A New Property of Random Regular Bipartite Graphs

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Abstract

One deals with r-regular bipartite graphs with $2n$ vertices. In a previous paper Butera, Pernici, and the author have introduced a quantity $d(i)$, a function of the number of i-matchings, and conjectured that as n goes to infinity the fraction of graphs that satisfy $\Delta^k d(i) \geq 0$, for all k and i, approaches 1. Here Δ is the finite difference operator. In this paper it is proved that for each r, i, and $k \leq 6$ that the probability that $\Delta^k d(i) \geq 0$ goes to 1 with n going to infinity. A formalism of Wanless as systematized by Pernici is central to the proof.

1 Introduction

We deal with r-regular bipartite graphs with $v = 2n$ vertices. We let m_i be the number of i-matchings. In [1], Butera, Pernici, and I introduced the quantity $d(i)$, in eq. (10) therein,

$$d(i) \equiv \ln\left(\frac{m_i}{r^i}\right) - \ln\left(\frac{\overline{m}_i}{(v-1)^i}\right) \tag{1.1}$$

where $\overline{m}_i$ is the number of i-matchings for the complete (not bipartite complete) graph on the same vertices,

$$\overline{m}_i = \frac{v!}{(v-2i)!\, i!\, 2^i} \tag{1.2}$$

We here have changed some of the notation from [1] to agree with notation in [2]. We then considered $\Delta^k d(i)$ where Δ is the finite difference operator, so

$$\Delta d(i) = d(i+1) - d(i) \tag{1.3}$$

A graph was defined to satisfy *graph positivity* if all the meaningful $\Delta^k d(i)$ were non-negative. That is

$$\Delta^k d(i) \geq 0 \tag{1.4}$$

for $k = 0, \ldots, v$ and $i = 0, \ldots, v-k$. We made the conjecture, supported by some computer evidence,

Conjecture. *As n goes to infinity the fraction of graphs that satisfy graph positivity approaches one.*

In this paper we prove a weaker result in the same direction.

Theorem 1.1. *For each r, i, and $k \leq 6$, one has*
$$\mathrm{Prob}\big(\Delta^k d(i) \geq 0\big) \xrightarrow[n\to\infty]{} 1$$

The paper relies heavily on the work of Wanless, [3], and Pernici, [2], that gives a nice representation of the m_i. The restriction to $k \leq 6$ is very disappointing. Much of the theoretical development herein is for all k. The current technique of proof certainly does not work for $k \geq 7$, and it is possible that the statement in Theorem 1.1 is not true for $k \geq 7$.

The restriction to bipartite graphs is mainly because this restriction is made in [1], and is certainly (it seems) necessary for the conjecture above. In this paper the bipartite nature appears used in two places. First, the number of vertices is assumed to be even, and second, in eq.(3.6) the lower limit 4 is replaced by 3 if one does not assume the graph is bipartite.

2 Idea of the proof

Suppose we want $\mathrm{Prob}(x > y)$ to be large. We have
$$\mathrm{Prob}(x < y) = \mathrm{Prob}(e^x < e^y) \tag{2.1}$$
Set
$$\mathrm{E}(e^x - e^y) \equiv \alpha \tag{2.2}$$
We will want α to be positive, and in the present work proving this positivity will be a major component. Let
$$\mathrm{E}\big((e^x - e^y)^2\big) - \alpha^2 \equiv \beta \tag{2.3}$$
Then, assuming $\alpha > 0$,
$$\mathrm{E}\big((e^x - e^y - \alpha)^2\big) \geq \alpha^2 \ \mathrm{P}(e^x - e^y < 0) \tag{2.4}$$
And so
$$\mathrm{Prob}(e^x < e^y) \leq \frac{\beta}{\alpha^2} \tag{2.5}$$
In our problem β and α will be functions of n, and we'll want probability to go to zero with n as n goes to infinity. We will guarantee this by finding twice the highest power of n in α to be greater than the highest power of n in β.

We turn to the object of study, and perform some simple manipulations, working from eq. (1.1)
$$\mathrm{Prob}\big(\Delta^k d(i) > 0\big) = \mathrm{Prob}\left((-1)^k \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \ln\left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}}\right) > 0\right) \tag{2.6}$$
$$= \mathrm{Prob}\left(\sum_{\ell\in\mathcal{L}^+} \binom{k}{\ell} \ln\left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}}\right) > \sum_{\ell\in\mathcal{L}^-} \binom{k}{\ell} \ln\left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}}\right)\right) \tag{2.7}$$
where $\mathcal{L}^+$ is the set of odd ℓ, $0 \leq \ell \leq k$, if k is odd and is the set of even ℓ, $0 \leq \ell \leq k$, if k is even, and $\mathcal{L}^-$ is defined vice versa.

Returning to the language of (2.1)–(2.5), we set
$$x \equiv \sum_{\ell\in\mathcal{L}^+} \binom{k}{\ell} \ln\left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}}\right) \tag{2.8}$$
$$y \equiv \sum_{\ell\in\mathcal{L}^-} \binom{k}{\ell} \ln\left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}}\right) \tag{2.9}$$

and so

$$e^x = \prod_{\ell \in \mathcal{L}^+} \left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}} \right)^{\binom{k}{\ell}} \tag{2.10}$$

$$e^y = \prod_{\ell \in \mathcal{L}^-} \left(\frac{m_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}} \right)^{\binom{k}{\ell}} \tag{2.11}$$

3 The Work of Wanless and Pernici

In [3] Wanless developed a formalism to compute the m_i of any regular graph. We here only give a flavor of this formalism, but present those consequences we will use in this paper. For each i there are defined a set of graphs $g_{i1}, g_{i2}, \ldots, g_{in(i)}$. Given a regular graph g, one computes for each j the number of subgraphs of g isomorphic to g_{ij}, call this $g \mathbin{/\!\!/} g_{ij}$. Then m_i for g is determined by the $n(i)$ values of $g \mathbin{/\!\!/} g_{ij}$. We define M_i to be the value of m_i assigned to a graph with all $n(i)$ values of $g \mathbin{/\!\!/} g_{ij}$ zero. $M_i = M_i(r, n)$ is an important object of study to us. Most of the paper we will approximate m_i by M_i. We will use subscript 0 as indicating m_i as it appears in any expression is replaced by M_i. Thus from (2.2), (2.10), and (2.11) we get

$$\alpha_0 = \mathrm{E}\left(\prod_{\ell \in \mathcal{L}^+} \left(\frac{M_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}} \right)^{\binom{k}{\ell}} - \prod_{\ell \in \mathcal{L}^-} \left(\frac{M_{i+\ell}}{\overline{m}_{i+\ell}} \cdot \frac{(v-1)^{i+\ell}}{r^{i+\ell}} \right)^{\binom{k}{\ell}} \right) \tag{3.1}$$

In [2] Pernici systematized the results of Wanless; we take a number of formulae from this paper. From eq. (12) and eq. (13) of [2] we write

$$M_j = \frac{n^j r^j}{j!}(1 + H_j) \tag{3.2}$$

$$H_j = \sum_{h=1}^{j-1} \frac{a_h(r,j)}{n^h} \tag{3.3}$$

a_h is a polynomial of degree at most $2h$ in j. We will use eq. (16) and (17) of [2]

$$[j^k n^{-h}] \ \ln\left(1 + \sum_{s=1}^{j-1} \frac{a_s(r,j)}{n^s} \right) = 0, \qquad k \geq h+2 \tag{3.4}$$

$$[j^{h+1} n^{-h}] \ \ln\left(1 + \sum_{s=1}^{j-1} \frac{a_s(r,j)}{n^s} \right) = \frac{1}{(h+1)h} \left(\frac{1}{r^h} - 2 \right) \tag{3.5}$$

where $[j^k n^{-h}]$ in front of an expression picks out the coefficient of the $\frac{j^k}{n^h}$ term, $c(k,h)$, in an expression $\sum_{\alpha,\beta} c(\alpha,\beta) \frac{j^\alpha}{n^\beta}$.

We set $M_0 = 1$ and $M_s = 0$ if $s < 0$. Then m_j is recovered from M_j by the formula

$$m_j = \exp\left(\sum_{s \geq 4} \frac{\varepsilon_s}{2s} (-\hat{x})^s \right) M_j \tag{3.6}$$

eq. (11) of [2]. Here

$$\hat{x} M_j = M_{j-1} \tag{3.7}$$

ε_s for a graph g is a linear function of a finite number of $g \mathbin{/\!/} \ell_i$, ℓ_i a set of given graphs. All we need to know is that for any given product of ε_s's, $\prod_i \varepsilon_{s(i)}$, one has that

$$\mathrm{E}\Big(\prod_i \varepsilon_{s(i)}\Big) \leq C \tag{3.8}$$

i.e. it is a bounded function of n. The main result one needs to see this is that the number of s-cycles are independent Poisson random variables of finite means in the fixed r, n goes to infinity limit, [4]. One then uses the fact that the ℓ_i and g_{ij} graphs discussed above all are either single cycle or multicycle in nature.

4 Some simple reorganization

We define

$$1 + K_i \equiv \frac{(v-1)^i}{r^i} \cdot \frac{(v-2i)!\, i!\, 2^i}{v!} \cdot \frac{r^i n^i}{i!} \tag{4.1}$$

using notably eq. (1.2). Then α_0 becomes

$$\alpha_0 = \mathrm{E}\Bigg(\prod_{\ell \in \mathcal{L}^+} \big((1+H_{i+\ell})(1+K_{i+\ell})\big)^{\binom{k}{\ell}} - \prod_{\ell \in \mathcal{L}^-} \big((1+H_{i+\ell})(1+K_{i+\ell})\big)^{\binom{k}{\ell}}\Bigg) \tag{4.2}$$

Further we set

$$1 + K_i \equiv e^{G_i} \tag{4.3}$$

where

$$G_i \equiv G_{i,1} + G_{i,2} + G_{i,3} + G_{i,4} + G_{i,5} \tag{4.4}$$

$$G_{i,1} \equiv i \, \ln\Big(1 - \frac{1}{2n}\Big) \tag{4.5}$$

$$G_{i,2} \equiv (2n - 2i) \, \ln\Big(1 - \frac{i}{n}\Big) \tag{4.6}$$

$$G_{i,3} \equiv 2i \tag{4.7}$$

$$G_{i,4} \equiv \frac{1}{2} \, \ln\Big(1 - \frac{1}{n}\Big) \tag{4.8}$$

$$G_{i,5} \equiv \sum_{j \text{ odd}} c_j \left(\frac{1}{n^j} - \frac{1}{(n-i)^j}\right) \tag{4.9}$$

We have used the Sterling series to expand $\ln n!$. We also note $c_1 = -\frac{1}{24}$. H_i and K_i are easily developed as a series in inverse powers of n.

5 $k = 1$ and $k = 0$

Not only is $k = 1$ the first case, but it is different from $k \geq 2$ in some essential ways. We proceed to compute α_0 for $k = 1$. From eq. 4.2 we have

$$\alpha_0 = \mathrm{E}\big((1 + H_{i+1})(1 + K_{i+1}) - (1 + H_i)(1 + K_i)\big) \tag{5.1}$$

From eq. (18) of [2], and eq. (45) of [2] for q_1 therein, one gets

$$H_i = i(i-1)\Big(-1+\frac{1}{2r}\Big)\frac{1}{n} + \mathcal{O}\Big(\frac{1}{n^2}\Big) \tag{5.2}$$

Using eq. (4.3)–(4.9) and the value of c_1 thereafter one gets

$$K_i = (i^2-i)\frac{1}{n} + \mathcal{O}\Big(\frac{1}{n^2}\Big) \tag{5.3}$$

There follows

Theorem 5.1. *For $k=1$*

$$\alpha_0 = \frac{i}{rn} + \mathcal{O}\Big(\frac{1}{n^2}\Big) \tag{5.4}$$

One easily gets

Theorem 5.2. *For $k=0$*

$$\alpha_0 = 1 + \mathcal{O}\Big(\frac{1}{n}\Big) \tag{5.5}$$

6 $k \geq 2$

The goal of this section is proving the following theorem.

Theorem 6.1. *For $k \geq 2$*

$$\left[\frac{1}{n^d}\right]\alpha_0 = \begin{cases} 0 & d < k-1 \\ \dfrac{(k-2)!}{r^{k-1}} & d = k-1 \end{cases} \tag{6.1}$$

where as elsewhere $[\frac{1}{n^d}]Q$ is the coefficient of the $\frac{1}{n^d}$ term in an expansion of Q into inverse powers of n.

From Section 8 using the Second Identity we have

$$\alpha_0 = \mathrm{E}\bigg(\Big(1+t_+ + \frac{1}{2}t_+^2 + \cdots\Big) - \Big(1+t_- + \frac{1}{2}t_-^2 + \cdots\Big)\bigg) \tag{6.2}$$

where with

$$1 + U_i = (1+H_i)(1+K_i) \tag{6.3}$$

one defines

$$t_+ = \sum_{\ell \in \mathcal{L}^+} \binom{k}{\ell}\Big(U_{i+\ell} - \frac{1}{2}(U_{i+\ell})^2 + \frac{1}{3}(U_{i+\ell})^3 \cdots\Big) \tag{6.4}$$

$$t_- = \sum_{\ell \in \mathcal{L}^-} \binom{k}{\ell}\Big(U_{i+\ell} - \frac{1}{2}(U_{i+\ell})^2 + \frac{1}{3}(U_{i+\ell})^3 \cdots\Big) \tag{6.5}$$

We treat the terms written explicitly in (6.2); the induction to higher powers of t is trivial.

$$\left[\frac{1}{n^d}\right]E(t_+ - t_-) = \left[\frac{1}{n^d}\right]\sum_{\ell}\binom{k}{\ell}(-1)^{k+\ell}\Big(U_{i+\ell} - \frac{1}{2}(U_{i+\ell})^2 + \frac{1}{3}(U_{i+\ell})^3 \cdots\Big) \tag{6.6}$$

$$= \begin{cases} 0 & d < k-1 \\ \dfrac{(k-2)!}{r^{k-1}} & d = k-1 \end{cases} \tag{6.7}$$

by the First Identity, Theorem 7.1. Thus we want to prove that the higher powers of t's make no contribution in (6.2)! This is amazing when one first sees it.

We want to show

$$\left[\frac{1}{n^d}\right](t_+^2 - t_-^2) = 0 \quad \text{for} \quad d \le k-1 \tag{6.8}$$

We proceed by looking at the powers of $\frac{1}{n}$.

$$\left[\frac{1}{n^d}\right](t_+^2 - t_-^2) = \sum_{s=1}^{d-1}\left[\left(\left[\frac{1}{n^s}\right]t_+\right)\left(\left[\frac{1}{n^{d-s}}\right]t_+\right) - \left(\left[\frac{1}{n^s}\right]t_-\right)\left(\left[\frac{1}{n^{d-s}}\right]t_-\right)\right] \tag{6.9}$$

All we need to complete a proof of (6.8) is to show

$$\left[\frac{1}{n^s}\right]t_+ = \left[\frac{1}{n^s}\right]t_- \qquad 1 \le s \le d-1 \tag{6.10}$$

But this follows from (6.6),(6.7) above. Pretty neat.

7 First Identity

For convenience we introduce

$$F_i \equiv (1+H_i)(1+K_i) \tag{7.1}$$

Theorem 7.1 (First Identity)**.** *For all r, $i \ge 0$, and $k \ge 2$*

$$\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell+k}\left[\frac{1}{n^{k-1}}\right]\sum_{m=1}^{k-1}(-1)^{m+1}\frac{1}{m}\big(F_{(i+\ell)}-1\big)^m = \frac{(k-2)!}{r^{k-1}} \tag{7.2}$$

Theorem 7.2. *For all r, and $k \ge 2$*

$$\left[\frac{1}{n^{k-1}}\right]\ln(F_i) \tag{7.3}$$

has highest power of $i = i^k$, and this term is

$$\frac{(k-2)!}{k!}\frac{i^k}{r^{k-1}} \tag{7.4}$$

For example for $k=3$

$$\left[\frac{1}{n^2}\right]\ln(F_s) = -\frac{1}{12}\frac{s\big(3r^2s - 3r^2 - 12rs - 2s^2 + 12r + 9s - 7\big)}{r^2} \tag{7.5}$$

We now note

$$\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell+k}\ell^d = \begin{cases} 0 & d<k \\ k! & d=k \end{cases} \tag{7.6}$$

that follows from

$$\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell+k}\ell^d = \Delta^k i^d \tag{7.7}$$

which like $(\frac{d}{dx})^k x^d$ has values in (7.6). From eq. (7.6) one can deduce that Theorem 7.1 follows from Theorem 7.2, which we proceed to prove.

$$\left[\frac{1}{n^{k-1}}\right]\ln(F_i) = \left[\frac{1}{n^{k-1}}\right]\ln(1+H_i) + \left[\frac{1}{n^{k-1}}\right]\ln(1+K_i) \tag{7.8}$$

From eqs. (3.4)–(3.5) with $k \geq 2$ we see that the highest power of i in $[\frac{1}{n^{k-1}}]\ln(1+H_i)$ is k and its coefficient is

$$\frac{1}{k(k-1)}\left(\frac{1}{r^{k-1}} - 2\right) \tag{7.9}$$

To study $[\frac{1}{n^{k-1}}]\ln(1+K_i)$ we turn to equations (4.3)–(4.9). We note the highest power of i arises from the expansion of the term $G_{i,2}$, eq. (4.6), and $[\frac{1}{n^{k-1}}]\ln(1+K_i)$ has highest power i

$$\frac{2}{k(k-1)}$$

So

$$\left[\frac{1}{n^{k-1}}\right]\ln(F_i) = \frac{(k-2)!}{k!}\frac{1}{r^{k-1}} i^k \tag{7.10}$$

Quod erat demonstrandum.

8 Second Identity

We start with some simple manipulations

$$\prod_i (1+x_i)^{e_i} = e^{\sum e_i \ln(1+x_i)} = 1 + \left(\sum e_i \ln(1+x_i)\right) + \frac{1}{2!}\left(\sum e_i \ln(1+x_i)\right)^2 + \cdots \tag{8.1}$$

We introduce a new notation

$$(1+H_i)(1+K_i) \equiv 1 + U_i \tag{8.2}$$

and substitute U_i for x_i and $\binom{k}{\ell}$ for e_i in (8.1)

$$\prod_{\ell \in \mathcal{L}^+} (1+U_{i+\ell})^{\binom{k}{\ell}} = 1 + t_+ + \frac{1}{2}t_+^2 + \frac{1}{3!}t_+^3 \cdots \tag{8.3}$$

where

$$t_+ \equiv \sum_{\ell \in \mathcal{L}^+} \binom{k}{\ell}\left(U_{i+\ell} - \frac{1}{2}(U_{i+\ell})^2 + \frac{1}{3}(U_{i+\ell})^3 \cdots\right) \tag{8.4}$$

The Second Identity consists of (8.3) and (8.4) and the same expressions with $\mathcal{L}^+$, t_+ replaced by $\mathcal{L}^-$, t_-.

9 Completion

If there were no "contributor" contributions to m_i, see Section 3, we would be through since then β would be zero, see eq. (2.5). We note α would then be equal to α_0 which is greater than zero, except for $k = 1$, $i = 0$. The single case $k = 1$, $i = 0$ need not be considered since in fact $\Delta d(0) = 0$.

It is a straight forward, if fussy, computation to show that for $k \leq 6$ the leading term in α in powers of n does not depend on "contributors." For $k = 7$ the leading term does depend on ε_4, and the current line of proof fails for $k \geq 7$.

References

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