On the existence of stationary solutions for some systems of non-Fredholm integro-differential equations with the bi-Laplacian

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Abstract. We establish the existence of stationary solutions for some systems of reaction -diffusion type equations in the appropriate H^4 spaces via the fixed point technique when the system of elliptic equations contains fourth order differential operators with and without Fredholm property, generalizing the results of our preceding work [22].

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1 Introduction

Let us recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. As a consequence, the problem $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. These properties of Fredholm operators are extensively used in many methods of linear and nonlinear analysis.

Elliptic equations in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Lopatinskii conditions are fulfilled (see e.g. [1], [9], [12]). This is the main result of the theory of linear elliptic problems. In the case of unbounded domains, these conditions may not be sufficient and the

Fredholm property may not be satisfied. For example, Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ does not satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic problems in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions stated above, limiting operators are invertible (see [14]). In some trivial cases, limiting operators can be explicitly constructed. For example, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinity,

$$a_\pm = \lim_{x\to\pm\infty} a(x), \quad b_\pm = \lim_{x\to\pm\infty} b(x), \quad c_\pm = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are:

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u.$$

Since the coefficients are constant, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ does not satisfy the Fredholm property, can be explicitly found using the Fourier transform:

$$\lambda_\pm(\xi) = -a_\pm \xi^2 + b_\pm i\xi + c_\pm, \quad \xi \in \mathbb{R}.$$

Invertibility of limiting operators is equivalent to the condition that the essential spectrum does not contain the origin.

In the case of general elliptic problems, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, these conditions may not be explicitly written.

In the case of non-Fredholm operators the usual solvability conditions may not be applicable and solvability conditions are, in general, not known. There are some classes of operators for which solvability conditions are obtained. Let us illustrate them with the following example. Consider the equation

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where a is a positive constant. The operator L coincides with its limiting operators. The homogeneous problem has a nonzero bounded solution. Hence the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can apply the Fourier transform and find the solution explicitly. Solvability conditions can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exists a solution of this equation in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [21]). Here and further down S_r^d denotes the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Hence, though the operator does not satisfy the Fredholm property, solvability conditions are formulated similarly. However, this similarity is only formal because the range of the operator is not closed.

In the case of the operator with a potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

Fourier transform is not directly applicable. Nevertheless, solvability conditions in $\mathbb{R}^3$ can be obtained by a rather sophisticated application of the theory of self-adjoint operators (see [18]). As before, solvability conditions are formulated in terms of orthogonality to solutions of the homogeneous adjoint problem. There are several other examples of linear elliptic operators without Fredholm property for which solvability conditions can be obtained (see [14]-[21]). The bi-Laplacian is relevant to the studies of the solvability conditions for a linearized Cahn-Hilliard problem (see e.g. [15]). The boundedness of the gradient of a solution for the biharmonic equation was proved in [10]. The behavior near the boundary of solutions to the Dirichlet problem for the biharmonic operator was investigated in [11].

Solvability relations play a crucial role in the analysis of nonlinear elliptic problems. In the case of non Fredholm operators, in spite of some progress in understanding of linear equations, there exist only few examples where nonlinear non-Fredholm operators are analyzed (see [4]-[6]). In the present article we consider another class of systems of nonlinear equations, for which the Fredholm property may not be satisfied:

$$\frac{\partial u_k}{\partial t} = -\Delta^2 u_k + a_k^2 u_k + \int_\Omega G_k(x-y)F_k(u_1(y,t), u_2(y,t), ..., u_N(y,t), y)dy, \quad 1 \le k \le N, \tag{1.2}$$

generalizing the results obtained in [22] for the system of equations analogous to (1.2) but containing the standard Laplace operator. Here all $a_k > 0$ and Ω is a domain in $\mathbb{R}^d$, $d = 1, 2, 3$, the more physically interesting dimensions. In population dynamics the integro-differential equations describe models with intra-specific competition and nonlocal consumption of resources (see e.g. [2], [3], [7]). The linear parts of the corresponding operators here are similar to problem (1.1) above, they only contain the negative bi-Laplacian. The nonlinear functions here $F_k(u_1, u_2, ..., u_N, y)$ depend on the vector function

$$u := (u_1, u_2, ..., u_N) \in \mathbb{R}^N. \tag{1.3}$$

We will use the explicit form of solvability relations and will study the existence of stationary solutions of the nonlinear system.

2 Formulation of the results

The nonlinear part of system (1.2) will satisfy the following regularity conditions.

Assumption 1. *Functions $F_k(u,x) : \mathbb{R}^N \times \Omega \to \mathbb{R},\ 1 \le k \le N$ are such that*

$$\sqrt{\sum_{k=1}^{N} F_k^2(u,x)} \le K|u|_{\mathbb{R}^N} + h(x) \quad for \quad u \in \mathbb{R}^N,\ x \in \Omega \tag{2.1}$$

with a constant $K > 0$ and $h(x) : \Omega \to \mathbb{R}^+, \quad h(x) \in L^2(\Omega)$. Moreover, they are Lipschitz continuous functions, such that for any $u^{(1),(2)} \in \mathbb{R}^N,\ x \in \Omega$

$$\sqrt{\sum_{k=1}^{N} (F_k(u^{(1)},x) - F_k(u^{(2)},x))^2} \le L|u^{(1)} - u^{(2)}|_{\mathbb{R}^N}, \tag{2.2}$$

with a constant $L > 0$.

Here and below the norm of a vector function given by (1.3) is

$$|u|_{\mathbb{R}^N} := \sqrt{\sum_{k=1}^{N} u_k^2}.$$

Obviously, the stationary solutions of (1.2), if they exist, will satisfy the system of nonlocal elliptic equation

$$-\Delta^2 u_k + \int_\Omega G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy + a_k^2 u_k = 0,\ a_k > 0,\ 1 \le k \le N.$$

Let us introduce the auxiliary problem

$$\Delta^2 u_k - a_k^2 u_k = \int_\Omega G_k(x-y)F_k(v_1(y), v_2(y), ..., v_N(y), y)dy,\ a_k > 0,\ 1 \le k \le N. \tag{2.3}$$

We denote

$$(f_1(x), f_2(x))_{L^2(\Omega)} := \int_\Omega f_1(x)\bar{f}_2(x)dx, \tag{2.4}$$

with a slight abuse of notations when these functions are not square integrable, like for instance those involved in orthogonality relations (2.5) and (2.6) below. Indeed, if $f_1(x) \in L^1(\Omega)$ and $f_2(x) \in L^\infty(\Omega)$, the integral in the right side of (2.4) makes sense. In the first part of the article we consider the case of $\Omega = \mathbb{R}^d,\ 1 \le d \le 3$, such that the appropriate Sobolev space is equipped with the norm

$$\|u\|^2_{H^4(\mathbb{R}^d,\mathbb{R}^N)} := \sum_{k=1}^{N} \|u_k\|^2_{H^4(\mathbb{R}^d)} = \sum_{k=1}^{N} \{\|u_k\|^2_{L^2(\mathbb{R}^d)} + \|\Delta^2 u_k\|^2_{L^2(\mathbb{R}^d)}\},$$

with $u(x) : \mathbb{R}^d \to \mathbb{R}^N$. The main issue for the problem above is that the operators $\Delta^2 - a_k^2 : H^4(\mathbb{R}^d) \to L^2(\mathbb{R}^d),\ a_k > 0,\ 1 \le k \le N$ fail to satisfy the Fredholm property, which is the obstacle to solve problem (2.3). The similar situations but in linear problems, both self- adjoint and non self-adjoint involving non Fredholm second or fourth order differential operators or even systems of equations with non Fredholm operators have been studied extensively in recent years (see [13], [14], [15], [16], [17], [18], [19], [20], [21]). However, we manage to prove that system (2.3) in this case defines a map $T_a : H^4(\mathbb{R}^d, \mathbb{R}^N) \to H^4(\mathbb{R}^d, \mathbb{R}^N)$, $a_k > 0$, $1 \leq k \leq N$, which is a strict contraction under the given technical conditions. Let us make the following assumption on the integral kernels involved in the nonlocal parts of (2.3).

Assumption 2. *Let* $1 \leq k \leq N$, $N \geq 2$, $G_k(x) : \mathbb{R}^d \to \mathbb{R}$, $G_k(x) \in L^1(\mathbb{R}^d)$, $xG_k(x) \in L^1(\mathbb{R}^d)$, $1 \leq d \leq 3$ *and all* $a_k > 0$. *Let*

$$\left(G_k(x), \frac{e^{\pm i\sqrt{a_k}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0, \quad d = 1 \tag{2.5}$$

and

$$\left(G_k(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a_k}}, \quad d = 2, 3. \tag{2.6}$$

Let us use the hat symbol to designate the standard Fourier transform

$$\widehat{G_k}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} G_k(x) e^{-ipx} dx, \ p \in \mathbb{R}^d, \tag{2.7}$$

such that

$$\|\widehat{G_k}(p)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|G_k\|_{L^1(\mathbb{R}^d)} \tag{2.8}$$

We define the following auxiliary quantities

$$M_k := max\left\{\left\|\frac{\widehat{G_k}(p)}{p^4 - a_k^2}\right\|_{L^\infty(\mathbb{R}^d)}, \quad \left\|\frac{p^4\widehat{G_k}(p)}{p^4 - a_k^2}\right\|_{L^\infty(\mathbb{R}^d)}\right\}, \quad 1 \leq k \leq N, \tag{2.9}$$

where all $a_k > 0$. In our notations p^4 stands for $|p|^4$ with $p \in \mathbb{R}^d$, $d \leq 3$. Note that expressions (2.9) are finite by means of the first lemma of the Appendix of [23] under our Assumption 2. This lemma is a trivial generalization of Lemmas A1 and A2 of [19] when the standard Laplace operator in the integro-differential equation is being replaced by the bi-Laplacian, under the same orthogonality conditions (2.5) and (2.6). Thus, we define

$$M = \max M_k, \quad 1 \leq k \leq N. \tag{2.10}$$

We have the following proposition.

Theorem 3. *Let* $\Omega = \mathbb{R}^d$, $d \leq 3$, *Assumptions 1 and 2 hold and* $\sqrt{2}(2\pi)^{\frac{d}{2}} ML < 1$.

Then the map $T_a v = u$ on $H^4(\mathbb{R}^d, \mathbb{R}^N)$ defined by the system of equations (2.3) has a unique fixed point $v^{(a)} : \mathbb{R}^d \to \mathbb{R}^N$, which is the only stationary solution of problem (1.2) in $H^4(\mathbb{R}^d, \mathbb{R}^N)$.

This fixed point $v^{(a)}$ is nontrivial provided the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0,x)}(p) \cap supp\widehat{G_k}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$ for some $1 \le k \le N$.

In the second part of the work we consider the analogous system on the finite interval $\Omega = I := [0,\ 2\pi]$ with periodic boundary conditions for the solution vector function and its first three derivatives. We assume the following about the integral kernels involved in the nonlocal parts of system (2.3) in this case.

Assumption 4. *Let $\Omega = I,\ 1 \le k \le N,\ N \ge 2,\ G_k(x) : I \to \mathbb{R},\ G_k(x) \in L^1(I)$ with $G_k(0) = G_k(2\pi)$ and $1 \le m \le N-1,\ m \in \mathbb{N}$.*

I) Let $a_k > 0$ and $a_k \ne n^2,\ n \in \mathbb{Z}$ for $1 \le k \le m$.

II) Let $a_k = n_k^2,\ n_k \in \mathbb{N}$ and

$$\left(G_k(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}} \right)_{L^2(I)} = 0, \quad m+1 \le k \le N \tag{2.11}$$

Let $F_k(u,0) = F_k(u,2\pi)$ for $u \in \mathbb{R}^N$ and $k = 1, ..., N$.

We introduce the Fourier transform for the functions on the $[0, 2\pi]$ interval as

$$G_{k,n} := \int_0^{2\pi} G_k(x) \frac{e^{-inx}}{\sqrt{2\pi}} dx, \quad n \in \mathbb{Z}. \tag{2.12}$$

Similarly to the whole space case we define

$$\mathcal{N}_k := \max\left\{ \left\| \frac{G_{k,n}}{n^4 - a_k^2} \right\|_{l^\infty}, \quad \left\| \frac{n^4 G_{k,n}}{n^4 - a_k^2} \right\|_{l^\infty} \right\}, \quad 1 \le k \le m, \tag{2.13}$$

$$\mathcal{N}_k := max\left\{ \left\| \frac{G_{k,n}}{n^4 - n_k^4} \right\|_{l^\infty}, \quad \left\| \frac{n^4 G_{k,n}}{n^4 - n_k^4} \right\|_{l^\infty} \right\}, \quad m+1 \le k \le N. \tag{2.14}$$

By means of the second lemma of the Appendix of [23] under Assumption 4 above the quantities given by (2.13) and (2.14) are finite. This lemma is an easy generalization of Lemma A3 of [19] when the standard Laplace operator considered on the interval I with periodic boundary conditions is replaced by the bi-Laplacian, under the same orthogonality relations (2.11). This allows us to define

$$\mathcal{N} := \max \mathcal{N}_k, \quad 1 \le k \le N \tag{2.15}$$

with $\mathcal{N}_k$ defined in formulas (2.13) and (2.14). For the studies of the existence of solutions of our problem we use the corresponding function spaces

$$H^4(I) = \{u(x) : I \to \mathbb{R} \mid u(x), u''''(x) \in L^2(I),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),$$

$$u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi)\}.$$

Then we introduce the following auxiliary constrained subspaces

$$H_k^4(I) := \{u \in H^4(I) \mid \Big(u(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}}\Big)_{L^2(I)} = 0\},\ n_k \in \mathbb{N},\ m+1 \le k \le N, \tag{2.16}$$

with the goal of having $u_k(x) \in H_k^4(I),\ m+1 \le k \le N$. The constrained subspaces defined above are Hilbert spaces as well (see e.g. Chapter 2.1 of [8]). The resulting space used for establishing the existence of solutions $u(x) : I \to \mathbb{R}^N$ of system (2.3) will be the direct sum of the spaces mentioned above, namely

$$H_c^4(I, \mathbb{R}^N) := \oplus_{k=1}^m H^4(I) \oplus_{k=m+1}^N H_k^4(I),$$

such that the corresponding Sobolev norm is given by

$$\|u\|^2_{H_c^4(I,\mathbb{R}^N)} = \sum_{k=1}^N \{\|u_k\|^2_{L^2(I)} + \|u_k''''\|^2_{L^2(I)}\},$$

with $u(x) : I \to \mathbb{R}^N$. Let us prove that the system of equations (2.3) in this case defines a map τ_a on the above mentioned space which will be a strict contraction under our assumptions.

Theorem 5. *Let $\Omega = I$, Assumptions 1 and 4 hold and $2\sqrt{\pi}\mathcal{N}L < 1$.*

Then the map $\tau_a v = u$ on $H_c^4(I, \mathbb{R}^N)$ defined by the system of equations (2.3) has a unique fixed point $v^{(a)} : I \to \mathbb{R}^N$, the only stationary solution of problem (1.2) in $H_c^4(I, \mathbb{R}^N)$.

This fixed point $v^{(a)}$ is nontrivial provided the Fourier coefficients $G_{k,n}F_k(0,x)_n \ne 0$ for some $k = 1, ..., N$ and some $n \in \mathbb{Z}$.

Remark. *We use the constrained subspaces $H_k^4(I)$, such that the operators*

$$\frac{d^4}{dx^4} - n_k^4 : H_k^4(I) \to L^2(I)$$

which possesses the Fredholm property, have empty kernels.

We conclude the article with the studies of our system on the product of sets, where one is the finite interval with periodic boundary conditions as before and another is the whole space of dimension not exceeding two. Thus, in our notations $\Omega = I \times \mathbb{R}^d = [0, 2\pi] \times \mathbb{R}^d,\ d = 1, 2$ and $x = (x_1, x_\perp)$ with $x_1 \in I$ and $x_\perp \in \mathbb{R}^d$. We make the following assumption about the integral kernels involved in the nonlocal parts of system (2.3) in such case.

Assumption 6. *Let* $\Omega = I \times \mathbb{R}^d,\ d = 1, 2,\ 1 \le k \le N,\ N \ge 2,\ G_k(x) : \Omega \to \mathbb{R},\ G_k(x) \in L^1(\Omega),\ G_k(0, x_\perp) = G_k(2\pi, x_\perp)$ *for* $x_\perp \in \mathbb{R}^d$ *a.e. and* $1 \le m \le N-1,\ m \in \mathbb{N}$.

I) Let $n_k^2 < a_k < (n_k+1)^2,\ n_k \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\},\ x_\perp G_k(x) \in L^1(\Omega)$,

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{a_k - n^2} x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n| \le n_k, \quad d = 1, \tag{2.17}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p \in S^2_{\sqrt{a_k - n^2}} \text{ a.e.}, \quad |n| \le n_k, \quad d = 2. \tag{2.18}$$

for $1 \le k \le m$.

II) Let $a_k = n_k^2,\ n_k \in \mathbb{N},\ x_\perp^2 G_k(x) \in L^1(\Omega)$,

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{n_k^2 - n^2} x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad |n| \le n_k - 1, \quad d = 1, \tag{2.19}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \quad p \in S^2_{\sqrt{n_k^2 - n^2}} \quad \text{a.e.}, \quad |n| \le n_k - 1, \quad d = 2, \tag{2.20}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{\pm in_k x_1}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \quad \left(G_k(x_1, x_\perp), \frac{e^{\pm in_k x_1}}{\sqrt{2\pi}} x_{\perp,\ s}\right)_{L^2(\Omega)} = 0, \tag{2.21}$$

for $1 \le s \le d,\ m+1 \le k \le N$. *Let* $F_k(u, 0, x_\perp) = F_k(u, 2\pi, x_\perp)$ *for* $x_\perp \in \mathbb{R}^d$ *a.e.,* $u \in \mathbb{R}^N$ *and* $1 \le k \le N$.

Let $G_k(x)$ be a function on our product of sets, $G_k(x) : \Omega = I \times \mathbb{R}^d \to \mathbb{R},\ d = 1, 2,\ G_k(0, x_\perp) = G_k(2\pi, x_\perp)$ for $x_\perp \in \mathbb{R}^d$ a.e., such that its Fourier transform on the product of sets is given by

$$\widehat{G}_{k,n}(p) := \frac{1}{(2\pi)^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} dx_\perp e^{-ipx_\perp} \int_0^{2\pi} G_k(x_1, x_\perp) e^{-inx_1} dx_1, \quad p \in \mathbb{R}^d,\ n \in \mathbb{Z}. \tag{2.22}$$

The norm

$$\|\widehat{G}_{k,n}(p)\|_{L^\infty_{n,p}} := \sup_{\{p \in \mathbb{R}^d,\ n \in \mathbb{Z}\}} |\widehat{G}_{k,n}(p)| \le \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G_k\|_{L^1(\Omega)} \tag{2.23}$$

and $G_k(x) = \dfrac{1}{(2\pi)^{\frac{d+1}{2}}} \displaystyle\sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^d} \widehat{G}_{k,n}(p) e^{ipx_\perp} e^{inx_1} dp$. It is also helpful to introduce the Fourier transform only in the first variable, namely

$$G_{k,n}(x_\perp) := \int_0^{2\pi} G_k(x_1, x_\perp) \frac{e^{-inx_1}}{\sqrt{2\pi}} dx_1, \quad n \in \mathbb{Z}. \tag{2.24}$$

Let us define $\xi_{k,n}^{(a)}(p) := \dfrac{\widehat{G}_{k,n}(p)}{(p^2+n^2)^2 - a_k^2}$ with $1 \le k \le N,\ n \in \mathbb{Z},\ p \in \mathbb{R}^d,\ d = 1, 2,\ a_k > 0$ and introduce

$$\mathcal{M}_k := max\{\|\xi_{k,n}^{(a)}(p)\|_{L^\infty_{n,p}}, \quad \|(p^2+n^2)^2 \xi_{k,n}^{(a)}(p)\|_{L^\infty_{n,p}}\}, \quad 1 \le k \le N. \tag{2.25}$$

Expressions (2.25) are finite by means of the third and the last lemmas of the Appendix of [23] under our Assumption 6. These lemmas are the trivial generalizations of Lemmas A5 and A6 of [19] when the Laplace operator in our domain Ω is replaced by the bi-Laplacian, under the same orthogonality conditions (2.17), (2.18), (2.19), (2.20), (2.21). This enables us to define

$$\mathcal{M} = max_{1 \le k \le N} \mathcal{M}_k. \tag{2.26}$$

The total Laplace operator in this context will be given by $\Delta := \dfrac{\partial^2}{\partial x_1^2} + \displaystyle\sum_{s=1}^{d} \dfrac{\partial^2}{\partial x_{\perp,s}^2}$. The corresponding Sobolev space for our problem is $H^4(\Omega, \mathbb{R}^N)$ of vector functions $u(x) : \Omega \to \mathbb{R}^N$, such that for $k = 1, ..., N$

$$u_k(x),\ \Delta^2 u_k(x) \in L^2(\Omega),\ u_k(0, x_\perp) = u_k(2\pi, x_\perp),$$

$$\frac{\partial u_k}{\partial x_1}(0, x_\perp) = \frac{\partial u_k}{\partial x_1}(2\pi, x_\perp),\ \frac{\partial^2 u_k}{\partial x_1^2}(0, x_\perp) = \frac{\partial^2 u_k}{\partial x_1^2}(2\pi, x_\perp),\ \frac{\partial^3 u_k}{\partial x_1^3}(0, x_\perp) = \frac{\partial^3 u_k}{\partial x_1^3}(2\pi, x_\perp),$$

with $x_\perp \in \mathbb{R}^d$ *a.e.* It is equipped with the norm

$$\|u\|^2_{H^4(\Omega, \mathbb{R}^N)} = \sum_{k=1}^{N} \{\|u_k\|^2_{L^2(\Omega)} + \|\Delta^2 u_k\|^2_{L^2(\Omega)}\}.$$

Analogously to the whole space case treated in Theorem 3 above, the operators $\Delta^2 - a_k^2 : H^4(\Omega) \to L^2(\Omega), \quad a_k > 0$ are non Fredholm. Let us establish that system (2.3) in this context defines a map $t_a : H^4(\Omega, \mathbb{R}^N) \to H^4(\Omega, \mathbb{R}^N)$, a strict contraction under the given technical conditions.

Theorem 7. *Let $\Omega = I \times \mathbb{R}^d,\ d = 1, 2$, Assumptions 1 and 6 hold and $\sqrt{2}(2\pi)^{\frac{d+1}{2}} \mathcal{M} L < 1$. Then the map $t_a v = u$ on $H^4(\Omega, \mathbb{R}^N)$ defined by system (2.3) has a unique fixed point $v^{(a)} : \Omega \to \mathbb{R}^N$, which the only stationary solution of the system of equations (1.2) in $H^4(\Omega, \mathbb{R}^N)$. This fixed point $v^{(a)}$ is nontrivial provided that for some $1 \le k \le N$ and a certain $n \in \mathbb{Z}$ the intersection of supports of the Fourier transforms of functions $supp \widehat{F_k(0,x)}_n(p) \cap supp \widehat{G}_{k,n}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$.*

Remark. *Note that the maps discussed above act on real valued vector functions by virtue of the assumptions on $F_k(u, x)$ and $G_k(x)$, $1 \le k \le N$ involved in the nonlocal terms of the system of equations (2.3).*

3 The Problem in the Whole Space

Proof of Theorem 3. Let us first suppose that in the case of $\Omega = \mathbb{R}^d$ for some $v \in H^4(\mathbb{R}^d, \mathbb{R}^N)$ there exist two solutions $u^{(1),(2)} \in H^4(\mathbb{R}^d, \mathbb{R}^N)$ of system (2.3). Then their difference $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H^4(\mathbb{R}^d, \mathbb{R}^N)$ will satisfy the homogeneous system of equations

$$\Delta^2 w_k = a_k^2 w_k, \quad 1 \le k \le N.$$

Since the bi-Laplacian acting in the whole space does not have any nontrivial square integrable eigenfunctions, $w(x)$ vanishes a.e. in $\mathbb{R}^d$.

Let us choose arbitrarily $v(x) \in H^4(\mathbb{R}^d, \mathbb{R}^N)$. We apply the standard Fourier transform (2.7) to both sides of (2.3). This gives us

$$\widehat{u_k}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G_k}(p)\widehat{f_k}(p)}{p^4 - a_k^2}, \quad k = 1, ..., N. \tag{3.1}$$

Here $\widehat{f_k}(p)$ denotes the Fourier image of $F_k(v(x), x)$. Clearly, we have the upper bounds

$$|\widehat{u_k}(p)| \le (2\pi)^{\frac{d}{2}} M_k |\widehat{f_k}(p)| \quad \textit{and} \quad |p^4 \widehat{u_k}(p)| \le (2\pi)^{\frac{d}{2}} M_k |\widehat{f_k}(p)|, \quad k = 1, ..., N,$$

where $M_k,\ k = 1, ..., N$ are finite by virtue of the first lemma of the Appendix of [23] under Assumption 2 above. This enables us to derive the estimate from above for the norm

$$\|u\|^2_{H^4(\mathbb{R}^d, \mathbb{R}^N)} = \sum_{k=1}^{N} \{\|\widehat{u_k}(p)\|^2_{L^2(\mathbb{R}^d)} + \|p^4 \widehat{u_k}(p)\|^2_{L^2(\mathbb{R}^d)}\} \le 2(2\pi)^d M^2 \sum_{k=1}^{N} \|F_k(v(x), x)\|^2_{L^2(\mathbb{R}^d)},$$

which is finite by means of (2.1) of Assumption 1 for $|v(x)|_{\mathbb{R}^N} \in L^2(\mathbb{R}^d)$. Hence, for any $v(x) \in H^4(\mathbb{R}^d, \mathbb{R}^N)$ there is a unique solution $u(x) \in H^4(\mathbb{R}^d, \mathbb{R}^N)$ of system (2.3) with its Fourier transform given by (3.1), such that the map $T_a : H^4(\mathbb{R}^d, \mathbb{R}^N) \to H^4(\mathbb{R}^d, \mathbb{R}^N)$ is well defined. This allows us to choose arbitrarily $v^{(1),(2)}(x) \in H^4(\mathbb{R}^d, \mathbb{R}^N)$ such that their images $u^{(1),(2)} = T_a v^{(1),(2)} \in H^4(\mathbb{R}^d, \mathbb{R}^N)$. Thus,

$$\Delta^2 u_k^{(1)} - a_k^2 u_k^{(1)} = \int_{\mathbb{R}^d} G_k(x-y) F_k(v_1^{(1)}(y), v_2^{(1)}(y), ..., v_N^{(1)}(y), y) dy, \quad 1 \le k \le N,$$

$$\Delta^2 u_k^{(2)} - a_k^2 u_k^{(2)} = \int_{\mathbb{R}^d} G_k(x-y) F_k(v_1^{(2)}(y), v_2^{(2)}(y), ..., v_N^{(2)}(y), y) dy, \quad 1 \le k \le N.$$

We apply standard Fourier transform (2.7) to both sides of these systems of equations. This gives us

$$\widehat{u_k^{(1)}}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G}_k(p)\widehat{f_k^{(1)}}(p)}{p^4 - a_k^2}, \quad \widehat{u_k^{(2)}}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G}_k(p)\widehat{f_k^{(2)}}(p)}{p^4 - a_k^2}, \quad 1 \le k \le N.$$

Here $\widehat{f_k^{(j)}}(p)$ stands for the Fourier transform of $F_k(v^{(j}(x), x),\ j = 1, 2$. Evidently, for $k = 1, ..., N$, we have

$$|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)| \leq (2\pi)^{\frac{d}{2}} M |\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}|, \quad |p^4\widehat{u_k^{(1)}}(p) - p^4\widehat{u_k^{(2)}}(p)| \leq (2\pi)^{\frac{d}{2}} M |\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}|.$$

Hence, for the appropriate norms of our vector functions we derive

$$\|u^{(1)} - u^{(2)}\|^2_{H^4(\mathbb{R}^d,\mathbb{R}^N)} = \sum_{k=1}^{N} \{\|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\|^2_{L^2(\mathbb{R}^d)} + \|p^4\widehat{u_k^{(1)}}(p) - p^4\widehat{u_k^{(2)}}(p)\|^2_{L^2(\mathbb{R}^d)}\} \leq$$

$$\leq 2(2\pi)^d M^2 \sum_{k=1}^{N} \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(\mathbb{R}^d)}.$$

Apparently, $v_k^{(1),(2)}(x) \in H^4(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d),\ d \leq 3$ by means of the Sobolev embedding. By virtue of condition (2.2), we easily arrive at

$$\|T_a v^{(1)} - T_a v^{(2)}\|_{H^4(\mathbb{R}^d,\mathbb{R}^N)} \leq \sqrt{2}(2\pi)^{\frac{d}{2}} ML \|v^{(1)} - v^{(2)}\|_{H^4(\mathbb{R}^d,\mathbb{R}^N)},$$

with the constant in the right side of this estimate less than one via the assumption of the theorem. Therefore, by means of the Fixed Point Theorem, there exists a unique function $v^{(a)} \in H^4(\mathbb{R}^d, \mathbb{R}^N)$ with the property $T_a v^{(a)} = v^{(a)}$, which is the only stationary solution of system (1.2) in $H^4(\mathbb{R}^d, \mathbb{R}^N)$. Suppose $v^{(a)}(x)$ vanishes a.e. in $\mathbb{R}^d$. This will imply the contradiction to our assumption that for some $k = 1, ..., N$ the Fourier images of $G_k(x)$ and $F_k(0, x)$ do not vanish on some set of nonzero Lebesgue measure in $\mathbb{R}^d$. ∎

4 The Problem on the Finite Interval

Proof of Theorem 5. First we suppose that for some certain $v \in H_c^4(I, \mathbb{R}^N)$ there exist two solutions $u^{(1),(2)} \in H_c^4(I, \mathbb{R}^N)$ of system (2.3) with $\Omega = I$. Then the vector function $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H_c^4(I, \mathbb{R}^N)$ will be a solution of the homogeneous system of equations

$$w_k'''' = a_k^2 w_k, \quad 1 \leq k \leq N.$$

In case I) of Assumption 4 above we have $a_k > 0,\ a_k \neq n^2,\ n \in \mathbb{Z}$ for $1 \leq k \leq m$. Therefore, a_k^2 is not an eigenvalue of the operator $\frac{d^4}{dx^4} : H^4(I) \to L^2(I)$, such that $w_k(x) = 0$ in I for $1 \leq k \leq m$. By the similar reasoning, using the constrained subspace (2.16), we show that $w_k(x)$ vanishes identically in I when $m + 1 \leq k \leq N$.

We choose arbitarily $v(x) \in H_c^4(I, \mathbb{R}^N)$ and apply Fourier transform (2.12) to system (2.3) considered on the interval I. This yields

$$u_{k,n} = \sqrt{2\pi}\frac{G_{k,n} f_{k,n}}{n^4 - a_k^2}, \quad 1 \leq k \leq N, \quad n \in \mathbb{Z}, \tag{4.1}$$

where $f_{k,n} := F_k(v(x),x)_n$. Clearly, for the transforms of the fourth derivatives we easily derive

$$(u_k'''')_n = \sqrt{2\pi}\frac{n^4 G_{k,n} f_{k,n}}{n^4 - a_k^2}, \quad 1 \le k \le N, \quad n \in \mathbb{Z}.$$

Hence,

$$|u_{k,n}| \le \sqrt{2\pi}\mathcal{N}_k |f_{k,n}|, \quad |n^4 u_{k,n}| \le \sqrt{2\pi}\mathcal{N}_k |f_{k,n}|, \quad 1 \le k \le N, \quad n \in \mathbb{Z},$$

with $\mathcal{N}_k,\ 1 \le k \le N$ finite by means of the second lemma of the Appendix of [23] under our Assumption 4 above. This enables us to estimate

$$\|u\|^2_{H_c^4(I,\mathbb{R}^N)} = \sum_{k=1}^{N}\left[\sum_{n=-\infty}^{\infty}|u_{k,n}|^2 + \sum_{n=-\infty}^{\infty}|n^4 u_{k,n}|^2\right] \le 4\pi\mathcal{N}^2 \sum_{k=1}^{N}\|F_k(v(x),x)\|^2_{L^2(I)} < \infty$$

due to (2.1) of Assumption 1 for $|v(x)|_{\mathbb{R}^N} \in L^2(I)$. Hence, for an arbitrary $v(x) \in H_c^4(I,\mathbb{R}^N)$ there is a unique $u(x) \in H_c^4(I,\mathbb{R}^N)$, which satisfies system (2.3) with its Fourier image given by (4.1), such that the map $\tau_a : H_c^4(I,\mathbb{R}^N) \to H_c^4(I,\mathbb{R}^N)$ is well defined.

Let us consider the arbitrary $v^{(1),(2)} \in H_c^4(I,\mathbb{R}^N)$. Their images under the map discussed above $u^{(1),(2)} = \tau_a v^{(1),(2)} \in H_c^4(I,\mathbb{R}^N)$, such that

$$u_k^{(1)''''} - a_k^2 u_k^{(1)} = \int_0^{2\pi} G_k(x-y) F_k(v_1^{(1)}(y), v_2^{(1)}(y), ..., v_N^{(1)}(y), y)dy, \quad 1 \le k \le N,$$

$$u_k^{(2)''''} - a_k^2 u_k^{(2)} = \int_0^{2\pi} G_k(x-y) F_k(v_1^{(2)}(y), v_2^{(2)}(y), ..., v_N^{(2)}(y), y)dy, \quad 1 \le k \le N.$$

Let us apply Fourier transform (2.12) to both sides of these systems to arrive at

$$u_{k,n}^{(1)} = \sqrt{2\pi}\frac{G_{k,n} f_{k,n}^{(1)}}{n^4 - a_k^2}, \quad u_{k,n}^{(2)} = \sqrt{2\pi}\frac{G_{k,n} f_{k,n}^{(2)}}{n^4 - a_k^2}, \quad 1 \le k \le N, \quad n \in \mathbb{Z},$$

where $f_{k,n}^{(j)} := F_k(v^{(j)}(x),x)_n,\ j = 1,2$. Obviously, for $1 \le k \le N,\ n \in \mathbb{Z}$ we have

$$|u_{k,n}^{(1)} - u_{k,n}^{(2)}| \le \sqrt{2\pi}\mathcal{N}|f_{k,n}^{(1)} - f_{k,n}^{(2)}|, \quad |n^4 u_{k,n}^{(1)} - n^4 u_{k,n}^{(2)}| \le \sqrt{2\pi}\mathcal{N}|f_{k,n}^{(1)} - f_{k,n}^{(2)}|.$$

We derive easily the estimates from above

$$\|u^{(1)} - u^{(2)}\|^2_{H_c^4(I,\mathbb{R}^N)} = \sum_{k=1}^{N}\left[\sum_{n=-\infty}^{\infty}|u_{k,n}^{(1)} - u_{k,n}^{(2)}|^2 + \sum_{n=-\infty}^{\infty}|n^4(u_{k,n}^{(1)} - u_{k,n}^{(2)})|^2\right] \le$$

$$\le 4\pi\mathcal{N}^2 \sum_{k=1}^{N}\|F_k(v^{(1)}(x),x) - F_k(v^{(2)}(x),x)\|^2_{L^2(I)}.$$

Clearly, $v_k^{(1),(2)}(x) \in H^4(I) \subset L^\infty(I),\ 1 \le k \le N$ do to the Sobolev embedding. By means of (2.2), we easily obtain

$$\|\tau_a v^{(1)} - \tau_a v^{(2)}\|_{H_c^4(I,\mathbb{R}^N)} \le 2\sqrt{\pi}\mathcal{N}L\|v^{(1)} - v^{(2)}\|_{H_c^4(I,\mathbb{R}^N)}.$$

The constant in the right side of this estimate is less than one by virtue of the one of our assumptions. Therefore, the Fixed Point Theorem implies the existence and uniqueness of a vector function $v^{(a)} \in H_c^4(I,\mathbb{R}^N)$ satisfying $\tau_a v^{(a)} = v^{(a)}$, which is the only stationary solution of system (1.2) in $H_c^4(I,\mathbb{R}^N)$. Suppose $v^{(a)}(x)$ vanishes in I. Then we arrive at the contradiction to our assumption that $G_{k,n}F_k(0,x)_n \neq 0$ for some $1 \le k \le N$ and a certain $n \in \mathbb{Z}$. ■

5 The Problem on the Product of Sets

Proof of Theorem 7. First we suppose that there exists $v(x) \in H^4(\Omega,\mathbb{R}^N)$ which generates $u^{(1),(2)}(x) \in H^4(\Omega,\mathbb{R}^N)$ which satisfies system (2.3). Then their difference $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H^4(\Omega,\mathbb{R}^N)$ will be a solution of the homogeneous system of equations

$$\Delta^2 w_k = a_k^2 w_k, \quad 1 \le k \le N$$

in the domain Ω. Let us apply the partial Fourier transform (2.24) to both sides of this system of equations. This yields

$$(n^2 - \Delta_\perp)^2 w_{k,n}(x_\perp) = a_k^2 w_{k,n}(x_\perp), \quad 1 \le k \le N, \quad n \in \mathbb{Z},$$

where $\Delta_\perp$ denotes the transversal Laplacian acting on $x_\perp$. Apparently,

$$\|w_k\|^2_{L^2(\Omega)} = \sum_{n=-\infty}^{\infty} \|w_{k,n}(x_\perp)\|^2_{L^2(\mathbb{R}^d)}, \quad 1 \le k \le N.$$

Thus, $w_{k,n}(x_\perp) \in L^2(\mathbb{R}^d),\ 1 \le k \le N,\ n \in \mathbb{Z}$. But each operator $(n^2 - \Delta_\perp)^2,\ n \in \mathbb{Z}$ does not have any nontrivial square integrable eigenfunctions belonging to $L^2(\mathbb{R}^d)$. Therefore, $w(x)$ vanishes in Ω.

Let us choose an arbitrary $v(x) \in H^4(\Omega,\mathbb{R}^N)$ and apply Fourier transform (2.22) to both sides of system (2.3). This gives us

$$\widehat{u}_{k,n}(p) = (2\pi)^{\frac{d+1}{2}} \frac{\widehat{G}_{k,n}(p)\widehat{f}_{k,n}(p)}{(p^2+n^2)^2 - a_k^2}, \quad 1 \le k \le N, \quad n \in \mathbb{Z}, \quad p \in \mathbb{R}^d, \quad d = 1, 2, \tag{5.1}$$

with $\widehat{f}_{k,n}(p)$ standing for the Fourier image of $F_k(v(x),x)$ under transform (2.22) for $1 \le k \le N$. Obviously, for $1 \le k \le N,\ n \in \mathbb{Z},\ p \in \mathbb{R}^d$, we have

$$|\widehat{u}_{k,n}(p)| \le (2\pi)^{\frac{d+1}{2}} \mathcal{M}_k |\widehat{f}_{k,n}(p)|, \quad |(p^2+n^2)^2 \widehat{u}_{k,n}(p)| \le (2\pi)^{\frac{d+1}{2}} \mathcal{M}_k |\widehat{f}_{k,n}(p)|,$$

with all $\mathcal{M}_k < \infty$ by means of the third and the last lemmas of the Appendix of [23] under Assumption 6 above. Hence,

$$\|u\|^2_{H^4(\Omega,\mathbb{R}^N)} = \sum_{k=1}^{N}\left[\sum_{n=-\infty}^{\infty}\int_{\mathbb{R}^d}|\widehat{u}_{k,n}(p)|^2dp + \sum_{n=-\infty}^{\infty}\int_{\mathbb{R}^d}|(p^2+n^2)^2\widehat{u}_{k,n}(p)|^2dp\right] \leq$$

$$\leq 2(2\pi)^{d+1}\mathcal{M}^2\sum_{k=1}^{N}\|F_k(v(x),x)\|^2_{L^2(\Omega)} < \infty$$

due to (2.1) of Assumption 1 for $|v(x)|_{\mathbb{R}^N} \in L^2(\Omega)$. Therefore, for any $v(x) \in H^4(\Omega,\mathbb{R}^N)$ there exists a unique $u(x) \in H^4(\Omega,\mathbb{R}^N)$, which satisfies system (2.3) with its Fourier transform given by (5.1), such that the map $t_a : H^4(\Omega,\mathbb{R}^N) \to H^4(\Omega,\mathbb{R}^N)$ is well defined.

Let us choose arbitrarily $v^{(1),(2)}(x) \in H^4(\Omega,\mathbb{R}^N)$, such that their images under our map are $u^{(1),(2)} = t_a v^{(1),(2)} \in H^4(\Omega,\mathbb{R}^N)$. Hence,

$$\Delta^2 u_k^{(1)} - a_k^2 u_k^{(1)} = \int_{\Omega} G_k(x-y)F_k(v_1^{(1)}(y), v_2^{(1)}(y), ..., v_N^{(1)}(y), y)dy, \quad 1 \leq k \leq N,$$

$$\Delta^2 u_k^{(2)} - a_k^2 u_k^{(2)} = \int_{\Omega} G_k(x-y)F_k(v_1^{(2)}(y), v_2^{(2)}(y), ..., v_N^{(2)}(y), y)dy, \quad 1 \leq k \leq N.$$

We apply Fourier transform (2.22) to both sides of these systems. Hence,

$$\widehat{u}_{k,n}^{(1)}(p) = (2\pi)^{\frac{d+1}{2}}\frac{\widehat{G}_{k,n}(p)\widehat{f}_{k,n}^{(1)}(p)}{(p^2+n^2)^2-a_k^2}, \quad \widehat{u}_{k,n}^{(2)}(p) = (2\pi)^{\frac{d+1}{2}}\frac{\widehat{G}_{k,n}(p)\widehat{f}_{k,n}^{(2)}(p)}{(p^2+n^2)^2-a_k^2}, \tag{5.2}$$

where $1 \leq k \leq N$, $n \in \mathbb{Z}$, $p \in \mathbb{R}^d$, $d = 1, 2$ and $\widehat{f}_{k,n}^{(j)}(p)$ denotes the Fourier image of $F_k(v^{(j)}(x), x)$ under transform (2.22) for $j = 1, 2$. This enables us to derive for $1 \leq k \leq N$, $n \in \mathbb{Z}$, $p \in \mathbb{R}^d$

$$|\widehat{u}_{k,n}^{(1)}(p) - \widehat{u}_{k,n}^{(2)}(p)| \leq (2\pi)^{\frac{d+1}{2}}\mathcal{M}|\widehat{f}_{k,n}^{(1)}(p) - \widehat{f}_{k,n}^{(2)}(p)|,$$

$$|(p^2+n^2)^2\widehat{u}_{k,n}^{(1)}(p) - (p^2+n^2)^2\widehat{u}_{k,n}^{(2)}(p)| \leq (2\pi)^{\frac{d+1}{2}}\mathcal{M}|\widehat{f}_{k,n}^{(1)}(p) - \widehat{f}_{k,n}^{(2)}(p)|.$$

Therefore, $\|u^{(1)} - u^{(2)}\|^2_{H^4(\Omega,\mathbb{R}^N)} =$

$$= \sum_{k=1}^{N}\left[\sum_{n=-\infty}^{\infty}\int_{\mathbb{R}^d}|\widehat{u}_{k,n}^{(1)}(p) - \widehat{u}_{k,n}^{(2)}(p)|^2dp + \sum_{n=-\infty}^{\infty}\int_{\mathbb{R}^d}|(p^2+n^2)^2(\widehat{u}_{k,n}^{(1)}(p) - \widehat{u}_{k,n}^{(2)}(p))|^2dp\right] \leq$$

$$\leq 2(2\pi)^{d+1}\mathcal{M}^2\sum_{k=1}^{N}\|F_k(v^{(1)}(x),x) - F_k(v^{(2)}(x),x)\|^2_{L^2(\Omega)}.$$

Apparently, $v_k^{(1),(2)}(x) \in H^4(\Omega) \subset L^\infty(\Omega),\ 1 \le k \le N$ via the Sobolev embedding theorem. By virtue of (2.2) we easily arrive at the estimate from above

$$\|t_a v^{(1)} - t_a v^{(2)}\|_{H^4(\Omega,\mathbb{R}^N)} \le \sqrt{2}(2\pi)^{\frac{d+1}{2}}\mathcal{M}L\|v^{(1)} - v^{(2)}\|_{H^4(\Omega,\mathbb{R}^N)},$$

with the constant in the right side of it less than due to the one of our assumptions. Thus, the Fixed Point Theorem yields the existence and uniqueness of a vector function $v^{(a)} \in H^4(\Omega,\mathbb{R}^N)$ satisfying $t_a v^{(a)} = v^{(a)}$. This is the only stationary solution of system (1.2) in $H^4(\Omega,\mathbb{R}^N)$. Suppose $v^{(a)}(x)$ vanishes in Ω. This will imply the contradiction to our assumption that there exist $1 \le k \le N$ and $n \in \mathbb{Z}$ for which $supp\widehat{G}_{k,n}(p) \cap supp\widehat{F}_k(0,x)_n(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$. ■

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