Multiple positive bound states for the subcritical NLS equation on metric graphs

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Abstract

We consider the Schrödinger equation with a subcritical focusing power nonlinearity on a noncompact metric graph, and prove that for every finite edge there exists a threshold value of the mass, beyond which there exists a positive bound state achieving its maximum on that edge only. This bound state is characterized as a minimizer of the energy functional associated to the NLS equation, with an additional constraint (besides the mass prescription): this requires particular care in proving that the minimizer satisfies the Euler?Lagrange equation. As a consequence, for a sufficiently large mass every finite edge of the graph hosts at least one positive bound state that, owing to its minimality property, is orbitally stable.

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1 Introduction

Given $p \in (2,6)$, the existence of ground states for the NLS energy functional

$$E(u,\mathcal{G}) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} = \frac{1}{2}\int_{\mathcal{G}} |u'|^2 dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p dx \tag{1}$$

on a noncompact metric graph $\mathcal{G}$, under the *mass constraint*

$$\|u\|^2_{L^2(\mathcal{G})} \;=\; \mu, \tag{2}$$

has been been investigated in a series of papers [1, 2, 3, 4, 5]. Several conditions on $\mathcal{G}$ and μ have been found, which may guarantee (or, on the contrary, rule out) the existence of absolute minimizers of $E(u,\mathcal{G})$ in the mass–constrained space

$$H^1_\mu(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) : \|u\|^2_{L^2(\mathcal{G})} = \mu \right\}. \tag{3}$$

In this paper, instead, we switch our focus from ground states to *bound states*, that is, to functions $u \in H^1_\mu(\mathcal{G})$ which are constrained *critical points* of the NLS energy functional, possibly without being absolute minimizers. More precisely, given a mass $\mu > 0$, a bound state is a function $u \in H^1_\mu(\mathcal{G})$ that satisfies the Euler–Lagrange equation

$$\int_{\mathcal{G}} \left(-u'\eta' + u|u|^{p-2}\eta\right) dx = \lambda \int_{\mathcal{G}} u\eta \, dx \quad \forall \eta \in H^1(\mathcal{G}), \tag{4}$$

where λ is a Lagrange multiplier due to the mass constraint (2). Of course, this means that u solves the nonlinear equation

$$u'' + u|u|^{p-2} = \lambda u \tag{5}$$

on every edge of $\mathcal{G}$, coupled with the homogeneous *Kirchhoff condition*

$$\sum_{e \succ \mathrm{v}} \frac{du}{dx_e}(\mathrm{v}) = 0 \tag{6}$$

at every vertex v of $\mathcal{G}$ (the sum is extended to all edges e incident at v, see [3] for more details). As usual, the existence of bound states is equivalent to the existence of stationary solutions of the corresponding NLS equation on $\mathcal{G}$, and this becomes all the more relevant if $\mathcal{G}$ has no ground state.

Before describing in detail our main results, we recall that a *noncompact metric graph* $\mathcal{G}$ is a connected metric space obtained by gluing together, by the identification of some of their endpoints (the "vertices" of $\mathcal{G}$), a finite number of closed line intervals (not necessarily bounded), according to the topology of a graph, self-loops and multiple edges being allowed. Each edge e, after choosing a coordinate x_e on it, can be regarded either as an interval $[0, \ell_e]$, or as a positive half-line $[0, +\infty)$ (in this case the edge is attached to $\mathcal{G}$ at $x_e = 0$), and the spaces $L^r(\mathcal{G})$, $H^1(\mathcal{G})$ etc. can be defined in a natural way (in particular, $u \in H^1(\mathcal{G})$ means that $u \in H^1(e)$ for every edge e of $\mathcal{G}$, with the additional requirement that u is *continuous* on $\mathcal{G}$, i.e. it has no jump at any vertex: we refer to [3, 14] for more details). Endowing $\mathcal{G}$ with the shortest path distance, one obtains a locally compact metric space, and $\mathcal{G}$ being noncompact is equivalent to one or more edges being unbounded: two very special cases are $\mathcal{G} = \mathbb{R}^+$ (i.e. $\mathcal{G}$ consists of just one unbounded edge) and $\mathcal{G} = \mathbb{R}$ (obtained by gluing together two unbounded edges).

It is within this framework that we look for bound states, i.e. solutions to (4) belonging to the space $H^1_\mu(\mathcal{G})$ defined in (3). In this investigation, we fix an exponent $p \in (2, 6)$ (the *critical case* where $p = 6$ is rather special and will not be considered here, see [5] for results on ground states), while we will focus our attention on the mass μ as a varying parameter of the problem.

We will prove (Theorem 4.3 and Corollary 4.4) that if $\mathcal{G}$ is any noncompact metric graph and μ is large enough, then there exist at least as many bound states of mass μ as the number of bounded edges of $\mathcal{G}$. This is in contrast with the case of ground states, which on some graphs may fail to exist for every value of μ (see [3, 4]).

Our strategy is a purely variational one, in that bound states are obtained as solutions of a doubly-constrained minimization procedure for the NLS energy (1). More precisely, we fix a bounded edge e of $\mathcal{G}$ (if every edge of $\mathcal{G}$ is a halfline, then $\mathcal{G}$ is a star–graph and in this case the existence of a bound state is already known, see [2]), and we minimize $E(u, \mathcal{G})$ among all functions $u \geq 0$ of mass μ which achieve their absolute maximum on the edge e (see (21), (22)). It turns out that a minimizer (i.e. a function u satisfying (23)) exists, provided μ is large enough (Theorem 3.3). Observe that the existence of a minimizer is nontrivial: indeed, even though the condition of achieving the maximum on e is weakly closed (see Remark 3.1), the class V_μ where we seek a minimizer is far from being compact, due to the mass constraint (2). Thus, to prove that a minimizer exists, we first prove a general result (Theorem 2.4), that gives quantitative lower bounds for the NLS energy along a sequence of functions $u_n \rightharpoonup u$ in $H^1_\mu(\mathcal{G})$, according to the discrepancy m between μ (the mass of every u_n) and the mass of the weak limit u. This can be applied to our minimization problem and, at least when μ is large, it reveals that minimizing sequences are in fact strongly compact in L^2 and converge to a minimizer. We point out that, beside the present application, Theorem 2.4 can be applied to any minimization problem involving the NLS energy on graphs, e.g. in the investigation of ground states (see Remark 2.5).

At this stage a minimizer $u \in V_\mu$ satisfying (23) exists, yet one would not expect u to solve the Euler equation (4) (i.e. to be a critical point of the NLS energy subject to the mass constraint (2)), since a generic variation of u (albeit in a direction that preserves its mass) might well violate the additional condition that u achieves its maximum on the edge e. In fact, we can prove that, if μ is large enough, then u achieves its maximum *only on* e (and away from branching points), so that u is strictly smaller elsewhere on $\mathcal{G}$ (Lemma 4.2): in other words, u lies in the *interior* of the additional constraint, which therefore becomes stable under small perturbations of u. This, in turn, allows for arbitrary small variations in any mass-preserving direction, which leads to the Euler equation (4) with a Lagrange multiplier λ (just as if u were a ground state, i.e. a minimizer subject to the mass constraint only).

We point out that, in general, the bound states obtained in this way are not ground states, and this is certainly so if $\mathcal{G}$ admits no ground state (see Example 5.1). More generally, if $\mathcal{G}$ has k bounded edges and μ is large, by the above procedure we obtain at least k distinct bound states (Corollary 4.4), and not all of them are necessarily ground states (see Example 5.2).

As far as we know, the results contained in this paper are the first to establish the existence of (many) positive bound states by variational techniques. In particular, the fact that such bound states arise as *local minimizers* of the constrained energy functional has a relevant consequence as regards the dynamics described by the Schrödinger Equation with an additional focusing power term:

$$i\partial_t u(t) = -\Delta u(t) - |u(t)|^{p-2}u(t),$$

where the symbol Δ is to be interpreted as the operator acting as the second derivative on functions satisfying Kirchhoff conditions (6) at every vertex. It is indeed well-known ([12]) that local minimizers of the constrained energy functional are *orbitally stable* so that, if u is any bound state described in Theorem 4.3, then for the set $\{e^{i\theta}u,\ \theta \in [0, 2\pi)\}$ a stability property in the sense of Lyapunov holds. This is relevant because, while for ground states orbital stability is guaranteed by classical results ([9]), for excited states (namely bound states that are *not* ground states) it is often an open issue. Indeed, existence of positive excited states was already proved for the tadpole graph in [7], but the problem of their stability was left untouched (nonetheless, our results do not apply to such states). On the other hand, in [15] many non-positive excited states of the tadpole graph were found to be unstable.

Finally, we remark that it would be interesting to investigate the existence of sign-changing bound states. This has been done in the papers [16, 17] but only in the context of localized nonlinearities, and in the already cited [7, 15] for the particular case of the tadpole graph. Moreover, it is an open problem to establish the existence of bound states without the assumption that the mass μ be large enough.

The paper can be summarized as follows: in Section 2 we give some results that work as preparation for the main theorems: in particular, in Theorem 2.4 we give a new, short proof of the beaviour of weakly convergent sequences. The core of Section 3 is Theorem 3.3, that establishes the existence of a solution to the double-constrained minimimum problem for a sufficiently large mass. Section 4 contains Theorem 4.3, stating that the solutions whose existence is ensured in Theorem 3.3 are actually bound states. A remarkable technical step is achieved by Lemma 4.2, where we prove that the constraint is actually an open subset of the hypersurface of constant mass. Finally, in Section 5 we provide further explanations and examples in order to make the scope of the results more precise.

2 Some basic tools (old and new)

The purpose of this section is twofold. First, we recall some known facts and inequalities on metric graphs which will be used throughout the paper. Second, we prove a new result which, we believe, may be of interest in any minimization procedure for the NLS energy on metric graphs.

Two basic tools in our analysis are the Gagliardo–Nirenberg inequality

$$\forall u \in H^1(\mathcal{G}) \qquad \|u\|_{L^p(\mathcal{G})}^p \leq C \|u\|_{L^2(\mathcal{G})}^{\frac{p}{2}+1} \|u'\|_{L^2(\mathcal{G})}^{\frac{p}{2}-1} \tag{7}$$

for some constant C depending only on p, and the L^∞ estimate

$$\|u\|_{L^\infty(\mathcal{G})}^2 \leq 2\|u\|_{L^2(\mathcal{G})} \|u'\|_{L^2(\mathcal{G})}. \tag{8}$$

Both are well known when $\mathcal{G} = \mathbb{R}^+$, and by a rearrangement argument they are proved to be valid on any noncompact metric graph $\mathcal{G}$, with the same constant as on $\mathbb{R}^+$ (see [4, 18] for more details).

In the following we will need to compare the achievable energy levels on $\mathcal{G}$ with the ground-state energy levels on the real line $\mathbb{R}$ and on the halfline $\mathbb{R}^+$, exploiting the fact that in the prototypical cases where $\mathcal{G} = \mathbb{R}$ or $\mathcal{G} = \mathbb{R}^+$ everything is known explicitly.

When $\mathcal{G} = \mathbb{R}$ (and for fixed $p \in (2, 6)$) the ground states of mass μ are called *solitons*, and we refer to [8] for more details. Here, we just record that they achieve a negative energy level given by

$$\min_{u \in H^1_\mu(\mathbb{R})} E(u, \mathcal{G}) = -\theta_p \, \mu^{2\beta+1}, \tag{9}$$

for some constant $\theta_p > 0$ and an exponent β given by

$$\beta = \frac{p-2}{6-p} > 0. \tag{10}$$

Similarly, when $\mathcal{G} = \mathbb{R}^+$ the unique ground state of mass μ is a "half soliton" (i.e. the restriction to $\mathbb{R}^+$ of a soliton of mass 2μ on $\mathbb{R}$), with an energy level given by

$$\min_{u \in H^1_\mu(\mathbb{R}^+)} E(u, \mathcal{G}) = -2^{2\beta} \theta_p \, \mu^{2\beta+1} \tag{11}$$

with the same θ_p and β as before. We point out that these two energy levels on $\mathbb{R}^+$ and $\mathbb{R}$ provide universal bounds for the ground-state energy level on any noncompact metric graph $\mathcal{G}$. More precisely, it was proved in [3] that

$$-2^{2\beta} \theta_p \, \mu^{2\beta+1} \leq \inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) \leq -\theta_p \, \mu^{2\beta+1}. \tag{12}$$

Later on we will need the following lemma, which gives a lower bound to the energy $E(w, \mathcal{G})$ of a function, in terms of the number N of its preimages. When $N = 2$ the result, based on a rearrangement technique, is well known (see e.g. [3], Proposition 3.1), while the general case is a little more tricky, though in the same spirit.

Lemma 2.1. *Let $w \in H^1(\mathcal{G})$ be a nonnegative function having at least N preimages ($N \geq 1$) for almost every value, i.e.*

$$\#\{x \in \mathcal{G} \mid w(x) = t\} \geq N \quad \textit{for a.e. } t \in (0, \|w\|_{L^\infty(\mathcal{G})}). \tag{13}$$

Then

$$E(w, \mathcal{G}) \geq -\theta_p \left(\frac{2}{N}\right)^{2\beta} \nu^{2\beta+1}, \qquad \nu := \|w\|^2_{L^2(\mathcal{G})}. \tag{14}$$

Proof. Let $w^* \in H^1(\mathbb{R}^+)$ denote the decreasing rearrangement of w on the positive halfline. Regardless of (13), it is well known (see [3]) that

$$\|w^*\|_{L^r(\mathbb{R}^+)} = \|w\|_{L^r(\mathcal{G})} \quad \forall r \in [1, +\infty], \qquad \|(w^*)'\|_{L^2(\mathbb{R}^+)} \leq \|w'\|_{L^2(\mathcal{G})}.$$

If, however, one has (13) (which is nontrivial only if $N > 1$), then the last inequality takes the stronger form

$$\|(w^*)'\|^2_{L^2(\mathbb{R}^+)} \leq \frac{1}{N^2}\|w'\|^2_{L^2(\mathcal{G})}, \tag{15}$$

as pointed out in [10]. This, however, follows immediately from a direct inspection of the proof of the Polya–Szegő inequality (see [13] or [11] for the metric graph setting), which is based on a slicing argument: at each level t in the range of w, letting $N(t) = \#w^{-1}(t)$, at a certain stage one uses the minorization $N(t)^2 \geq 1$ (which is trivial, but optimal for a generic w). In fact, the proof yields the sharp bound (15), if one exploits (13) and uses $N(t)^2 \geq N^2$ for a.e. t.

Now define $v \in H^1(\mathbb{R}^+)$ by letting $v(x) = w^*(Nx)$, and observe that

$$\|v\|^p_{L^p(\mathbb{R}^+)} = \frac{1}{N}\|w^*\|^p_{L^p(\mathbb{R}^+)} = \frac{1}{N}\|w\|^p_{L^p(\mathcal{G})},$$

while

$$\|v'\|^2_{L^2(\mathbb{R}^+)} = N\|(w^*)'\|^2_{L^2(\mathbb{R}^+)} \leq \frac{1}{N}\|w'\|^2_{L^2(\mathcal{G})}.$$

Therefore, we have

$$E(w, \mathcal{G}) = \frac{1}{2}\|w'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|w\|^p_{L^p(\mathcal{G})} \geq \frac{N}{2}\|v'\|^2_{L^2(\mathbb{R}^+)} - \frac{N}{p}\|v\|^p_{L^p(\mathbb{R}^+)} = N \cdot E(v, \mathbb{R}^+).$$

On the other hand,

$$\|v\|^2_{L^2(\mathbb{R}^+)} = \frac{1}{N}\|w^*\|^2_{L^2(\mathbb{R}^+)} = \frac{1}{N}\|w\|^2_{L^2(\mathcal{G})} = \frac{\nu}{N}$$

so that we can use (9) (with $\mu = \nu$) to estimate

$$E(v, \mathbb{R}^+) \geq -2^{2\beta}\theta_p \left(\frac{\nu}{N}\right)^{2\beta+1},$$

and (14) is established. □

Remark 2.2. As suggested by the proof, the bound in (14) can be written as $-N\theta_p 2^{2\beta}(\nu/N)^{2\beta+1}$. Due to (11), we can interpret this quantity as N times the energy of a half-soliton of mass ν/N on $\mathbb{R}^+$. In particular, when $N = 1$ this is the energy level in (11), while when $N = 2$ this is the energy level in (9). When $N = 3$ (or bigger), this is the lowest energy level achievable on the star-graph S_N (the union of N half-lines joined at the origin), among functions which respect the N-fold symmetry of the graph (see [2]).

Now we show that if a sequence of functions tends to vanish locally on $\mathcal{G}$, while dispersing a certain mass m at infinity along the halflines of $\mathcal{G}$, then the limit energy level is not smaller than the energy level of a soliton of mass m.

Lemma 2.3 (energy loss at infinity)**.** *Let $\mathcal{G}$ be a noncompact metric graph, and assume that a sequence $\{v_n\} \subset H^1(\mathcal{G})$ satisfies*

$$v_n \to 0 \quad \text{in } L^\infty_{loc}(\mathcal{G}), \qquad \lim_n \|v_n\|^2_{L^2(\mathcal{G})} = m \in (0, +\infty).$$

Then

$$\liminf_n E(v_n, \mathcal{G}) \geq -\theta_p m^{2\beta+1}.$$

Proof. Replacing v_n with $|v_n|$, we may assume that $v_n \geq 0$. Then, setting $M_n := \max_{\mathcal{G}} v_n$, two cases may occur according to where M_n is attained.
(i) If M_n is attained in the interior of a halfline $\mathcal{H}$ of $\mathcal{G}$, putting a coordinate $x \in [0, +\infty)$ on $\mathcal{H}$, and setting $\delta_n = v_n(0)$, we define on $\mathcal{H}$ the modified function

$$\widehat{v}_n(x) = \begin{cases} |x - \delta_n| & \text{if } x \in [0, 2\delta_n], \\ v_n(x - 2\delta_n) & \text{if } x > 2\delta_n, \end{cases}$$

and observe that $\widehat{v}_n \in H^1(\mathcal{H})$ and $\widehat{v}_n(0) = v_n(0) = \delta_n$. Then we set $\widehat{v}_n(x) = v_n(x)$ for $x \in \mathcal{G} \setminus \mathcal{H}$, so that $\widehat{v}_n \in H^1(\mathcal{G})$. Now, considering that $\widehat{v}_n(\delta_n) = 0$, that $\widehat{v}_n(x) \to 0$ as $x \to +\infty$ on $\mathcal{H}$, and that $\widehat{v}_n$ attains its maximum M_n somewhere along $\mathcal{H}$, we have

$$\#\{x \mid \widehat{v}_n(x) = t\} \geq 2 \qquad \forall t \in (0, M_n).$$

Therefore, applying Lemma 2.1 with $N = 2$, we have

$$E(\widehat{v}_n, \mathcal{G}) \geq -\theta_p m_n^{2\beta+1}, \quad m_n := \|\widehat{v}_n\|^2_{L^2(\mathcal{G})}.$$

But since $v_n \to 0$ in $L^\infty_{\text{loc}}(\mathcal{G})$, we have $\delta_n \to 0$, so that $m_n = m + o(1)$ as $n \to \infty$. Similarly, also $E(v_n, \mathcal{G}) = E(\widehat{v}_n, \mathcal{G}) + o(1)$, so that the last inequality yields

$$E(v_n, \mathcal{G}) \geq -\theta_p m^{2\beta+1} + o(1) \qquad \text{as } n \to \infty.$$

(ii) If M_n is *not* attained in the interior of any halfline of $\mathcal{G}$, then it is attained in the complementary set, which is a compact set independent of n: since $v_n \to 0$ in $L^\infty_{\text{loc}}(\mathcal{G})$, we must have $M_n \to 0$. But then a stronger inequality holds, namely

$$E(v_n, \mathcal{G}) \geq -\frac{1}{p} \int_{\mathcal{G}} |v_n(x)|^p \, dx \geq -\frac{M_n^{p-2}}{p} \int_{\mathcal{G}} |v_n(x)|^2 \, dx$$

which is $o(1)$ as $n \to \infty$. Thus, the claim is proved in either case. $\square$

Theorem 2.4. *Let $\{u_n\} \subset H^1_\mu(\mathcal{G})$ be a sequence such that $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$ and a.e. on $\mathcal{G}$, and let*

$$m := \mu - \|u\|^2_{L^2(\mathcal{G})} \in [0, \mu] \tag{16}$$

be the loss of mass in the limit. Then, letting $\Lambda := \liminf_n E(u_n, \mathcal{G})$, one of the following alternatives occurs, depending on the value of m:

(i) $m = 0$. Then $u_n \to u$ strongly *in $L^2(\mathcal{G}) \cap L^p(\mathcal{G})$, $u \in H^1_\mu(\mathcal{G})$ and $E(u, \mathcal{G}) \leq \Lambda$.*

(ii) $0 < m < \mu$. *Then*

$$\Lambda > \min\left\{-\theta_p \mu m^{2\beta},\, E(w,\mathcal{G})\right\}, \tag{17}$$

where $w \in H^1_\mu(\mathcal{G})$ *is the renormalized limit*

$$w(x) = \sqrt{\frac{\mu}{\mu - m}}\, u(x) \quad \forall x \in \mathcal{G}. \tag{18}$$

(iii) $m = \mu$. *Then* $u \equiv 0$ *and* $\Lambda \geq -\theta_p \mu^{2\beta+1}$.

Proof. If $m = 0$, then $u_n \to u$ strongly in $L^2(\mathcal{G})$ and therefore $u \in H^1_\mu(\mathcal{G})$. Moreover, since u_n is bounded in $H^1(\mathcal{G})$ hence in $L^\infty(\mathcal{G})$, $u_n \to u$ strongly also in $L^p(\mathcal{G})$. Then $E(u,\mathcal{G}) \leq \Lambda$ by semicontinuity of the kinetic term.

If, on the other hand, $m = \mu$, then $u \equiv 0$ so that $u_n \to 0$ in $L^\infty_{\mathrm{loc}}(\mathcal{G})$. In this case, $\Lambda \geq -\theta_p \mu^{2\beta+1}$ follows from Lemma 2.3, applied with $v_n = u_n$ and $m = \mu$.

Finally, assume $0 < m < \mu$. According to the Brezis–Lieb Lemma ([6]) we may split

$$E(u_n,\mathcal{G}) = E(u_n - u,\mathcal{G}) + E(u,\mathcal{G}) + o(1) \quad \text{as } n \to \infty. \tag{19}$$

Now, from $u_n \rightharpoonup u$ in $L^2(\mathcal{G})$ we have

$$\|u_n - u\|^2_{L^2(\mathcal{G})} = \|u_n\|^2_{L^2(\mathcal{G})} + \|u\|^2_{L^2(\mathcal{G})} - 2\langle u_n, u\rangle_{L^2(\mathcal{G})} \to \mu - \|u\|^2_{L^2(\mathcal{G})} = m$$

as $n \to \infty$, hence Lemma 2.3 applied with $v_n = u_n - u$ yields

$$\liminf_n E(u_n - u,\mathcal{G}) \geq -\theta_p m^{2\beta+1}.$$

Thus, taking the liminf in (19), we find

$$\Lambda \geq -\theta_p m^{2\beta+1} + E(u,\mathcal{G}).$$

On the other hand, since $p > 2$ and $u \not\equiv 0$, we have

$$E(w,\mathcal{G}) = \frac{1}{2}\frac{\mu}{\mu - m}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\left(\frac{\mu}{\mu - m}\right)^{\frac{p}{2}}\|u\|^p_{L^p(\mathcal{G})} < \frac{\mu}{\mu - m}E(u,\mathcal{G})$$

and thus from the previous inequality we obtain

$$\Lambda > -\theta_p m^{2\beta+1} + \left(1 - \frac{m}{\mu}\right)E(w,\mathcal{G}) = -\left(\frac{m}{\mu}\right)\theta_p \mu m^{2\beta} + \left(1 - \frac{m}{\mu}\right)E(w,\mathcal{G}),$$

and (17) follows. □

Remark 2.5. The typical application of the previous theorem is when u_n is a minimizing sequence for some variational problem, i.e. $\Lambda = \inf_{w \in V} E(w,\mathcal{G})$ where $V \subseteq H^1_\mu(\mathcal{G})$ is a certain class of functions of mass μ. For instance, in [4] it was proved that, when $V = H^1_\mu(\mathcal{G})$, the condition $\Lambda < -\theta_p \mu^{2\beta+1}$ (i.e. the strict inequality in the upper bound in (12)) is sufficient for the existence of a ground state of mass μ on $\mathcal{G}$. Now Theorem 2.4 yields a short proof of this result. Indeed, the assumption on Λ immediately rules out case (iii), and (17) is also impossible since w is an admissible competitor: then case (i) must occur, and the limit u is a ground state.

3 Energy minimization with a localized maximum

In this section we investigate the existence of a function with least energy, among all functions of prescribed mass that achieve their maximum on a prescribed bounded edge of $\mathcal{G}$.

Throughout, $\mathcal{G}$ denotes a noncompact, connected metric graph, having at least one bounded edge (if every edge of $\mathcal{G}$ is a halfline, the existence of bound states has already been investigated in [2]). We also denote by e a *fixed* bounded edge of $\mathcal{G}$, with vertices v_1 and v_2, with the only assumption that e is maximal, i.e. $\deg(\mathrm{v}_1) \neq 2$ and $\deg(\mathrm{v}_2) \neq 2$ (there is no loss of generality in this assumption, since any vertex v of degree two can a priori be *eliminated* from any metric graph, by melting the two edges incident at v into a single edge). Observe that the two conditions $\deg(\mathrm{v}_1) = 1$ and $\deg(\mathrm{v}_2) = 1$ cannot be satisfied simultaneously, otherwise e would be an isolated edge and $\mathcal{G}$, being connected, would be compact. Thus, swapping if necessary v_1 with v_2, we can concretely assume that

$$\deg(\mathrm{v}_1) \neq 2, \qquad \deg(\mathrm{v}_2) \geq 3. \tag{20}$$

If $\deg(\mathrm{v}_1) = 1$ then e is called a *terminal edge* of $\mathcal{G}$. Finally, we do not exclude the case where e forms a *self loop*: in other words, it may well happen that $\mathrm{v}_1 = \mathrm{v}_2$.

We shall focus our attention on the class of functions

$$V = \left\{u \in H^1(\mathcal{G}) \ : \ \|u\|_{L^\infty(e)} = \|u\|_{L^\infty(\mathcal{G})}\right\}, \tag{21}$$

characterized by the fact that their L^∞ norm is achieved on the edge e, and, for every $\mu > 0$, on the subclass

$$V_\mu = V \cap H^1_\mu(\mathcal{G}) \tag{22}$$

of functions satisfying the additional mass constraint (2).

Remark 3.1. The set V is closed in the weak topology of $H^1(\mathcal{G})$. Indeed, if $u_n \in V$ and $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$, then by semicontinuity

$$\|u\|_{L^\infty(\mathcal{G})} \leq \liminf_n \|u_n\|_{L^\infty(\mathcal{G})} = \liminf_n \|u_n\|_{L^\infty(e)} = \|u\|_{L^\infty(e)}$$

(the last equality follows since $u_n \to u$ uniformly on e).

We shall study the following minimization problem: find $u \in V_\mu$ such that

$$E(u, \mathcal{G}) = \inf_{v \in V_\mu} E(v, \mathcal{G}). \tag{23}$$

The first crucial step is to estimate the infimum in (23) in terms of μ. Now we show that, for large μ, this infimum is very close to the ground-state energy level on $\mathbb{R}$ given by (9). Moreover, if e is a terminal edge, then the bound can be improved, and the infimum approaches the ground-state energy level on $\mathbb{R}^+$ given by (11).

Lemma 3.2. *For every $\varepsilon > 0$ there exists μ_ε (depending only on ε and on the length of the edge e) such that, for all $\mu \geq \mu_\varepsilon$,*

$$\inf_{v \in V_\mu} E(v, \mathcal{G}) \leq -\theta_p (1 - \varepsilon) \mu^{2\beta+1}. \tag{24}$$

Moreover, if e is a terminal edge, then the previous inequality can be replaced with

$$\inf_{v \in V_\mu} E(v, \mathcal{G}) \leq -\theta_p (1 - \varepsilon) 2^{2\beta} \mu^{2\beta+1}. \tag{25}$$

Proof. Let $\phi \in H^1(\mathbb{R})$ denote the soliton of unitary mass centered at the origin. Since ϕ solves the minimization problem (9) when $\mu = 1$, $E(\phi, \mathbb{R}) = -\theta_p$ and hence, given $\varepsilon > 0$, by standard density arguments there exists $\phi_\varepsilon \in H^1(\mathbb{R})$ with *compact support*, such that

$$E(\phi_\varepsilon, \mathbb{R}) \le -(1-\varepsilon)\theta_p, \qquad \|\phi_\varepsilon\|^2_{L^2(\mathbb{R})} = 1.$$

Then, recalling (10), for $\mu > 0$ we define the following rescaled version of ϕ_ε

$$v_\mu(x) = \mu^\alpha \phi_\varepsilon(\mu^\beta x), \quad x \in \mathbb{R}, \quad \alpha = \frac{2}{6-p}$$

(this is the natural scaling for the NLS energy, see Remark 2.2 in [4]), and one can check that

$$\|v_\mu\|^2_{L^2(\mathbb{R})} = \mu \|\phi_\varepsilon\|^2_{L^2(\mathbb{R})} = \mu, \qquad E(v_\mu, \mathbb{R}) = \mu^{2\beta+1} E(\phi_\varepsilon, \mathbb{R}) \le -(1-\varepsilon)\theta_p \mu^{2\beta+1}.$$

Now, if μ is large enough, the diameter of the support of v_μ becomes smaller than the length of the edge e: thus, we can fit v_μ on e (so that it vanishes at v_1 and v_2) and, setting it equal to zero on $\mathcal{G} \setminus e$, we may regard v_μ as a function in $H^1_\mu(\mathcal{G})$. This produces an admissible competitor in (24) and, since in the last inequality we can interpret $E(v_\mu, \mathbb{R})$ as $E(v_\mu, \mathcal{G})$, (24) is proved.

Finally, assume e is a terminal edge (according to (20), e is attached to $\mathcal{G}$ at v_2 while v_1 is the tip of the edge, with $\deg(\mathrm{v}_1) = 1$). The previous argument remains valid, but a variant of it gives a better bound. Indeed, now one can start with $v_{2\mu} \in H^1(\mathbb{R})$ instead of v_μ, and exploit the fact that $v_{2\mu}(-x) = v_{2\mu}(x)$, so that on $\mathbb{R}^+$ one has

$$\|v_{2\mu}\|^2_{L^2(\mathbb{R}^+)} = \frac{1}{2}\|v_{2\mu}\|^2_{L^2(\mathbb{R})} = \mu, \qquad E(v_{2\mu}, \mathbb{R}^+) = \frac{1}{2} E(v_{2\mu}, \mathbb{R}) \le -(1-\varepsilon)\theta_p 2^{2\beta} \mu^{2\beta+1}$$

(the same bound as in (25)). Then, for large μ, one can *restrict* $v_{2\mu}$ to $\mathbb{R}^+$, fit it on e so that it vanishes at v_2, and set it equal to zero on $\mathcal{G} \setminus e$ as before (but now $v_{2\mu}$, once fitted on $\mathcal{G}$, will satisfy a Neumann condition at v_1, rather than Dirichlet).

□

Theorem 3.3. *There exists a mass threshold $\overline{\mu}$ such that, for every $\mu \ge \overline{\mu}$, the minimization problem* (23) *has a solution $u \in H^1_\mu(\mathcal{G})$.*

Proof. The mass threshold will be obtained from Lemma 3.2 letting $\overline{\mu} := \mu_\varepsilon$, for a suitable choice of $\varepsilon \in (0, 1/2)$ to be specified later. Now, given $\mu \ge \mu_\varepsilon$, let $\{u_n\} \subset V_\mu$ be a minimizing sequence for problem (23), that is

$$\lim_{n\to\infty} E(u_n, \mathcal{G}) = \Lambda := \inf_{v \in V_\mu} E(v, \mathcal{G}).$$

Since $\varepsilon < 1/2$, (24) entails $\Lambda < -\theta_p \mu^{2\beta+1}/2$, so we may assume that $E(u_n, \mathcal{G}) \le -\theta_p \mu^{2\beta+1}/2$ for every n, which, using (1) and rearranging terms, can be written as

$$\frac{\theta_p}{2}\mu^{2\beta+1} + \frac{1}{2}\|u_n'\|^2_{L^2(\mathcal{G})} \le \frac{1}{p}\|u_n\|^p_{L^p(\mathcal{G})}. \tag{26}$$

This will be used in two ways. First, ignoring u_n' and using $u_n \in V_\mu$, we have

$$\frac{p\theta_p}{2}\mu^{2\beta+1} \le \|u_n\|^p_{L^p(\mathcal{G})} \le \|u_n\|^{p-2}_{L^\infty(\mathcal{G})} \|u_n\|^2_{L^2(\mathcal{G})} = \mu \|u_n\|^{p-2}_{L^\infty(e)},$$

which (recalling (10)) we may record for future usage as

$$C_p\mu^{\beta+1} \le \|u_n\|^2_{L^\infty(e)}. \tag{27}$$

Second, if we drop the first summand in (26), and use (7) combined with $\|u_n\|^2_{L^2} = \mu$ to estimate the right hand side, we obtain

$$\|u_n'\|^2_{L^2(\mathcal{G})} \le C_p\mu^{2\beta+1}. \tag{28}$$

Therefore $\{u_n\}$ is bounded in $H^1(\mathcal{G})$, and we may assume that $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$ and $u_n \to u$ pointwise a.e. on $\mathcal{G}$, for some $u \in H^1(\mathcal{G})$. Thus Theorem 2.4 applies, and we shall prove that case (i) occurs.

Since $u_n \to u$ uniformly on the edge e, recalling (27) we have

$$\|u\|^2_{L^\infty(\mathcal{G})} \ge \|u\|^2_{L^\infty(e)} = \lim_n \|u_n\|^2_{L^\infty(e)} \ge C_p\mu^{\beta+1}.$$

In particular, $u \not\equiv 0$, and this (defining m as in (16)) rules out case (iii) of Theorem 2.4. Moreover, using (27), (8), (16) and (28), we have

$$\begin{aligned} C_p\mu^{\beta+1} &\le \|u\|^2_{L^\infty(\mathcal{G})} \le 2\|u\|_{L^2(\mathcal{G})}\|u'\|_{L^2(\mathcal{G})} = 2\sqrt{\mu - m}\ \|u'\|_{L^2(\mathcal{G})} \\ &\le 2\sqrt{\mu - m}\ \liminf_n \|u_n'\|_{L^2(\mathcal{G})} \le C_p'\sqrt{\mu - m}\ \mu^{\beta+\frac{1}{2}}. \end{aligned}$$

Thus we find $\mu - m \ge \delta_p\mu$ for some universal constant $\delta_p > 0$, that is

$$m \le (1 - \delta_p)\mu. \tag{29}$$

Now we now show that (17) is violated. Indeed, since $u \in V$ by Remark 3.1, the renormalized function w defined in (18) belongs to V_μ, and therefore $\Lambda \le E(w, \mathcal{G})$, so that (17) simplifies to $\Lambda > -\theta_p\mu\, m^{2\beta}$, which combined with (24) gives

$$-\theta_p(1 - \varepsilon)\mu^{2\beta+1} > -\theta_p\mu\, m^{2\beta}.$$

But this is incompatible with (29), if ε is small enough. Summing up, for small ε, we are in case (i) of Theorem 2.4, that is, u is a solution to problem (23). □

4 Bound states

Now we prove that, for large μ, every solution to the minimization problem (23) achieves its maximum *only* on the edge e, and nowhere else.

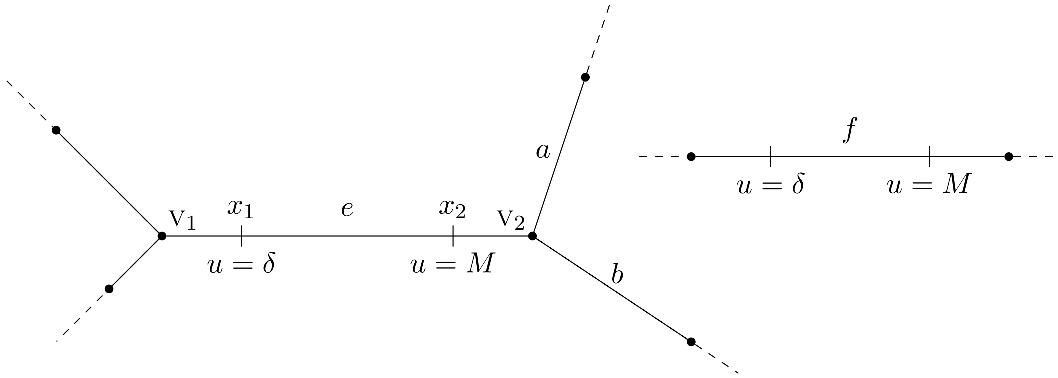
Remark 4.1. If $u \in V_\mu$ is a solution to (23), then also $|u|$ is a solution, since $|u| \in V_\mu$ and $E(|u|, \mathcal{G}) = E(u, \mathcal{G})$. In fact, as we will prove later, for large μ every solution is either strictly positive o strictly negative on $\mathcal{G}$.

Lemma 4.2. *Let $u \in V_\mu$ be a solution to (23), found according to Theorem 3.3. If μ is large enough (depending only on p and on the graph $\mathcal{G}$), then*

$$\|u\|_{L^\infty(e)} > \|u\|_{L^\infty(\mathcal{G}\setminus e)}. \tag{30}$$

Proof. Considering $|u|$ instead of u, by Remark 4.1 we may assume that $u \ge 0$ (note that (30) depends only on $|u|$). Let $M = \|u\|_{L^\infty(e)}$. Since $u \in V_\mu$, we clearly have $M \ge \|u\|_{L^\infty(\mathcal{G}\setminus e)}$. To prove the claim, we assume that

$$M = \|u\|_{L^\infty(\mathcal{G}\setminus e)} \tag{31}$$



and we obtain a contradiction, if μ is large enough. To start with, we certainly assume $\mu \geq \overline{\mu}$ (the mass threshold of Theorem 3.3) but, throughout the proof, a possibly larger mass threshold will arise.

The proof is divided into two steps.

Step 1). Denoting by $\mathcal{B}$ the set of all the bounded edges of $\mathcal{G}$, and letting

$$\delta := \max_{h \in \mathcal{B}} \min_{x \in h} u(x) \qquad (0 \leq \delta \leq M), \tag{32}$$

we will prove a nontrivial lower bound on the number of preimages in $u^{-1}(t)$, for every value $t \in (\delta, M)$ achieved by u on $\mathcal{G}$. The precise lower bound depends on whether e is a terminal edge or not. More precisely, recalling (20), we will show that

$$\deg(\mathrm{V}_1) = 1 \qquad \Rightarrow \qquad \#\{x \in \mathcal{G} \mid u(x) = t\} \geq 2 \quad \text{for every } t \in (\delta, M), \tag{33}$$

while

$$\deg(\mathrm{V}_1) \geq 3 \qquad \Rightarrow \qquad \#\{x \in \mathcal{G} \mid u(x) = t\} \geq 3 \quad \text{for every } t \in (\delta, M) \tag{34}$$

We point out that $\delta \geq 0$ since $u \geq 0$. Moreover we may assume $\delta < M$, otherwise the claims are trivial.

Now, (31) means that u (which already attains its maximum M on e) achieves the value M also on *another* edge f (possibly at V_1 or V_2, if e and f share a vertex). Moreover, since u is continuous, in each edge g (bounded or not) where u attains M, u also attains *every intermediate value* $t \in (\delta, M)$: indeed, if g is a bounded edge this follows immediately from (32), while if g is a halfline, then $u(x) \to 0$ as x tends to infinity along g. In particular, for every $t \in (\delta, M)$, u has at least two preimages on $\mathcal{G}$ (one on e and one on f), and (33) is established (regardless of $\deg(\mathrm{V}_1)$).

Now assume that e is not a terminal edge, i.e. $\deg(\mathrm{V}_1) \geq 3$, and consider two points $x_1, x_2 \in e$ with $u(x_1) = \delta$ and $u(x_2) = M$. By swapping (if necessary) V_1 and V_2, we may assume that, on the edge e, x_2 lies between x_1 and V_2, as shown in the picture (of course, we do not exclude that $x_2 = \mathrm{V}_2$ or $x_1 = \mathrm{V}_1$).

Let a, b denote two edges (other than e) emanating from V_2: if $a \neq f$ then, for every value $t \in (\delta, M)$, a *third preimage* can be found (in addition to those already found on e in the interval $[x_1, x_2]$ and, similarly, on f) by travelling on e from x_2 (where $u = M$) to V_2 and, if $u(\mathrm{V}_2) > t$, travelling further along a, until $u = t$ (if a is a bounded edge then, somewhere on a, $u = t$ by (32), while if a is a halfline, then u achieves t since $u(\mathrm{V}_2) > t$ while $u \to 0$ at infinity along a). Of course, the same can be done if $b \neq f$. Finally, if $f = a = b$, then f is necessarily a bounded edge, that forms a self-loop attached at V_2:

since on this loop u attains the values M and δ, every intermediate value $t \in (\delta, M)$ is attained *at least twice* on f, so that in this case a third preimage can be found on f itself. Therefore, also (34) is established.

Step 2) Now, in order to apply Lemma 2.1, we would like the inequalities in (33) and (34) to be valid for every $t \in (0, M)$, without the restriction $t > \delta$. This can be achieved by *extending* u to a graph $\mathcal{G}'$ larger than $\mathcal{G}$, as follows (if by any chance $\delta = 0$, then no extension is necessary: our argument, however, formally works also when $\delta = 0$).

Given $\lambda > 0$ (to be specified later), let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$, with the addition of *two extra edges* e_1 and e_2, both of length λ, and both with one endpoint attached to $\mathcal{G}$ at x_1 (recall $x_1 \in e$ and $u(x_1) = \delta$). Putting a coordinate $s \in [0, \lambda]$ on each e_i (so that e_i is attached to x_1 at $s = 0$), we extend u to $\mathcal{G}'$ letting

$$u(s) = \frac{\delta}{\lambda}(\lambda - s), \quad \forall s \in e_1 \sim [0, \lambda], \quad \forall s \in e_2 \sim [0, \lambda].$$

Clearly, $u \in H^1(\mathcal{G}')$. Moreover, u now achieves every value $t \in (0, \delta)$ no less than three times on $\mathcal{G}'$: once on e_1, once on e_2, and at least once on the original graph $\mathcal{G}$ (since $u \to 0$ along every halfline of $\mathcal{G}$). This, combined with (33) and (34), gives

$$\begin{aligned} \deg(\mathrm{v}_1) = 1 \quad &\Rightarrow \quad \#\left\{x \in \mathcal{G}' \mid u(x) = t\right\} \geq 2 \quad \text{for almost every } t \in (0, M), \\ \deg(\mathrm{v}_1) \geq 3 \quad &\Rightarrow \quad \#\left\{x \in \mathcal{G}' \mid u(x) = t\right\} \geq 3 \quad \text{for almost every } t \in (0, M). \end{aligned}$$

Now Lemma (2.1) can be applied to u on $\mathcal{G}'$ in either case (choosing $N = 2$ or $N = 3$ according to $\deg(\mathrm{v}_1)$), thus obtaining

$$\deg(\mathrm{v}_1) = 1 \quad \Rightarrow \quad E(u, \mathcal{G}') \geq -\theta_p \, \nu^{2\beta+1}, \tag{35}$$

$$\deg(\mathrm{v}_1) \geq 3 \quad \Rightarrow \quad E(u, \mathcal{G}') \geq -\theta_p \left(\frac{2}{3}\right)^{2\beta} \nu^{2\beta+1} \tag{36}$$

where

$$\nu := \|u\|^2_{L^2(\mathcal{G}')} = \|u\|^2_{L^2(\mathcal{G})} + 2\frac{\delta^2}{\lambda^2} \int_0^\lambda (\lambda - s)^2 \, ds \leq \mu + \lambda\delta^2. \tag{37}$$

Moreover,

$$E(u, \mathcal{G}') = E(u, \mathcal{G}) + 2\, E(u, e_1) \leq E(u, \mathcal{G}) + 2 \times \frac{1}{2}\|u'\|^2_{L^2(e_1)} = E(u, \mathcal{G}) + \frac{\delta^2}{\lambda},$$

so that $E(u, \mathcal{G}) \geq E(u, \mathcal{G}') - \delta^2/\lambda$. Plugging this inequality into (35) and (36), and then using (37), yields

$$\deg(\mathrm{v}_1) = 1 \quad \Rightarrow \quad E(u, \mathcal{G}) \geq -\theta_p \, (\mu + \lambda\delta^2)^{2\beta+1} - \frac{\delta^2}{\lambda}, \tag{38}$$

$$\deg(\mathrm{v}_1) \geq 3 \quad \Rightarrow \quad E(u, \mathcal{G}) \geq -\theta_p \left(\frac{2}{3}\right)^{2\beta} (\mu + \lambda\delta^2)^{2\beta+1} - \frac{\delta^2}{\lambda}. \tag{39}$$

Moreover, since $E(u, \mathcal{G})$ coincides with the infimum in (24) (and in (25)), for every small $\varepsilon > 0$ (assuming that $\mu \geq \mu_\varepsilon$) we can use Lemma (3.2) to estimate $E(u, \mathcal{G})$ from above, and plug (24) and (25), respectively, into (39) and (38). Reversing signs, this gives

$$\begin{aligned} \deg(\mathrm{v}_1) = 1 \quad &\Rightarrow \quad \theta_p(1 - \varepsilon)2^{2\beta}\mu^{2\beta+1} \leq \theta_p \, (\mu + \lambda\delta^2)^{2\beta+1} + \frac{\delta^2}{\lambda}, \\ \deg(\mathrm{v}_1) \geq 3 \quad &\Rightarrow \quad \theta_p(1 - \varepsilon)\mu^{2\beta+1} \leq \theta_p \left(\frac{2}{3}\right)^{2\beta} (\mu + \lambda\delta^2)^{2\beta+1} + \frac{\delta^2}{\lambda}. \end{aligned}$$

Of these two estimates, the latter (being weaker) is valid in either case: therefore, from now on, we will focus on the latter, regardless of $\deg(\mathrm{v}_1)$. Now, to estimate δ, observe that for every bounded edge h of $\mathcal{G}$ we have

$$\left(\min_{x\in h} u(x)\right)^2 \le \frac{1}{\operatorname{length}(h)} \int_h u(x)^2\, dx \le \frac{\mu}{\ell},$$

where $\ell > 0$ is the length of the shortest edge of $\mathcal{G}$ (i.e. a constant that depends only on $\mathcal{G}$). Therefore, from (32) we have $\delta^2 \le \mu/\ell$, and hence

$$\theta_p(1-\varepsilon)\mu^{2\beta+1} \le \theta_p \left(\frac{2}{3}\right)^{2\beta} \mu^{2\beta+1}(1+\lambda/\ell)^{2\beta+1} + \frac{\mu}{\lambda\ell}.$$

Now, regardless of μ, we can fix ε and λ small enough, so that the coefficient of $\mu^{2\beta+1}$ on the left is *strictly bigger* than the corresponding coefficient on the right: then, a contradiction is easily found if μ is large enough. ☐

As a consequence of this lemma, we can prove that any solution $u \in V_\mu$ to (23) is in fact a bound state.

Theorem 4.3. *Let $u \in V_\mu$ be a solution to* (23)*, found according to Theorem 3.3. If μ is large enough (depending only on p and on the graph $\mathcal{G}$), then u is a bound state of mass μ, i.e. u satisfies* (2) *and* (4) *for a suitable $\lambda > 0$. Moreover, either $u > 0$ or $u < 0$ on $\mathcal{G}$.*

Proof. Since $u \in V_\mu \subset H^1_\mu(\mathcal{G})$, (2) is obvious. Moreover, by Lemma 4.2, we may rely on (30) which, due to the *strict* inequality, is stable under perturbations of u that are sufficiently small in the L^∞ norm on $\mathcal{G}$. But since $H^1(\mathcal{G}) \hookrightarrow L^\infty(\mathcal{G})$, we see that (30) is stable also under small perturbations of u in the H^1 norm: in other words, recalling (21) and (22), u lies in the *interior* of V, so that u is not only a global minimizer in V_μ, but also a local minimizer in $H^1_\mu(\mathcal{G})$. Then u is a critical point of the NLS energy subject to (2), and (4) (with λ as a Lagrange multiplier) follows by standard arguments.

The fact that $\lambda > 0$ is due to the non-compactness of $\mathcal{G}$ (i.e. to the presence of at least one halfline): indeed, it is well known that any solution of (5) in $L^2(\mathbb{R}^+)$ is (the restriction of) a soliton, which forces $\lambda > 0$.

Finally, by Remark 4.1, all we have proved for u is valid also for $|u|$, including the analogues of (5) and (6). Then (since $|u|$ is not identically zero) $|u| > 0$ can be proved exactly as in [3] (see (iii) of Proposition 3.3). This shows that u does not vanish on $\mathcal{G}$, therefore either $u > 0$ or $u < 0$ since $\mathcal{G}$ is connected. ☐

Corollary 4.4. *Let $\mathcal{G}$ be a noncompact metric graph, having k bounded edges. If μ is large enough, then $\mathcal{G}$ admits at least k positive bound states of mass μ.*

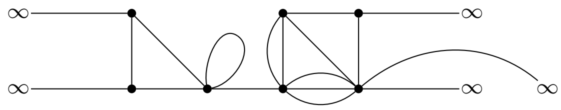
Proof. Since the edge e (that determines the set V defined in (21)) can be chosen in k different ways, for large μ one obtains k bound states (and replacing each u with $-u$, we may assume that each of them is positive). Since each of them satisfies (30) relative to the corresponding edge e, we see that these k bound states are, in all respects, distinct. ☐

5 Some examples

In this section we discuss some examples of metric graphs, in order to illustrate how our results can be concretely applied.

In the pictures, the unbounded edges (halflines) of the graph can be recognized at a glance, since their terminal edge (the "point at infinity" of the halfline) is labeled with the symbol "∞". The bullet symbol "$\bullet$", in contrast, is used to label the ordinary vertices of the graph.

Example 5.1. The following graph

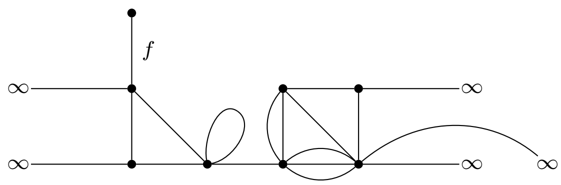


consists of 5 unbounded edges (halflines) and 13 bounded edges. Therefore, according to Corollary 4.4, for μ large enough there at least 13 (positive) bound states of mass μ.

On the other hand, it follows from the results of [3] (see in particular Theorem 2.5 and the remark after Theorem 2.3 therein) that, regardless of μ, this graph admits *no ground state*: as a consequence, none of these 13 bound states is a ground state.

More generally (replacing 13 with the actual number of bounded edges) this applies to any graph $\mathcal{G}$ satisfying the so called "assumption (H)" introduced in [3], a topological assumption which rules out ground states of any prescribed mass.

Example 5.2. The following graph



is obtained from the graph of Example 5.1 by adding one *terminal* edge (the one labelled "f" in the picture), so that there are 14 bounded edges and therefore, according to Corollary 4.4, for μ large enough $\mathcal{G}$ has at least 14 bound states of mass μ. Here, however, in contrast with Example 5.1, for large μ the presence of a terminal edge guarantees the existence of a ground state of mass μ (see Proposition 4.1 of [4]).

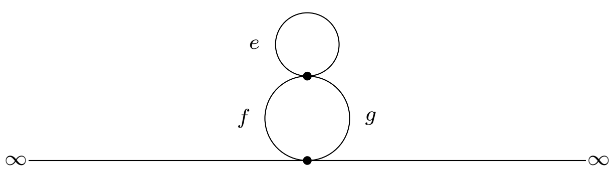
Now let us denote by $\{u_e\}$ the 14 bound states (one for every bounded edge e of $\mathcal{G}$) provided by Corollary 4.4, and recall that u_e achieves its maximum on the edge e.

With this notation we claim that, among the bound states $\{u_e\}$, *at most one* can be a ground state, the unique candidate being u_f (the one that achieves its maximum on the terminal edge f).

Indeed, when μ is large we can apply Lemma 3.2 with $e = f$: since f is a terminal edge and u_f achieves the infimum in (25), in particular we obtain that $E(u_f, \mathcal{G}) < -\theta_p \mu^{2\beta+1}$.

Now consider any bounded edge $e \neq f$, and pick a point $x_0 \in e$ where u_e achieves its maximum. Since $x_0 \notin f$, exploiting the topology of $\mathcal{G}$ (more precisely, of $\mathcal{G} \setminus f$) and the fact $u_e(x) \to 0$ as $x \to \infty$ along any halfline, the reader can easily check that (13) is satisfied if $w = u_e$ and $N = 2$. Then (14) gives $E(u_e, \mathcal{G}) \geq -\theta_p \mu^{2\beta+1}$, so that $E(u_e, \mathcal{G}) > E(u_f, \mathcal{G})$ and hence u_e cannot be a ground state.

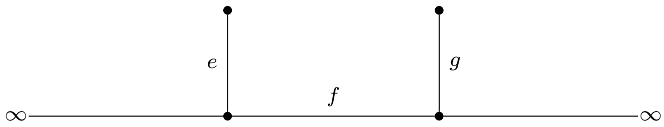
Example 5.3. Consider the graph



with two halflines and three bounded edges e, f and g (note that e forms a self-loop, while f and g are assumed to have the same length).

For large μ, Corollary 4.4 yields three positive bound states u_e, u_f and u_g, each achieving its maximum on the corresponding edge. Of these bound states, however, only u_e is a ground state: indeed, it is known (see Example 2.4 in [3]) that for every mass μ this graph has a unique positive ground state, obtained by fitting a soliton of mass μ on $\mathcal{G}$ with the maximum in the middle of the edge e (thus respecting the symmetry of the graph). Therefore, in this graph, the minimization problem (23) (whose solution is of course u_e) is equivalent to the minimization of the NLS energy with the mass constraint only, since the additional requirement that u achieves its maximum on e is a *natural constraint*, already satisfied by the ground state. More generally, these considerations remain valid for all the graphs discussed in Example 2.4 of [3].

Example 5.4. The graph



consists of two halflines and three bounded edges e, f and g, hence for large μ one can find three bound states u_e, u_f and u_g, each achieving its maximum on the corresponding edge. We claim that, if e and g have the same length, then both u_e and e_g are ground states, while u_f is not.

Indeed, since the two edges e and g are terminal, for large μ there is at least one ground state of mass μ, with an energy level *lower* than the soliton threshold given by (9) (see [4], Proposition 4.1).

On the other hand, since u_f achieves its maximum on f and f is attached to a halfline at each endpoint, it is clear that (13) holds true with $w = u_f$ and $N = 2$: then, we see from (14) that $E(u_f, \mathcal{G})$ cannot be smaller than the energy threshold in (9), so that u_f cannot be a ground state. Moreover, the same applies to any nonnegative function $w \in H^1_\mu(\mathcal{G})$ that achieves its maximum on a halfline of $\mathcal{G}$, so that any ground state must necessarily achieve its maximum on e or on g. Since u_e (resp. u_g) has least energy among all functions in $H^1_\mu(\mathcal{G})$ with the maximum on e (resp. on g), we see that one of them is a ground state: finally, by the symmetry of the graph, $E(u_e, \mathcal{G}) = E(u_g, \mathcal{G})$, so that both functions are a ground state.

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