Persistence Properties of Normally Hyperbolic Tori

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Abstract

Near-resonances between frequencies notoriously lead to small denominators when trying to prove persistence of invariant tori carrying quasi-periodic motion. In dissipative systems external parameters detuning the frequencies are needed so that Diophantine conditions can be formulated, which allow to solve the homological equation that yields a conjugacy between perturbed and unperturbed quasi-periodic tori. The parameter values for which the Diophantine conditions are not fulfilled form the so-called resonance gaps. Normal hyperbolicity can guarantee invariance of the perturbed tori, if not their quasi-periodicity, for larger parameter ranges. For a 1–dimensional parameter space this allows to close almost all resonance gaps.

1 Introduction

Unlike equilibria or periodic orbits, invariant tori of dynamical systems need at least one parameter to be dynamically persistent under small perturbations of the dynamics. Indeed, a dense set of resonances obstructs persistence: every resonance among its frequencies foliates a torus into lower dimensional tori and a small parameter variation is needed to re-obtain a dense orbit, which turns the torus into a dynamical object.

Imposing Diophantine conditions on the frequencies of the torus yields a strong form of non-resonance and for such Diophantine tori Kolmogorov–Arnol'd–Moser (KAM) theory allows to prove persistence under — very — small perturbations, providing a conjugacy with the unperturbed torus. For proving only invariance of the perturbed torus — with possibly a different dynamics *on* the torus — normal hyperbolicity supplies an alternative approach. It has the added benefit that persistence proofs using normal hyperbolicity do not lead to as pessimistic bounds as KAM-proofs do.

This article combines these two approaches, by 'fattening' the Cantor-like subset of parameters for which the torus as well as its quasi-periodic dynamics persist to a larger and open subset of parameters for which only the torus persists as an invariant manifold. In case of a single parameter in the Cantorised direction, this leads to the complement of the parameter set of persisting tori consisting of only finitely many 'low order resonance gaps'.

In the following, we first shortly review separately both KAM theory and the theory of normally hyperbolic invariant manifolds before using the latter to fill the gaps left open by the former. Then we apply this to three examples, among which the quasi-periodic Hopf bifurcation where we prove explicit estimates on the — finite — number of gaps still left open after fattening the Cantor set of Diophantine tori by hyperbolicity.

2 Results

Both KAM theory and normally hyperbolic invariant manifold theory are well-established approaches that yield persistence under sufficiently small perturbations. Before exploring how their combination allows to better understand the persistence properties of invariant tori, we present the two approaches in their own right, tailored to the situation at hand.

2.1 KAM Theory

Starting point is a vector field $X = X_\mu$ on $\mathbb{T}^n \times \mathbb{R}^m$, depending on a parameter $\mu \in \Gamma$ where $\Gamma \subseteq \mathbb{R}$ open, and a torus $\mathbb{T}^n \times \{0\}$ invariant under X_μ for all μ. We require X to be equivariant with respect to the $\mathbb{T}^n$–action

$$\begin{array}{rcl} \mathbb{T}^n \times (\mathbb{T}^n \times \mathbb{R}^m) & \longrightarrow & \mathbb{T}^n \times \mathbb{R}^m \\ (\xi, (x, z)) & \mapsto & (x + \xi, z) \end{array} \tag{1}$$

on the phase space, whence in co-ordinates we have that

$$X(x, z; \mu) \;=\; f(z; \mu)\,\partial_x \;+\; h(z; \mu)\,\partial_z \tag{2}$$

does not depend on the toral angles x. Expanding

$$f(z; \mu) \;=\; \omega(\mu) \;+\; \mathcal{O}(z) \tag{3a}$$

$$h(z; \mu) \;=\; \Omega(\mu) \cdot z \;+\; \mathcal{O}(z^2) \tag{3b}$$

we see that the invariant torus $\mathbb{T}^n \times \{0\}$ is automatically in Floquet form, with Ω independent of x. Consequently, the dynamics defined by the 'normally linear' part

$$NX_\mu(x, z) \;=\; \omega(\mu)\,\partial_x \;+\; \Omega(\mu) \cdot z\,\partial_z \tag{4}$$

is the superposition of the conditionally periodic motion

$$x(t) \;=\; x(0) \;+\; t\omega(\mu) \tag{5}$$

with frequency vector $\omega(\mu) \in \mathbb{R}^n$ and the linear behaviour

$$z(t) \;=\; \mathrm{e}^{t\Omega(\mu)} \cdot z(0)$$

which is governed by the eigenvalues of $\Omega(\mu) \in \mathfrak{gl}(m, \mathbb{R})$. In particular, the torus $\mathbb{T}^n \times \{0\}$ is normally hyperbolic if $\Omega(\mu)$ has no eigenvalue on the imaginary axis.

Our interest is in the dynamics defined by a small perturbation

$$\widetilde{X} \;=\; X \;+\; P$$

of the integrable dynamics, in particular whether there is an invariant torus of $\widetilde{X}$ close to the invariant torus $\mathbb{T}^n \times \{0\}$ of X. Normally hyperbolic invariant manifold theory asks for persistence of the invariant manifold; KAM theory asks additionally for persistence of the dynamics. For a dynamical persistence result to hold true the invariant torus of X had better be a dynamical object — with a dense orbit — rather than a union of other invariant sets, in this case of lower dimensional tori. Such a partition into lower dimensional tori occurs if there are resonances

$$\langle k \mid \omega \rangle \;:=\; k_1\omega_1 \;+\; \ldots \;+\; k_n\omega_n \;=\; 0$$

among the frequencies of the torus.

The KAM procedure consists of repeated steps of 'averaging out' of perturbations along the unperturbed flow. At resonances, the averaging is relative to the lower dimensional tori, resulting in obstructions for the persistence of the n–torus itself. To prevent this we impose Diophantine conditions

$$|\langle k \mid \omega \rangle| \;\geq\; \frac{\gamma}{|k|^\tau} \quad \text{for all } 0 \neq k \in \mathbb{Z}^n \tag{6}$$

where $|k| = |k_1| + \ldots + |k_n|$ while $\gamma > 0$ is the 'gap-parameter', and we choose $\tau > n - 1$ to ensure that the size of all gaps has small relative measure that moreover tends to 0 as $\gamma \to 0$. This strong non-resonance condition allows for polynomial estimates of the small denominators that arise if the frequencies are near the dense set of resonances, which are then outweighed by the exponential decay of Fourier coefficients (Lemma of Paley–Wiener, see e.g. [5]), a consequence of the regularity of the perturbation.

The occurrence of r normal frequencies α is an additional risk for persistence and leads to extended Diophantine conditions

$$|\langle k \mid \omega \rangle + \langle \ell \mid \alpha \rangle| \;\geq\; \frac{\gamma}{|k|^\tau} \quad \text{for all } 0 \neq k \in \mathbb{Z}^n,\, \ell \in \mathbb{Z}^r,\, |\ell| \leq 2 \tag{7}$$

to avoid normal-internal resonances. Note that already the imaginary parts α of hyperbolic eigenvalues $\beta \pm \mathrm{i}\alpha$ count as normal frequencies.

Locally in the frequency space $\mathbb{R}^{n+r}$, the set $\mathbb{R}^{n+r}_{\tau,\gamma}$ of all Diophantine frequency vectors has a product structure: half lines times a Cantor set. Indeed, when $(\omega, \alpha) \in \mathbb{R}^{n+r}_{\tau,\gamma}$ then also $(s\omega, s\alpha) \in \mathbb{R}^{n+r}_{\tau,\gamma}$ for all $s \geq 1$. The intersection $\mathbb{R}^{n+r}_{\tau,\gamma} \cap \mathbb{S}^{n+r-1}_R$ with a sphere of radius $R > 0$ is closed and totally disconnected; by the theorem of Cantor–Bendixson [15] it is the union of a countable and a perfect set, the latter necessarily being a Cantor set. For non-degenerate parameter dependence the 1–dimensional subset

$$\Gamma_{\tau,\gamma} \quad := \quad \omega^{-1}(\mathbb{R}^{n+r}_{\tau,\gamma}) \quad = \quad \{\, \mu \in \Gamma \mid \omega(\mu) \, \text{Diophantine} \,\}$$

of $\Gamma \subseteq \mathbb{R}$ then is a Cantor set as well. To allow for small shifts in the parameter we furthermore need the Cantor set

$$\Gamma^{\gamma}_{\tau,\gamma} \quad := \quad \{\, \mu \in \Gamma_{\tau,\gamma} \mid \operatorname{dist}(\mu, \partial\Gamma) \geq \gamma \,\} \quad .$$

We formulate the persistence results for analytic vector fields, highlighting where loss of regularity is inevitable. There are C^∞ — and even C^r — counterparts of these results, and sometimes the seeming strength of a theorem is obtained only by accepting the inevitable already among the assumptions. In the analytic context it is possible to weaken the Diophantine conditions (7) to so-called Bryuno conditions, replacing the polynomial bounds of small denominators by acceptable exponential bounds; see [21] and references therein. We refrain here from such improvements as they do not carry over to less regular situations.

Theorem 2.1 (Dissipative KAM theorem) *Let X_μ be a family of analytic vector fields as in* (2) *on $\mathbb{T}^n \times \mathbb{R}^m$, $n \geq 2$, with the eigenvalues of $\Omega(\mu)$ in* (3b) *bounded away from the imaginary axis. Assume that the frequency mapping $\sigma := (\omega, \alpha) : \Gamma \longrightarrow \mathbb{R}^{n+r}$ in* (3) *and its derivatives span the frequency space at every $\mu \in \Gamma \subseteq \mathbb{R}$, i.e.*

$$<\sigma(\mu), \sigma'(\mu), \sigma''(\mu), \ldots, \sigma^{(n+r-1)}(\mu)> \quad = \quad \mathbb{R}^{n+r} \quad . \tag{8}$$

Then for given $\gamma > 0$, $\tau > n^2 + nr - n - 1$ there exists a subset $\mathcal{G}_{\tau,\gamma} \subseteq \Gamma$, diffeomorphic to $\Gamma^{\gamma}_{\tau,\gamma}$, such that for a sufficiently close perturbation $\widetilde{X}_\mu = X_\mu + P_\mu$ of X_μ there exist normally affine diffeomorphisms

$$\begin{array}{rcccc} \Phi_\mu & : & \mathbb{T}^n \times \mathbb{R}^m & \longrightarrow & \mathbb{T}^n \times \mathbb{R}^m \\ & & (x, z) & \mapsto & (x + \xi(x;\mu), z + \eta(x;\mu) + \zeta(x;\mu) \cdot z) \end{array}$$

close to the identity and depending analytically on the angles x, but only Gevrey-smoothly (in the sense of Whitney) on the parameter μ, that for $\mu_0 \in \mathcal{G}_{\tau,\gamma}$ conjugate $\widetilde{X}_{\mu_0}$ to

$$(\Phi_{\mu_0})^{-1}_* \widetilde{X}_{\mu_0} \quad = \quad [\bar{\omega}(\mu_0) + \mathcal{O}(z)] \; \partial_x \; + \; \left[\bar{\Omega}(\mu_0) \cdot z + \mathcal{O}(z^2)\right] \; \partial_z$$

with x–dependent higher order terms $\mathcal{O}(z)$ and $\mathcal{O}(z^2)$, while $\bar{\omega}$ and $\bar{\Omega}$ are close to ω and Ω, respectively, and $\bar{\omega}(\mu_0)$ is Diophantine.

In particular $\bar{\Omega}(\mu_0)$ is again hyperbolic. For the proof of Theorem 2.1 see [5, 21, 24] or references therein. As discussed in [21] the non-degeneracy condition (8) can be weakened to only concern derivatives of the internal frequency vector ω; see also [5].

Moser [18] achieves the passage from ω and Ω to $\bar{\omega}$ and $\bar{\Omega}$ through addition of so-called modifying terms, see also [25, 4]. This can be avoided if there are more parameters, with $\mu \in \Gamma \subseteq \mathbb{R}^s$ and s large enough to accomodate for the finite-dimensional space of all possible modifications. For $s \geq n + r$ parameters the restriction of X to $\mathbb{T}^n \times \{0\} \times \Gamma^\gamma_{\tau,\gamma}$ is conjugate to a subsystem of $\widetilde{X}$, under preservation of the normal linear behaviour, and for $s = n + r - 1$ one can still relate the perturbed tori to the unperturbed ones as they have the same frequency ratio; this applies in particular to the case $n = 2$ of invariant 2–tori with no normal frequencies. However, in general a single parameter cannot yield persistence of individual tori, but only persistence of the 1–parameter family of tori as a Cantorised family.

For $s \geq n + r - 1$ parameters it suffices to take $\tau > n - 1$ in (7), as the image of a non-degenerate frequency mapping $\sigma = (\omega, \alpha) : \Gamma \subseteq \mathbb{R}^s \longrightarrow \mathbb{R}^{n+r}$ is transverse to the half lines of Diophantine frequencies. Conditions on higher order derivatives can compensate for a lack of parameters, by making the image of (ω, α) sufficiently curved to prevent it to remain in a — linearly defined — resonance gap. Taking $\tau > nL - 1$ ensures that the size of all gaps within $\Gamma^\gamma_{\tau,\gamma}$ has small relative measure that moreover tends to 0 as $\gamma \to 0$; here L is the highest derivative needed. In applications one often can do with $L = 2$, see e.g. [19] for examples. In the case $s = 1$ of a single parameter, all derivatives up to $L = n + r - 1$ are needed in (8), leading to the condition $\tau > nL - 1 = n^2 + nr - n - 1$ in Theorem 2.1.

There is a vast amount of results concerning dynamical persistence of invariant tori using KAM theory. In particular, Theorem 2.1 has counterparts for conservative dynamical systems and also for non-hyperbolic tori, see again [5] or references therein.

2.2 Normally hyperbolic invariant manifolds

The tori $\mathbb{T}^n \times \{0\}$ in Theorem 2.1 are normally hyperbolic invariant manifolds and as such they allow for an alternative approach to prove persistence, see [14, 9] and references therein. While Diophantine conditions are needed when proving dynamical persistence of tori via KAM theory, a persistence proof using normal hyperbolicity needs precise bounds on the ratio of normal attraction and expansion to that of internal attraction and expansion of the flow restricted to the invariant manifold. Such bounds also involve the regularity with which one wants the invariant manifold to persist.

Recall from [14, 11] that hyperbolicity of a given vector field X on a manifold M means that at each point $p \in M$ there exists an X–invariant direct sum splitting of the tangent space

$$T_pM \quad = \quad E^s(p) \;\oplus\; E^c(p) \;\oplus\; E^u(p) \tag{9}$$

in a stable (s), an unstable (u) and a central direction (c). At points $p \in M$ where $X(p) \neq 0$ the splitting depends continuously on p; the central direction $E^c(p)$ is the 1–dimensional vector space generated by $X(p)$. Let φ_t denote the flow of X. For hyperbolicity we require moreover that

- the splitting is X–invariant, amounting to invariance under the derivative $D\varphi_t$ in the sense that for each $p \in M$ and $t \in \mathbb{R}$ we have

$$D\varphi_t(E^u(p)) \;=\; E^u(\varphi_t(p)) \;, \quad D\varphi_t(E^s(p)) \;=\; E^s(\varphi_t(p)) \tag{10a}$$

$$\text{and} \quad D\varphi_t(E^c(p)) \;=\; E^c(\varphi_t(p)) \;; \tag{10b}$$

- the vectors $v \in E^u(p)$ and $w \in E^s(p)$ increase and decrease exponentially, respectively, under application of $D\varphi_t$ as a function of t, in the following sense: there are constants $C \geq 1$ and $\lambda > 0$ such that for all $0 < t \in \mathbb{R}$, $p \in M$, $v \in E^u(p)$ and $w \in E^s(p)$ we have

$$\|D\varphi_t(v)\| \;\geq\; C^{-1}\mathrm{e}^{\lambda t}\|v\| \quad \text{and} \quad \|D\varphi_t(w)\| \;\leq\; C\mathrm{e}^{-\lambda t}\|w\| \tag{11}$$

  where $\|..\|$ is the norm of tangent vectors with respect to a Riemannian metric.

For an X–invariant manifold $V \subset M$, we define *normal hyperbolicity* as a variation on the above notion of hyperbolicity. We restrict to giving a simplified description, referring to [14, 9] for details. Normal hyperbolicity means that a splitting (9) exists that is X–invariant in the sense of (10) and for which exponential estimates of the form (11) hold true. In this case the central subspace $E^c(p) = T_pV$ is more involved since it has to account for the dynamics inside V. In particular, we have a further invariant splitting

$$E^c(p) \;=\; E^{cs}(p) \,\oplus\, E^{cc}(p) \,\oplus\, E^{cu}(p)$$

into centre-stable, centre-centre and centre-unstable subspaces on which the invariance-property (10) and the hyperbolicity-property (11) are valid — there exists a constant $\lambda_c > 0$ that bounds from above the expansion and contraction rates within V. For any

$$r \;<\; \frac{\lambda}{\lambda_c} \tag{12}$$

the invariant manifold V is *r–normally hyperbolic*. This roughly means that the normal expansions and contractions are r times stronger than the internal expansions and contractions; the inequality (12) enforces a spectral gap for all $p \in V$ between the eigenvalues of eigenvectors in $E^{cs}(p) \oplus E^{cu}(p)$ and in $E^s(p) \oplus E^u(p)$, respectively. The Normally Hyperbolic Invariant Manifold Theorem, Theorem 2.2 below, then ensures that r–normal hyperbolicity leads to the persistence of V as a C^r–manifold under sufficiently small perturbations. The theorem provides a C^r–diffeomorphism between V and its perturbation, which preserves normal hyperbolicity.

Theorem 2.2 (Normally hyperbolic invariant manifolds) *Let X_μ be a family of analytic vector fields on M that admits a compact connected submanifold V_μ as an r–normally hyperbolic invariant manifold. Then a sufficiently small perturbation $\widetilde{X}_\mu = X_\mu + P_\mu$ has a unique C^r–manifold $\widetilde{V}_\mu$ close to V_μ that is C^r–diffeomorphic to V_μ and invariant under the flow of $\widetilde{X}_\mu$.*

The parameter μ is not needed here and can simply be fixed; on the other hand, for parameter-dependent $\widetilde{X}_\mu$ the resulting $\widetilde{V}_\mu$ is C^r–smooth in μ. For the proof of Theorem 2.2 see [14].

We want to apply Theorem 2.2 for the case when V is an invariant n–torus of an integrable dissipative system. Since the integrable dynamics is conditionally periodic, there are no internal contractions or expansions, and λ_c can be taken as close to 0 as desired. Therefore, if such a torus is r–normally hyperbolic for any $r > 0$ it is in fact r–normally hyperbolic for all $r > 0$, see e.g. [3]. This implies that for every value of r the invariant torus is persistent as a C^r–manifold under small perturbations, the maximal perturbation size depending on r. In particular, it follows that for a smooth family $X = X_\mu$, $\mu \in \mathbb{R}^s$ of vector fields the μ–regime $\mathcal{G}^r$ with r–normally hyperbolic tori is open in $\mathbb{R}^s$ and shrinks as the regularity r increases.

Theorem 2.2 not only applies to invariant tori, but also to other compact manifolds and even extends to non-compact normally hyperbolic invariant manifolds. The latter requires uniform estimates, which is automatic in the compact case; for details see [9] or references therein.

2.3 Fattening by hyperbolicity

Theorem 2.2 in particular applies to invariant tori $V = \mathbb{T}^n \times \{0\} \subseteq M = \mathbb{T}^n \times \mathbb{R}^m$ of a vector field as in (2). By unicity the n–tori carrying quasi-periodic dynamics with a Diophantine frequency vector must coincide with those obtained from Theorem 2.1. Hence these are analytic invariant tori of the perturbed vector field $\widetilde{X}$.

The tori $V = \mathbb{T}^n \times \{0\}$ carry conditionally periodic motion irrespective of the strength of the normal expansions or contractions. So even when the latter are very weak, that is, when the constant λ bounding the normal expansion and contraction rates from below is close to 0, the constant $\lambda_c > 0$ in the spectral gap condition (12) can always be chosen much smaller. Hence (12) effectively puts no bound on the regularity r that can be achieved by applying Theorem 2.2. However, the allowed size of the perturbation P decreases as $\lambda \to 0$ and $r \to \infty$, and in general the regularity of the perturbed tori is not C^∞; compare with [23].

Thus, by normal hyperbolicity any $\mu_0 \in \mathcal{G}_{\tau,\gamma}$, where $\mathcal{G}_{\tau,\gamma}$ is given by Theorem 2.1, is contained in a connected open neighbourhood $\mathcal{G}^r_{\tau,\gamma}(\mu_0) \subseteq \Gamma \subseteq \mathbb{R}^s$, such that $\widetilde{X}_\mu$ has an r–normally hyperbolic invariant n–torus for every $\mu \in \mathcal{G}^r_{\tau,\gamma}(\mu_0)$. The neighbourhoods $\mathcal{G}^r_{\tau,\gamma}(\mu_0)$ form an open cover of $\mathcal{G}_{\tau,\gamma}$, the union of which we denote by $\mathcal{G}^r = \mathcal{G}^r_{\tau,\gamma}$. This yields a normally hyperbolic *fattening* of the Diophantine invariant n–tori found by Theorem 2.1.

Assuming $\Gamma \subseteq \mathbb{R}^s$ to be bounded, the set $\Gamma^\gamma_{\tau,\gamma}$ and thus $\mathcal{G}_{\tau,\gamma}$ is compact and there exists a finite subcover, which consists of finitely many connected components. By making $\mathcal{G}^r$ a bit smaller, if necessary, we then choose $\mathcal{G}^r = \mathcal{G}^r_{\tau,\gamma}$ to be the finite union of this subcover. For the special case $s = 1$ of a single parameter $\mu \in \Gamma \subseteq \mathbb{R}$ this yields the following consequence of Theorems 2.1 and 2.2.

Corollary 2.3 (Gap-closing by hyperbolicity) *Let X_μ be a family of analytic vector fields as in* (2) *on $M = \mathbb{T}^n \times \mathbb{R}^m$, $n \geq 2$, satisfying the conditions of Theorem 2.1 with relative compact parameter space $\Gamma \subseteq \mathbb{R}$. Then the normally hyperbolic fattening procedure as described above leads for given regularity $r \in \mathbb{N}$ to only finitely many intervals in the complement $\Gamma \setminus \mathcal{G}^r$.*

For higher regularity r the number of remaining gaps may increase, and for $r \to \infty$ their number may even increase to ∞.

Proof. From Theorem 2.1 we obtain a compact set $\mathcal{G}_{\tau,\gamma}$ of parameters μ for which the vector field $\widetilde{X}_\mu$ perturbed from X_μ has a normally hyperbolic invariant n–torus with quasi-periodic flow (5) with Diophantine frequency vector $\bar{\omega}(\mu)$. For given $r \in \mathbb{N}$, Theorem 2.2 yields an open neighbourhood $\mathcal{G}^r$ of $\mathcal{G}_{\tau,\gamma}$ that has finitely many connected components. In the present case $s = 1$ these components divide the parameter space Γ and the complement $\Gamma \setminus \mathcal{G}^r$ has finitely many connected components as well. $\square$

Only comparatively large resonance gaps are not completely filled up by normally hyperbolic tori and these are the gaps opened by the resonances with the smallest denominators. One might therefore speak of 'low order' resonances when not all gaps are filled up by normal hyperbolicity.

Remarks.

- Note that for non-compact $\mathcal{G}_{\tau,\gamma}$ it is possible that infinitely many low order resonance gaps remain, but these are discrete and no longer dense within $\Gamma \subseteq \mathbb{R}$.

- In higher dimension $s \geq 2$ of multiple parameters the fattening of a compact set $\Gamma^\gamma_{\tau,\gamma}$ of Diophantine frequencies still has only finitely many connected components, but the complement — if not empty — is expected to have a single connected component, the web of low order resonances. The finitely many connected components of $\mathcal{G}^r$ is what one actually 'sees' when trying to simulate aspects of the dynamics on a computer.

- Finitely many connected components parametrising non-resonant tori is also what one 'sees' in the resonance web of a Hamiltonian system [16], but here this is due to 'finite pixel size' and a corresponding 'coarse-grained' treatment of tori as being 'non-resonant'.

3 Examples

We illustrate the effect of fattening by hyperbolicity with applications to a normally attracting torus, a torus losing normal attraction as a pair of eigenvalues passes through the imaginary axis, and a hyperbolic torus of a conservative dynamical system acquiring a normal frequency as its eigenvalues pass through 0.

3.1 Coupled van der Pol oscillators

We consider two autonomous van der Pol type oscillators

$$\begin{aligned} \ddot{z}_1 + c_1\dot{z}_1 + a_1 z_1 \; + \; f_1(z_1, \dot{z}_1) &= \varepsilon g_1(z_1, z_2, \dot{z}_1, \dot{z}_2) \\ \ddot{z}_2 + c_2\dot{z}_2 + a_2 z_2 \; + \; f_2(z_2, \dot{z}_2) &= \varepsilon g_2(z_1, z_2, \dot{z}_1, \dot{z}_2) \ , \end{aligned}$$

$z_1, z_2 \in \mathbb{R}$, with a weak coupling, that is, with ε small. We assume the 'damping' c_1 and c_2 to be negative, as happens for the values studied by van der Pol [11]. This yields a vector field $\widetilde{X}_\mu$ on the 4–dimensional state space $\mathbb{R}^2 \times \mathbb{R}^2$, which reads in co-ordinates $z = (z_1, z_3, z_2, z_4)$ as

$$\begin{aligned} \dot{z}_1 &= z_3 \\ \dot{z}_2 &= z_4 \\ \dot{z}_3 &= -a_1 z_1 - c_1 z_3 - f_1(z_1, z_3) + \varepsilon g_1(z_1, z_2, z_3, z_4) \\ \dot{z}_4 &= -a_2 z_2 - c_2 z_4 - f_2(z_2, z_4) + \varepsilon g_2(z_1, z_2, z_3, z_4) \end{aligned}$$

and which constitutes an ε–small perturbation of the vector field X_μ obtained by putting $\varepsilon = 0$. Here $\mu \in \mathbb{R}^s$ stands for a multi-parameter that includes coefficients like a_1 and a_2 such that the periods of the periodic orbits in the free oscillators vary with μ. Note that for $\varepsilon = 0$ the system decouples to 2 independent oscillators and has an attractor in the form of a 2–dimensional torus T_μ; here we use that the f_i consist of higher order terms, having no constant or linear part in z, and the assumption that the coefficients c_1 and c_2 are negative. This torus arises as the product of two circles, along each of which each of the free oscillators has its periodic motion. The circles lie in the 2–dimensional planes given by $z_2 = z_4 = 0$ and $z_1 = z_3 = 0$, respectively. The vector field X_μ has a restriction $X_\mu|_{T_\mu}$ of the format

$$\begin{aligned} \dot{x}_1 &= \omega_1(\mu) \\ \dot{x}_2 &= \omega_2(\mu) \ . \end{aligned}$$

The non-degeneracy assumption on the μ–dependence means that the frequency mapping $\omega : \mu \mapsto (\omega_1(\mu), \omega_2(\mu))$ is a submersion. This allows us to apply Theorem 2.1 and to conclude that for $|\varepsilon| \ll 1$ there is quasi-periodicity (with Diophantine frequencies) on a set $\mathcal{G}_{\tau,\gamma} \subseteq \Gamma$ of positive measure in parameter space.

Restricting parameters to $c_1, c_2 \leq -2\lambda$, for a positive constant λ, we may invoke Theorem 2.2. As a consequence it follows that if ε is sufficiently small, depending on λ, then for the corresponding parameter values $\mu = (a_1, a_2, c_1, c_2)$ a normally hyperbolic 2–torus attractor persists. If $\omega(\mu_0)$ is a non-resonant frequency vector also the quasi-periodic flow generated by $\dot{x} = \omega(\mu_0)$ persists, without the need for $\omega(\mu_0)$ to be Diophantine. The reason is that on a 2–dimensional torus $\widetilde{T}_\mu$ persistent by normal hyperbolicity, the perturbation of the dynamics on T_μ to the dynamics on $\widetilde{T}_\mu$ is subject to the Theorem of Denjoy [11].

In order to apply Corollary 2.3 we have to reduce $\mu \in \mathbb{R}^4$ to a single parameter ν. This prevents $\nu \mapsto (\omega_1(\nu), \omega_2(\nu))$ from being a submersion, but by allowing for a rescaling of time we may use $\nu \in \Gamma \subseteq \mathbb{R}$ to control the ratio $[\omega_1(\nu) : \omega_2(\nu)]$ of the internal frequencies. Then non-degeneracy amounts to requiring the resulting mapping $[\omega] : \Gamma \longrightarrow \mathbb{RP}^1$ to be a submersion. For the decoupled van der Pol oscillators the eigenvalues of $\Omega(\nu)$ in (3b) are all real whence no normal frequencies have to be excluded by the extended Diophantine conditions (7).

Hence, taking ν as a linear combination of the components a_1, a_2, c_1, c_2 of μ, the gaps resulting from the resonances are open neighbourhoods of hyperplanes defined by

$$[\omega_1(\nu) : \omega_2(\nu)] \quad = \quad \text{const.} \ ,$$

slightly distorted by the diffeomorphism of Theorem 2.1 that relates $\mathcal{G}_{\tau,\gamma}$ to $\Gamma^{\gamma}_{\tau,\gamma}$. Applying Corollary 2.3 now closes all but the low order resonance gaps.

Note that a large part of the above discussion can be generalised to n coupled oscillators, yielding a persistent n–torus attractor with Diophantine quasi-periodic dynamics corresponding to a set of positive measure in parameter space and a fattening to an open set of parameters with normally hyperbolic invariant tori.

3.2 The quasi-periodic Hopf bifurcation

The $\mathbb{T}^n$–symmetry (1) on $\mathbb{T}^n \times \mathbb{R}^m$ of the unperturbed integrable system, with a conditionally periodic invariant torus $\mathbb{T}^n \times \{0\}$, allows us to reduce to $\mathbb{R}^m$ and consider bifurcations of relative equilibria. The present interest is with small non-integrable perturbations of such integrable models; we focus on a Hopf bifurcation of the relative equilibria, whence we may choose $m = 2$. Paraphrasing the discussion in [8], the unperturbed, integrable family $X = X_\mu(x, z)$ on $\mathbb{T}^n \times \mathbb{R}^2$ has the form

$$X_\mu(x,z) \quad = \quad [\omega(\mu) + f(z;\mu)]\ \partial_x \ + \ [\Omega(\mu)\cdot z + h(z;\mu)]\ \partial_z \ , \tag{13}$$

where $f = \mathcal{O}(z)$ and $h = \mathcal{O}(z^2)$, while $\omega : \Gamma \longrightarrow \mathbb{R}^n$ and $\Omega : \Gamma \longrightarrow \mathfrak{gl}(2,\mathbb{R})$ are smooth mappings. Moreover, we first take $\mu \in \Gamma \subseteq \mathbb{R}^s$ as a multi-parameter. A full description of the normal linear behaviour

$$\Omega(\mu) \quad = \quad \begin{pmatrix} \beta & -\alpha \\ \alpha & \beta \end{pmatrix}$$

of the planar Hopf bifurcation requires two parameters, the normal frequency α and the bifurcation parameter β, whose passage through 0 is what triggers the Hopf bifurcation in the first place. Non-degeneracy now requires that the 'amended' frequency mapping

$$\begin{array}{rccc} (\omega,\alpha,\beta) \ : & \Gamma & \longrightarrow & \mathbb{R}^{n+2} \\ & \mu & \mapsto & (\omega(\mu),\alpha(\mu),\beta(\mu)) \end{array}$$

is a submersion, compare with [4]. We may even assume that μ is replaced by

$$\mu \ = \ (\omega,\alpha,\beta) \ \in \ \mathbb{R}^{n+2} \ , \tag{14}$$

thereby suppressing additional mute parameters. If the nonlinearity h satisfies the Hopf non-degeneracy conditions, as stated in e.g. [11, 4], then the — relative — equilibrium $z = 0$ undergoes a Hopf bifurcation. Here β plays the part of bifurcation parameter and a — relative — periodic orbit branches off at $\beta = 0$.

Let us assume that we are in the supercritical case of the Hopf bifurcation where the origin $z = 0$ is attracting for $\beta < 0$, while the branching periodic orbit is present for $\beta > 0$ and is attracting as well. For the integrable family X we have to superpose this planar scenario with the flows on $\mathbb{T}^n$; in this way equilibria become attracting or repelling invariant n–tori and attracting periodic orbits become attracting invariant $(n+1)$–tori. In the following we study what happens to both the n–tori and the $(n+1)$–tori under a small non-integrable perturbation of X to $\widetilde{X} = X + P$.

By applying Theorem 2.1 to the present setting we obtain persistent invariant n–tori, using the Diophantine conditions (7) with $\gamma > 0$ and $\tau > n - 1$ and restricting to a bounded set $\Gamma \subseteq \mathbb{R}^{n+2}$ of parameters (ω, α, β). In fact, we use a multi-parameter generalisation of Theorem 2.1, e.g. the version used in [5] to actually prove Theorem 2.1, and do not restrict the eigenvalues of $\Omega(\mu)$ to be bounded away from the imaginary axis, see [1, 2, 4] for more details. We conclude that for any family $\widetilde{X}$ sufficiently close to X a near-identity C^∞–diffeomorphism

$$\Phi \ : \ \mathbb{T}^n \times \mathbb{R}^2 \times \Gamma \ \longrightarrow \ \mathbb{T}^n \times \mathbb{R}^2 \times \Gamma$$

exists, defined near $\mathbb{T}^n \times \{0\} \times \Gamma$, that conjugates X to $\widetilde{X}$ when further restricting to $\mathbb{T}^n \times \{0\} \times \Gamma^\gamma_{\tau,\gamma}$, under preservation of the normal linear behaviour of the n–tori.

This allows us to consider the perturbed family $\widetilde{X}$ in the co-ordinates provided by the inverse Φ^{-1}. As $\Phi^*\widetilde{X}$ coincides on $\mathbb{T}^n \times \{0\} \times \Gamma^\gamma_{\tau,\gamma}$ with the integrable family X, we directly conclude that on the Cantor-like set $\Gamma^\gamma_{\tau,\gamma}$ the pull-back $\Phi^*\widetilde{X}$ has $\mathbb{T}^n \times \{0\}$ as a quasi-periodic invariant n–torus, attracting for $\beta < 0$ and repelling for $\beta > 0$. Moreover, we have the normal form splitting

$$(\Phi^*\widetilde{X} - X)_{\omega,\alpha,\beta} \ = \ \mathcal{O}(z)\,\partial_x \ + \ \mathcal{O}(z^2)\,\partial_z \ + \ Q_{\omega,\alpha,\beta}(x,z) \tag{15}$$

as $z \to 0$ with uniform estimates in ω, α, β and x. The Gevrey regular family of vector fields Q, say in Gevrey class $G^{1+\nu}$ with $\nu > 0$, is uniformly flat on $\mathbb{T}^n \times \Delta \times \Gamma^\gamma_{\tau,\gamma} \subseteq \mathbb{T}^n \times \mathbb{R}^2 \times \Gamma$, where Δ is a small neighbourhood of the origin in $\mathbb{R}^2$ — its Taylor series completely vanishes — which implies the exponential estimate (18) below. Indeed, for Δ sufficiently small we can arrange that Q vanishes identically on the Cantor-like set $\mathbb{T}^n \times \Delta \times \Gamma^\gamma_{\tau,\gamma}$, whence by perfectness of Cantor sets we conclude that all derivatives vanish as well. See [2, 5, 8] for a proof and more details.

For $\beta \neq 0$ the invariant n–tori are normally hyperbolic. By Theorem 2.2 we conclude that the parameter domain where invariant n–tori exist is open inside $\Gamma \cap \{\beta \neq 0\}$. This means that $\Gamma^\gamma_{\tau,\gamma} \cap \{\beta \neq 0\}$ can be fattened to an open subset of Γ. For parameter values outside of $\Gamma^\gamma_{\tau,\gamma}$ the invariant n–tori do not have to be quasi-periodic.

Theorem 3.1 (Order of contact) *Take $r_0 \in \mathbb{N}$ and (ω_0, α_0) satisfying* (7)*. There are positive constants A, B and ν such that for all $1 \leq r \leq r_0$ and all $\mu = (\omega, \alpha, \beta)$ in the set*

$$\left\{ (\omega, \alpha, \beta) \in \Gamma \;\middle|\; p(\|\omega - \omega_0\| + |\alpha - \alpha_0|) \;<\; \frac{|\beta|}{(1+r)} \right\} \;, \tag{16}$$

where

$$p(d) \;=\; A \exp(-B d^{-\frac{1}{\nu}}) \;, \tag{17}$$

the vector field $\Phi^* \widetilde{X}$ has an invariant r–normally hyperbolic torus.

Proof. Fix a regularity r_0. The term $Q = Q_{\omega,\alpha,\beta}$ is in Gevrey class $G^{1+\nu}$ and depends continuously on the perturbation $P_\mu = \widetilde{X}_\mu - X_\mu$. Hence if that perturbation is sufficiently small, then Q is C^{r_0}–close to 0. As Q is uniformly flat on $\mathbb{T}^n \times \Delta \times \Gamma^\gamma_{\tau,\gamma}$, the Taylor formula implies for $m \geq 0$ that

$$|Q| \,,\, |\partial Q| \;\leq\; \frac{1}{m!} |\partial^{m+1} Q| d^m \;\leq\; m^{\nu m} \|Q\|_{G^{1+\nu}} h^{-m} d^m$$

for some $h > 0$ and $d = \|\omega - \omega_0\| + |\alpha - \alpha_0|$. The right hand side is minimised at $\log m = -1 - \log(d/h)/\nu$, yielding the exponential estimate

$$|Q| \,,\, |\partial Q| \;\leq\; q(d) \;:=\; C_1 \exp(-C_2 d^{-\frac{1}{\nu}}) \tag{18}$$

for some positive constants C_1 and C_2. Fixing $\beta > 0$ we obtain

$$\lambda \;=\; |\beta| - C_3 q(d) \quad \text{and} \quad \lambda_c \;=\; C_3 q(d) \;,$$

with C_3 a positive constant, for the bounds λ on the minimal normal expansion and inverse contraction rate and λ_c on the maximal internal expansion and inverse contraction rate of the perturbed system. Since (12) requires

$$r \;<\; \frac{\lambda}{\lambda_c} \;=\; \frac{|\beta|}{C_3 q(d)} \;-\; 1$$

then r–normal hyperbolicity is guaranteed if

$$C_3 q(d) \;<\; \frac{|\beta|}{1+r} \;.$$

Now define $p := C_3 q$, $A := C_1 C_3$, $B := C_2$ and apply Theorem 2.2. □

Remarks.

- Note that for fixed Diophantine (ω_0, α_0) the set (16) is the union of two open disks — or blunt cusps — with a boundary that has infinite order of contact with the bifurcation hyperplane $\{\beta = 0\}$ at $(\omega, \alpha, \beta) = (\omega_0, \alpha_0, 0)$. In particular, the intersection of the closure of the set (16) with $\{\beta = 0\}$ reduces to the set of Diophantine (ω_0, α_0).

- The Gevrey regular dependence of Φ_μ on μ in Theorem 2.1 is what allows to replace the flat p in [2, 5, 8] by the explicit exponential p in (17); here the high regularity of the perturbed vector field $\widetilde{X}_\mu$ is used.

Given a regularity $r \in \mathbb{N}$ this yields $A, B, \nu > 0$ such that for all (ω_0, α_0) satisfying (7), the disks (16) are contained in the parameter domain with normally hyperbolic $\Phi^*\widetilde{X}$–invariant n–tori of class C^r. These tori are attracting for $\beta < 0$ and repelling for $\beta > 0$. Note that the disks grow larger as the degree r of differentiability decreases.

This directly concerns the perturbed family $\widetilde{X} = \Phi_*\Phi^*\widetilde{X}$ of vector fields. The union of the disks is uncountable, leaving open gaps 'centered' around the pure resonances $(\omega, \alpha, 0) \in \Gamma$ with $\langle k \mid \omega\rangle + \ell\alpha = 0$ for some $0 \neq k \in \mathbb{Z}^n$ and $\ell \in \{0, \pm1, \pm2\}$. Almost all of these resonance gaps can be closed when $\beta \neq 0$, as explained below.

For finding invariant $(n + 1)$–tori one has to develop a $\mathbb{T}^{n+1}$–symmetric normal form, related to both the normal form of the Hopf bifurcation of equilibria and the quasi-periodic normal form (15). This higher order normalisation requires more resonances to be excluded, so consider for given $N \geq 2$ the subset $\Gamma^{(N)}_{\tau,\gamma} \subseteq \Gamma$ obtained by a further extension of the Diophantine conditions (7) to all $\ell \in \mathbb{Z}$ with $|\ell| \leq N$. The resulting normal form is a small variation on Theorem 2.1. Indeed, after application of Theorem 2.1 one carries out a formal normal form procedure pushing the $\mathbb{T}^{n+1}$–symmetry of the normal linear part for $\beta = 0$ through the formal series in z. For the quasi-periodic Hopf bifurcation we follow [2, 8] and take $N = 7$. The invariant $(n + 1)$–tori can then be found by applying Theorem 2.2.

Given a regularity $r \in \mathbb{N}$ this yields $A, B, \nu > 0$ such that if $(\omega_0, \alpha_0) \in \Gamma^{(7)}_{\tau,\gamma}$, then the 'corresponding disk (16)' — which lies inside $\{\beta > 0\}$ — is contained in the parameter domain with normally hyperbolic $\Phi^*\widetilde{X}$–invariant $(n + 1)$–tori of class C^r. These tori are attracting, while there are no invariant $(n + 1)$–tori with $\beta \leq 0$. See [2, 5, 8] for a proof and more details.

Also for $(n+1)$–tori gaps are left out, 'centered' around the pure resonances $(\omega, \alpha, 0) \in \Gamma$ such that for some $0 \neq k \in \mathbb{Z}^n$ and $\ell \in \{0, \pm1, \ldots, \pm7\}$ one has $\langle k \mid \omega\rangle + \ell\alpha = 0$. The only difference to the resonance gaps concerning the n–tori, apart from the higher order $|\ell| \leq 7$ of normal-internal resonances, is that the resonance gaps related to the $(n+1)$–tori are situated only in the half plane space $\beta > 0$.

In a parameter space $\Gamma \subseteq \mathbb{R}^s$ of dimension $s = n+2$ the Diophantine conditions determine a Cantor-like set which is continuous in the β–direction, just as the half lines of constant frequency ratios in the (ω, α)–direction. The latter half lines allow for a 'preliminary' parameter reduction by means of time reparametrisation, see [5]. In order to really speak of an (ω, α)–direction it is best to use a 'final' parameter reduction as performed in [5], also used to prove Theorem 2.1, which amounts to using the dependence of ω on the parameter μ to essentially let ω and its derivatives span the internal frequency space $\mathbb{R}^n$; effectively we then can work with $s = 2$ and $\mu = (\alpha, \beta)$. For ease of argumentation we here prefer to achieve $\mu = (\alpha, \beta)$ by simply fixing $\omega = \omega_0$ Diophantine. Under variation of β the quasi-periodic Hopf bifurcation of the integrable family X is perturbed to the following dynamics of $\widetilde{X}$.

For values of $\mu = (\alpha, \beta)$ with $\beta \ll 0$ fixed all orbits are attracted to an invariant n–torus, which moreover carries a quasi-periodic flow with Diophantine frequency vector. As β increases towards 0, low order resonance strips open up — values of α for which the normally hyperbolic torus no longer survives the perturbation. As $\beta \nearrow 0$ the number of resonance strips increases, but for any fixed $\beta < 0$ these result in only finitely many gaps in the α–line.

Theorem 3.2 (Gaps left open) *For given $\beta \neq 0$ the number N of low order resonance gaps can be estimated as*

$$N \;\leq\; \left(E \;+\; F \log \frac{1}{|\beta|}\right)^{\nu} \;, \tag{19}$$

where E, F and ν are positive constants.

Proof. Fix ω and β at ω_0 and β_0 respectively, where ω_0 satisfies the Diophantine conditions (6). The normal frequency α is restricted by (7) and the resulting set

$$\Gamma_{\tau,\gamma} \,\cap\, \{\omega = \omega_0,\, \beta = \beta_0\} \quad\subset\quad \mathbb{R}$$

is closed and nowhere dense, hence its complement consists of infinitely many open intervals. Given a positive distance $d > 0$, we first estimate the number of those intervals whose length exceeds d. This then readily yields the estimate of the number of low order resonance gaps.

Throughout the proof we denote by C_i positive constants that do not depend on d. Introduce for $0 \neq k \in \mathbb{Z}^n$ and $0 < |\ell| \leq 2$ the interval

$$\mathcal{R}(k,\ell) \;=\; \left\{ \alpha \in \mathbb{R} \;\middle|\; |\langle k \mid \omega_0\rangle + \ell\alpha| < \frac{\gamma}{|k|^{\tau}} \right\}$$

containing the normal frequencies α that violate the extended Diophantine conditions (7) for (k, ℓ) where, however, we have to take $\tau > n$ (next to $\gamma > 0$) as $\omega = \omega_0$ is now fixed. Since $\mathcal{R}(k, -\ell) = \mathcal{R}(-k, \ell)$, it suffices to consider only $\ell = 1$ and $\ell = 2$. The length of the interval $\mathcal{R}(k, \ell)$ is $2\gamma|k|^{-\tau}$; this is larger than d if and only if

$$|k| \;\leq\; K(d) \;:=\; \left(\frac{2\gamma}{d}\right)^{\frac{1}{\tau}} .$$

There are at most

$$N_1 \;\leq\; 2(2K(d)+1)^n \;\leq\; C_1 d^{-\frac{n}{\tau}} \tag{20}$$

of these intervals.

The total measure m of the intervals $\mathcal{R}(k, \ell)$ for which $|k| > K(d)$ can be estimated as

$$m \;\leq\; \sum_{\substack{|k|>K(d)\\ \ell\in\{1,2\}}} \frac{2\gamma}{|k|^{\tau}} \;\leq\; C_2 K(d)^{n-\tau} \;\leq\; C_3 d^{1-\frac{n}{\tau}} \;,$$

where in the second inequality we use our assumption $\tau > n$. Although individually each interval $\mathcal{R}(k,\ell)$ has length less than d if $|k| > K(d)$, two or more of these intervals may overlap to form an interval of length at least d. However, the total number N_2 of such intervals is bounded by m/d, yielding

$$N_2 \;\leq\; C_3 d^{-\frac{n}{\tau}} \;. \tag{21}$$

In an interval of length d, the midpoint α is at least at distance $d/2$ from the extended Diophantine set $\Gamma_{\tau,\gamma}$. In order for α to be located in a resonance gap, it has to be in the complement of the intersection of (16) and $\{\omega = \omega_0,\ \beta = \beta_0\}$. Thus Theorem 3.1 leads to the necessary condition

$$p(d/2) \;\geq\; p(|\alpha - \alpha_0|) \;\geq\; C_4|\beta_0|$$

which implies

$$d^{-1/\nu} \;\leq\; C_5 \;+\; C_6 \log|\beta_0|^{-1} \;.$$

Combining this with (20) and (21) yields

$$N \;\leq\; N_1 \;+\; N_2 \;\leq\; (C_7 \;-\; C_8 \log|\beta_0|)^{\nu n/\tau} \;\leq\; (C_7 \;-\; C_8 \log|\beta_0|)^{\nu}$$

(using again $n/\tau < 1$) and now we take $E = C_7$, $F = C_8$. □

Remarks.

- Keeping track of r in the constants C_i shows that E depends on r logarithmically and F is independent of r — one could also write (19) as

$$N \;\leq\; \left(D\log(1+r) \;+\; F\log\frac{1}{|\beta|}\right)^{\nu} \;.$$

- Note that we did not have to restrict α to a compact line segment to ensure finitely many resonance gaps. Indeed, there is no repetition of low order resonances as for larger and larger $|\alpha|$ also $|k|$ has to be larger and larger.

- In applications it is rare that the internal frequency vector ω remains fixed when varying the parameters μ that the system depends upon. Therefore, when achieving to identify $\mu = (\alpha,\beta)$ the parameter dependence $\omega = \omega(\alpha)$ becomes important. We expect that a non-degeneracy condition like (8) on $\omega(\alpha)$ and its derivatives $\omega'(\alpha),\ldots,\omega^{(n-1)}(\alpha)$ at every α–value together with the assumption $\tau > n^2 - n$ should suffice to nevertheless obtain the estimate (19) on the number of gaps left open.

The n–tori perturbed from $\beta = 0$ are attracting, but not normally hyperbolic. Here only tori $\mathbb{T}^n \times \{0\}$ with Diophantine frequency vectors persist and there are infinitely many resonance gaps between the Diophantine α, giving rise to the 'bubbles' in the frayed

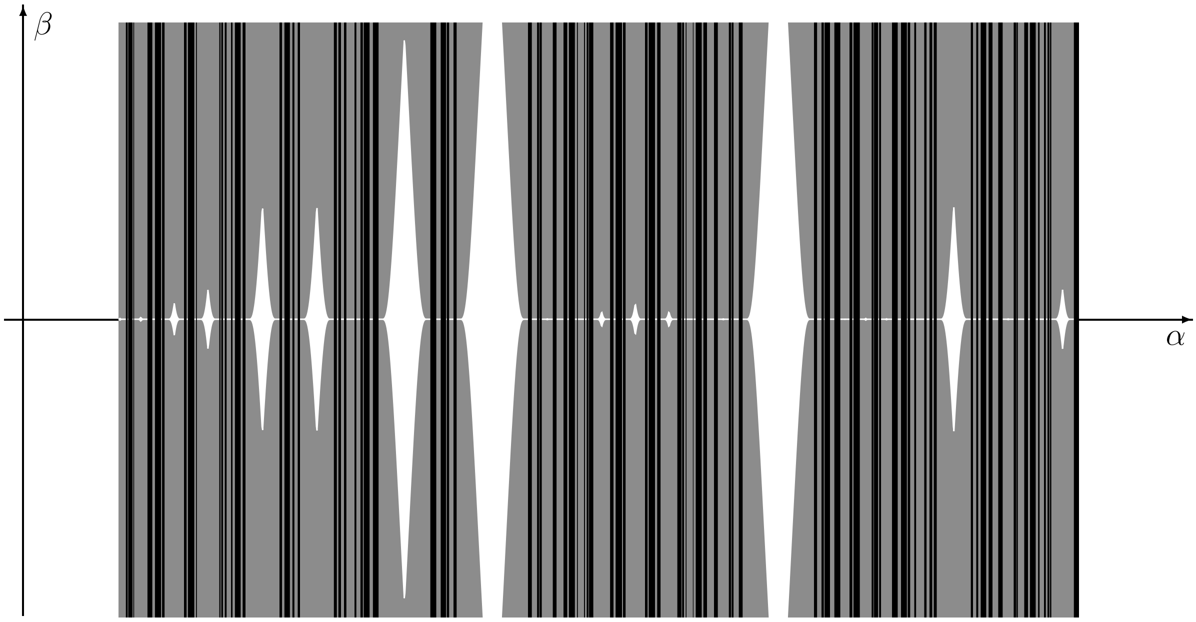

Figure 1: Parameter sets with persistent invariant tori and the subset of persistent quasi-periodic invariant tori. The latter set is the product of a Cantor set in the α–direction and the real line in the β–direction: it is nowhere dense. The complement of the former set (which includes the resonance set on the β–axis) is a set of ignorance, in particular persistent invariant tori may exist there as well. At the frayed boundary around the β–axis the figure is self-similar: it repeats itself when zoomed into (outside of a resonance bubble).

boundary between the parametrisations of attracting and repelling n–tori; see Figure 1. Repelling n–tori perturbed from $\mathbb{T}^n \times \{0\}$ with $\beta > 0$ also have resonance strips between the persistent Diophantine tori with their normally hyperbolic fattening, finitely many for any fixed $\beta > 0$, and those are one by one closed as β increases until none are left as $\beta \gg 0$ — a process which is the reverse to the one as β increases from $\beta \ll 0$ to $\beta = 0$, again governed by (19).

For $\beta > 0$ there are furthermore attracting $(n + 1)$–tori that are perturbed from the $(n + 1)$–tori $\mathbb{T}^n \times \mathbb{S}^1$ of the unperturbed dynamics that have been created in the Hopf bifurcation. Starting with $\beta \gg 0$ the situation is similar to that of the attracting n–tori. Where the normal attraction is sufficiently strong all orbits, except for those on the repelling n–torus, are attracted to the invariant $(n + 1)$–torus, which moreover for most values of α — the bifurcation parameter $\beta \gg 0$ is fixed — carries a quasi-periodic flow with Diophantine frequency vector. As β decreases towards 0, more and more resonance strips open up, but only finitely many for any fixed $\beta > 0$. The precise position of these resonance strips does not necessarily coincide with those between the repelling n–tori — they concern different tori with slightly different frequencies. Still, the resonance strips between the $(n+1)$–tori are expected to be larger than the resonance strips between the n–tori as the set of Diophantine frequencies of the latter is larger than the set of Diophantine frequencies of the former. The disks corresponding to (16) limit to the frayed boundary of non-hyperbolic Diophantine n–tori. The number of resonance strips obeys a similar estimate as (19), but with different values of the constants E and F.

The quasi-periodic Hopf bifurcation is triggered by variation of a single parameter and it is instructive to have a look at the change of dynamics under variation of a single parameter $\mu \in \Gamma \subseteq \mathbb{R}$. To avoid that the bifurcation itself disappears into a resonance gap we follow [5] and assume that the bifurcation value $\mu = \mu^*$ is an element of $\mathcal{G}_{\tau,\gamma}$ whence the unperturbed $\mathbb{T}^n \times \{0\}$ has Diophantine frequencies and thus persists as an invariant n–torus V_{μ^*}. Furthermore we assume that the path $\mu \mapsto (\alpha(\mu), \beta(\mu))$ crosses the frayed boundary transversely — this is achieved through the condition $\beta'(\mu^*) \neq 0$; in fact this inequality could be used for a reparametrisation after which $\mu = \beta$. Then the order of contact in (16) ensures that both for the n–tori and for the $(n+1)$–tori the path stays outside of the resonance strips whence along the path an attracting n–torus, not necessarily but 'often' with a quasi-periodic flow, becomes repelling, with an attracting $(n+1)$–torus, again not necessarily but 'often' with a quasi-periodic flow, bifurcating off. One also speaks of a Hopf–Landau bifurcation [5].

Out of all paths that transversely cross the frayed boundary, only the ones passing through some $\mu^* \in \mathcal{G}_{\tau,\gamma}$ undergo the above Hopf–Landau bifurcation, but since $\mathcal{G}_{\tau,\gamma}$ has large relative measure this happens with high probability. For a given path that passes the frayed boundary through a resonance bubble, in the complement of the disks (16), a further analysis of the dynamics inside that resonance bubble is needed to clarify the fine details of the bifurcation along that particular path, see [22] and references therein.

We refer to [2, 5, 3, 8] for more details on the quasi-periodic Hopf bifurcation. For a treatment of torus bifurcations with a single parameter see [7].

3.3 The quasi-periodic centre-saddle bifurcation

In an integrable Hamiltonian system the necessary parameters to control the internal frequencies of an invariant torus are built-in into the system as tori are parametrised by the actions conjugate to the toral angles. We restrict to the case $n = 2$ of invariant 2–tori in 3 degrees of freedom. Using Kolmogorov's non-degeneracy condition on the frequency mapping we may simply use the 2 frequencies as parameters, compare with (14). In the frequency space $\mathbb{R}^2$ the Diophantine conditions (6) yield one discontinuous direction transverse to the continuous direction of the half lines, putting us effectively into the situation of a 1–dimensional parameter μ.

During a centre-saddle bifurcation a centre and a saddle meet under variation of a parameter at a parabolic equilibrium — with linear part given by a nilpotent matrix $\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$, $a \neq 0$ — and vanish. In the quasi-periodic version of this bifurcation the equilibria are replaced by invariant tori and using the parametrisation by the actions conjugate to the toral angles one can also say that the family of hyperbolic tori has eigenvalues approaching 0 — where the tori become parabolic — and then forming a pair of purely imaginary eigenvalues with the tori turning elliptic with normal frequency given by the absolute value, or the positive imaginary part, of the pair of eigenvalues.

Such a quasi-periodic centre-saddle bifurcation persists a sufficiently small perturbation from the integrable Hamiltonian system to a non-integrable one, see [12, 13, 4]. Again Diophantine conditions are needed to achieve this persistence result. The parabolic tori are defined by the additional equation

$$\det \Omega \;=\; 0$$

and thus form a 1–parameter subfamily. Variation of the frequency ratio is needed along this 1–parameter family of parabolic 2–tori to ensure that the whole family does not disappear in a resonance gap. Under this requirement the continuous half lines in the set of Diophantine frequencies are transverse to the 1–dimensional subset of frequencies of parabolic tori.

On the hyperbolic side the situation is similar to that of the quasi-periodic Hopf bifurcation — next to Theorem 2.1 we can again apply Theorem 2.2 to fill the resonance gaps between the half lines emanating from the persistent parabolic tori. Sufficiently far away from the parabolic tori the union of the unperturbed hyperbolic tori forms a normally hyperbolic invariant manifold that persists the small perturbation away from integrability, see [10]. Note, however, that the actions conjugate to the toral angles are not external parameters, but phase space variables to which the non-integrable perturbation assigns a slow drift. The individual unperturbed 2–tori do *not* constitute normally hyperbolic invariant manifolds, it is only their union to which Theorem 2.2 can be applied.

This makes the passage from hyperbolic to parabolic tori less transparant. While Diophantine tori can still be continuated up until the parabolic tori of the same internal frequency ratio, it is not clear whether again only finitely many 'low order' resonance gaps appear at some small but fixed distance to the parabolic boundary. While the zig-zag lines obtained in [10] can in principle be constructed as close to this boundary as desired, the resulting persistence is only achieved for correspondingly small perturbations. Recall that the fine structure seen in the quasi-periodic Hopf bifurcation has been obtained for a *fixed* sufficiently small perturbation.

Remarks.

- Restricted to the normally hyperbolic invariant manifold obtained from Theorem 2.2 the flow is again Hamiltonian, see [17]. The dynamics is that of an integrable system with Lagrangean 2–tori for the unperturbed system and that of a non-integrable 2–degree-of-freedom system for the perturbed one.

- The dynamics within the gaps that are filled by normal hyperbolicity does not partition into invariant 2–tori, but consists of (slow) non-integrable Hamiltonian dynamics confined inside an energy shell of the perturbed normally hyperbolic invariant manifold between two Diophantine 2–tori obtained from Theorem 2.1.

On the elliptic side the situation is even more dramatic. Here no form of hyperbolicity allows us to fill the resonance gaps. Moreover, the continuous half lines of internal frequencies with the same frequency ratio are broken into a discontinuous Cantor set through the extended Diophantine conditions (7) which are needed for bounds away from resonances between the internal frequencies and the newly born normal frequency of the elliptic tori.

Thus, when following the 2–parameter family of invariant 2–tori involved in a quasi-periodic centre-saddle bifurcation from hyperbolic to elliptic, a 4–dimensional normally hyperbolic invariant C^r–manifold, which consists for a measure-theoretically large part of 1–parameter families of quasi-periodic tori with Diophantine frequency ratios, breaks up at the frayed parabolic boundary into normally elliptic tori, parametrised by 'Cantor dust' of (Hausdorff) dimension 2.

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