BCS-Hubbard model, in the limit $L \to \infty$: At inverse temperature $\beta > 0$,

$$\begin{aligned}&\lim_{L\to\infty}\frac{1}{\beta|\Lambda_L|}\ln \mathrm{Trace}_{\mathcal{F}_{\Lambda_L}}\left(e^{-\beta\mathrm{H}_L}\right) \\ &= \sup_{c\in\mathbb{C}}\left\{-\gamma|c|^2 + \lim_{L\to\infty}\left\{\frac{1}{\beta|\Lambda_L|}\ln\mathrm{Trace}_{\mathcal{F}_{\Lambda_L}}\left(e^{-\beta\mathrm{H}_L(c)}\right)\right\}\right\}\end{aligned} \tag{9}$$

and the (exact) Gibbs state ω_L converges[4] to a convex combination of the thermodynamic limit $L \to \infty$ of the (approximating) Gibbs state $\omega^{(L,\mathfrak{d})}$ defined by

$$\omega^{(L,\mathfrak{d})}(A) \doteq \mathrm{Trace}_{\mathcal{F}_{\Lambda_L}}\left(A\frac{e^{-\beta \mathrm{H}_L(\mathfrak{d})}}{\mathrm{Trace}_{\mathcal{F}_{\Lambda_L}}\left(e^{-\beta \mathrm{H}_L(\mathfrak{d})}\right)}\right), \qquad A \in \mathcal{B}(\mathcal{F}_{\Lambda_L}), \tag{10}$$

the complex number $\mathfrak{d} \in \mathbb{C}$ being a solution to the variational problem (9). Since $\gamma \geq 0$, this can be heuristically be seen from the inequality

$$\gamma|\Lambda_L||c|^2 + \mathrm{H}_L(c) - \mathrm{H}_L = \gamma\left(\mathfrak{c}_0^* - \sqrt{|\Lambda_L|}\bar{c}\right)\left(\mathfrak{c}_0 - \sqrt{|\Lambda_L|}c\right) \geq 0,$$

[3] In the weak* topology.

[4] In the weak* topology.

where

$$\mathfrak{c}_0 \doteq \frac{1}{\sqrt{|\Lambda_L|}} \sum_{x \in \Lambda_L} a_{x,\downarrow} a_{x,\uparrow} \tag{11}$$

(resp. $\mathfrak{c}_0^*$) annihilates (resp. creates) one Cooper pair within the condensate, i.e., in the zero-mode for fermion pairs. This suggests the proven fact [41, Theorem 3.1] that

$$|\mathfrak{d}| = \lim_{L \to \infty} \frac{\omega^{(L)}\left(\mathfrak{c}_0^* \mathfrak{c}_0\right)}{|\Lambda_L|} \tag{12}$$

for any[5] $\mathfrak{d} \in \mathbb{C}$ solution to the variational problem (9). The parameter $|\mathfrak{d}|$ is the condensate density of Cooper pairs and so, $|\mathfrak{d}| > 0$ corresponds to the existence of a superconducting phase, which is shown to exist for sufficiently large $\gamma \geq 0$. See also [41, Figs. 1,2,3].

2.3 Dynamical Problem in the Thermodynamic Limit

An Hamiltonian like the strong-coupling BCS-Hubbard Hamiltonian drives a dynamics in the Heisenberg picture of quantum mechanics: As is usual, the corresponding time-evolution is, for $L \in \mathbb{N}_0$, a continuous group $\{\tau_t^{(L)}\}_{t \in \mathbb{R}}$ of $*$-automorphisms of the algebra $\mathcal{B}(\mathcal{F}_{\Lambda_L})$ of linear operators acting on the Fermion Fock space $\mathcal{F}_{\Lambda_L}$ (see (5)), defined by

$$\tau_t^{(L)}(A) \doteq \mathrm{e}^{it\mathrm{H}_L} A \mathrm{e}^{-it\mathrm{H}_L} , \qquad A \in \mathcal{B}(\mathcal{F}_{\Lambda_L}),\ t \in \mathbb{R} .$$

The generator of this time evolution is the linear operator δ_L defined on $\mathcal{B}(\mathcal{F}_{\Lambda_L})$ by

$$\delta_L(A) \doteq i[\mathrm{H}_L, A] \doteq i\left(\mathrm{H}_L A - A\mathrm{H}_L\right) , \qquad A \in \mathcal{B}(\mathcal{F}_{\Lambda_L}) ,$$

that is,

$$\tau_t^{(L)}(A) = \exp\left(it\delta_L\right)(A) , \qquad A \in \mathcal{B}(\mathcal{F}_{\Lambda_L}),\ t \in \mathbb{R} .$$

If $\gamma = 0$ then it is well-known that the thermodynamic limit of $\{\tau_t^{(L)}\}_{t \in \mathbb{R}}$ exist as a strongly continuous group $\{\tau_t\}_{t \in \mathbb{R}}$ of $*$-automorphisms of the CAR algebra of the infinite lattice, as explained in Section 4.1.2. If $\gamma > 0$ then the situation is not that obvious. A first guess is to approximate $\{\tau_t^{(L)}\}_{t \in \mathbb{R}}$ by $\{\tau_t^{(L,c)}\}_{t \in \mathbb{R}}$, where

$$\tau_t^{(L,c)}(A) \doteq \mathrm{e}^{it\mathrm{H}_L(c)} A \mathrm{e}^{-it\mathrm{H}_L(c)} , \qquad A \in \mathcal{B}(\mathcal{F}_{\Lambda_L}),\ t \in \mathbb{R} , \tag{13}$$

for any $L \in \mathbb{N}_0$ and some complex number $c \in \mathbb{C}$. In this case, the linear operator

$$\delta_{L,c}(A) \doteq i[\mathrm{H}_L(c), A] , \qquad A \in \mathcal{B}(\mathcal{F}_{\Lambda_L}) , \tag{14}$$

is the generator of the dynamics $\{\tau_t^{(L,c)}\}_{t \in \mathbb{R}}$. A natural choice for $c \in \mathbb{C}$ would be a solution to the variational problem (9), but what about if the solution is not unique ? As a matter of fact, as explained in Section 4.1.3, in the thermodynamic limit $L \to \infty$, the finite-volume dynamics $\{\tau_t^{(L)}\}_{t \in \mathbb{R}}$ does *not* converge within the CAR C^*-algebra of the infinite lattice for $\gamma > 0$, even if $\mathfrak{d} = 0$ would be the unique solution to the variational problem (9)!

The validity of the approximation with respect to the primordial dynamics was an open question that Thirring and Wehrl [10, 11] solve in 1967 for the special case

$$\mathrm{H}_L|_{\mu=\lambda=h=0} = -\frac{\gamma}{|\Lambda_L|} \sum_{x,y \in \Lambda_L} a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow} , \tag{15}$$

[5] This implies that any solution $|d|$ to the variational problem (9) must have the same absolute value.

which is an exactly solvable permutation-invariant model for any $\gamma \in \mathbb{R}$. An attempt to generalize Thirring and Wehrl's results to a general class of fermionic models, including the BCS theory, has been done in 1978 [12], but at the cost of technical assumptions that are difficult to verify in practice. This research direction has been strongly developed by many authors until 1992, see [14–33]. All these papers study dynamical properties of *permutation-invariant* quantum-spin systems with mean-field interactions. Our results [1–3], summarized in Section 4, represent a significant generalization of such previous results to possibly non-permutation-invariant lattice-fermion or quantum-spin systems. To understand what's going on in the infinite-volume dynamics, we now come back to our pedagogical example, that is, the strong-coupling BCS-Hubbard model.

2.4 Self-Consistency Equations

Instead of considering the Heisenberg picture, let us consider the Schrödinger picture of quantum mechanics. In this case, recall that, at fixed $L \in \mathbb{N}_0$, a finite-volume state $\rho^{(L)}$ is a positive and normalized functional acting on the algebra $\mathcal{B}(\mathcal{F}_{\Lambda_L})$ of linear operators on the fermion Fock space $\mathcal{F}_{\Lambda_L}$. By finite dimensionality of $\mathcal{F}_{\Lambda_L}$,

$$\rho^{(L)}(A) \doteq \mathrm{Trace}_{\mathcal{F}_{\Lambda_L}}\left(\mathrm{d}^{(L)} A\right) , \qquad A \in \mathcal{B}(\mathcal{F}_{\Lambda_L}) ,$$

for a uniquely defined positive operator $\mathrm{d}^{(L)} \in \mathcal{B}(\mathcal{F}_{\Lambda_L})$ satisfying $\mathrm{Trace}_{\mathcal{F}_{\Lambda_L}}(\mathrm{d}^{(L)}) = 1$ and named the density matrix of $\rho^{(L)}$. Compare with (7) and (10). See also Section 4.2.1. At $L \in \mathbb{N}_0$, the time evolution of any finite-volume state is

$$\rho_t^{(L)} \doteq \rho^{(L)} \circ \tau_t^{(L)} , \qquad t \in \mathbb{R} , \tag{16}$$

which corresponds to a time-dependent density matrix equal to $\mathrm{d}_t^{(L)} = \tau_{-t}^{(L)}(\mathrm{d}^{(L)})$. Compare with (71).

The thermodynamic limit of (16) for periodic states can be explicitly computed, as explained in Section 4.3.2. It refers to a *non-linear* state-dependent dynamics related to *self-consistency*: By (5) with $\Lambda = \Lambda_0 = \{0\}$, recall that

$$\mathcal{F}_{\{0\}} \doteq \bigwedge \mathbb{C}^{\{0\}\times\{\uparrow,\downarrow\}} \equiv \mathbb{C}^4 \tag{17}$$

is the fermion Fock space associated with the lattice site $(0, \ldots, 0) \in \mathbb{Z}^d$ and so, $\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)$ can be identified with the set $\mathrm{Mat}(4, \mathbb{C})$ of complex 4×4 matrices, in some orthonormal basis[6]. For any continuous family $\omega \doteq (\omega_t)_{t\in\mathbb{R}}$ of states acting on $\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)$, we define the finite-volume *non-autonomous* dynamics $(\tau_{t,s}^{(L,\omega)})_{s,t\in\mathbb{R}}$ by the Dyson-Phillips series

$$\tau_{t,s}^{(L,\omega)} \equiv \text{“}\exp\left(\int_s^t \delta_L^{\omega_u} \mathrm{d}u\right)\text{”} \doteq \mathbf{1}_{\mathcal{B}(\mathcal{F}_{\Lambda_L})} + \sum_{k\in\mathbb{N}} \int_s^t \mathrm{d}t_1 \cdots \int_s^{t_{k-1}} \mathrm{d}t_k \delta_L^{\omega_{t_k}} \circ \cdots \circ \delta_L^{\omega_{t_1}}$$

acting on $\mathcal{B}(\mathcal{F}_{\Lambda_L})$ for any $s, t \in \mathbb{R}$, with $\mathbf{1}_{\mathcal{B}(\mathcal{F}_{\Lambda_L})}$ being the identity mapping of $\mathcal{B}(\mathcal{F}_{\Lambda_L})$ and where δ_L^ρ is the generator of the group $\{\tau_t^{(L,c)}\}_{t\in\mathbb{R}}$, defined by (14) for $c = \rho(a_{0,\uparrow}a_{0,\downarrow})$, i.e.,

$$\delta_L^\rho(A) \doteq i[\mathrm{H}_L(\rho(a_{0,\uparrow}a_{0,\downarrow})), A] , \qquad A \in \mathcal{B}(\mathcal{F}_{\Lambda_L}) .$$

Compare with (12)-(11). Note that, for every continuous family $\omega \doteq (\omega_t)_{t\in\mathbb{R}}$ of on-site (even) states acting on $\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)$, $s, t \in \mathbb{R}$, $L_0 \in \mathbb{N}_0$ and all integers $L \geq L_0$,

$$\tau_{t,s}^{(L,\omega)}(A) = \tau_{t,s}^{(L_0,\omega)}(A) , \qquad A \in \mathcal{B}(\mathcal{F}_{\Lambda_{L_0}}) . \tag{18}$$

[6] For instance, $(1,0,0,0)$ is the vacuum; $(0,1,0,0)$ and $(0,0,1,0)$ correspond to one fermion with spin $\uparrow$ and $\downarrow$, respectively; $(0,0,0,1)$ refers to two fermions with opposite spins.

It follows that the family $\{\tau_{t,s}^{(L,\omega)}\}_{s,t\in\mathbb{R}}$ strongly converges in the thermodynamic limit $L\rightarrow\infty$ to a strongly continuous two-parameter family $\{\tau_{t,s}^{\omega}\}_{s,t\in\mathbb{R}}$ of $*$-automorphisms of the CAR algebra $\mathcal{U}$ of the lattice. With these observations, we are in a position to give the self-consistency equations: By (69) and (94), for any fixed initial (even) state ρ_0 on $\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)$ at $t=0$, there is a unique family $(\boldsymbol{\varpi}(t;\rho_0))_{t\in\mathbb{R}}$ of on-site states acting on $\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)$ such that

$$\boldsymbol{\varpi}(t;\rho_0)=\rho_0\circ\tau_{t,0}^{\boldsymbol{\varpi}(\cdot;\rho_0)}\ ,\qquad t\in\mathbb{R}\ . \tag{19}$$

Remark that $\tau_{t,0}^{\boldsymbol{\varpi}(\cdot;\rho_0)}\left(\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)\right)\subseteq\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)$, because this infinite-volume dynamics is constructed from local Hamiltonians (8), which are sums of on-site terms. See, e.g., (18). Observe that (19) is an equation on a finite-dimensional space, see (17).

2.5 Infinite-Volume Dynamics of Product States

For simplicity, as initial state (at $t=0$), take a finite-volume product state

$$\rho^{(L)}\doteq\otimes_{\Lambda_L}\rho_0 \tag{20}$$

associated with an even[7] state ρ_0 on $\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)$: Recall that even means that the expectation value of any odd monomials in $\{a_{0,\mathrm{s}}^*,a_{0,\mathrm{s}}\}_{\mathrm{s}\in\{\uparrow,\downarrow\}}$ with respect to the on-site state ρ_0 is zero, while the product state $\rho^{(L)}$ is (well-) defined by

$$\rho^{(L)}(\alpha_{x_1}(A_1)\cdots\alpha_{x_n}(A_n))=\rho_0(A_1)\cdots\rho_0(A_n) \tag{21}$$

for all $A_1,\ldots,A_n\in\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)$ and all $x_1,\ldots,x_n\in\Lambda_L$ such that $x_i\neq x_j$ for $i\neq j$, where $\alpha_{x_j}(A_j)\in\mathcal{B}\left(\mathcal{F}_{\{x_j\}}\right)$ is the x_j-translated copy of A_j for all $j\in\{1,\ldots,n\}$, see (38) for more details. An example of finite-volume product states is given by the approximating Gibbs states (10). Then, in this case, as explained in Section 4.4, for any $t\in\mathbb{R}$, $L_0\in\mathbb{N}_0$ and $A\in\mathcal{B}(\mathcal{F}_{\Lambda_{L_0}})$, one has that

$$\lim_{L\rightarrow\infty}\rho_t^{(L)}(A)=\lim_{L\rightarrow\infty}\rho\circ\tau_t^{(L)}(A)=\rho\circ\tau_{t,0}^{\boldsymbol{\varpi}(\cdot;\rho_0)}(A)\ , \tag{22}$$

with $\rho_t^{(L)},\boldsymbol{\varpi}(\cdot;\rho_0)$ being respectively defined by (16) and (19) and where $\rho\doteq\otimes_{\mathbb{Z}^d}\rho_0$ is the (infinite-volume) product state associated with the even state ρ_0 on $\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)$, see either (20)-(21) with $\Lambda_L=\mathbb{Z}^d$ or (89). Note that the restriction to $\mathcal{B}(\mathcal{F}_{\Lambda_L})$ of ρ is equal to $\rho^{(L)}$.

For any $t\in\mathbb{R}$, the limit state

$$\rho_t^{(\infty)}\doteq\rho\circ\tau_{t,0}^{\boldsymbol{\varpi}(\cdot;\rho_0)}$$

is again a product state and hence, it is completely determined by its restriction to the single lattice site $(0,\ldots,0)\in\mathbb{Z}^d$, that is, by the on-site state

$$\rho_{t,\{0\}}^{(\infty)}\doteq\rho\circ\tau_{t,0}^{\boldsymbol{\varpi}(\cdot;\rho_0)}|_{\mathcal{B}(\mathcal{F}_{\{0\}})}=\rho_0\circ\tau_{t,0}^{\boldsymbol{\varpi}(\cdot;\rho_0)}|_{\mathcal{B}(\mathcal{F}_{\{0\}})}=\boldsymbol{\varpi}(t;\rho_0)$$

for all $t\in\mathbb{R}$.

If one sees quantum states as elements of a state space in classical mechanics, it is natural to consider complex or real-valued function of states. By using the self-consistency equation (19), we thus define a classical space of observables as being the (commutative C^*-)algebra $C(E_{\{0\}}^+;\mathbb{C})$ of continuous functions on the space $E_{\{0\}}^+$ of all even states acting on $\mathcal{B}(\mathcal{F}_{\{0\}})$. As explained in Section 4.4.4, the self-consistency equation leads to a *group* $(V_t)_{t\in\mathbb{R}}$ of automorphisms of $C(E_{\{0\}}^+;\mathbb{C})$ defined by

$$\left[V_tf\right](\rho)\doteq f\left(\boldsymbol{\varpi}(t;\rho)\right)\ ,\qquad\rho\in E_{\{0\}}^+,\ f\in C(E_{\{0\}}^+;\mathbb{C}),\ t\in\mathbb{R}\ .$$

[7]Observe that even states are the physically relevant ones.

This dynamics can be written in terms of Poisson brackets, i.e., as some *Liouville's equation* of classical mechanics: A polynomial in $C(E_{\{0\}}^{+};\mathbb{C})$ is a function f of the form

$$f(\rho) \doteq g(\rho(A_1),\ldots,\rho(A_n)) , \qquad \rho \in E_{\{0\}}^{+} ,$$

for some polynomial g of $n \in \mathbb{N}$ variables and elements $A_1,\ldots,A_n \in \mathcal{B}(\mathcal{F}_{\{0\}})$. Such a polynomial has (convex) derivative $\mathrm{D}f(\rho)$ equal to (75) for $\rho \in E_{\{0\}}^{+}$. The classical hamiltonian $\mathrm{h} \in C(E_{\{0\}}^{+};\mathbb{C})$ related to the strong-coupling BCS-Hubbard model is a polynomial defined by

$$\mathrm{h}(\rho) \doteq -\mu\rho(n_{\uparrow}+n_{\downarrow}) - h\rho(n_{\uparrow}-n_{\downarrow}) + 2\lambda\rho(n_{\uparrow}n_{\downarrow}) - \gamma|\rho(a_{\uparrow}a_{\downarrow})|^2 ,$$

the 0 indices of operators acting on $\mathcal{F}_{\{0\}}$ having been omitted for notational simplicity. It leads to a state-dependent Hamiltonian equal to

$$\begin{aligned}\mathrm{Dh}(\rho) &= -\mu(n_{\uparrow}+n_{\downarrow}) - h(n_{\uparrow}-n_{\downarrow}) + 2\lambda(n_{\uparrow}n_{\downarrow}) - \gamma\left(a_{\uparrow}^{*}a_{\downarrow}^{*}\rho(a_{\downarrow}a_{\uparrow}) + \rho\left(a_{\uparrow}^{*}a_{\downarrow}^{*}\right)a_{\downarrow}a_{\uparrow}\right) \\ &\quad + \left(\mu\rho(n_{\uparrow}+n_{\downarrow}) + h\rho(n_{\uparrow}-n_{\downarrow}) - 2\lambda\rho(n_{\uparrow}n_{\downarrow}) + 2\gamma|\rho(a_{\downarrow}a_{\uparrow})|^2\right)\mathbf{1} .\end{aligned} \tag{23}$$

Like (8), it is an Hamiltonian generating the restriction to $\mathcal{B}(\mathcal{F}_{\{0\}})$ of the quantum dynamics $\{\tau_t^{(L,c)}\}_{t\in\mathbb{R}}$ defined by (13) with $c = \rho(a_{\downarrow}a_{\uparrow})$: For every $s,t \in \mathbb{R}$ and all $L \in \mathbb{N}_0$,

$$\tau_{t,s}^{(L,\rho(a_{\downarrow}a_{\uparrow}))}(A) = \tau_{t,s}^{(1,\rho(a_{\downarrow}a_{\uparrow}))}(A) = \mathrm{e}^{it\mathrm{Dh}(\rho)}A\mathrm{e}^{-it\mathrm{Dh}(\rho)} , \qquad A \in \mathcal{B}(\mathcal{F}_{\{0\}}) ,$$

similar to (18). Then, using (97) we obtain Liouville's equation:

$$\partial_t V_t(f) = V_t(\{\mathrm{h},f\}) = \{\mathrm{h},V_t(f)\} , \qquad t \in \mathbb{R} , \tag{24}$$

where, by (76) and (98),

$$\{\mathrm{h},f\}(\rho) \doteq \rho(i[\mathrm{Dh}(\rho),\mathrm{D}f(\rho)]) , \qquad \rho \in E_{\{0\}}^{+} . \tag{25}$$

Liouville's equation is written here on a finite-dimensional state space and can easily be studied analytically. Its solution at fixed initial state gives access to all dynamical properties of product states driven by the strong-coupling BCS-Hubbard model in the thermodynamic limit. Below, we give the explicit computations of the time evolution of the most important physical quantities related to this model, in this situation:

Lemma 1 (Dynamical properties)
Fix any on-site state $\rho \in E_{\{0\}}^{+}$.
(i) *Electron density: Let $\mathrm{d}(\rho) \doteq \rho(n_{\uparrow}+n_{\downarrow})$. Then, for any $t \in \mathbb{R}$, $V_t(\mathrm{d}) = \mathrm{d}$.*
(ii) *Magnetization density: Let $\mathrm{m}(\rho) \doteq \rho(n_{\uparrow}-n_{\downarrow})$. Then, for any $t \in \mathbb{R}$, $V_t(\mathrm{m}) = \mathrm{m}$.*
(iii) *Coulomb correlation density: Let $\mathrm{w}(\rho) \doteq \rho(n_{\uparrow}n_{\downarrow})$. Then, for any $t \in \mathbb{R}$, $V_t(\mathrm{w}) = \mathrm{w}$.*
(iv) *Cooper pair condensate density: Let $\kappa(\rho) \doteq |\rho(a_{\downarrow}a_{\uparrow})|^2$. Then, for any $t \in \mathbb{R}$, $V_t(\kappa) = \kappa$.*
(v) *Cooper field densities: Let $\varphi(\rho) \doteq \mathrm{Re}(\rho(a_{\downarrow}a_{\uparrow}))$ and $\psi(\rho) \doteq \mathrm{Im}\rho(a_{\downarrow}a_{\uparrow})$. Then, for any $t \in \mathbb{R}$, the functions $\varphi_t \doteq V_t(\varphi)$ and $\psi_t \doteq V_t(\psi)$ on $E_{\{0\}}^{+}$ satisfy*

$$\left\{\begin{array}{l}\varphi_t(\rho) = \sqrt{\kappa}\cos(t\nu(\rho)+\theta_\rho) , \\ \psi_t(\rho) = \sqrt{\kappa}\sin(t\nu(\rho)+\theta_\rho) ,\end{array}\right.$$

with $\nu(\rho) \doteq 2(\mu-\lambda) + \gamma(1-\mathrm{d}(\rho))$ and $\kappa \in [0,1]$, $\theta_\rho \in [-\pi,\pi)$ such that $\rho(a_{\downarrow}a_{\uparrow}) = \sqrt{\kappa}\mathrm{e}^{i\theta_\rho}$.

Proof. We start with elementary computations using the CAR (36): Recall that $[A, B] \doteq AB - BA$ is the commutator and, for any $\mathrm{s} \in \{\uparrow, \downarrow\}$, $n_{\mathrm{s}} \doteq a_{\mathrm{s}}^* a_{\mathrm{s}}$ is the spin-s-particle number operator on the lattice site 0 (with "$\{0\}$" being omitted in the notation for simplicity). By (36), for any $\mathrm{s}, \mathrm{t} \in \{\uparrow, \downarrow\}$,

$$[n_\uparrow, a_\downarrow] = [n_\downarrow, a_\uparrow] = [n_{\mathrm{s}}, n_{\mathrm{t}}] = 0 \ , \quad [n_{\mathrm{s}}, a_{\mathrm{s}}] = -a_{\mathrm{s}} \ , \quad \left[a_\uparrow^* a_\downarrow^*, a_\uparrow\right] = -a_\downarrow^* \ , \quad \left[a_\uparrow^* a_\downarrow^*, a_\downarrow\right] = a_\uparrow^* \ . \tag{26}$$

It follows that, for any $\mathrm{s} \in \{\uparrow, \downarrow\}$,

$$[n_{\mathrm{s}}, a_\downarrow a_\uparrow] = -a_\downarrow a_\uparrow \ , \quad \left[n_{\mathrm{s}}, a_\uparrow^* a_\downarrow^*\right] = a_\uparrow^* a_\downarrow^* \ , \quad \left[a_\uparrow^* a_\downarrow^*, a_\downarrow a_\uparrow\right] = n_\uparrow + n_\downarrow - 1 \ . \tag{27}$$

From the CAR (36) note also that

$$n_\uparrow a_\uparrow^* a_\downarrow^* = a_\uparrow^* a_\downarrow^* \ , \quad n_\uparrow a_\downarrow a_\uparrow = a_\uparrow^* a_\downarrow^* n_\downarrow = n_\uparrow a_\downarrow a_\uparrow = 0 \ , \quad a_\downarrow a_\uparrow n_\downarrow = a_\downarrow a_\uparrow \ . \tag{28}$$

Now, we are in a position to prove Assertions (i)-(v): Let $\mathrm{f}_{\mathrm{s}}(\rho) \doteq \rho(n_{\mathrm{s}})$ for all $\rho \in E_{\{0\}}^+$ and $\mathrm{s} \in \{\uparrow, \downarrow\}$. Then, by using (23), (26)-(27) and (75) for $\rho \in E_{\{0\}}^+$, we compute that

$$[\mathrm{Dh}(\rho), \mathrm{Df}_{\mathrm{s}}(\rho)] = [\mathrm{Dh}(\rho), n_{\mathrm{s}}] = \gamma\left(\rho(a_\downarrow a_\uparrow) a_\uparrow^* a_\downarrow^* - \rho\left(a_\uparrow^* a_\downarrow^*\right) a_\downarrow a_\uparrow\right) \ .$$

By (24)-(25), we then deduce from the last equality that, for any $t \in \mathbb{R}$ and $\mathrm{s} \in \{\uparrow, \downarrow\}$, $V_t(\mathrm{f}_{\mathrm{s}}) = \mathrm{f}_{\mathrm{s}}$, which implies Assertions (i)-(ii). To get Assertion (iii), note from (23), (26)-(28) and (75) that, for any $\rho \in E_{\{0\}}^+$,

$$[\mathrm{Dh}(\rho), \mathrm{Dw}(\rho)] = [\mathrm{Dh}(\rho), n_\uparrow n_\downarrow] = \gamma\left(a_\uparrow^* a_\downarrow^* \rho(a_\downarrow a_\uparrow) - \rho\left(a_\uparrow^* a_\downarrow^*\right) a_\downarrow a_\uparrow\right) \ .$$

We then arrive at Assertion (iii) by combining this last computation with (24)-(25). To obtain Assertions (iv)-(v), it suffices to study the time evolution of the function $\mathfrak{z}$ defined, for any $\rho \in E_{\{0\}}^+$, by $\mathfrak{z}(\rho) \doteq \rho(a_\downarrow a_\uparrow)$. Then, we use again (23), (26)-(28) and (75) to obtain that

$$[\mathrm{Dh}(\rho), \mathrm{D}\mathfrak{z}(\rho)] = [\mathrm{Dh}(\rho), a_\downarrow a_\uparrow] = 2(\mu - \lambda) a_\downarrow a_\uparrow - \gamma\rho(a_\downarrow a_\uparrow)(n_\uparrow + n_\downarrow - 1)$$

and, by (24)-(25) and Assertion (i), the function $\mathfrak{z}_t \doteq V_t(\mathfrak{z})$, $t \in \mathbb{R}$, evaluated at $\rho \in E_{\{0\}}^+$ satisfies the elementary ODE

$$\forall t \in \mathbb{R}: \qquad \partial_t \mathfrak{z}_t(\rho) = i\left(2(\mu - \lambda) + \gamma(1 - \mathrm{d}(\rho))\right) \mathfrak{z}_t(\rho) \ , \qquad \mathfrak{z}_0(\rho) = \rho(a_\downarrow a_\uparrow) \ ,$$

from which Assertions (iv)-(v) obviously follow. Note that the time evolution $V_t(\kappa)$ of the *non-affine* polynomial $\kappa(\rho) \doteq |\rho(a_\downarrow a_\uparrow)|^2$, $\rho \in E_{\{0\}}^+$, could also have been obtained by using, from (75), that

$$\mathrm{D}\kappa(\rho) = a_\uparrow^* a_\downarrow^* \rho(a_\downarrow a_\uparrow) + \rho\left(a_\uparrow^* a_\downarrow^*\right) a_\downarrow a_\uparrow - 2|\rho(a_\downarrow a_\uparrow)|^2 \mathbf{1} \ .$$

■

In the special case $\lambda = 0$, i.e., without the Hubbard interaction, Lemma 1 reproduces the results of [21, Section A] on the strong-coupling BCS model, written in that paper as a permutation-invariant quantum-spin model.

From Lemma 1 observe that we recover the equation of a symmetric *rotor* in classical mechanics: Fix $\rho \in E_{\{0\}}^+$. For any $t \in \mathbb{R}$, let $\Omega_1(t) \doteq \varphi_t(\rho)$, $\Omega_2(t) \doteq \psi_t(\rho)$ and $\Omega_3(t) \doteq V_t(\nu)(\rho)$. Then, by Lemma 1, or by using directly Liouville's equation (24)-(25) as it is done in the proof of Lemma 1, one easily checks that the 3D vector $(\Omega_1(t), \Omega_2(t), \Omega_3(t))$ satisfies, for any time $t \in \mathbb{R}$, the following system of ODEs:

$$\begin{cases} \dot{\Omega}_1(t) = -\Omega_3(t)\Omega_2(t) \ , \\ \dot{\Omega}_2(t) = \Omega_3(t)\Omega_1(t) \ , \\ \dot{\Omega}_3(t) = 0 \ , \end{cases}$$

which describes the time evolution of the angular momentum of a symmetric rotor in classical mechanics.

Lemma 1 leads to the exact dynamics of a physical system prepared in a product state at initial time, driven by the strong-coupling BCS-Hubbard Hamiltonian. This set of states is still restrictif and our results [1–3], summarized in Section 4, go beyond this simple case, by allowing us to consider general periodic states as initial states, in contrast with all previous results on lattice Fermi, or quantum-spin, systems with long-range, or mean-field, interactions.

2.6 From Product to Periodic States as Initial States

The strong-coupling BCS-Hubbard model is permutation-invariant, which means that it is invariant under the transformation $\mathfrak{p}_\pi : a_{x,\mathrm{s}} \mapsto a_{\pi(x),\mathrm{s}}$ with $x \in \mathbb{Z}^d$ and $\mathrm{s} \in \{\uparrow, \downarrow\}$, for all bijective mappings $\pi : \mathbb{Z}^d \to \mathbb{Z}^d$ which leave all but finitely many elements invariant. See Section 4.4.1. First, take a permutation-invariant state, which means that $\rho \circ \mathfrak{p}_\pi = \rho$, again for all bijective mappings $\pi : \mathbb{Z}^d \to \mathbb{Z}^d$ which leave all but finitely many elements invariant. As is explained in Section 4.4.2, any permutation-invariant state can be written (or approximated to be more precise) as a convex combination of product states. For instance, let $\rho_1, \ldots, \rho_n$ be $n \in \mathbb{N}$ product states and $u_1, \ldots, u_n \in [0, 1]$ such that $u_1 + \cdots + u_n = 1$, and

$$\rho = \sum_{j=1}^{n} u_j \rho_j \ , \tag{29}$$

which is a permutation-invariant state. At fixed $L \in \mathbb{N}_0$, we take the restriction $\rho^{(L)}$ of ρ to $\mathcal{B}(\mathcal{F}_{\Lambda_L})$, which is thus a finite-volume permutation-invariant state, like the Gibbs state (7) associated with the strong-coupling BCS-Hubbard model. Then, in this case, for any $t \in \mathbb{R}$, $L_0 \in \mathbb{N}_0$ and $A \in \mathcal{B}(\mathcal{F}_{\Lambda_{L_0}})$, we infer from (22) that

$$\lim_{L\to\infty} \rho_t^{(L)}(A) = \lim_{L\to\infty} \rho \circ \tau_t^{(L)}(A) = \sum_{j=1}^{n} u_j \rho_j \circ \tau_{t,0}^{\boldsymbol{\varpi}(\cdot;\rho_j)}(A) \ , \tag{30}$$

where, by a slight abuse of notation, $\boldsymbol{\varpi}(\cdot;\rho) = \boldsymbol{\varpi}(\cdot;\rho|_{\mathcal{B}(\mathcal{F}_{\{0\}})})$. For general permutation-invariant states, one has to replace the finite sum (29) by an integral with respect to a probability measure on the set of product states to generalize (30). See again (96) for more details. As a consequence, by combining Lemma 1 with such a decomposition of permutation-invariant states into product states, we obtain all dynamical properties of the strong-coupling BCS-Hubbard model, in any permutation-invariant initial state: For instance, taking the state (29) and combining (30) with Lemma 1, the time-evolution $\varphi_t \doteq V_t(\varphi)$ and $\psi_t \doteq V_t(\psi)$ of Cooper fields are given, for any $t \in \mathbb{R}$, by

$$\begin{cases} \varphi_t(\rho) = \sum_{j=1}^{n} u_j \sqrt{\kappa_j} \cos(t\nu(\rho_j) + \theta_{\rho_j}) \ , \\ \psi_t(\rho) = \sum_{j=1}^{n} u_j \sqrt{\kappa_j} \sin(t\nu(\rho_j) + \theta_{\rho_j}) \ , \end{cases}$$

where

$$\nu(\rho_j) \doteq 2(\mu - \lambda) + \gamma\left(1 - \rho_j(n_\uparrow + n_\downarrow)\right) \qquad \text{and} \qquad \rho_j(a_\downarrow a_\uparrow) = \sqrt{\kappa_j}\mathrm{e}^{i\theta_{\rho_j}}$$

for any $j \in \{1, \ldots, n\}$. Then, the Cooper pair condensate density defined by $\kappa(\rho) \doteq |\rho(a_\downarrow a_\uparrow)|^2$ is *not anymore necessarly constant* and can have a complicated, highly non-trivial, behavior, in particular when ρ is not a finite sum like (29), but only the barycenter of a probability measure on the set of product states.

The permutation-invariant case already applies to the (weak*) limit $\omega^{(\infty)}$ of the Gibbs state $\omega^{(L)}$ (7) which is proven to exist as a permutation-invariant state $\omega^{(\infty)}$ because, by [41, Theorem 6.5], away from the superconducting critical point,

$$\omega^{(\infty)}(\cdot)=\frac{1}{2\pi}\int_0^{2\pi}\omega^{(\infty,r\mathrm{e}^{i\theta})}(\cdot)\,\mathrm{d}\theta \tag{31}$$

with $\{\mathfrak{d}=r\mathrm{e}^{i\theta},\theta\in[0,2\pi]\}$ being all solutions to the variational problem (9) and where the product state $\omega^{(\infty,\mathfrak{d})}$ is the thermodynamic limit $L\rightarrow\infty$ of the Gibbs state $\omega^{(L,\mathfrak{d})}(\cdot)$ defined by (10). In this case, by [41, Theorem 6.4 and previous discussions],

$$\omega^{(\infty,r\mathrm{e}^{i\theta})}(a_{\downarrow}a_{\uparrow})=r\mathrm{e}^{i\theta}=\mathfrak{d}\ ,\qquad \theta\in[0,2\pi]\ , \tag{32}$$

and if one has a superconducting phase, i.e., $r>0$, then, by [41, Eq. (3.3) and Theorem 6.4 (i)], one always has the equality

$$\omega^{(\infty,r\mathrm{e}^{i\theta})}(n_{\downarrow}+n_{\uparrow})=1+2\gamma^{-1}(\mu-\lambda)\ ,\qquad \theta\in[0,2\pi]\ . \tag{33}$$

Any equilibrium state is a state in the closed convex hull of $\{\omega^{(\infty,r\mathrm{e}^{i\theta})},\theta\in[0,2\pi]\}$. Equations (32)-(33) imply that, for any equilibrium state ω, like $\omega^{(\infty)}$, the frequency $\nu(\omega)$, defined in Lemma 1 (v), vanishes, i.e., $\nu(\omega)=0$. Hence, in this case, by Lemma 1, all densities are constant in time for any equilibrium state. The same property is also true at the superconducting critical point, by [41, Theorem 6.5 (ii)]. This is of course coherent with the well-known stationnarity of equilibrium states.

The results presented above could still have been deduced from Bóna's ones, as it is done in [21, Section A] for the strong-coupling BCS model. Of course, in this case, one has to represent the lattice Fermi systems as a permutation-invariant quantum-spin system and a permutation-invariant state would again be required as initial state.

Using [1,2] one can easily extend this study of the strong-coupling BCS-Hubbard model to a much larger set of initial states: Indeed, if the initial finite-volume state $\rho^{(L)}$ is the restiction to $\mathcal{B}(\mathcal{F}_{\Lambda_L})$ of an *extreme* (or ergodic) translation-invariant state on the CAR algebra of the lattice, which means in particular that it is invariant, for any $x\in\mathbb{Z}^d$, under the transformation $a_{x,\mathrm{s}}\mapsto a_{x+y,\mathrm{s}}$, $y\in\mathbb{Z}^d$ and $\mathrm{s}\in\{\uparrow,\downarrow\}$, then Equation (96) also tells us that, for any $t\in\mathbb{R}$, $L_0\in\mathbb{N}_0$ and $A\in\mathcal{B}(\mathcal{F}_{\Lambda_{L_0}})$,

$$\lim_{L\rightarrow\infty}\rho_t^{(L)}(A)=\rho\circ\tau_{t,0}^{\boldsymbol{\varpi}(\cdot;\rho)}(A)$$

where, again by a slight abuse of notation, $\boldsymbol{\varpi}(\cdot;\rho)=\boldsymbol{\varpi}(\cdot;\rho|_{\mathcal{B}(\mathcal{F}_{\{0\}})})$. What's more, since

$$\tau_{t,0}^{\boldsymbol{\varpi}(\cdot;\rho)}\left(\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)\right)\subseteq\mathcal{B}\left(\mathcal{F}_{\{0\}}\right)\ ,$$

because of its construction from local Hamiltonians (8) which are only a sum of on-site terms, the electron, magnetization, Coulomb correlation, Cooper pair condensate and the Cooper field densities can in this case directly be deduced for extreme, translation-invariant, initial states from Lemma 1, and, similar to (29)-(30), these quantities for general translation-invariant states can be derived by using their decompositions (66) in terms of extreme translation-invariant states.

All these outcomes can be extended to the case of general periodic initial states, via straightforward modications: for any $(\ell_1,\ldots,\ell_d)\in\mathbb{N}^d$ and initial $(\ell_1,\ldots,\ell_d)$-periodic state ρ (see Section 4.2.3 for more details) replace in all the above discussions on translation-invariant initial states terms like $\rho(a_{\downarrow}a_{\uparrow})=\rho(a_{0,\downarrow}a_{0,\uparrow})$ by

$$\frac{1}{\ell_1\cdots\ell_d}\sum_{x=(x_1,\ldots,x_d),\ x_i\in\{0,\ldots,\ell_i-1\}}\rho(a_{x,\downarrow}a_{x,\uparrow})\ . \tag{34}$$

Cf. (67)-(68). This goes beyond all previous studies on lattice Fermi, or quantum-spin, systems with long-range, or mean-field, interactions.

3 Long-Range Interactions in Physics

Long-range, or mean-field, effective models are essential in condensed matter physics to study, from microscopic considerations, macroscopic phenomena like superconductivity. They come from different approximations or Ansätze like the choice $\gamma_{k,q} \doteq \gamma > 0$ for the (effective) BCS interaction in (3). The general form of the (effective) BCS Hamiltonian in (3) comes from the celebrated Fröhlich electron-phonon interactions. What's more, they are possibly not merely effective models.

Long-range, or mean-field, models capture surprisingly well many phenomena in condensed matter physics. For instance, recall that the BCS interaction (1) allows us to qualitatively display most of basic properties of conventional superconductors [59, Chapter VII, Section 4]. Ergo, one could wonder whether such interactions may have a more fundamental physical relevance. Such a question is usually not addressed, because these interactions seem to break the spacial locality of Einstein's relativity. For instance, the BCS interaction (6) can be seen as a kinetic term for fermion pairs that can hop from $y \in \Lambda_L$ to *any* other lattice site $x \in \Lambda_L$, for each $L \in \mathbb{N}_0$.

This non-locality property is reminiscent of the inherent non-locality of quantum mechanics, highlighted by Einstein, Podolsky and Rosen with the celebrated EPR paradox. Philosophically, this general issue challenges causality, in its local sense, as well as the notion of a material object[8]. In [70], Einstein said the following:

"*If one asks what, irrespective of quantum mechanics, is characteristic of the world of ideas of physics, one is first of all struck by the following: the concepts of physics relate to a real outside world... it is further characteristic of these physical objects that they are thought of as a range in a space-time continuum. An essential aspect of this arrangement of things in physics is that they lay claim, at a certain time, to an existence independent of one another, provided these objects "are situated in different parts of space".*

The following idea characterizes the relative independence of objects far apart in space (A and B): external influence on A has no direct influence on B...

There seems to me no doubt that those physicists who regard the descriptive methods of quantum mechanics as definitive in principle would react to this line of thought in the following way: they would drop the requirement... for the independent existence of the physical reality present in different parts of space; they would be justified in pointing out that the quantum theory nowhere makes explicit use of this requirement.

I admit this, but would point out: when I consider the physical phenomena known to me, and especially those who are being so successfully encompassed by quantum mechanics, I still cannot find any fact anywhere which would make it appear likely that (that) requirement will have to be abandoned.

I am therefore inclined to believe that the description of quantum mechanics... has to be regarded as an incomplete and indirect description of reality, to be replaced at some later date by a more complete and direct one."

The debate on non-locality in Physics, experimentally shown, refers to the existence of quantum entanglement, essential in quantum information theory. For a discussion on locality and realism in quantum mechanics, see, e.g., [71] by Alain Aspect, who is one of the main initiators of experimental studies on quantum entanglement, in the beginning of the 1980s.

The non-locality of long-range, or mean-field, interactions like the BCS interaction[9] (1) can be seen as an instance of the (controversial) intrinsic non-locality of quantum physics. Long-range interactions are thus usually not considered by the physics community as being fundamental interactions,

[8]According to the spatio-temporal identity of classical mechanics, the same physical object cannot be at the same time on two distinct points of the phase space. This refers to Leibniz's Principle of Identity of Indiscernibles [69, p. 1]. The spatio-temporal identity of classical mechanics is questionable in quantum mechanics. See, e.g., [69].

[9]The strength of the BCS interaction (1) between two points of the space does not decay at large distances.

in order to avoid polemics. We partially agree with this position and see long-range interactions as possibly resulting from (more fundamental) interactions with (bosonic) mediators, like phonons in conventional superconductivity.

Nonetheless, a long-range interaction like (1), being quantum mechanical, does not refer to an actuality, *but only to a potentiality*. Physical properties of any (energy-conserving) physical system do not just depend on its Hamiltonian but also on its state which accounts for the "environmental" part of the system: This situation is analogous to the epigenetics[10] showing that the DNA sequence is only a set of constraints and potentialities, the physical realizations of which depend on the history and environment of the corresponding organism. For instance, in a lattice-fermion system described by the so-called (reduced) BCS Hamiltonian with $\gamma_{k,q} = \gamma$, pairs of particles may (almost) never hop in arbitrarily large distances if the state[11] ρ of the corresponding system is such that

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{x,y\in\Lambda_L} \rho\left(a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow}\right) = 0 \ .$$

This is the case for equilibrium states of this model at sufficiently high temperatures. It is thus too reductive to a priori eliminate such kind of interaction from "fundamental" Hamiltonians of physical systems.

On the top of that, as is well-known, the thermodynamic limit of mean-field dynamics is representation-dependent. This is basically Haag's original argument proposed in 1962 [6] in order to give a precise mathematical sense to the BCS model. In fact, the description of the full dynamics requires an extended quantum framework [3], which refers to an intricate combination of classical and quantum dynamics, as observed by Bóna already thirty years ago [26]. In [3], the emergence of a classical dynamics defined from Poisson brackets on state spaces is shown, without necessarily the disappearance of the quantum world, offering a general mathematical framework to understand physical phenomena with macroscopic quantum coherence. In the context of lattice-fermion systems, it is explained in detail in [1]. Such an entanglement of classical and quantum worlds is noteworthy, opening new theoretical perspectives, and is a direct consequence of the highly non-local character of long-range, or mean-field, interactions.

4 Mathematical Foundations

4.1 Algebraic Structures

4.1.1 CAR Algebra of Lattices

Let $\mathbb{Z}^d$ be the d-dimensional cubic lattice and $\mathcal{P}_f \subseteq 2^{\mathbb{Z}^d}$ the set of all non-empty finite subsets of $\mathbb{Z}^d$. In order to define the thermodynamic limit, for simplicity, we use cubic boxes

$$\Lambda_L \doteq \{\mathbb{Z} \cap [-L, L]\}^d \ , \qquad L \in \mathbb{N}_0 \ . \tag{35}$$

Let S be a fixed (once and for all) finite set of spins. For any $\Lambda \in \mathcal{P}_f \cup \{\mathbb{Z}\}$, $\mathcal{U}_\Lambda$ is the separable unital C^*-algebra[12] generated by the elements $\{a_{x,\mathrm{s}}\}_{x\in\Lambda,\mathrm{s}\in\mathrm{S}}$ satisfying the canonical anti-commutation relations (CAR): for any $x, y \in \mathbb{Z}^d$ and $\mathrm{s}, \mathrm{t} \in \mathrm{S}$,

$$a_{x,\mathrm{s}} a_{y,\mathrm{t}} + a_{y,\mathrm{t}} a_{x,\mathrm{s}} = 0 \ , \qquad a_{x,\mathrm{s}} a_{y,\mathrm{t}}^* + a_{y,\mathrm{t}}^* a_{x,\mathrm{s}} = \delta_{\mathrm{s},\mathrm{t}} \delta_{x,y} \mathbf{1} \ . \tag{36}$$

[10]Quoting [72]: "*Epigenetics is typically defined as the study of heritable changes in gene expression that are not due to changes in DNA sequence. Diverse biological properties can be affected by epigenetic mechanisms: for example, the morphology of flowers and eye colour in fruitflies.*"

[11]I.e., a positive and normalized continuous functional on the CAR algebra.

[12]$\mathcal{U}_\Lambda \equiv \mathcal{B}(\mathbb{C}^{2^{\Lambda\times\mathrm{S}}})$ is equivalent to the algebra of $2^{|\Lambda\times\mathrm{S}|} \times 2^{|\Lambda\times\mathrm{S}|}$ complex matrices, when $\Lambda \in \mathcal{P}_f$.

Here, $\delta_{k,l}$ is the Kronecker delta, that is, the function of two variables defined by $\delta_{k,l} \doteq 1$ if $k = l$ and $\delta_{k,l} = 0$ otherwise. Note that we use the notation $\mathcal{U} \equiv \mathcal{U}_{\mathbb{Z}^d}$ and define

$$\mathcal{U}_0 \doteq \bigcup_{\Lambda \in \mathcal{P}_f} \mathcal{U}_\Lambda \ , \tag{37}$$

which is a dense normed $*$-algebra of $\mathcal{U}$. Elements of $\mathcal{U}_0$ are called local elements. The (real) Banach subspace of all self-adjoint elements of $\mathcal{U}$ is denoted by $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$.

Translations are represented by a group homomorphism $x \mapsto \alpha_x$ from $\mathbb{Z}^d$ to the group of $*$-automorphisms of $\mathcal{U}$, which is uniquely defined by the condition

$$\alpha_x(a_{y,\mathrm{s}}) = a_{y+x,\mathrm{s}} \ , \quad y \in \mathbb{Z}^d, \ \mathrm{s} \in \mathrm{S} \ . \tag{38}$$

The mapping $x \mapsto \alpha_x$ is used below to define symmetry groups of states as well as translation-invariant interactions of lattice-fermion systems.

The results presented in the current paper also hold true in the context of quantum-spin systems, but we focus on lattice Fermi systems which are, from a technical point of view, slightly more difficult because of the non-commutativity of their elements on different lattice sites. Indeed, the additional difficulty in Fermi systems is that, for any finite subsets $\Lambda^{(1)}, \Lambda^{(2)} \in \mathcal{P}_f$ with $\Lambda^{(1)} \cap \Lambda^{(2)} = \emptyset$, the commutator

$$[B_1, B_2] \doteq B_1 B_2 - B_2 B_1 = 0 \ , \qquad B_1 \in \mathcal{U}_{\Lambda^{(1)}}, \ B_2 \in \mathcal{U}_{\Lambda^{(2)}} \ ,$$

may not be zero, in general. For instance, the CAR (36) trivially yield $[a_{x,\mathrm{s}}, a_{y,\mathrm{t}}] = 2a_{x,\mathrm{s}}a_{y,\mathrm{t}} \neq 0$ for any $x, y \in \mathfrak{L}$ and $\mathrm{s}, \mathrm{t} \in \mathrm{S}$, $(x, \mathrm{s}) \neq (y, \mathrm{t})$. Because of the CAR (36), such a commutation property is satisfied for all *even* local elements defined as follows: The condition

$$\sigma(a_{x,\mathrm{s}}) = -a_{x,\mathrm{s}}, \qquad x \in \Lambda, \ \mathrm{s} \in \mathrm{S} \ , \tag{39}$$

defines a unique $*$-automorphism σ of the C^*-algebra $\mathcal{U}$. The subspace

$$\mathcal{U}^+ \doteq \{A \in \mathcal{U} : A = \sigma(A)\} \tag{40}$$

is the C^*-subalgebra of so-called even elements of $\mathcal{U}$. Then, for any subsets $\Lambda^{(1)}, \Lambda^{(2)} \in \mathcal{P}_f$ with $\Lambda^{(1)} \cap \Lambda^{(2)} = \emptyset$,

$$[B_1, B_2] = 0 \ , \qquad B_1 \in \mathcal{U}_{\Lambda^{(1)}} \cap \mathcal{U}^+, \ B_2 \in \mathcal{U}_{\Lambda^{(2)}} \ .$$

This last condition is the expression of the local causality in quantum field theory. Using well-known constructions[13], the C^*-algebra $\mathcal{U}$, generated by anticommuting elements, can be recovered from $\mathcal{U}^+$. As a consequence, the C^*-algebra $\mathcal{U}^+$ should thus be seen as more fundamental than $\mathcal{U}$ in Physics. In fact, $\mathcal{U}$ corresponds in this context to the so-called local field algebra. See, e.g., [73, Sections 4.8 and 6].

The fact that the local causality in quantum field theory can be invoked to see $\mathcal{U}^+$ as being more fundamental than $\mathcal{U}$ in Physics does not prevent us from considering long-range interactions as possibly fundamental interactions, as explained in Section 3. The choice of $\mathcal{U}^+$ only compel us to consider (local) observables satisfying the local causality as measurable physical quantities, the full energy of lattice Fermi systems with short-range or long-range interactions being generally inaccessible in infinite volume. In fact, the long-range part yields possibly non-vanishing background fields, in the spirit of the Higgs mechanism of quantum field theory, in a given representation of the observable algebra, which is fixed by the initial state.

[13] More precisely, the so-called sector theory of quantum field theory.

4.1.2 Short-Range Interactions

A (complex) *interaction* is a mapping $\Phi : \mathcal{P}_f \rightarrow \mathcal{U}^+$ such that $\Phi_\Lambda \in \mathcal{U}_\Lambda$ for all $\Lambda \in \mathcal{P}_f$. The set of all interactions can be naturally endowed with the structure of a complex vector space and using the norm

$$\|\Phi\|_{\mathcal{W}} \doteq \sup_{x,y\in\mathbb{Z}^d} \sum_{\Lambda\in\mathcal{P}_f,\, \Lambda\supseteq\{x,y\}} (1+|x-y|)^{(d+\epsilon)} \|\Phi_\Lambda\|_{\mathcal{U}} \ , \tag{41}$$

for some fixed $\epsilon > 0$, we then define a separable Banach space $\mathcal{W}$ of short-range interactions. Here $|\cdot|$ is the Euclidean metric. Note that the particular positive-valued decay function

$$\mathbf{F}(x,y) = (1+|x-y|)^{-(d+\epsilon)} \ , \qquad x,y \in \mathbb{Z}^d,\ \epsilon > 0 \ ,$$

in (41) is used for simplicity and other choices can be made, as discussed in [1, Section 3.1]. We use in the sequel three important properties on short-range interactions:

(i) Self-adjointness: There is a natural involution $\Phi \mapsto \Phi^* \doteq (\Phi_\Lambda^*)_{\Lambda\in\mathcal{P}_f}$ defined on the Banach space $\mathcal{W}$ of short-range interactions. Self-adjoint interactions are, by definition, interactions Φ satisfying $\Phi = \Phi^*$. The (real) Banach subspace of all self-adjoint short-range interactions is denoted by $\mathcal{W}^{\mathbb{R}} \subsetneq \mathcal{W}$, similar to $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$.

(ii) Translation invariance: By definition, the interaction Φ is translation-invariant if

$$\Phi_{\Lambda+x} = \alpha_x(\Phi_\Lambda) \ , \qquad x \in \mathbb{Z}^d,\ \Lambda \in \mathcal{P}_f \ ,$$

where

$$\Lambda + x \doteq \left\{ y + x \in \mathbb{Z}^d : y \in \Lambda \right\} \ .$$

Here, $\{\alpha_x\}_{x\in\mathbb{Z}^d}$ is the family of (translation) $*$-automorphisms of $\mathcal{U}$ defined by (38). We then denote by $\mathcal{W}_1 \subsetneq \mathcal{W}$ the (separable) Banach subspace of translation-invariant, short-range interactions on $\mathbb{Z}^d$.

(iii) Finite range: For any $\mathrm{R} \in [0,\infty)$, we define the closed subspace

$$\mathcal{W}^{\mathrm{R}} \doteq \left\{ \Phi \in \mathcal{W}_1 : \Phi_\Lambda = 0 \text{ for } \Lambda \in \mathcal{P}_f \text{ such that } \max_{x,y\in\Lambda}\{|x-y|\} > \mathrm{R} \right\} \tag{42}$$

of finite-range, translation-invariant interactions. When $\mathrm{R} = 0$, we obtain the space $\mathcal{W}_\Pi \doteq \mathcal{W}^0$ of permutation-invariant interactions described in Section 4.4.

Short-range interactions define sequences of local energy elements: For any $\Phi \in \mathcal{W}$ and $L \in \mathbb{N}_0$,

$$U_L^\Phi \doteq \sum_{\Lambda\subseteq\Lambda_L} \Phi_\Lambda \in \mathcal{U}_{\Lambda_L} \cap \mathcal{U}^+ \ , \tag{43}$$

where we recall that $\Lambda_L \doteq \{\mathbb{Z} \cap (-L, L)\}^d$ is the cubic box used to define the thermodynamic limit (see (35)). The energy elements U_L^Φ, $L \in \mathbb{N}_0$, refer to an extensive quantity since their norm are proportional to the volume of the region they correspond to: For any $L \in \mathbb{N}_0$ and $\Phi \in \mathcal{W}$,

$$\left\|U_L^\Phi\right\|_{\mathcal{U}} \leq \mathbf{C}\,|\Lambda_L|\,\|\Phi\|_{\mathcal{W}} \ , \tag{44}$$

where

$$\mathbf{C} \doteq \sum_{x\in\mathbb{Z}^d} \frac{1}{(1+|x|)^{d+\epsilon}} < \infty \ . \tag{45}$$

Moreover, for any self-adjoint interaction $\Phi \in \mathcal{W}^{\mathbb{R}}$ and $L \in \mathbb{N}_0$, $U_L^\Phi \in \mathcal{U}^{\mathbb{R}}$, i.e., $U_L^\Phi = (U_L^\Phi)^*$ is a local Hamiltonian.

Each local Hamiltonian associated with $\Phi \in \mathcal{W}^{\mathbb{R}}$ leads to a local dynamics on the full C^*-algebra $\mathcal{U}$ via the group $\{\tau_t^{(L,\Phi)}\}_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$ defined by

$$\tau_t^{(L,\Phi)}(A) = \mathrm{e}^{itU_L^\Phi} A \mathrm{e}^{-itU_L^\Phi} , \qquad A \in \mathcal{U} . \tag{46}$$

It is the continuous group which is the solution to the evolution equation

$$\forall t \in \mathbb{R} : \qquad \partial_t \tau_t^{(L,\Phi)} = \tau_t^{(L,\Phi)} \circ \delta_L^\Phi , \qquad \tau_0^{(L,\Phi)} = \mathbf{1}_{\mathcal{U}} ,$$

with $\mathbf{1}_{\mathcal{U}}$ being the identity mapping of $\mathcal{U}$ and δ_L^Φ defined on $\mathcal{U}$, for any $L \in \mathbb{N}_0$ and $\Phi \in \mathcal{W}^{\mathbb{R}}$, by

$$\delta_L^\Phi(A) \doteq i \left[U_L^\Phi, A\right] \doteq i \left(U_L^\Phi A - A U_L^\Phi\right) , \qquad A \in \mathcal{U} .$$

This corresponds to a quantum dynamics, in the Heisenberg picture. Note that, for every $L \in \mathbb{N}_0$ and $\Phi \in \mathcal{W}^{\mathbb{R}}$, δ_L^Φ is a so-called symmetric derivation which belongs to the Banach space $\mathcal{B}(\mathcal{U})$ of bounded operators acting on the C^*-algebra $\mathcal{U}$, see, e.g., [1, Section 3.3].

More generally, for possibly time-dependent interactions, the non-autonomous local dynamics is defined, for any continuous function $\Psi \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$ and $L \in \mathbb{N}_0$, as the unique (fundamental) solution $(\tau_{t,s}^{(L,\Psi)})_{s,t\in\mathbb{R}}$ in the Banach space $\mathcal{B}(\mathcal{U})$ to the (finite-volume) non-autonomous evolution equation[14]

$$\forall s, t \in \mathbb{R} : \qquad \partial_t \tau_{t,s}^{(L,\Psi)} = \tau_{t,s}^{(L,\Psi)} \circ \delta_L^{\Psi(t)} , \qquad \tau_{s,s}^{(L,\Psi)} = \mathbf{1}_{\mathcal{U}} . \tag{47}$$

The solution to (47) can be explicitly written as a Dyson–Phillips series: For any $s, t \in \mathbb{R}$,

$$\tau_{t,s}^{(L,\Psi)} = \mathbf{1}_{\mathcal{U}} + \sum_{k\in\mathbb{N}} \int_s^t \mathrm{d}t_1 \cdots \int_s^{t_{k-1}} \mathrm{d}t_k \delta_L^{\Psi(t_k)} \circ \cdots \circ \delta_L^{\Psi(t_1)} . \tag{48}$$

By [74, Corollary 5.2], in the thermodynamic limit $L \to \infty$, for any $\Psi \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$, the group $(\tau_{t,s}^{(L,\Psi)})_{s,t\in\mathbb{R}}$, $L \in \mathbb{N}_0$, strongly converges, at any fixed s, t, to a strongly continuous two-parameter family $(\tau_{t,s}^\Psi)_{s,t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{U}$:

$$\lim_{L\to\infty} \tau_{t,s}^{(L,\Psi)}(A) \doteq \tau_{t,s}^{\Psi}(A) , \qquad A \in \mathcal{U}, \ s, t \in \mathbb{R} . \tag{49}$$

In other words, (time-dependent) self-adjoint interactions lead to an infinite-volume (possibly non-autonomous) dynamics on the CAR algebra of the lattice.

4.1.3 Long-Range Models

We start with some preliminary definitions: Let $\mathbb{S}$ be the unit sphere of $\mathcal{W}_1$. For any $n \in \mathbb{N}$ and finite signed Borel measure $\mathfrak{a}$ on the Cartesian product $\mathbb{S}^n$ (endowed with its product topology), we define the finite signed Borel measure $\mathfrak{a}^*$ to be the pushforward of $\mathfrak{a}$ through the self-homeomorphism

$$\left(\Psi^{(1)}, \ldots, \Psi^{(n)}\right) \mapsto \left((\Psi^{(n)})^*, \ldots, (\Psi^{(1)})^*\right) \in \mathbb{S}^n \tag{50}$$

of $\mathbb{S}^n$. A finite signed Borel measure $\mathfrak{a}$ on $\mathbb{S}^n$ is, by definition, *self-adjoint* whenever $\mathfrak{a}^* = \mathfrak{a}$. For any $n \in \mathbb{N}$, the real Banach space of self-adjoint, finite, signed Borel measures on $\mathbb{S}^n$ endowed with the norm

$$\|\mathfrak{a}\|_{\mathcal{S}(\mathbb{S}^n)} \doteq |\mathfrak{a}|(\mathbb{S}^n) , \qquad n \in \mathbb{N} ,$$

[14] Let $\mathcal{H}$ be some Hilbert space and $(H_t)_{t\in\mathbb{R}}$ some continuous family of bounded Hamiltonians acting on $\mathcal{H}$. The corresponding Schrödinger equation with $\hbar = 1$ reads $i\partial_t \varphi_t = H_t \varphi_t$ and so, $\varphi_t = \mathrm{U}_{t,s}\varphi_s$ with $\mathrm{U}_{t,s}$ being the solution to $\partial_t \mathrm{U}_{t,s} = H_t \mathrm{U}_{t,s}$. Then, in the Heisenberg picture, the time-evolution of any (bounded) observable B acting on $\mathcal{H}$ at initial time $t = s \in \mathbb{R}$ is $B_t = \tau_{t,s}(B_s) \doteq \mathrm{U}_{t,s}^* B_s \mathrm{U}_{t,s}$ for $s, t \in \mathbb{R}$, which yields $\partial_t \tau_{t,s} = \tau_{t,s} \circ \delta_t$ with $\delta_t(A) \doteq i[H_t, A]$.

is denoted by $\mathcal{S}(\mathbb{S}^n)$. The set of all sequences $\mathfrak{a} \equiv (\mathfrak{a}_n)_{n\in\mathbb{N}}$ of finite signed Borel measures $\mathfrak{a}_n \in \mathcal{S}(\mathbb{S}^n)$ gives origin to a (real) Banach space $\mathcal{S}$ by using the norm

$$\|\mathfrak{a}\|_{\mathcal{S}} \doteq \sum_{n\in\mathbb{N}} n^2 \mathbf{C}^{n-1} \|\mathfrak{a}_n\|_{\mathcal{S}(\mathbb{S}^n)} , \qquad \mathfrak{a} \equiv (\mathfrak{a}_n)_{n\in\mathbb{N}} \in \mathcal{S} , \tag{51}$$

with the constant $\mathbf{C} > 0$ defined by (45).

The separable Banach space of long-range models is defined by

$$\mathcal{M} \doteq \left\{ \mathfrak{m} \in \mathcal{W}^{\mathbb{R}} \times \mathcal{S} : \|\mathfrak{m}\|_{\mathcal{M}} < \infty \right\} \tag{52}$$

with the norm of $\mathcal{M}$ being defined from (41) and (51) by

$$\|\mathfrak{m}\|_{\mathcal{M}} \doteq \|\Phi\|_{\mathcal{W}} + \|\mathfrak{a}\|_{\mathcal{S}} , \qquad \mathfrak{m} \doteq (\Phi, \mathfrak{a}) \in \mathcal{M} . \tag{53}$$

The spaces $\mathcal{W}^{\mathbb{R}}$ and $\mathcal{S}$ are seen as subspaces of $\mathcal{M}$. In particular, $\Phi \equiv (\Phi, 0) \in \mathcal{M}$ for $\Phi \in \mathcal{W}^{\mathbb{R}}$. Using the subspace $\mathcal{W}^{\mathrm{R}}$ of finite-range interactions defined by (42) for $\mathrm{R} \in [0,\infty)$, we introduce the subspace

$$\mathcal{S}^{\infty} \doteq \bigcup_{\mathrm{R}\in[0,\infty)} \left\{ (\mathfrak{a}_n)_{n\in\mathbb{N}} \in \mathcal{S} : \forall n \in \mathbb{N},\ |\mathfrak{a}_n|(\mathbb{S}^n) = |\mathfrak{a}_n|((\mathbb{S} \cap \mathcal{W}^{\mathrm{R}})^n) \right\} . \tag{54}$$

Long-range dynamics are studied for models in the subspaces

$$\mathcal{M}^{\infty} \doteq \mathcal{W}^{\mathbb{R}} \times \mathcal{S}^{\infty} \qquad \text{and} \qquad \mathcal{M}_1^{\infty} \doteq \left(\mathcal{W}^{\mathbb{R}} \cap \mathcal{W}_1 \right) \times \mathcal{S}^{\infty}. \tag{55}$$

Clearly, $\mathcal{W}^{\mathbb{R}} \subseteq \mathcal{M}^{\infty} \subseteq \mathcal{M}$ and $\left(\mathcal{W}^{\mathbb{R}} \cap \mathcal{W}_1 \right) \subseteq \mathcal{M}_1^{\infty} \subseteq \mathcal{M}^{\infty}$.

Similar to self-adjoint short-range interactions, each long-range model leads to a sequence of local Hamiltonians: For any $L \in \mathbb{N}_0$ and $\mathfrak{m} \in \mathcal{M}$,

$$U_L^{\mathfrak{m}} \doteq U_L^{\Phi} + \sum_{n\in\mathbb{N}} \frac{1}{|\Lambda_L|^{n-1}} \int_{\mathbb{S}^n} U_L^{\Psi^{(1)}} \cdots U_L^{\Psi^{(n)}} \mathfrak{a}_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)} \right) \tag{56}$$

with U_L^{Φ} and $U_L^{\Psi^{(k)}}$ been defined by (43). Note that $U_L^{\mathfrak{m}} = (U_L^{\mathfrak{m}})^*$ and straightforward estimates yield the upper bound

$$\|U_L^{\mathfrak{m}}\|_{\mathcal{U}} \leq \mathbf{C} \, |\Lambda_L| \, \|\mathfrak{m}\|_{\mathcal{M}} , \qquad L \in \mathbb{N}_0 . \tag{57}$$

Compare with (44).

Similar to (46), each model $\mathfrak{m} \in \mathcal{M}$ leads to finite-volume dynamics defined, for any $L \in \mathbb{N}_0$, by

$$\tau_t^{(L,\mathfrak{m})}(A) = \mathrm{e}^{itU_L^{\mathfrak{m}}} A \mathrm{e}^{-itU_L^{\mathfrak{m}}} , \qquad A \in \mathcal{U} . \tag{58}$$

In contrast with short range interactions (see (49)), for any fixed $A \in \mathcal{U}$ and $t \in \mathbb{R}$, the thermodynamic limit $L \to \infty$ of $\tau_t^{(L,\mathfrak{m})}(A)$ does *not* necessarly exist in the C^*-algebra $\mathcal{U}$. However, by [2, Theorem 4.3], for any $\mathfrak{m} \in \mathcal{M}_1^{\infty}$, it converges in the σ-weak topology within some represention of $\mathcal{U}$. This is reminiscent of the fact that the energy-density observable $U_L^{\Phi} / |\Lambda_L|$ does not converges in $\mathcal{U}$, as $L \to \infty$, but its expectation value with respect to any periodic state does. See Section 4.3 for more details.

4.2 State Spaces

4.2.1 Finite-Volume State Space

Let $\mathcal{U}_\Lambda^*$ be the dual space of the local C^*-algebra $\mathcal{U}_\Lambda$ for any (non-empty) finite subset $\Lambda \subseteq \mathbb{Z}^d$, i.e., for $\Lambda \in \mathcal{P}_f$. We denote by

$$E_\Lambda \doteq \{ \rho_\Lambda \in \mathcal{U}_\Lambda^* : \rho_\Lambda \geq 0,\ \rho_\Lambda(\mathfrak{1}) = 1 \} , \qquad \Lambda \in \mathcal{P}_f , \tag{59}$$

the space of all states on $\mathcal{U}_\Lambda$. By finite dimensionality of $\mathcal{U}_\Lambda$ for $\Lambda \in \mathcal{P}_f$, the space E_Λ is a norm-compact convex subset of the dual space $\mathcal{U}_\Lambda^*$ and, for any $\rho_\Lambda \in E_\Lambda$, there is a unique, positive, trace-one operator $\mathrm{d}_\Lambda \in \mathcal{B}(\mathcal{F}_{\Lambda_L})$ satisfying

$$\rho_\Lambda (A) \doteq \mathrm{Trace}\,(\mathrm{d}_\Lambda A) \ , \qquad A \in \mathcal{U}_\Lambda \ , \tag{60}$$

named the density matrix of ρ_Λ. In fact, E_Λ is affinely equivalent to the set of all states acting on the algebra of $2^{|\Lambda|\times|\mathrm{S}|} \times 2^{|\Lambda|\times|\mathrm{S}|}$ matrices. In comparison, the structure of the set of states for infinite systems is more subtle, as demonstrated in [3, 75].

Note that the physically relevant finite-volume states ρ_Λ, $\Lambda \in \mathcal{P}_f$, are *even*, i.e., $\rho_\Lambda \circ \sigma|_{\mathcal{U}_\Lambda} = \rho_\Lambda$ with $\sigma|_{\mathcal{U}_\Lambda}$ being the restiction to $\mathcal{U}_\Lambda$ of the unique $*$-automorphism σ of $\mathcal{U}$ satisfying (39). It means that ρ_Λ vanishes on all *odd* monomials in $\{a_{x,\mathrm{s}}, a_{x,\mathrm{s}}^*\}_{x\in\Lambda,\mathrm{s}\in\mathrm{S}}$. We define by

$$E_\Lambda^+ \doteq \{\rho_\Lambda \in E_\Lambda : \rho_\Lambda \circ \sigma|_{\mathcal{U}_\Lambda} = \rho_\Lambda\} \subseteq E_\Lambda \ , \qquad \Lambda \in \mathcal{P}_f \ , \tag{61}$$

the space of all finite-volume even states.

4.2.2 Infinite-Volume State Spaces

For the infinite system, let $\mathcal{U}^* \equiv \mathcal{U}_{\mathbb{Z}^d}^*$ be the dual space of $\mathcal{U} \equiv \mathcal{U}_{\mathbb{Z}^d}$. In contrast with $\mathcal{U}_\Lambda$ for $\Lambda \in \mathcal{P}_f$, $\mathcal{U}$ has infinite dimension and the natural topology on $\mathcal{U}^*$ is the weak* topology[15] (and not the norm topology). The topology used here on $\mathcal{U}^*$ is always the weak* topology and, in this case, $\mathcal{U}^*$ is a locally convex space, by [76, Theorem 3.10].

Similar to (59), the state space on $\mathcal{U}$ is defined by

$$E \equiv E_{\mathbb{Z}^d} \doteq \{\rho \in \mathcal{U}^* : \rho \geq 0,\ \rho(\mathfrak{1}) = 1\} \ . \tag{62}$$

It is a metrizable, convex and compact subset of $\mathcal{U}^*$, by [76, Theorems 3.15 and 3.16]. It is also the state space of the classical dynamics we define in [3]. By the Krein-Milman theorem [76, Theorem 3.23], E is the closure of the convex hull of the (non-empty) set of its extreme points, which are meanwhile dense in E, by [1, Eq. (13)].

As explained below Equation (40), recall that the C^*-algebra $\mathcal{U}^+$ should be considered as more fundamental than $\mathcal{U}$ in Physics, because of the local causality in quantum field theory. As a consequence, states on the C^*-algebra $\mathcal{U}^+$ should be seen as being the physically relevant ones. The set of states on $\mathcal{U}^+$ is canonically identified with the metrizable, convex and compact set of *even* states defined by

$$E^+ \equiv E_{\mathbb{Z}^d}^+ \doteq \{\rho \in \mathcal{U}^* : \rho \geq 0,\ \rho(\mathfrak{1}) = 1,\ \rho \circ \sigma = \rho\} \ , \tag{63}$$

σ being the unique $*$-automorphism of $\mathcal{U}$ satisfying (39). This space has the same geometrical structure than the full state space E, i.e., E^+ and E are equivalent by an affine homeomorphism. In particular, E^+ is the closure of the convex hull of the (non-empty) set of its extreme points, which are dense in E^+. See [1, Proposition 2.1] and its proof.

Note that the spaces E and E^+, having a dense set of extreme points – or equivalently having dense extreme boundary – has a much more peculiar geometrical structure than the finite-volume state space E_Λ for $\Lambda \in \mathcal{P}_f$. At first glance, this structure may look very strange for a non-expert on convex analysis, but it should not be that surprising: For instance, the unit ball of any infinite-dimensional Hilbert space has a dense extreme boundary in the weak topology. In fact, the existence of convex compact sets with dense extreme boundary *is not an accident* in infinite-dimensional spaces. This has been first proven [77] in 1959 for convex norm-compact sets within a separable Banach

[15]The weak* topology of $\mathcal{U}^*$ is the coarsest topology on $\mathcal{U}^*$ that makes the mapping $\rho \mapsto \rho\,(A)$ continuous for every $A \in \mathcal{U}$. See [76, Sections 3.8, 3.10, 3.14] for more details.

space. Recently, in [3, Section 2.3] and more generally in [75], the property of having dense extreme boundary is proven to be generic for weak*-compact convex sets within the dual space of an infinite-dimensional topological space. As a matter of fact, all state spaces of infinite-volume systems one is going to encounter in the current paper have dense extreme boundary, except the set of permutation-invariant states described in Section 4.4, because it can be encoded within a finite-dimensional space.

4.2.3 Periodic State Spaces

Consider the sub-groups $(\mathbb{Z}_{\vec{\ell}}^d, +) \subseteq (\mathbb{Z}^d, +)$, $\vec{\ell} \in \mathbb{N}^d$, where

$$\mathbb{Z}_{\vec{\ell}}^d \doteq \ell_1 \mathbb{Z} \times \cdots \times \ell_d \mathbb{Z} \ .$$

At fixed $\vec{\ell} \in \mathbb{N}^d$, a state $\rho \in E$ satisfying $\rho \circ \alpha_x = \rho$ for all $x \in \mathbb{Z}_{\vec{\ell}}^d$ is called $\mathbb{Z}_{\vec{\ell}}^d$*-invariant* on $\mathcal{U}$ or $\vec{\ell}$*-periodic*, α_x being the unique $*$-automorphism of $\mathcal{U}$ satisfying (38). Translation-invariant states refer to $(1, \ldots, 1)$-periodic states. For any $\vec{\ell} \in \mathbb{N}^d$, let

$$E_{\vec{\ell}} \doteq \left\{ \rho \in E : \rho \circ \alpha_x = \rho \quad \text{for all } x \in \mathbb{Z}_{\vec{\ell}}^d \right\} \tag{64}$$

be the space of $\vec{\ell}$-periodic states. By [43, Lemma 1.8], periodic states are always even and, by [1, Proposition 2.3], the set of all periodic states

$$E_{\mathrm{p}} \doteq \bigcup_{\vec{\ell} \in \mathbb{N}^d} E_{\vec{\ell}} \subseteq E^+ \tag{65}$$

is dense in the space E^+ of even states.

For each $\vec{\ell} \in \mathbb{N}^d$, $E_{\vec{\ell}}$ is metrizable, convex and compact and, by the Krein-Milman theorem [76, Theorem 3.23], it is the closure of the convex hull of the (non-empty) set $\mathcal{E}_{\vec{\ell}}$ of its extreme points. In fact, by [43, Theorem 1.9] (which uses the Choquet theorem [78, p. 14]), for any $\rho \in E_{\vec{\ell}}$, there is a unique probability measure μ_ρ on $E_{\vec{\ell}}$ with support in $\mathcal{E}_{\vec{\ell}}$ such that[16]

$$\rho = \int_{\mathcal{E}_{\vec{\ell}}} \hat{\rho} \, \mathrm{d}\mu_\rho (\hat{\rho}) \ . \tag{66}$$

The set $\mathcal{E}_{\vec{\ell}}$ can be characterized by an ergodicity property of states, see [43, Theorem 1.16]. Moreover, $\mathcal{E}_{\vec{\ell}}$ is dense in $E_{\vec{\ell}}$, by [43, Corollary 4.6]. In other words, like the sets E and E^+, $E_{\vec{\ell}}$ has dense extreme boundary for any $\vec{\ell} \in \mathbb{N}^d$.

4.3 Long-Range Dynamics

4.3.1 Self-Consistency Equations

Generically, long-range dynamics in infinite volume are intricate combinations of a classical and quantum dynamics. Similar to [3, Theorem 4.1], both dynamics are consequences of the existence of a solution to a *self-consistency equation*. In order to present such equations we need some preliminary definitions: For $\vec{\ell} \in \mathbb{N}^d$, $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}$ and $\rho \in E$, we define the approximating (self-adjoint, short-range) interaction $\Phi^{(\mathfrak{m},\rho)} \in \mathcal{W}^{\mathbb{R}}$ by

$$\Phi^{(\mathfrak{m},\rho)} \doteq \Phi + \sum_{n \in \mathbb{N}} \int_{\mathbb{S}^n} \mathfrak{a}_n \left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)} \right) \sum_{m=1}^{n} \Psi^{(m)} \prod_{j \in \{1,\ldots,n\}, j \neq m} \rho(\mathfrak{e}_{\Psi^{(j)},\vec{\ell}}) \ , \tag{67}$$

[16] The integral in (66) only means that $\rho \in E_{\vec{\ell}}$ is the (unique) barycenter of the normalized positive Borel regular measure μ_ρ on $E_{\vec{\ell}}$. See, e.g., [43, Definition 10.15, Theorem 10.16, and also Lemma 10.17].

where

$$\mathfrak{e}_{\Phi,\vec{\ell}} \doteq \frac{1}{\ell_1 \cdots \ell_d} \sum_{x=(x_1,\ldots,x_d),\, x_i \in \{0,\ldots,\ell_i - 1\}} \sum_{\Lambda \in \mathcal{P}_f,\, \Lambda \ni x} \frac{\Phi_\Lambda}{|\Lambda|} . \tag{68}$$

Recall meanwhile that $\mathcal{M}^\infty \doteq \mathcal{W}^{\mathbb{R}} \times \mathcal{S}^\infty$, see (54)-(55). Then, by [1, Theorem 6.5], for any $\mathfrak{m} \in \mathcal{M}^\infty$, there is a unique continuous[17] mapping $\boldsymbol{\varpi}^{\mathfrak{m}}$ from $\mathbb{R}$ to the space of automorphisms[18] (or self-homeomorphisms) of E such that

$$\boldsymbol{\varpi}^{\mathfrak{m}}(t;\rho) = \rho \circ \tau_{t,0}^{\Psi^{(\mathfrak{m},\rho)}} , \qquad t \in \mathbb{R},\ \rho \in E , \tag{69}$$

with $\Psi^{(\mathfrak{m},\rho)} \in C(\mathbb{R};\mathcal{W}^{\mathbb{R}})$, $\rho \in E$, defined by

$$\Psi^{(\mathfrak{m},\rho)}(t) \doteq \Phi^{(\mathfrak{m},\boldsymbol{\varpi}^{\mathfrak{m}}(t;\rho))} , \qquad t \in \mathbb{R} , \tag{70}$$

and where the strongly continuous two-parameter family $(\tau_{t,s}^{\Psi^{(\mathfrak{m},\rho)}})_{s,t\in\mathbb{R}}$ is the strong limit, at any fixed $s,t \in \mathbb{R}$, of the local dynamics $(\tau_{t,s}^{(L,\Psi^{(\mathfrak{m},\rho)})})_{s,t\in\mathbb{R}}$ defined by (47) for $\Psi = \Psi^{(\mathfrak{m},\rho)}$, see (49) and [74, Corollary 5.2]. Equation (69) is named here the self-consistency equation.

4.3.2 Quantum Part of Long-Range Dynamics

Recall that any model $\mathfrak{m} \in \mathcal{M}$ leads to finite-volume dynamics $(\tau_t^{(L,\mathfrak{m})})_{t\in\mathbb{R}}$, $L \in \mathbb{N}_0$, defined by (58). Therefore, at $L \in \mathbb{N}_0$, the time-evolution $(\rho_t^{(L)})_{t\in\mathbb{R}}$ of any finite-volume state $\rho^{(L)} \in E_{\Lambda_L}$ is given by

$$\rho_t^{(L)} \doteq \rho^{(L)} \circ \tau_t^{(L,\mathfrak{m})} . \tag{71}$$

The corresponding time-dependent density matrix is $\mathrm{d}_t^{(L)} = \tau_{-t}^{(L,\mathfrak{m})}(\mathrm{d}^{(L)})$. Equation (71) refers to the Schrödinger picture of quantum mechanics.

As already mentioned, for any fixed $A \in \mathcal{U}$ and $t \in \mathbb{R}$, the thermodynamic limit $L \to \infty$ of $\tau_t^{(L,\mathfrak{m})}(A)$ does not necessarly exist in $\mathcal{U}$, but the limit $L \to \infty$ of $\rho_t^{(L)}$ can still make sense: Fix once and for all a translation-invariant model $\mathfrak{m} \in \mathcal{M}_1^\infty$, see (55). Take $\vec{\ell} \in \mathbb{N}^d$ and recall that $E_{\vec{\ell}}$ is the space of $\vec{\ell}$-periodic states defined by (64), with set of extreme points denoted by $\mathcal{E}_{\vec{\ell}}$. Recall also (66), i.e., that, for any $\rho \in E_{\vec{\ell}}$, there is a unique probability measure μ_ρ on $E_{\vec{\ell}}$ with support in $\mathcal{E}_{\vec{\ell}}$ such that

$$\rho = \int_{\mathcal{E}_{\vec{\ell}}} \hat{\rho}\, \mathrm{d}\mu_\rho(\hat{\rho}) .$$

Since the set $\mathcal{E}_{\vec{\ell}}$ is characterized by an ergodicity property (see [43, Theorem 1.16]), one can prove that, for any $A \in \mathcal{U}$,

$$\lim_{L\to\infty} \rho \circ \tau_t^{(L,\mathfrak{m})}(A) = \int_{\mathcal{E}_{\vec{\ell}}} \boldsymbol{\varpi}^{\mathfrak{m}}(t;\hat{\rho})(A)\ \mathrm{d}\mu_\rho(\hat{\rho}) = \int_{\mathcal{E}_{\vec{\ell}}} \hat{\rho} \circ \tau_{t,0}^{\Psi^{(\mathfrak{m},\hat{\rho})}}(A)\ \mathrm{d}\mu_\rho(\hat{\rho}) \tag{72}$$

with $\boldsymbol{\varpi}^{\mathfrak{m}}$ being the solution to the self-consistency equation (69). See [2, Proposition 4.2, Theorem 4.3]. Using in particular, for any $L \in \mathbb{N}_0$, the restriction $\rho^{(L)} \doteq \rho|_{\mathcal{U}_{\Lambda_L}}$ of a state $\rho \in E_{\vec{\ell}}$ to $\mathcal{U}_{\Lambda_L}$ then (72) can also be seen as the thermodynamic limit $L \to \infty$ of the expectation value $\rho_t^{(L)}(A)$ of any local element $A \in \mathcal{U}_0$, the time-dependent state $\rho_t^{(L)}$ been defined by (71).

[17] We endow the set $C(E;E)$ of continuous functions from E to itself with the topology of uniform convergence. See [1, Eq. (100)] for more details.

[18] I.e., elements of $C(E;E)$ with continuous inverse.

Equation (72) means in fact that the thermodynamic limit $L \to \infty$ of $\tau_t^{(L,\mathfrak{m})}(A)$ exists in the GNS representation[19] $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ of $\mathcal{U}$ associated with the initial state ρ. In other words, one obtains a dynamics $(T_t^{\mathfrak{m}})_{t\in\mathbb{R}}$ defined by

$$T_t^{\mathfrak{m}} \circ \pi_\rho (A) = \lim_{L\to\infty} \pi_\rho \circ \tau_t^{(L,\mathfrak{m})} (A) \ , \qquad A \in \mathcal{U} \ ,$$

on the subalgebra $\pi_\rho(\mathcal{U})$ of the algebra $\mathcal{B}(\mathcal{H}_\rho)$ of bounded operators on the Hilbert space $\mathcal{H}_\rho$. The above limit has to be understood in the σ-weak topology within $\mathcal{B}(\mathcal{H}_\rho)$ (and in many cases one could even prove strong convergence). This refers to the quantum part of the long-range dynamics (in some representation), which is generally *non-autonomous*, although the primordial local dynamics is autonomous.

4.3.3 Classical Part of Long-Range Dynamics

For any $\vec{\ell} \in \mathbb{N}^d$, the infinite-volume long-range dynamics of $\vec{\ell}$-periodic states, as given by (72), involves the knowledge of a continuous flow[20] on $\mathcal{E}_{\vec{\ell}}$. Seeing $\mathcal{E}_{\vec{\ell}}$ or $E_{\vec{\ell}} = \overline{\mathcal{E}_{\vec{\ell}}}$ as a (classical) phase space, it becomes natural to study the classical Hamiltonian dynamics associated with this flow, as is usual in classical mechanics. Note that, for a (possibly non-translation-invariant) model $\mathfrak{m} \in \mathcal{M}^\infty$, $\varpi^{\mathfrak{m}}(t; \mathcal{E}_{\vec{\ell}})$ conserves the space E^+ of *even* states defined by (63), but not necessary $E_{\vec{\ell}}$. If $\mathfrak{m} \in \mathcal{M}_1^\infty$ then the flow conserves the sets $\mathcal{E}_{\vec{\ell}}$ and $E_{\vec{\ell}}$ for any $\vec{\ell} \in \mathbb{N}^d$, see below (80). Here, we adopt a broader perspective by taking the full state space E, defined by (62), because the classical dynamics described below can be easily pushed forward, through the restriction map, from $C(E;\mathbb{C})$ to $C(E^+;\mathbb{C})$ for general $\mathfrak{m} \in \mathcal{M}^\infty$, and also to $C(E_{\vec{\ell}};\mathbb{C})$ for any $\vec{\ell} \in \mathbb{N}^d$, when $\mathfrak{m} \in \mathcal{M}_1^\infty$ is translation-invariant.

Note that $C(E;\mathbb{C})$, $C(E^+;\mathbb{C})$ and $C(E_{\vec{\ell}};\mathbb{C})$, endowed with the point-wise operations and complex conjugation as well as the supremum norm, are unital commutative C^*-algebras. For any model $\mathfrak{m} \in \mathcal{M}^\infty$, the mapping $\varpi^{\mathfrak{m}}$, the solution to the self-consistency equation (69), yields a family $(V_t^{\mathfrak{m}})_{t\in\mathbb{R}}$ of $*$-automorphisms on $C(E;\mathbb{C})$ defined by

$$V_t^{\mathfrak{m}} (f) \doteq f \circ \varpi^{\mathfrak{m}} (t) \ , \qquad f \in C(E;\mathbb{C}) \, , \ t \in \mathbb{R} \, . \tag{73}$$

It is a Feller group: $(V_t^{\mathfrak{m}})_{t\in\mathbb{R}}$ is a strongly continuous group of $*$-automorphisms of $C(E;\mathbb{C})$, which is thus positivity preserving with operator norm equal to one, by [1, Proposition 6.7]. When it is restricted to the dense subspace $E_{\mathrm{p}} \subseteq E^+$ (65) of all periodic states, the ones we are interested in (cf. (72)), the group $(V_t^{\mathfrak{m}})_{t\in\mathbb{R}}$ for any translation-invariant model $\mathfrak{m} \in \mathcal{M}_1^\infty$ is generated by a Poissonian symmetric derivation:

(i) Local polynomials: Elements of the C^*-algebra $\mathcal{U}$ naturally define continuous and affine functions $\hat{A} \in C(E;\mathbb{C})$ by

$$\hat{A}(\rho) \doteq \rho(A) \ , \qquad \rho \in E, \ A \in \mathcal{U} \, .$$

This is the well-known Gelfand transform. Recall that $\mathcal{U}_0$ is the normed $*$-algebra of local elements of $\mathcal{U}$ defined by (37). We denote by

$$\mathbb{P} \doteq \mathbb{C}[\{\hat{A} : A \in \mathcal{U}_0\}] \subseteq C(E;\mathbb{C}) \tag{74}$$

the subspace of (local) polynomials in the elements of $\{\hat{A} : A \in \mathcal{U}_0\}$, with complex coefficients.

[19] Recall that $\mathcal{H}_\rho$ is an Hilbert space, $\pi_\rho : \mathcal{U} \to \mathcal{B}(\mathcal{H}_\rho)$ is a representation of $\mathcal{U}$ and $\Omega_\rho \in \mathcal{H}_\rho$ is a cyclic vector for $\pi_\rho(\mathcal{U})$.

[20] That is, the continuous mapping $\varpi^{\mathfrak{m}}$ from $\mathbb{R}$ to the space of automorphisms (or self-homeomorphisms) of E defined by (69).

(ii) Poisson structure: For any $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathcal{U}$ and $g \in C^1(\mathbb{R}^n, \mathbb{C})$ we define the function $\Gamma_g \in C(E;\mathbb{C})$ by

$$\Gamma_g(\rho) \doteq g(\rho(A_1), \ldots, \rho(A_n)) , \qquad \rho \in E .$$

Functions of this type are known in the literature as cylindrical functions. For such a function and any $\rho \in E$, define

$$\mathrm{D}\Gamma_g(\rho) \doteq \sum_{j=1}^{n} (A_j - \rho(A_j)\mathfrak{1}) \partial_{x_j} g(\rho(A_1), \ldots, \rho(A_n)) , \qquad \rho \in E . \tag{75}$$

This definition comes from a convex weak*-continuous Gâteaux derivative, as explained in [1, Section 5.2]. Then, for any $n, m \in \mathbb{N}$, $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathcal{U}$, $g \in C^1(\mathbb{R}^n, \mathbb{C})$ and $h \in C^1(\mathbb{R}^m, \mathbb{C})$, we define the continuous function $\{\Gamma_h, \Gamma_g\} \in C(E;\mathbb{C})$ by

$$\{\Gamma_h, \Gamma_g\}(\rho) \doteq \rho(i[\mathrm{D}\Gamma_h(\rho), \mathrm{D}\Gamma_g(\rho)]) , \qquad \rho \in E . \tag{76}$$

This defines a Poisson bracket on the space $\mathbb{P}$ of all (local) polynomial functions acting on E. By contruction, for any $\vec{\ell} \in \mathbb{N}^d$,

$$\{\Gamma_h|_{E^+}, \Gamma_g|_{E^+}\} \doteq \{\Gamma_h, \Gamma_g\}|_{E^+}, \quad \{\Gamma_h|_{E_{\vec{\ell}}}, \Gamma_g|_{E_{\vec{\ell}}}\} \doteq \{\Gamma_h, \Gamma_g\}|_{E_{\vec{\ell}}}, \quad \{\Gamma_h|_{\mathcal{E}_{\vec{\ell}}}, \Gamma_g|_{\mathcal{E}_{\vec{\ell}}}\} \doteq \{\Gamma_h, \Gamma_g\}|_{\mathcal{E}_{\vec{\ell}}} \tag{77}$$

also define a Poisson bracket on polynomials of $C(E^+;\mathbb{C})$, $C(E_{\vec{\ell}};\mathbb{C})$ and $C(\mathcal{E}_{\vec{\ell}};\mathbb{C})$, respectively. This definition can be extended to the space

$$\mathfrak{Y} \equiv C^1(E;\mathbb{C}) \subsetneq C(E;\mathbb{C})$$

of continuously differentiable functions. See [1, Section 5.2] and [3, Section 3] for a more detailed construction of such Poisson structures in quantum mechanics.

(iii) Liouville's equation: Local classical energy functions [1, Definition 6.8] associated with $\mathfrak{m} \in \mathcal{M}$ are defined, for any $L \in \mathbb{N}_0$, by

$$\mathrm{h}_L^{\mathfrak{m}} \doteq \widehat{U_L^{\Phi}} + \sum_{n \in \mathbb{N}} \frac{1}{|\Lambda_L|^{n-1}} \int_{\mathbb{S}^n} \widehat{U_L^{\Psi^{(1)}}} \cdots \widehat{U_L^{\Psi^{(n)}}} \, \mathfrak{a}_n\left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right) \in C^1(E;\mathbb{C}) . \tag{78}$$

Compare with the local Hamiltonian $U_L^{\mathfrak{m}}$ defined by (56). Then, by [1, Corollary 6.11], for each translation-invariant model $\mathfrak{m} \in \mathcal{M}_1^{\infty}$, any time $t \in \mathbb{R}$ and all local polynomials $f \in \mathbb{P}$, one has $V_t^{\mathfrak{m}}(f) \in C^1(E;\mathbb{C})$ and

$$\partial_t V_t^{\mathfrak{m}}(f) = V_t^{\mathfrak{m}}\left(\lim_{L \to \infty} \{\mathrm{h}_L^{\mathfrak{m}}, f\}\right) = \lim_{L \to \infty} \{\mathrm{h}_L^{\mathfrak{m}}, V_t^{\mathfrak{m}}(f)\} , \tag{79}$$

where all limits have to be understood point-wise on the *dense* subspace $E_{\mathrm{p}} \subsetneq E^+$ of all periodic states. We thus obtain the usual (autonomous) dynamics of classical mechanics written in terms of Poisson brackets. See, e.g., [79, Proposition 10.2.3]. This corresponds to *Liouville's equation*.

By [1, Eq. (110)], observe additionally that, for any $\mathfrak{m} \in \mathcal{M}_1^{\infty}$ and $\vec{\ell} \in \mathbb{N}^d$, the flow conserves the sets E^+, $E_{\vec{\ell}}$ and $\mathcal{E}_{\vec{\ell}}$, i.e.,

$$\bigcup_{t \in \mathbb{R}} \boldsymbol{\varpi}^{\mathfrak{m}}(t; E^+) \subseteq E^+, \quad \bigcup_{t \in \mathbb{R}} \boldsymbol{\varpi}^{\mathfrak{m}}(t; E_{\vec{\ell}}) \subseteq E_{\vec{\ell}}, \quad \bigcup_{t \in \mathbb{R}} \boldsymbol{\varpi}^{\mathfrak{m}}(t; \mathcal{E}_{\vec{\ell}}) \subseteq \mathcal{E}_{\vec{\ell}} . \tag{80}$$

Therefore, $V_{t,s}^{\mathfrak{m}}$ can be seen as a mapping from $C(E^+;\mathbb{C})$, $C(E_{\vec{\ell}};\mathbb{C})$ or $C(\mathcal{E}_{\vec{\ell}};\mathbb{C})$ to itself:

$$V_t^{\mathfrak{m}}(f|_{E^+}) \doteq (V_t^{\mathfrak{m}} f)|_{E^+}, \quad V_t^{\mathfrak{m}}(f|_{E_{\vec{\ell}}}) \doteq (V_t^{\mathfrak{m}} f)|_{E_{\vec{\ell}}}, \quad V_t^{\mathfrak{m}}(f|_{\mathcal{E}_{\vec{\ell}}}) \doteq (V_t^{\mathfrak{m}} f)|_{\mathcal{E}_{\vec{\ell}}} \tag{81}$$

for any $t \in \mathbb{R}$ and $f \in C(E;\mathbb{C})$. By using the Poisson brackets (77), Liouville's equation (79) can be written on $C(E^+;\mathbb{C})$, $C(E_{\vec{\ell}};\mathbb{C})$ or $C(\mathcal{E}_{\vec{\ell}};\mathbb{C})$ for any $\mathfrak{m} \in \mathcal{M}_1^{\infty}$ and $\vec{\ell} \in \mathbb{N}^d$.

Remark 2

The mathematically rigorous derivation of Liouville's equation (79) is non-trivial and results from Lieb-Robinson bounds for multi-commutators [74], first derived in 2017.

4.3.4 Entanglement of Quantum and Classical Dynamics

In the thermodynamic limit, the "primordial" algebra is the separable unital C^*-algebra $\mathcal{U}$, generated by fermionic annihilation and creation operators satisfying the canonical anti-commutation relations, as explained in Section 4.1.1. Fix once and for all $\mathfrak{m} \in \mathcal{M}_1^\infty$. Let $K = E$, E^+ or $E_{\vec{\ell}} = \overline{\mathcal{E}_{\vec{\ell}}}$ for any $\vec{\ell} \in \mathbb{N}^d$, which is, in each case, a metrizable, convex (weak*-) compact subset of the dual space $\mathcal{U}^*$.

(i) Classical dynamics: The classical (i.e., commutative) unital C^*-algebra is the algebra $C(K;\mathbb{C})$ of continuous and complex-valued functions on K. The mapping $\varpi^\mathfrak{m}$, the solution to the self-consistency equation (69), yields a strongly continuous group $(V_t^\mathfrak{m})_{t\in\mathbb{R}}$ of $*$-automorphisms of $C(K;\mathbb{C})$, satisfying Liouville's equation as previously explained.

(ii) Quantum dynamics: Similar to quantum-classical hybrid theories of theoretical physics, described for instance in [80–85], consider now a secondary quantum algebra $C(K;\mathbb{C}) \otimes \mathcal{U}$, which is nothing else (up to isomorphism) than the C^*-algebra $C(K,\mathcal{U})$ of all (weak*) continuous $\mathcal{U}$-valued functions on states. By [1, Proposition 6.2] and (80), the mapping $\varpi^\mathfrak{m}$ from $\mathbb{R}$ to the space of automorphisms (or self-homeomorphisms) of K leads to a (state-dependent) quantum dynamics $\mathfrak{T}^\mathfrak{m} \doteq (\mathfrak{T}_t^\mathfrak{m})_{t\in\mathbb{R}}$ on

$$C(K,\mathcal{U}) \equiv C(K;\mathbb{C}) \otimes \mathcal{U} ,$$

via the strongly continuous, state-dependent two-parameter family $(\tau_{t,s}^{\Psi^{(\mathfrak{m},\rho)}})_{s,t\in\mathbb{R}}$ with $\Psi^{(\mathfrak{m},\rho)}$ defined by (70):

$$[\mathfrak{T}_t^\mathfrak{m}(f)](\rho) \doteq \tau_{t,0}^{\Psi^{(\mathfrak{m},\rho)}}(f(\rho)) , \qquad \rho \in K,\ f \in C(K,\mathcal{U}),\ t \in \mathbb{R} .$$

(iii) Entangledment of quantum and classical dynamics: By following arguments of [3, End of Section 5.2], any (state-dependent) quantum dynamics on $C(K,\mathcal{U})$ preserving each element of $C(K;\mathbb{C}\mathfrak{1}) \subseteq C(K,\mathcal{U})$ yields a classical dynamics, which, in the case of $\mathfrak{T}^\mathfrak{m}$, is exactly $(V_t^\mathfrak{m})_{t\in\mathbb{R}}$. More interestingly, as we remark in [3, Section 4.2], each *classical* Hamiltonian, i.e., a continuously differentiable function of $C(K;\mathbb{R})$, leads to a state-dependent quantum dynamics. If the classical Hamiltonian equals (78) then the limit quantum dynamics, when $L \to \infty$, is precisely $\mathfrak{T}^\mathfrak{m}$. In other words, on can recover the classical dynamics from the quantum one, and vice versa. The classical and quantum systems are completely interdependent, i.e., *entangled*. This view point is very different from the common understanding[21] of the relation between quantum and classical mechanics, which is seen as a limiting case of quantum mechanics, even if there exist physical features (such as the spin of quantum particles) which do not have a clear classical counterpart.

The physical relevance of this mathematical structure comes from the fact that it encodes the infinite-volume dynamics of general long-range models with periodic initial state. In fact, the classical part of the long-range dynamics explicitly appears in the time evolution of extreme periodic states in (72) while the quantum part can be found in the last integral over extreme states of (72). The fact that the initial state must be a periodic state does not represent a serious constraint since any initial even state ρ can be approximated by a periodic state constructed[22] from its restriction $\rho|_{\mathcal{U}_{\Lambda_l}}$ to $\mathcal{U}_{\Lambda_l}$ for sufficiently large $l \in \mathbb{N}_0$. See, e.g., [1, Proof of Proposition 2.3]. Since $l \in \mathbb{N}_0$ is arbitrarily large, hence there is no real physical restriction in assuming that the initial state is a periodic one, noting that the physical states are always even[23].

[21] At least in many textbooks on quantum mechanics. See for instance [86, Section 12.4.2, end of the 4th paragraph of page 178].

[22] This is possible because of [87, Theorem 11.2].

[23] If the initial state is not even, we cannot a priori construct a periodic state from its restriction $\rho|_{\mathcal{U}_\Lambda}$ for any $\Lambda \in \mathcal{P}_f$.

4.4 Permutation-Invariant Lattice Fermi Systems

4.4.1 Permutation-Invariant Long-Range Models

Recall that $\mathcal{W}_\Pi \doteq \mathcal{W}^0$ is the space of permutation-invariant (or on-site) interactions, defined by Equation (42) for $\mathrm{R} = 0$. Define

$$\mathcal{M}_\Pi \doteq \left(\mathcal{W}^{\mathbb{R}} \cap \mathcal{W}_\Pi\right) \times \mathcal{S}^0 \ . \tag{82}$$

We name it the space of permutation-invariant long-range models, because associated local Hamiltonians are all invariant under permutations: Let Π be the set of all bijective mappings from $\mathbb{Z}^d$ to itself which leave all but finitely many elements invariant. It is a group with respect to the composition of mappings. The condition

$$\mathfrak{p}_\pi : a_{x,\mathrm{s}} \mapsto a_{\pi(x),\mathrm{s}}, \quad x \in \mathbb{Z}^d,\ \mathrm{s} \in \mathrm{S} \ , \tag{83}$$

defines a group homomorphism $\pi \mapsto \mathfrak{p}_\pi$ from Π to the group of $*$-automorphisms of the C^*-algebra $\mathcal{U}$. Then, for any $\mathfrak{m} \in \mathcal{M}_\Pi$ and $L \in \mathbb{N}_0$, the local Hamiltonian $U_L^{\mathfrak{m}}$ defined by (56) is permutation-invariant, that is,

$$\mathfrak{p}_\pi\left(U_L^{\mathfrak{m}}\right) = U_L^{\mathfrak{m}} \ , \qquad \pi \in \Pi,\ \pi\left(\Lambda_L\right) = \Lambda_L \ . \tag{84}$$

An example of permutation-invariant model is given by the strong-coupling BCS-Hubbard model: Fix $\mathrm{S} = \{\uparrow, \downarrow\}$. Let $\Phi^{Hubb}, \Psi^{BCS} \in \mathcal{W}_\Pi \cap \mathcal{W}^{\mathbb{R}}$ be defined by

$$\begin{aligned}
\Phi_{\{x\}}^{Hubb} &\doteq -\mu\left(n_{x,\uparrow} + n_{x,\downarrow}\right) - h\left(n_{x,\uparrow} - n_{x,\downarrow}\right) + 2\lambda n_{x,\uparrow} n_{x,\downarrow} \\
\Psi_{\{x\}}^{BCS} &\doteq a_{x,\downarrow} a_{x,\uparrow}
\end{aligned}$$

for $x \in \mathbb{Z}^d$ and $\Phi_\Lambda^{Hubb} \doteq 0 \doteq \Psi_\Lambda^{BCS}$ otherwise. Let $\mathfrak{a}^{BCS} \in \mathcal{S}^0$ be defined, for all Borel subset $\mathfrak{B} \subseteq \mathbb{S}$, by

$$\mathfrak{a}^{BCS}\left(\mathfrak{B}\right) = -\gamma \mathbf{1}\left[\Psi^{BCS} \in \mathfrak{B}\right] \ . \tag{85}$$

for some $\gamma \geq 0$, with $\mathbf{1}\left[\cdot\right]$ being the indicator function[24]. Then,

$$\mathfrak{m}_0 \doteq \left(\Phi^{Hubb}, \mathfrak{a}^{BCS}\right) \in \mathcal{M}_\Pi$$

is the strong-coupling BCS-Hubbard model since, in this case, the local Hamiltonian $U_L^{\mathfrak{m}_0}$ is equal to the strong-coupling BCS-Hubbard Hamiltonian H_L defined by (4).

4.4.2 Permutation-Invariant State Space

The set of all permutation-invariant states is defined by

$$E_\Pi \doteq \{\rho \in E : \rho = \rho \circ \mathfrak{p}_\pi \quad \text{for all } \pi \in \Pi\} \ , \tag{86}$$

$\mathfrak{p}_\pi$ being the unique $*$-automorphism of $\mathcal{U}$ satisfying (83). Obviously,

$$E_\Pi \subseteq \bigcap_{\vec{\ell} \in \mathbb{N}^d} E_{\vec{\ell}} \subseteq E^+ \ .$$

Furthermore, E_Π is metrizable, convex and compact and, by [43, Theorem 5.3], for any $\rho \in E_\Pi$, there is a unique probability measure μ_ρ on E_Π with support in the (non-empty) set $\mathcal{E}_\Pi$ of its extreme points such that

$$\rho = \int_{\mathcal{E}_\Pi} \hat{\rho} \, \mathrm{d}\mu_\rho\left(\hat{\rho}\right) \ . \tag{87}$$

[24] $\mathbf{1}\left[p\right] = 1$ if the proposition p holds true and $\mathbf{1}\left[p\right] = 0$ otherwise.

The set $\mathcal{E}_\Pi$ can be characterized by a version of the Størmer theorem for permutation-invariant states on the C^*-algebra $\mathcal{U}$. This theorem is a non-commutative version of the celebrated de Finetti theorem of (classical) probability theory. It is proven in the case of quantum-spin systems in [88] and for the fermion algebra $\mathcal{U}$ in [41, Lemmata 6.6-6.8]. It asserts that extreme permutation-invariant states $\rho \in \mathcal{E}_\Pi$ are product states defined as follows: First recall that the space E_Λ^+ of finite-volume even states is defined by (61) for any $\Lambda \in \mathcal{P}_f$. Then, via [87, Theorem 11.2], for any $\rho_0 \in E_{\{0\}}^+$, there is a unique even state

$$\rho \doteq \otimes_{\mathbb{Z}^d} \rho_0 \in E^+ \tag{88}$$

satisfying

$$\rho(\alpha_{x_1}(A_1) \cdots \alpha_{x_n}(A_n)) = \rho_0(A_1) \cdots \rho_0(A_n) \tag{89}$$

for all $A_1 \ldots A_n \in \mathcal{U}_{\{0\}}$ and all $x_1, \ldots x_n \in \mathbb{Z}^d$ such that $x_i \neq x_j$ for $i \neq j$. Recall that α_x, $x \in \mathbb{Z}^d$, defined by (38), are the $*$-automorphisms of $\mathcal{U}$ that represent translations. The set of all states of the form (88), called *product states*, is denoted by $E_\otimes$. It is nothing else but the set $\mathcal{E}_\Pi$ of extreme points of E_Π, i.e.,

$$E_\otimes = \mathcal{E}_\Pi\ . \tag{90}$$

This identity refers to the Størmer theorem, see, e.g., [43, Theorem 5.2].

Since product states are particular extremal states[25] of $E_{\vec{\ell}}$ for any $\vec{\ell} \in \mathbb{N}^d$, it follows from (90) that

$$\mathcal{E}_\Pi = E_\otimes \subseteq \bigcap_{\vec{\ell} \in \mathbb{N}^d} \mathcal{E}_{\vec{\ell}} \tag{91}$$

and the set $E_\Pi \subseteq E_{\vec{\ell}}$ is thus a closed metrizable face[26] of $E_{\vec{\ell}}$. For a more thorough exposition on this subject, see [43, Section 5.1]. By (90), the extreme boundary $\mathcal{E}_\Pi$ of E_Π is also closed and, in contrast with E, E^+ and $E_{\vec{\ell}}$ for any $\vec{\ell} \in \mathbb{N}^d$, $\mathcal{E}_\Pi$ is not a dense subset of E_Π. This is not surprising since states of $\mathcal{E}_\Pi = E_\otimes$ are in one-to-one correspondence with even states on the finite-dimensional C^*-algebra $\mathcal{U}_{\{0\}}$.

4.4.3 Quantum Part of Permutation-Invariant Long-Range Dynamics

Fix once and for all $\mathfrak{m} \in \mathcal{M}_\Pi$. If $\rho \in E_1 \doteq E_{(1,\cdots,1)}$, i.e., it is translation-invariant, then the approximating interaction (67) satisfies

$$\Phi^{(\mathfrak{m},\rho)} = \Phi^{(\mathfrak{m},\rho|_{\mathcal{U}_{\{0\}}})} \in \mathcal{W}_\Pi \cap \mathcal{W}^{\mathbb{R}} \tag{92}$$

and the infinite-volume dynamics constructed from this interaction, as defined by (49), preserves the local C^*-algebra $\mathcal{U}_\Lambda$ for any $\Lambda \in \mathcal{P}_f$. By (47)-(49) and (69)-(70), it also follows that

$$\bigcup_{t \in \mathbb{R}} \boldsymbol{\varpi}^{\mathfrak{m}}(t; E_\Pi) \subseteq E_\Pi \subseteq E_1\ , \qquad \bigcup_{t \in \mathbb{R}} \boldsymbol{\varpi}^{\mathfrak{m}}(t; E_\otimes) \subseteq E_\otimes \subseteq E_\Pi \tag{93}$$

(compare with (80)) and, for any $\Lambda \in \mathcal{P}_f$, $t \in \mathbb{R}$ and translation-invariant state $\rho \in E_1 \supseteq E_\Pi$,

$$\boldsymbol{\varpi}^{\mathfrak{m}}(t; \rho)|_{\mathcal{U}_\Lambda} = \boldsymbol{\varpi}^{\mathfrak{m}}(t; \rho|_{\mathcal{U}_\Lambda})|_{\mathcal{U}_\Lambda} \in E_\Lambda^+ \tag{94}$$

with E_Λ^+ being the space of finite-volume even states defined by (61) for any $\Lambda \in \mathcal{P}_f$.

[25] By [43, Theorem 5.2], all product states are strongly mixing, which means [43, Eq. (1.10)]. They are, in particular, strongly clustering and thus ergodic with respect to any sub-groups $(\mathbb{Z}_{\vec{\ell}}^d, +) \subseteq (\mathbb{Z}^d, +)$, where $\vec{\ell} \in \mathbb{N}^d$. By [43, Theorem 1.16], all product states belong to $\mathcal{E}_{\vec{\ell}}$ for any $\vec{\ell} \in \mathbb{N}^d$.

[26] A face F of a convex set K is defined to be a subset of K with the property that, if $\rho = \lambda_1 \rho_1 + \cdots + \lambda_n \rho_n \in F$ with $\rho_1, \ldots, \rho_n \in K$, $\lambda_1, \ldots, \lambda_n \in (0,1)$ and $\lambda_1 + \cdots + \lambda_n = 1$, then $\rho_1, \ldots, \rho_n \in F$.

If the initial state $\rho \in E_\Pi$ is permutation-invariant, then, by (72), (87) and (91), there is a unique probability measure μ_ρ on E_Π with support in $\mathcal{E}_\Pi = E_\otimes$ such that, for any $A \in \mathcal{U}$,

$$\lim_{L\to\infty} \rho \circ \tau_t^{(L,\mathfrak{m})}(A) = \int_{E_\otimes} \boldsymbol{\varpi}^{\mathfrak{m}}(t;\hat{\rho})(A)\ \mathrm{d}\mu_\rho(\hat{\rho}) = \int_{E_\otimes} \hat{\rho} \circ \tau_{t,0}^{\Psi^{(\mathfrak{m},\hat{\rho})}}(A)\ \mathrm{d}\mu_\rho(\hat{\rho}) \tag{95}$$

with $\boldsymbol{\varpi}^{\mathfrak{m}}$ being the solution to the self-consistency equation (69). In particular, by (93), the time-evolution of a permutation-invariant state is uniquely determined by its restriction to the finite-dimensional subalgebra $\mathcal{U}_{\{0\}}$ (dimension $2^{2|\mathrm{S}|}$).

If the initial state $\rho \in E_1 \supseteq E_\Pi$ is translation-invariant, then Equation (72) restricted to the finite-dimensional C^*-algebra $\mathcal{U}_\Lambda$ with $\Lambda \in \mathcal{P}_f$ reads[27]

$$\lim_{L\to\infty} \rho|_{\mathcal{U}_\Lambda} \circ \tau_t^{(L,\mathfrak{m})}(A) = \int_{E_\Lambda^+} \boldsymbol{\varpi}^{\mathfrak{m}}(t;\hat{\rho})(A)\ \mathrm{d}\mu_\rho(\hat{\rho}) \ , \qquad A \in \mathcal{U}_\Lambda \ . \tag{96}$$

For each fixed $\Lambda \in \mathcal{P}_f$, this gives now a family of equations on the finite-dimensional algebra $\mathcal{U}_\Lambda$ (dimension $2^{2|\Lambda|\times|\mathrm{S}|}$). These equations completly determine the time-evolution of a translation-invariant initial states.

For any $\vec{\ell}$-periodic state $\rho \in E_{\vec{\ell}}$ ($\vec{\ell} \in \mathbb{N}^d$), the approximating interaction (67) also belongs to $\mathcal{W}_\Pi \cap \mathcal{W}^{\mathbb{R}}$. The only difference with respect to translation-invariant states is that the on-site state $\rho|_{\mathcal{U}_{\{0\}}}$ in (92) has to be replaced with the finite-volume state $\rho|_{\mathcal{U}_{\mathcal{Z}_{\vec{\ell}}}}$, where, for $\vec{\ell} = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$,

$$\mathcal{Z}_{\vec{\ell}} \doteq \left\{(x_1, \ldots, x_d) \in \mathbb{Z}^d : x_i \in \{0, \ldots, \ell_i - 1\}\right\} \in \mathcal{P}_f \ .$$

Compare, as an example, with (34). Hence, if the initial state is periodic then Equation (72) leads again to a family of equations on the finite-dimensional algebra $\mathcal{U}_\Lambda$ (dimension $2^{2|\Lambda|\times|\mathrm{S}|}$) for each $\Lambda \in \mathcal{P}_f$ such that[28] $\Lambda \supseteq \mathcal{Z}_{\vec{\ell}}$. These equations again determine the time-evolution of a periodic initial state.

4.4.4 Classical Part of Permutation-Invariant Long-Range Dynamics

Fix again once and for all $\mathfrak{m} \in \mathcal{M}_\Pi$. By (93), the strongly continuous group $(V_t^{\mathfrak{m}})_{t\in\mathbb{R}}$ of $*$-automorphisms defined by (73) can be restricted to the unital C^*-algebra $C(E_\otimes;\mathbb{C})$ of continuous functions on the compact space $E_\otimes$ of product states. See also [3, Section 5.4 with $\mathcal{B} = \mathcal{U}_{\{0\}}$]. Without any risk of confusion, we denote the restriction of $(V_t^{\mathfrak{m}})_{t\in\mathbb{R}}$ to $E_\otimes$ again by $(V_t^{\mathfrak{m}})_{t\in\mathbb{R}}$.

Using (88)-(90) we identify $E_\otimes$ with the space $E_{\{0\}}^+$ of on-site even states and see now $(V_t^{\mathfrak{m}})_{t\in\mathbb{R}}$ as acting on the algebra $C(E_{\{0\}}^+;\mathbb{C})$. Similar to (74), the set of polynomials in this space of functions is denoted by

$$\mathbb{P}_{\{0\}} \doteq \mathbb{C}[\{\hat{A}|_{E_{\{0\}}^+} : A \in \mathcal{U}_{\{0\}}\}] \subseteq C(E_{\{0\}}^+;\mathbb{C}) \ .$$

Local classical energy functions [1, Definition 6.8] on $\mathcal{U}_{\{0\}}$ are defined by $\mathrm{h}_0^{\mathfrak{m}}|_{E_{\{0\}}^+}$, where, by (43) and (78),

$$\mathrm{h}_0^{\mathfrak{m}} = \widehat{\Phi_{\{0\}}} + \sum_{n\in\mathbb{N}} \int_{\mathbb{S}^n} \widehat{\Psi_{\{0\}}^{(1)}} \cdots \widehat{\Psi_{\{0\}}^{(n)}}\ \mathfrak{a}_n\left(\mathrm{d}\Psi^{(1)}, \ldots, \mathrm{d}\Psi^{(n)}\right) \ .$$

Then, for any time $t \in \mathbb{R}$ and polynomials $f \in \mathbb{P}_{\{0\}}$, Liouville's equation (79) restricted to the algebra $C(E_{\{0\}}^+;\mathbb{C})$ equals

$$\partial_t V_t^{\mathfrak{m}}(f) = V_t^{\mathfrak{m}}(\{\mathrm{h}_0^{\mathfrak{m}}, f\}) = \{\mathrm{h}_0^{\mathfrak{m}}, V_t^{\mathfrak{m}}(f)\} \ , \tag{97}$$

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[27] Note that μ_ρ in (72) is a probability measure on $E_1 \subseteq E^+$, but since the restriction mapping $\rho \mapsto \rho|_{\mathcal{U}_\Lambda}$ is continuous for any $\Lambda \in \mathcal{P}_f$, μ_ρ can be pushed forward to a probability measure on E_Λ^+.

[28] The restriction $\Lambda \supseteq \mathcal{Z}_{\vec{\ell}}$ can also be easily understood by seeing $\vec{\ell}$-periodic states as a translation-invariant state on the CAR C^*-algebra with new spin set $\mathcal{Z}_{\vec{\ell}} \times \mathrm{S}$.

where, for any $n, m \in \mathbb{N}$, $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathcal{U}$, $g \in C^1(\mathbb{R}^n, \mathbb{C})$ and $h \in C^1(\mathbb{R}^m, \mathbb{C})$,

$$\{\Gamma_h|_{\mathcal{U}_{\{0\}}}, \Gamma_g|_{\mathcal{U}_{\{0\}}}\} \doteq \{\Gamma_h, \Gamma_g\}|_{\mathcal{U}_{\{0\}}} \in C(E_{\{0\}}^+, \mathbb{C}) \tag{98}$$

defines again a Poisson bracket, which can be extended to the space $C^1(E_{\{0\}}^+; \mathbb{C})$ of continuously differentiable functions. Similar to (95), Liouville's equation (97) is now written on the finite-dimensional algebra $\mathcal{U}_{\{0\}}$ (dimension $2^{2|\mathrm{S}|}$) and completly determines a continuous flow on the compact space $E_\otimes$ of product states.

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