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Nonlinear dynamics on branched structures and networks

Abstract. In these lectures we review on a recently developed line of research, concerning the existence of ground states with prescribed mass (i.e. L^2-norm) for the *focusing* nonlinear Schrödinger equation with a power nonlinearity, *on noncompact quantum graphs.*
Nonlinear dynamics on graphs has rapidly become a topical issue with many physical applications, ranging from nonlinear optics to Bose-Einstein condensation. Whenever in a physical experiment a ramified structure is involved (e.g. in the propagation of signals, in a circuit of quantum wires or in trapping a boson gas), it can prove useful to approximate such a structure by a *metric graph*, or *network*.
For the Schrödinger equation it turns out that the *sixth* power in the nonlinear term of the energy (corresponding to the *quintic* nonlinearity in the evolution equation) is *critical* in the sense that below that power the constrained energy is lower bounded irrespectively of the value of the mass (*subcritical case)*. On the other hand, if the nonlinearity power equals six, then the lower boundedness depends on the value of the mass: below a *critical mass*, the constrained energy is lower bounded, beyond it, it is not. For powers larger than six the constrained energy functional is never lower bounded, so that it is meaningless to speak about ground states (*supercritical case*). These results are the same as in the case of the nonlinear Schrödinger equation on the real line. In fact, as regards the existence of ground states, the results for systems on graphs differ, in general, from the ones for systems on the line even in the subcritical case: in the latter case, whenever the constrained energy is lower bounded there always exist ground states (the *solitons*, whose shape is explicitly known), whereas for graphs the existence of a ground state is not guaranteed.
More precisely, we show that the existence of such constrained ground states is strongly conditioned by the topology of the graph. In particular, in the subcritical case we single out a topological hypothesis that prevents a graph from having ground states for every value of the mass. For the critical case, our results show a phenomenology much richer than the analogous on the line: if some topological assumptions are fulfilled, then there may exist a whole interval of masses for which a ground state exist. This behaviour is highly non-standard for L^2-critical nonlinearities.

Keywords. Minimization, metric graphs, critical growth, nonlinear Schrödinger Equation.

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1 - Introduction: why dynamics on networks?

Evolution on *metric graphs*, or *networks*, is a mathematical model used in order to approximate the dynamics of systems located on *branched* spatial structures. Such structures are characterized by the fact that locally only one direction is important, except for some points where several directions are available. Such special points in the structure are called *vertices, nodes* or *bifurcation points*, and the (possible) connections between two of them are called *edges*.

The research on dynamics of networks started in 1953 with the seminal work by Ruedenberg and Scherr ([55]) where the dynamics of valence electrons in organic molecules was approximated by defining a suitable Schrödinger operator on the molecular bonds, treated as edges of a metric graph. This paper initiated the research line nowadays known as *evolution on quantum graphs* (see the milestone paper by Kostrykin and Schrader [44] and the treatise by Berkolaiko and Kuchment [17]). By definition, a quantum graph is a network, made of edges and vertices, on which functions are defined and a linear differential operator acts.

More recently, several papers appeared, in which a nonlinear evolution on branched structure was proposed ([19, 1, 2, 3, 5, 22, 52, 51, 35, 50, 59, 56, 23, 61, 58, 37]). The first systematic study of nonlinear dynamics on networks is contained in [11], however it is only in the last two years that a great deal of efforts in this direction has been carried out.

Here we focus on the problem of establishing the existence of ground states for the Nonlinear Schrödinger (NLS) Equation on metric graphs. In particular, we shall review the results given in [8, 9, 10].

The problem generalizes along two directions the issue of finding the ground state of a Bose-Einstein condensate: first, the chosen domain is not standard, as it consists of a network instead of a three-dimensional regular region, or a disc, or a "cigar"; second, the nonlinearity we consider in the energy functional displays an arbitrary power, whilst typically for condensates in the so-called Gross-Pitaevskii regime the quartic power emerges as the effective one. Furthermore, we limit our analysis to the *focusing* case, i.e. the case in which the net effect of the nonlinearity on the time evolution is the concentration of the wave packets.

The present note is organized as follows: in the rest of the introduction we give a historical overview on the mean-field limit for a many-boson system and draw a link between the problem of minimizing the constrained energy, possibly on graphs, and the Bose-Einstein condensation; we finally give the basic definitions and notation and state the problem we shall focus on. In Section 2 we illustrate in a rather formal way the role of the so-called *critical nonlinearity power*, then we write down the Euler-Lagrange equation and the Kirchhoff condition. In Section 3 we give some well-known and some others less known examples in which the problem is solved, trying to convey some general ideas. Section 4 is devoted to the introduction of a topological hypothesis (Assumption (H)) that is the core of the key nonexistence result given in Theorem 5.1, to which Section 5 is devoted together with a review of rearrangement theory (for non-experts). In Section 6 we give many examples in which Assumption (H) is not satisfied and show how to prove existence of ground states through a technique of *graph surgery*. The point we stress here is that in all cases where topological information is not sufficient to solve the problem, analysis needs to be carried out case by case; to this aim, Theorem 6.2 with its operative Corollary 6.1 can give a great help, as it states that in order to ensure the existence of a ground state it is sufficient to find a state that *does better than a soliton on the line*, i.e. a state whose energy level is lower than the level of the solution to the same problem on the line. Finally, in Section 7 we treat the case of the critical power nonlinearity, where the role of topology overwhelms that of the metric, at least for simple cases, but the lack of compactness of minimizing sequences is much more serious. The analysis becomes thus more involved but, as a result, graphs can be classified in four disjoint categories, for each of those we give an exhaustive result (Theorems 7.1-7.4).

1.1 - *Nonlinearity and Condensation*

It is nowadays well-established both theoretically ([20, 28, 53]) and experimentally ([27, 26]) that, at a critical (usually very low, amounting to few Kelvin) temperature, an ultracold gas of identical bosons (e.g. atoms and ions like sodium, rubidium and potassium) in a magnetic and/or optical trap experiences a phase transition that turns the system into a *Bose-Einstein condensate*, i.e. a phase in which a macroscopic fraction of the elementary components acquires a one-particle quantum state (i.e. a *wave function* φ): furthermore, the bosonic symmetry imposes that such a state *is the same for all particles.* The system then can be thought of as a unique *giant quantum particle* lying in the state φ, called *ground state of the condensate.* Such a state can be found as a solution to the variational problem

$$\min_{u\in H^1(\Omega),\int_\Omega u^2=N} E_{GP}(u)$$

where the *Gross-Pitaevskii functional* E_{GP} reads

$$E_{GP}(u) \;=\; \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx + 8\pi\alpha\int_\Omega |u(x)|^4\,dx. \tag{1}$$

Here α is the *scattering length* of the two-body interaction between the particles of the gas, Ω is the spatial domain defined by the trap, and N is the number of particles in the condensate.

Sometimes, the presence of the trap is not modeled just by bounding the integral to the domain Ω, but rather endowing the functional with an additional term that takes into account the presence of a confining potential, often a harmonic one.

The rigorous derivation of the functional (1) is the core result of the *Gross-Pitaevskii theory*, that is an *effective* theory used in order to describe the behaviour of a Bose-Einstein condensate. As we shall point out later in some more detail, the main merit of such a theory is that it reduces the complexity of the problem from N-body to one-body, even though the resulting system is nonlinear.

In the first experimental realizations of condensation, the shape of the trap was definitely three-dimensional and regular. Since then, the technology of traps underwent an impressive development, so that nowadays disc-shaped and cigar-shaped traps are currently produced, and some indication of the occurrence of a Bose-Einstein condensation on a ramified structure (i.e. in a Josephson junction) has been recently provided ([49]).

1.2 - *From a linear N-body to a nonlinear 1-body problem*

The dynamics of the Bose-Einstein condensates naturally raises a question: given that quantum mechanics is a *linear* theory, where does the nonlinearity (i.e. the quartic power in the energy functional (1)) come from?

Such a problem can be formulated in the time-independent or in the time-dependent framework.

The time-independent formulation is closer to the problem of the ground state. The validity of the Gross-Pitaevskii theory and of the functional (1) was rigorously established in a series of works by E.-H. Lieb, R. Seiringer and J. Yngvason (see e.g. [46, 47, 48]). Among their achievements, we recall that they found the correct scaling for the potential describing the pair interaction between the particles, namely

$$V(x_i - x_j) \longrightarrow V_N(x_i - x_j) \ := \ N^2 V(N(x_i - x_j)), \tag{2}$$

so that the scattering length of the interaction scales as $1/N$.

A celebrated result in the cited paper is the proof of the Bose-Einstein condensation *in the ground state* of the N-body system of bosons. This means that in the ground state of the N-body Hamiltonian operator

$$H_N \ = \ \sum_{j=1}^{N}(-\Delta_{x_j} + W(x_j)) + \sum_{i<j} V_N(x_i - x_j)$$

representing the energy of the boson gas, the k-particle correlation function converges to the factorized state $\varphi(x_1)\dots\varphi(x_k)$ as the number of particles N grows to infinity. The resulting function φ minimizes the constrained functional (1).

In physical terms, in the limit $N \to \infty$ all particles collapse in the same quantum state, represented by a wave function φ minimizing the Gross-Pitaevskii energy functional (1). Thus, one has *condensation in the ground state*.

On the other hand, the emergence of factorized states and of a nonlinearity out of a linear dynamics can be described also in the time-dependent framework. The time-dependent formulation of the problem of the description of a dilute boson gas in the Gross-Pitaevskii regime historically arises from one of the most topical and active fields of research of the mathematical physics in the last two decades, namely the *mean field limit* for the dynamics of many-body systems. The problem can be summarized as follows: one is interested in studying the evolution in time of a system made of a huge number N of identical particles. According to the basics of quantum mechanics, the state of the whole system at time t is represented by a wave function $\Psi_N(t, x_1, \dots, x_N)$, where x_i is the position variable of the i.th particle and the evolution of Ψ_N is described by the N-body, linear Schrödinger equation

$$\begin{aligned} i\partial_t \Psi_N(t, x_1, \ldots, x_N) \;&=\; -\sum_{j=1}^{N} \Delta_{x_j} \Psi_N(t, x_1, \ldots, x_N) \\ &\quad + \sum_{i<j} V(x_i - x_j) \Psi_N(t, x_1, \ldots, x_N) \end{aligned} \tag{3}$$

where the potential V models the (two-body) interaction between the particles.

Such an equation is in general impossible to solve or even to study numerically, due to the fact that N is often very large (from around millions to 10^{23}). However, it is well-known that physicists are used to deal with such a systems by reducing the equation from N-body to one-body, but paying the price of introducing a nonlinear term in the equation. Justifying such an approximation and providing an estimate for the error made when employing it, has been the main task of the research on mean-field limit for large systems of identical interacting bosons. In its simplest version, the problem of the mean-field limit can be expressed as follows: prove that the evolution provided by the equation (3) with the potential modified through a *weak coupling scaling* $V \longrightarrow V/N$, i.e.

$$\begin{aligned} i\partial_t \Psi_N(t, x_1, \ldots, x_N) \;&=\; -\sum_{j=1}^{N} \Delta_{x_j} \Psi_N(t, x_1, \ldots, x_N) \\ &\quad + \frac{1}{N} \sum_{i<j} V(x_i - x_j) \Psi_N(t, x_1, \ldots, x_N) \end{aligned}$$

with the *factorized* initial data

$$\Psi_0(x_1, \ldots . x_N) \;=\; \phi_0(x_1) \ldots \phi_0(x_N)$$

can be approximated by the evolution of N independent particles, following each the dynamics given by

$$i\partial_t \phi(t, x) = -\Delta\phi(t, x) + (V \star |\phi(t, x)|^2)\phi(t, x). \tag{4}$$

More precisely, the task is to prove that in the limit $N \to \infty$ the correlation functions of the k-particle subsystems converge to $\phi(t, x_1) \ldots \phi(t, x_k)$ where $\phi(t, x)$ solves (4). The history of the main achievements of this research line starts with the work of K. Hepp ([41]), who stated the problem (even though his celebrated work is rather devoted to the classical limit of quantum mechanics). Then, J. Ginibre and G. Velo ([36]) treated the problem of mean field for Coulombian systems in the second-quantized framework. On the other hand, H. Spohn ([60]) proved the mean-field limit for systems of particles interacting through a bounded potential, using a first quantization formalism. In 2000, C. Bardos, F. Golse and N. Mauser ([14]) re-obtained Spohn's result by splitting the problem in the issue of the convergence as N goes to infinity of the N-body Schrödinger equation to an infinite hierarchy, and in the problem of the uniqueness of the solution to such resulting hierarchy. For Coulombian systems, the latter problem was solved one year later by L. Erdős and H.-T. Yau ([33]).

All the cited works are purely mean-field, so that the main result they get is equation (4), that bears a non-local nonlinearity. The first result that opened the road to the derivation of an effective equation with a local nonlinearity was given in [29], where a mixed scaling $V \longrightarrow N^{3\gamma}V(N^{\gamma}\cdot)/N$ was introduced. This can be *formally* interpreted as a mean-field theory for a smoothened Dirac's delta. The effective equation yielded by the limit is then (4) with the replacement of V with the Dirac's delta potential, so that

$$i\partial_t\phi(t,x) = -\Delta\phi(t,x) + \left(\int V dx\right)|\phi(t,x)|^2\phi(t,x)$$

is the target dynamics: this is the time-dependent Gross-Pitaevskii equation that describes the evolution of Bose-Einstein condensates. The derivation was not complete, since the problem of the uniqueness of the solution to the limit hierarchy, stated in ([14]) was not solved.

Later, R.A., F. Golse and A. Teta ([6]) derived the cubic Schrödinger equation in dimension one by studying a scaling limit for a system of one-dimensional identical bosons interacting through a repulsive pointwise interaction: morally, again, the limit whose existence they proved is a mean-field limit with a Dirac's delta potential. Since the result is limited to one-dimensional systems, its validity is restricted to *cigar-shaped* condensates. A more general result, valid for attractive interaction too, was given later in [25].

Finally, in a series of works dating from 2006 to 2010 ([30, 31, 32]), L. Erdős, B. Schlein and H.-T. Yau derived the Gross-Pitaevskii equation for three-dimensional systems. The scaling they adopted is the one discovered by E.-H. Lieb and R. Seiringer for the time-independent framework (2).

Once derived the effective equation, it remained to estimate the error made by replacing the N-body linear Schrödinger equation by the one-body nonlinear equation: for the pure mean-field scaling, the breakthrough came by I. Rodnianski and B. Schlein [54], who provided a new proof of the mean field limit, inspired to the work of Ginibre and Velo [36] and gave also the first estimate of the error. In 2010, A. Knowles and P. Pickl [43] gave a further proof of the limit, in the first-quantized formalism but avoiding use of hierarchies. The method allowed to deal with more singular potentials and produced a new estimate of the error.

Further improvements on the rate of convergence for the mean field have been achieved in [12, 13, 38, 39, 21, 45], while for the Gross-Pitaevskii regime a similar estimate has been proved in [15]. For a complete review on most recent results see the monography [16]

In this review there are no proofs of the condensation or of the Gross-Pitaevskii regime for systems on graphs. In fact, it is widely known that no condensation can occur in one-dimensional systems, in the sense that the phase transition that defines the condensation cannot take place: however, in 2015

J. Bolte and Kerner [18] proved condensation for *free* gas and no condensation for interacting gases in graphs, considered as quasi-dimensional systems, in the sense of the presence of the phase transition. Moreover, in 1996 W. Ketterle and N.J. van Druten [42] gave an evidence of a concentration phenomenon under some aspects analogous to condensation. Finally, one-dimensional condensates can be considered as squeezing limits of three-dimensional condensates, as proved by R. Seiringer and Lin [57].

1.3 - *The problem*

In these lectures we search for the simplest solutions to the *nonlinear focusing* Schrödinger equation on graphs, avoiding the problem of the rigorous derivation from first principles. Simplest solutions are particular cases of *standing waves*, that minimize the energy under some physical constraints: in particular, we consider the so-called *mass constraint*.

Before stating the problem precisely, we need some definitions and notation.

- A *graph* $\mathcal{G}$ i.e. a couple of sets $(\mathcal{V}, \mathcal{B})$, where $\mathcal{B}$ is a subset of $\mathcal{V} \times \mathcal{V}$.

  The set $\mathcal{V}$ is interpreted as the set of the *vertices*, i.e. points in the space, while $\mathcal{B}$ is the set of the *edges* or *bonds*, e.g. links between vertices: every edge is then identified with the couple of vertices it connects.

  The *degree* of a vertex $\mathrm{v} \in \mathcal{V}$ is the number of edges starting from or ending at v.

  Both $\mathcal{V}$ and $\mathcal{B}$ are *finite* sets, so that we shall always deal with graphs with a finite number of edges and vertices.

  There is a non-empty subset $\mathcal{V}_\infty$ of $\mathcal{V}$, made of *vertices at infinity*. Two vertices at infinity cannot be connected by edges, and every vertex at infinity has degree equal to one. We shall refer to edges ending at a vertex at infinity as to *halflines*.

  Two graphs are topologically equivalent if they can be deformed into each other without changing the sets $\mathcal{V}, \mathcal{V}_\infty$ and $\mathcal{B}$.

- In order to construct a *metric graph*, an edge e is identified with an interval $I_e := [0, \ell_e]$, where $\ell_e \in [0, +\infty]$. This correspondence fixes the *metric* of the graph.

  Given a topology, several metrics are possible, as every finite edge can have an arbitrary length. Conversely, the metric on the halflines is fixed.

  For a pictorial idea of a generic graph see Fig. 1, that show an example where selfloops, multiple connections, and haflines are present.

- A function $u : \mathcal{G} \to \mathbb{C}$ is a bunch of functions $u = (u_e)_{e \in \mathcal{B}}$, with $u_e : I_e \to \mathbb{C}$.

  A function u is continuous on $\mathcal{G}$ if every u_e is continuous in I_e and if u is continuous at vertices, namely, if the value attained at a vertex v is independent of the edge chosen to reach v.

- A function $u : \mathcal{G} \to \mathbb{C}$ is integrable if every function u_e is integrable on I_e, and
$$\int_{\mathcal{G}} u\,dx \ := \ \sum_{e\in\mathcal{B}} \int_0^{\ell_e} u_e(x_e)\,dx_e$$
The usual functional spaces can be defined as
$$L^p(\mathcal{G}) \ := \ \bigoplus_{e\in\mathcal{B}} L^p(I_e), \qquad \|u\|^p_{L^p(\mathcal{G})} := \sum_{e\in\mathcal{B}} \|u_e\|^p_{L^p(I_e)}.$$
The space $H^1(\mathcal{G})$ is defined as the set of continuous functions $u = (u_e)_{e\in\mathcal{B}}$ such that
$$u_e \in H^1(I_e) \quad \forall e \in \mathcal{B}, \qquad \|u\|^2_{H^1(\mathcal{G})} = \sum_{e\in\mathcal{B}} \|u_e\|^2_{H^1(I_e)}.$$
We stress that continuity is imposed at vertices too, so that no jump can occur.
- Fixed $\mu > 0$, we define
$$H^1_\mu(\mathcal{G}) := \{u \in H^1(\mathcal{G}),\ \|u\|^2_{L^2(\mathcal{G})} = \mu\},$$
that is the space of the functions in $H^1(\mathcal{G})$ that fulfil the *mass constraint*.

On the metric graph $\mathcal{G}$ let us define
$$E(u, \mathcal{G}) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})},$$
that is a functional in $C^1(H^1_\mu(\mathcal{G}), \mathbb{R})$ for all $p \in [2, +\infty)$.

The problem we treat is the following:

Problem P. Given a *connected, non-compact* metric graph $\mathcal{G}$ and fixed $\mu > 0$, does there exist a *ground state at mass* μ, namely a minimizer of $E(\cdot, \mathcal{G})$ in the space $H^1_\mu(\mathcal{G})$?

In other words, we look for functions $u \in H^1_\mu(\mathcal{G})$ such that
$$E(u, \mathcal{G}) \ = \ \mathcal{E}_{\mathcal{G}}(\mu)$$
where we introduced the notation
$$\mathcal{E}_{\mathcal{G}}(\mu) \ := \ \inf_{v\in H^1_\mu(\mathcal{G})} E(v, \mathcal{G}). \tag{5}$$

Notice that, as regards the problem of the existence of a minimizer, if $\mathcal{G}$ is compact, then every minimizing sequence is compact, so that a minimizer always exists. For this reason we restrict to non-compact graphs, i.e. graphs that contain at least one halfline. On the other hand, it will be clear from the analysis that if a graph is not connected, then minimizers concentrate on the most convenient connected component, so that the hypothesis that the graph is connected is not restrictive.

We end this introductory part by noticing that, due to the shape of the energy functional, we shall limit the analysis to nonnegative functions.

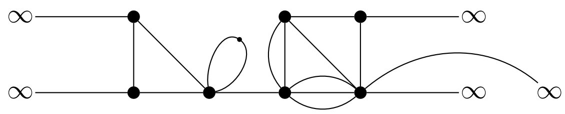

Figure 1: How a connected, non-compact metric graph may look like.

2 - Preliminary remarks

2.1 - *Mass constraint and lower boundedness*

First of all observe that, regardless of the chosen graph $\mathcal{G}$, if no constraint is imposed then the functional $E(\cdot, \mathcal{G})$ is *not* lower bounded. Indeed, fixed $u \in H^1(\mathcal{G})$, one has

$$E(\lambda u, \mathcal{G}) = \frac{\lambda^2}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{\lambda^p}{p}\|u\|^p_{L^p(\mathcal{G})} \ \to\ -\infty, \quad \lambda \to +\infty.$$

On the other hand, let us restrict to the case of star graphs made of halflines, and impose the mass constraint $\|u\|^2_{L^2(\mathcal{G})} = \mu$. On these particular graphs, one can perform mass preserving transformations $u(x) \to \sqrt{\lambda}u(\lambda x) := u_\lambda(x)$, so that

$$E(u_\lambda, \mathcal{G}) = \frac{\lambda^2}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{\lambda^{\frac{p}{2}-1}}{p}\|u\|^p_{L^p(\mathcal{G})}.$$

Now,

- if $p < 6$, then kinetic energy prevails and this suggests that the energy is *lower bounded.* Indeed by using Gagliardo-Nirenberg estimates one can prove that this is actually the case. The problem with $2 < p < 6$ is referred to as the *subcritical case.*
- If $p > 6$, then the potential term overwhelms the kinetic one and $E(u_\lambda, \mathcal{G}) \to -\infty$, as $\lambda \to +\infty$, so the energy is not lower bounded. The problem with $p > 6$ is then referred to as the *supercritical case.*
- If $p = 6$, then there is a delicate balance between kinetic and nonlinear term. As we shall see, lower boundedness of E depends on the value of μ. The problem with $p = 6$ is referred to as the *critical case.*

2.2 - *The Euler-Lagrange equation: Kirchhoff's rule*

As a minimum of the *constrained* functional, every ground state u must satisfy the Lagrange Multiplier Theorem

$$\nabla E(u, \mathcal{G})(u) = \frac{\omega}{2}\nabla(\mu - \|u\|^2_{L^2(\mathcal{G})})$$

for some $\omega \in \mathbb{R}$ (notice that in order to simplify notation here we called $\frac{\omega}{2}$ the Lagrange multiplier). Then, for every $\eta \in H^1(\mathcal{G})$,

$$
\begin{aligned}
0 &= \nabla E(u, \mathcal{G})\eta - \frac{\omega}{2}\nabla(\mu - \|u\|^2)\eta \;=\; \int_{\mathcal{G}} (u'\eta' - u^{p-1}\eta + \omega u \eta)\, dx \\
&= \sum_{e \in \mathcal{B}} \int_0^{\ell_e} (u_e'\eta_e' - u_e^{p-1}\eta_e + \omega u_e \eta_e)\, dx_e \\
&= \sum_{e \in \mathcal{B}} u_e'\eta_e|_0^{\ell_e} + \sum_{e \in \mathcal{B}} \int_0^{\ell_e} (-u_e'' - u_e^{p-1} + \omega u_e)\eta_e \, dx_e
\end{aligned}
$$

Notice that the first term concerns vertices, while the second is determined by the values of the integrand inside the edges.

Now pick an edge $\bar{e}$ and consider a function $\eta \in C_0^\infty(\bar{e})$. Then the second term only survives and forces the *Stationary Nonlinear Schrödinger equation*

$$
u_{\bar{e}}'' + u_{\bar{e}}^{p-1} = \omega u_{\bar{e}} \tag{6}
$$

to hold inside $\bar{e}$, and, by arbitrarity of $\bar{e}$, on every edge.

Now consider $\eta \in C_0^\infty(\mathcal{G})$ that vanishes *in all vertices except one*, say $\bar{\mathrm{v}}$. Then, boundary terms survive *if and only if they refer to edges starting from or ending at* $\bar{\mathrm{v}}$. Their contribution reads

$$
\begin{aligned}
\sum_{e \in E} u_e'\eta_e|_0^{\ell_e} &= \sum_{e \to \bar{\mathrm{v}}} u_e'(\ell_e)\eta(\ell_e) - \sum_{e \leftarrow \bar{\mathrm{v}}} u_e'(0)\eta(0) \\
&= \left(\sum_{e \to \bar{\mathrm{v}}} u_e'(\ell_e) - \sum_{e \leftarrow \bar{\mathrm{v}}} u_e'(0) \right) \eta(\bar{\mathrm{v}})
\end{aligned}
$$

where we denoted $e \to \bar{\mathrm{v}}$ the edges ending at $\bar{\mathrm{v}}$ and $e \leftarrow \bar{\mathrm{v}}$ the edges starting from $\bar{\mathrm{v}}$.
Thus

$$
\sum_{e \to \bar{\mathrm{v}}} u_e'(\ell_e) - \sum_{e \leftarrow \bar{\mathrm{v}}} u_e'(0) \;=\; 0
$$

that is called Kirchhoff's rule, and is often expressed by saying that at every vertex the global ingoing (or outgoing) derivative vanishes. A common compact form of Kirchhoff's rule is

$$
\sum_{e \succ \mathrm{v}} \frac{du_e}{dx_e}(\mathrm{v}) = 0. \tag{7}
$$

Let us finally recall that the Euler-Lagrange equations (6) together with the Kirchhoff's conditions can be summarized in the equation

$$
\Delta_K u(x) + u^{p-1}(x) = \omega u(x) \tag{8}
$$

where Δ_K is the operator acting as the laplacian on functions that are H^2 on every edge and fulfil Kirchhoff's rule at all vertices. It is immediately seen

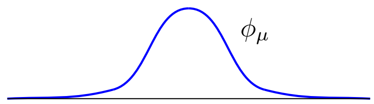

Figure 2: The soliton ϕ_μ.

that equation (8) is the stationary equation associated to the time-dependent nonlinear Schrödinger equation

$$(9)\qquad i\partial_t \psi(x) \;=\; -\Delta_K \psi(t) - |\psi(t)|^{p-2}\psi(t),$$

and it is well-known that for such equation the dynamics preserves the L^2-norm and the value of the energy $E(\cdot,\mathcal{G})$.

We stress that (8) is equivalent to the condition of stationarity of the functional, so that it is satisfied not only by ground states, but also by every standing wave of the nonlinear Schrödinger equation, namely by all solutions to (9) of the type

$$\psi(t,x) = e^{i\omega t}u(x),$$

so that it is clear that the Lagrange multiplier ω has the dynamical meaning of a *frequency*.

3 - Examples

In this section we give some basic examples of graphs and the related results concerning the existence or the nonexistence of ground states.

3.1 - *The real line*

It is well-known ([62, 24, 40]) that for $p \in (2,6)$ and $\mu > 0$ ground states exist and are all translated of the soliton (Fig. 2)

$$\phi_\mu(x) \;=\; C\mu^{\frac{2}{6-p}}\operatorname{sech}^{\frac{2}{p-2}}(c\mu^{\frac{p-2}{6-p}}x).$$

where C and c are irrelevant constants dependent on p only and not on μ.
If $p = 4$, i.e. in the case of the cubic Schrödinger equation, one gets

$$\phi_\mu(x) \;=\; \frac{\mu}{2\sqrt{2}}\operatorname{sech}\left(\frac{\mu}{4}x\right), \qquad E(\phi_\mu,\mathbb{R}) = -\frac{\mu^3}{96}.$$

Let us point out that the solitons and their translated are the only stationary solutions to (9) on the line. In order to prove it, notice that the Euler-Lagrange equation on the line

$$u'' + u^{p-1} = \omega u,$$

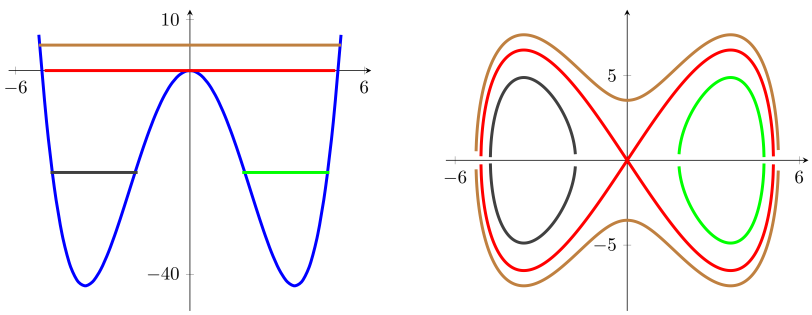

Figure 3: Left: The profile of the potential V. Right: The phase portrait induced by the potential V.

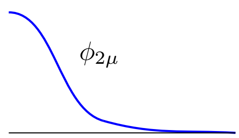

Figure 4: Half a soliton.

can be rewritten as

$$u'' = -\frac{dV}{du}(u) \tag{10}$$

where we defined

$$V(u) = \frac{u^p}{p} - \omega\frac{u^2}{2}$$

Equation (10) can be interpreted as a mechanical conservative problem, whose phase portrait is displayed in Fig. 3.
It is then clear that all solutions are periodic (and therefore not in $H^1(\mathbb{R})$) except the non-constant ones contained in the *separatrix*, corresponding to solutions to (10) with vanishing mechanical energy. They turn out to be the solitons and their translated.

3.2 - *The halfline*

In the case $\mathcal{G} = \mathbb{R}^+$, $p \in (2,6)$ and $\mu > 0$, there is exactly *one* ground state given by "half a soliton" (Fig. 4) of mass 2μ (notice that in this case the translational symmetry is broken).
If $p = 4$, then

$$\phi_{2\mu}(x) \; = \frac{\mu}{\sqrt{2}}\text{sech}\left(\frac{\mu}{2}x\right), \qquad E(\phi_{2\mu}, \mathbb{R}^+) = -\frac{\mu^3}{24}.$$

3.3 - *A star-graph made of halflines*

The cases of the halfline and of the line naturally generalize to the case of the star-graph $\mathcal{S}_n$ made of n halflines (the case $n = 4$ is represented in Fig. 5). Yet the result changes, as proved in [2, 3, 4, 5].

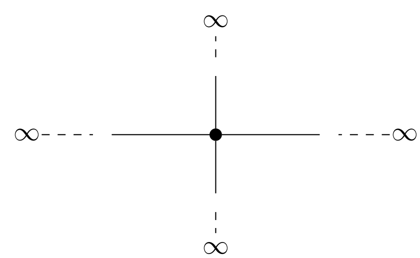

Figure 5: A star-graph made of four halflines.

Indeed, for $p \in (2, 6)$ and $\mu > 0$,

$$\mathcal{E}_{\mathcal{S}_n}(\mu) \;=\; \mathcal{E}_{\mathbb{R}}(\mu)$$

but the infimum is not achieved, so that there is no ground state.

In order to prove it, restrict to the simplest case $p = 4$ and consider the star-graph $\mathcal{S}_3$ made of three halflines $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3$, and a function $u \in H^1_\mu(\mathcal{S}_3)$. Let us introduce the notation $u = (u_1, u_2, u_3)$, where u_j is the restriction of u to the halfline $\mathcal{H}_j$, and $\mu_j := \|u_j\|^2_{L^2(H_j)}$. With no loss of generality, suppose $\mu_1 = \min(\mu_1, \mu_2, \mu_3)$ and construct a function $\widetilde{u} \in H^1_\mu(\mathcal{S}_3)$ such that $E(\widetilde{u}, \mathcal{S}_3) \leq E(u, \mathcal{S}_3)$ proceeding as follows:

1. On $\mathcal{H}_1$, replace u_1 with the half-soliton $\chi_{\mathbb{R}^+}\phi_{2\mu_1}$

2. On $\mathcal{H}_2 \cup \mathcal{H}_3$, replace the couple of functions (u_2, u_3) with the soliton $\phi_{\mu_2+\mu_3}$. At this point, on $\mathcal{S}_3$ set the function $(\chi_{\mathbb{R}^+}\phi_{\mu_1}, \chi_{\mathbb{R}^+}\phi_{\mu_2+\mu_3}, \chi_{\mathbb{R}^+}\phi_{\mu_2+\mu_3})$, that may not be continuous.

3. *Translate* the soliton sat on $\mathcal{H}_1 \cup \mathcal{H}_2$ in order to obtain a continuous function $\widetilde{u}$ on $\mathcal{S}_3$. It is immediate that $\widetilde{u}$ belongs to $H^1_\mu(\mathcal{S}_3)$.

Then, exploiting the minimum properties of the half-soliton on the half-line, and of the soliton on the line,

$$\begin{aligned} E(u, \mathcal{S}_3) &= E(u_1, \mathcal{H}_1) + E((u_2, u_3), \mathcal{H}_2 \cup \mathcal{H}_3) \\ &\geq E(\chi_{\mathbb{R}^+}\phi_{2\mu_1}, \mathbb{R}^+) + E(\phi_{\mu_2+\mu_3}, \mathbb{R}) \\ &= -\frac{\mu_1^3}{24} - \frac{(\mu - \mu_1)^3}{96} \;=\; E(\widetilde{u}, \mathcal{S}_3). \end{aligned}$$

By construction $\mu \geq 3\mu_1$, then the minimum is attained for

$$\mu_1 \;=\; 0.$$

Therefore, minimizing sequences concentrate on *one halfline only*, and *reconstruct a soliton at infinity.* An analogous result can be obtained for every $p < 6$.

This simple example provides at least two important messages: first, as regards the minimization of the energy, it is not convenient to spread the wave functions on many edges. Second, for every candidate ground state, the soliton is a serious competitor!

3.4 - *The 3-Bridge* $\mathcal{B}_3$

The first non-star cases treated in literature were the so-called *bridge graphs,* first dealt with in [7] (the triple bridge $\mathcal{B}_3$ is represented in Fig. 6). For $p \in (2, 6)$

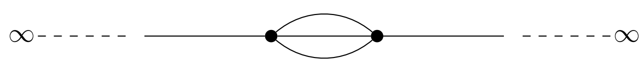

Figure 6: The 3-bridge $\mathcal{B}_3$.

and $\mu > 0$,

$$\mathcal{E}_{\mathcal{B}_3}(\mu) \; = \; \mathcal{E}_{\mathbb{R}}(\mu)$$

and again the infimum is not achieved, so that there is no ground state.

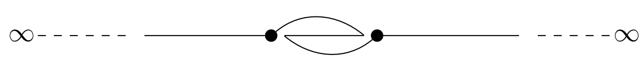

Figure 7: Unfolding $\mathcal{B}_3$ into a line.

Fixed $\mu > 0$ It turns out that for every function $u \in H^1_\mu(\mathcal{B}_3)$ one gets

$$E(u, \mathcal{B}_3) \; > \; E(\phi_\mu, \mathbb{R}).$$

To prove it, the key observation is that $\mathcal{B}_3$ is semi-Eulerian, namely, it can be unfolded into a line together with every function $u \in H^1(\mathcal{B}_3)$ (Fig. 7).
So, let $u \in H^1_\mu(\mathcal{B}_3)$ and $\widetilde{u} \in H^1_\mu(\mathbb{R})$ its unfolded version on the line. Then, denoting by $\mathrm{v}_1, \mathrm{v}_2$ the two vertices, on the line they correspond to points

$$x_1 \; < \; x_2 \; < \; y_1 \; < \; y_2$$

with x_i, y_i associated to the vertex v_i. Therefore, by continuity

$$\widetilde{u}(x_1) = \widetilde{u}(y_1), \qquad \widetilde{u}(x_2) = \widetilde{u}(y_2).$$

Now, if $\widetilde{u}(x_1) < \widetilde{u}(x_2)$, then there is a minimum in the interval (x_2, y_2).

If $\widetilde{u}(x_1) > \widetilde{u}(x_2)$, then there is a minimum in the interval (x_1, y_1).
If $\widetilde{u}(x_1) = \widetilde{u}(x_2)$, then u takes the same value in four points.

In all cases $\widetilde{u}$ cannot be a soliton, then

$$E(u, \mathcal{B}_3) \;=\; E(\widetilde{u}, \mathbb{R}) \;>\; E(\phi_\mu, \mathbb{R}) \;=\; \mathcal{E}_{\mathbb{R}}(\mu).$$

Nevertheless, it is possible to define a sequence that asymptotically reconstructs a soliton on a halfline, so that

$$\mathcal{E}_{\mathcal{B}_3}(\mu) \;=\; \mathcal{E}_{\mathbb{R}}(\mu)$$

but such a minimum is not attained.

3.5 - *The double bridge $\mathcal{B}_2$*

The double bridge $\mathcal{B}_2$ (Fig. 8) is not semi–Eulerian, and the problem of establishing the existence or the nonexistence of ground states becomes much more difficult than in $\mathcal{B}_3$.

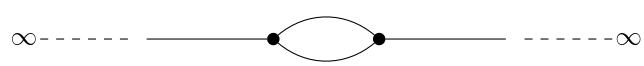

Figure 8: The two-bridge $\mathcal{B}_2$.

However, once again, as we shall see

$$\mathcal{E}_{\mathcal{B}_2}(\mu) \;=\; \mathcal{E}_{\mathbb{R}}(\mu)$$

and there is *no ground state*.
We can then conclude that on bridge–graphs the infimum is never achieved.

4 - A key assumption

From the examples of existence and of nonexistence given in the last section, one can single out some observations:

1. It seems convenient to escape "intricated" zones, e.g. vertices with high degree: at least, this is what happens for the star-graphs and for the bridges.
2. It is always possible to construct a soliton (possibly at infinity, as the asymptotics of a sequence), so, in order to be a ground state, a function must reach an energy level which is lower than the level of the soliton.
3. Therefore, in order to ensure existence of a ground state, the graph must exhibit *structures* able to trap functions that do better than the soliton.

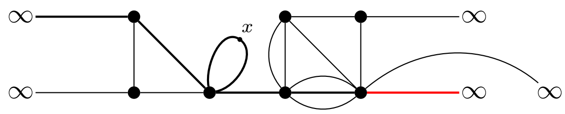

Figure 9: A point x in the graph and a trail containing x and two halflines.

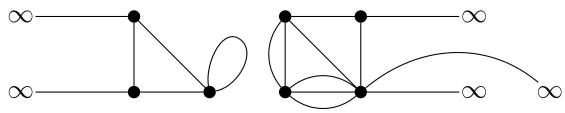

Figure 10: Once removed the only finite cut-edge of the graph, each connected component contains a halfline, and therefore a vertex at infinity.

4. On the other hand, in order not to have minimizers, it seems sufficient for the graph to be, in some sense, *more intricated than a line*.

The last observation is embodied in a topological assumption, that we call (H). We give three alternative formulations of such assumption. The proof of the equivalence of the three formulations is not completely straightforward, and will not be given here.

The first formulation is based on the graph-theoretical notion of *trail* (Fig. 9). A path made of adjacent edges, in which every edge is run through exactly once, is called a trail. Notice that in a trail vertices can be run through more than once.

Assumption (H), first formulation. *Every $x \in \mathcal{G}$ lies on a* trail *that contains* two halflines.

Assumption (H) can also be expressed as the absence of structure like "bottlenecks":

Assumption (H), second formulation. *After removing an arbitrary edge from $\mathcal{G}$, every resulting connected component contains a vertex at infinity* (Fig. 10, 11).

The last formulation we give is more pictorial and considers the possibility of covering the graph (vertex at infinity included) by cycles.

Assumption (H), third formulation. *After identifying all vertices at infinity, the graph $\mathcal{G}$ admits a* cycle covering (Fig. 12).

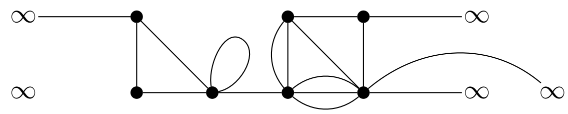

Figure 11: Assumption (H) in the second formulation applies to halflines too: After removing a halfline, two connected components result: one contains some halflines, and therefore vertices at infinity; the other is made of one vertex at infinity.

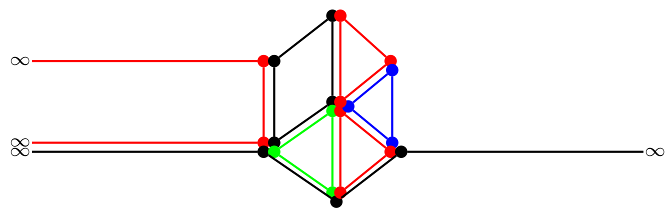

Figure 12: A possible cycle covering.

4.1 - *If (H) is violated*

Assumption (H) can be violated in several ways. Notice indeed that (H) implies that $\mathcal{G}$ has at least two vertices at infinity, so it is violated by every graph having less than two halflines. Furthermore, it is immediately seen that (H) is violated not only by having less than two halflines, but also by the presence of a *terminal edge* or *pendant* (Fig. 13):

It is also possible to violate assumption (H) without having a pendant, like in the *signpost graph* (Fig. 14).

5 - A nonexistence result

The first general result we give on graphs is negative:

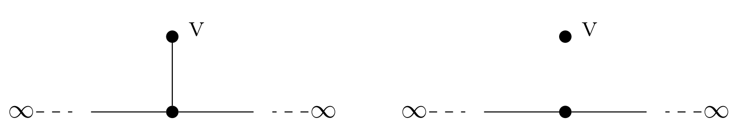

Figure 13: Left: line with a pendant. Right: after the removal of the pendant, there remains a connected compact component made of a vertex, violating (H) (see the second formulation).

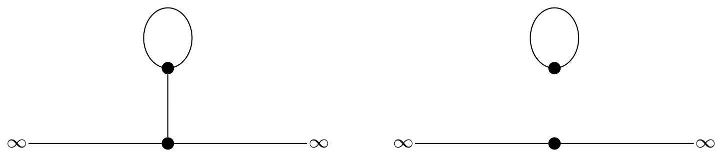

Figure 14: Left: Signpost graph. Right: after the removal of the pendant, there remains a connected compact component made of a loop, violating (H) (see the second formulation).

T h e o r e m 5.1 (Nonexistence). *Assume that $\mathcal{G}$ satisfies assumption (H). Then, for any $\mu > 0$*

$$\mathcal{E}_{\mathcal{G}}(\mu) \ = \mathcal{E}_{\mathbb{R}}(\mu)$$

and the infium is never attained, so that a ground state does not exist, except if $\mathcal{G}$ is a "bubble tower" (Fig. 15).

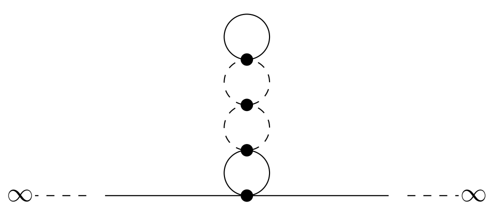

Figure 15: A bubble tower.

The key idea of the theorem is strictly related to *rearrangement theory*.

5.1 - *Rearrangements*

The technique of rearrangements is nowadays classical in calculus of variations, and is widely used in order to show the existence of minimizers and possibly to establish some of their features, like for instance the symmetries. Its first extension to graphs is due to L. Friedlander ([34]). In our proofs we use rearrangements to show the nonexistence of ground states. In what follows we give an intuitive and tutorial summary of the results we will use, for readers non familiar with rearrangements.

Given a nonnegative function $u \in H^1_\mu(\mathcal{G})$, we aim at constructing another nonnegative function $v \in H^1(\mathbb{R}^+)$ s.t. $E(v, \mathbb{R}^+) \leq E(u, \mathcal{G})$. To this purpose, one can construct the so-called *monotone rearrangement*. The idea behind it

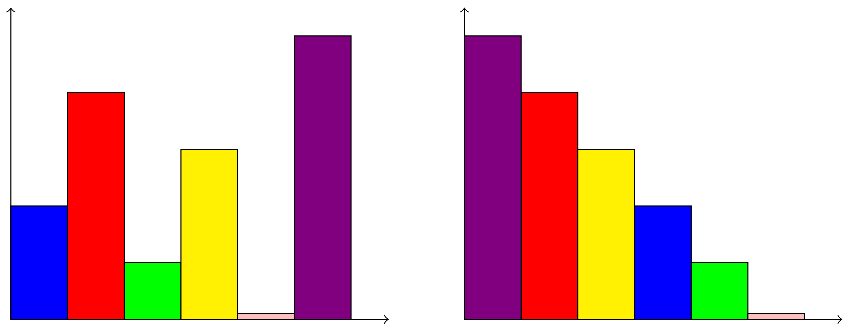

Figure 16: Monotone rearrangement of a piecewise constant function: this example shows pictorially the fact that integral norms are preserved while oscillations are reduced, so that the nonlinear term in the energy is left untouched, while the kinetic term is dumped, so that the energy has diminished.

can be roughly summarized as to *cutting the graph of the function in vertical slices and locating them on a halfline in order of decreasing height.*

As it appears from Fig.16, L^p-norms are preserved, while oscillations are suppressed, so that one can argue that, generalizing the procedure to regular functions, after rearranging a function *the kinetic energy diminishes*.

Of course, one can give a more formal and general definition of monotone rearrangement: let $(\Omega, \mathcal{F}, m)$ be a measure space, and consider a nonnegative function $f : \Omega \longrightarrow \mathbb{R}^+$. One can define the *distribution function* ρ_f of the function f as

$$\rho_f(t) := m(\{x \in \Omega,\ f(x) > t\}).$$

Clearly, ρ_f is defined $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and is monotonically decreasing. The monotone rearrangement f^* of f is a function defined on $\mathbb{R}^+$ with values in $\mathbb{R}^+$, given by

$$f^*(x) := \inf\{t \geq 0,\ \rho_f(t) \leq x\}.$$

It is straightforward that, if ρ_f is invertible, then

$$f^* = \rho_f^{-1}$$

Since $\rho_f = \rho_{f^*}$, f and f^* have the same level sets, so that

$$\|f\|_{L^p(X)} = \|f^*\|_{L^p(\mathbb{R}^+)}.$$

As an example, consider the measure space $X = [-\pi, \pi]$, and the function $f(x) = |\sin x|$. Then, one can directly compute $\rho_f(t) = 2\pi - 4\text{arcsin}x$ and $f^*(x) = \cos(x/4)$ (Fig. 17, 18).
The main result that we borrow from rearrangement theory consists in quantitatively estimating the decrease of the kinetic energy induced by a rearrangement.

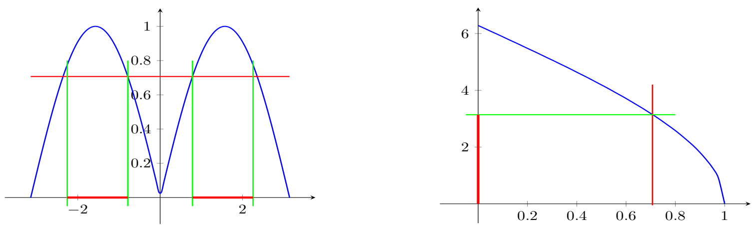

Figure 17: Left: $f(x) = |\sin x|$. Right: $\rho_f(t) = 2\pi - 4\text{arcsin}x$.

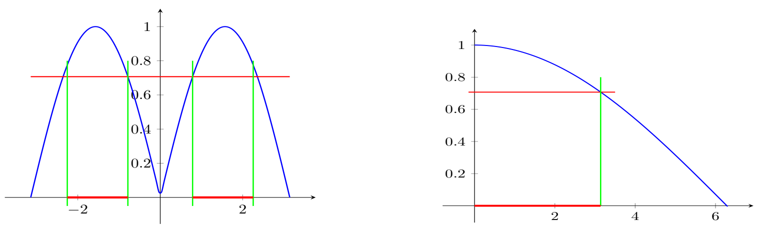

Figure 18: Left: $f(x) = |\sin(x)|$. Right: $f^*(x) = \cos(x/4)$

To this aim, we first show that the kinetic energy of the monotone rearrangement $u^* \in H^1(\mathbb{R}^+)$ cannot exceed the energy of the original function $u \in H^1(\mathcal{G})$. We limit ourselves to the case of a function u regular enough, so that ρ_u is differentiable and $\mathcal{G}$ can be partitioned in intervals $I_j = [a_j, b_j]$ such that u is monotone in every I_j (see Fig. 19). Then, it is easily seen that

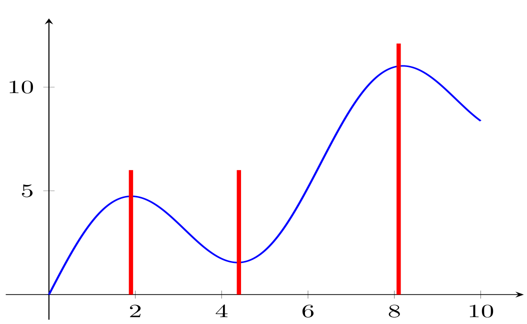

Figure 19: Partition of the domain of a function in intervals of monotonicity.

$$
\begin{aligned}
\rho(t+h)-\rho(t) &= m(\{u(x)>t+h\})-m(\{u(x)>t\}) \\
&= \sum_j \left(m(\{x\in I_j,\ u(x)>t+h\})-m(\{x\in I_j,\ u(x)>t\})\right) \\
&\simeq h\sum_j \frac{1}{|u'(x_j(t))|},
\end{aligned}
$$

where $x_j(t)$ is the only point in I_j where $u=t$. Then

$$
\rho'(t)=\sum_{x,\,\mathrm{s.t.}\,u(x)=t}\frac{1}{|u'(x)|}.
$$

Thus, computing the kinetic energy one finds

$$
\begin{aligned}
\int_{\mathcal{G}}|u'(x)|^2\,dx &= \sum_j\int_{a_j}^{b_j}|u'(x)|^2\,dx = \sum_j\int_{\min_{[a_j,b_j]}u(x)}^{\max_{[a_j,b_j]}u(x)}|u'(x_j(t))|\,dt \\
&= \int_0^{\|u\|_\infty}\sum_{x,\,\mathrm{s.t.}\,u(x)=t}|u'(x)|\,dt
\end{aligned}
$$

where, in every interval I_j, $t=u(x)$.

Let $a_j>0$. By Cauchy-Schwarz inequality,

$$
N = \sum_{j=1}^N 1 = \sum_{j=1}^N a_j^{1/2}a_j^{-1/2} \le \left(\sum_{j=1}^N a_j\right)^{1/2}\left(\sum_{j=1}^N a_j^{-1}\right)^{1/2},
$$

so that, replacing a_j with $|u'(x)|$,

$$
\begin{aligned}
\int_{\mathcal{G}}|u'(x)|^2\,dx &\ge \int_0^{\|u\|_\infty}N^2(t)\left(\sum_{x,\,\mathrm{s.t.}\,u(x)=t}\frac{1}{|u'(x)|}\right)^{-1} \\
&= \int_0^{\|u\|_\infty}N^2(t)\frac{1}{|\rho_u'(t)|},
\end{aligned}
$$

where, for every t in the range of u, we defined the *number of preimages of* t

$$
N(t):=\sharp u^{-1}(t).
$$

Now, as $u^*=\rho^{-1}$, one gets

$$
\begin{aligned}
\int_{\mathcal{G}}|u'(x)|^2\,dx &\ge \int_0^{\|u\|_\infty}\frac{N^2(t)}{|\rho_u'(t)|}\,dt \ge \int_0^{\|u\|_\infty}\frac{dt}{|\rho_u'(t)|} \\
&= \int_0^{\|u\|_\infty}|(u^*)'(x(t))|\,dt \\
&= \int_{\mathbb{R}^+}|(u^*)'(x)|^2\,dx
\end{aligned}
$$

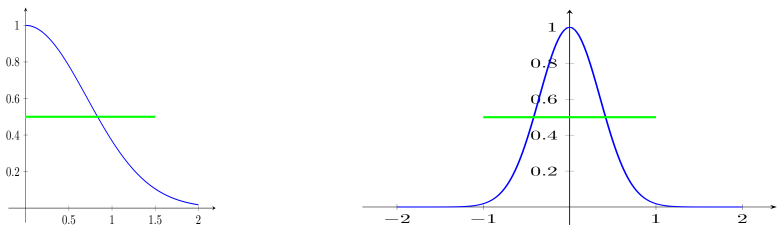

Figure 20: Left: The monotone rearrangement u^*. Right: The symmetric rearrangement $\widehat{u}$.

where equality holds iff $N(t) = 1$ for almost every t.
We then proved the *Pólya-Szegő inequality*:

$$\|(u^*)'\|_{L^2(\mathbb{R}^+)} < \|u'\|_{L^2(\mathcal{G})}.$$

Therefore, as the monotone rearrangement lowers the kinetic energy and preserves the nonlinear term, one has

$$E(u, \mathcal{G}) \geq E(u^*, \mathbb{R}^+).$$

Notice that this implies that the halfline is optimal among non-compact graphs:

$$\mathcal{E}_{\mathbb{R}+}(\mu) \leq \mathcal{E}_{\mathcal{G}}(\mu),$$

for every graph $\mathcal{G}$ containing at least one halfline.

5.2 - *Symmetric rearrangement*

To prove Theorem 5.1 we need to introduce the *symmetric rearrangement* (Fig. 20), defined as

$$\widehat{u}(x) := u^*(2|x|), \qquad x \in \mathbb{R}$$

One immediately has that $\widehat{u}$ is even and

$$\rho_{\widehat{u}} = \rho_{u^*} \ (= \ \rho_u).$$

By an elementary change of variable,

$$\int_{\mathbb{R}} \widehat{u}^p \, dx = \int_{\mathbb{R}^+} (u^*)^p \, dx \ \left(= \int_{\mathcal{G}} u^p \, dx \right)$$

and, analogously to the case of the monotone rarrangement, one finally has

$$\begin{aligned} \int_{\mathbb{R}} (\widehat{u}')^2 \, dx &= 4 \int_{\mathbb{R}+} [(u^*)']^2 \, dx = 4 \int_0^{\|u\|_\infty} \frac{dt}{|\rho_u'(t)|} \, dt \\ &\leq \int_0^{\|u\|_\infty} \frac{N(t)^2}{|\rho_u'(t)|} \, dt = \int_{\mathcal{G}} (u')^2 \, dx \end{aligned}$$

provided that $N(t) \geq 2$ for almost every t.

We finally proved the following

Proposition 5.1. *Let $\mathcal{G}$ be a connected non-compact metric graph, and u be a nonnegative function in $H^1(\mathcal{G})$. Then, denoted*

$$N(t) := \#\{x \in \mathcal{G} \,:\, u(x) = t\}, \qquad t \in (0, \max u],$$

the following inequality holds true:

$$\int_{\mathbb{R}^+} |(u^*)'|^2 \, dx \leq \int_{\mathcal{G}} |u'|^2 \, dx,$$

with strict inequality unless $N(t) = 1$ almost everywhere.
Moreover, if $N(t) \geq 2$ almost everywhere, then

$$\int_{\mathbb{R}} |(\widehat{u})'|^2 \, dx \;\leq\; \int_{\mathcal{G}} |u'|^2 \, dx,$$

where equality implies that $N(t) = 2$ almost everywhere, and thus

$$E(u, \mathcal{G}) \;\geq\; E(\widehat{u}, \mathbb{R}) \;\geq\; \mathcal{E}_{\mathbb{R}}(\mu) \;=\; E(\phi_\mu, \mathbb{R}).$$

We are now ready to prove the Theorem 5.1.

Proof of Theorem 5.1. Let $u \in H^1_\mu(\mathcal{G})$, and let x_0 be a global maximum point for u.
Owing to Assumption (H) (see e.g. the first formulation), there exists a trail $\mathcal{T}$ passing through x_0 and containing two halflines. Clearly, the restriction of u to $\mathcal{T}$ belongs to $H^1(\mathcal{T})$ and $\max_{\mathcal{T}} u = \max_{\mathcal{G}} u$.
Furthermore, since $\mathcal{T}$ connects two vertices at infinity and $x_0 \in \mathcal{T}$,

$$\#\{x \in \mathcal{G} \,:\, u(x) = t\} \geq \#\{x \in \mathcal{T} \,:\, u(x) = t\} \geq 2 \quad \text{for} \quad \text{a.e. } t.$$

Due to the proposition on symmetric rearrangement, and to the existence of runaway soliton sequences mimicking the soliton, the infimum can be attained by a function u, namely a ground state may exist, if and only if

1. Almost every point in $Ran\, u$ has exactly two preimages.
2. $E(u, \mathcal{G}) = E(\phi_\mu, \mathbb{R})$

Suppose that such a ground state u exists, and call x_0 a maximum point of u. Then, by assumption (H), there is a trail $\mathcal{T}$ passing through x_0 and containing two halflines. On this trail *every value in $Ran\, u$ is attained twice.*

If there were other edges starting from (or arriving to) the trail, then further counterimages would be created, and some interval in $Ran\, u$ would be made of points with at least *three* preimages, so that

$$E(u, \mathcal{G}) \;>\; \mathcal{E}_{\mathcal{G}}(\mu)$$

contradicting the hypothesis of u being a ground state.

Then, $\mathcal{G} = \mathcal{T}$, i.e., $\mathcal{G}$ must be the real line (up to some possible identification of vertices), and u must be a soliton.
The only identification of vertices that preserve the symmetry of the soliton gives rise to the family of *tower of bubbles*. For an extended explanation of this point, see [8].

□

6 - Ground states in the subcritical case

In this section, given an exponent $p \in (2,6)$ and a mass $\mu > 0$, we continue our study on the existence of absolute minimizers (ground states) for the functional

$$E(u,\mathcal{G}) \quad = \quad \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p\,dx,$$

subject to the *mass constraint*

$$\int_{\mathcal{G}} |u|^2\,dx = \mu.$$

Here $\mathcal{G}$ is an arbitrary noncompact metric graph (see Figure 1), and the range $(2,6)$ for the exponent p is called the "subcritical case" (see Section 7 for the critical case where $p = 6$).

Therefore, in trying to investigate ground states, we shall be concerned with the case where $\mathcal{G}$ does *not* satisfy assumption (H).

A first result, regardless of ground states, is that the *ground state energy level* is always *intermediate* between the half-soliton's (on the real halfline) and the soliton's (on real the line) of the same mass μ. More precisely, we have the following

Theorem 6.1 (Level-pinching). *For every non–compact graph $\mathcal{G}$,*

$$E(\phi_{2\mu},\mathbb{R}^+) \quad \le \quad \inf_{v\in H^1_\mu(\mathcal{G})} E(v,\mathcal{G}) \quad \le \quad E(\phi_\mu,\mathbb{R})$$

The first inequality is due to rearrangements: as explained in Sec. 5.1, given $v \in H^1(\mathcal{G})$, its decreasing rearrangement v^* (over $\mathbb{R}^+$) has a lower (possibly equal) energy. In other words, no function v on $\mathcal{G}$ can ever beat the half-soliton on $\mathbb{R}^+$.

The second inequality (as explained in Section 4) is due to the possibility of constructing "quasi-solitons" escaping at ∞, along any half-line of $\mathcal{G}$ (since $\mathcal{G}$ is noncompact, at least one of its edges must be unbounded, i.e. a half-line). More precisely, $\mathcal{G}$ contains arbitrarily large intervals (in any half-line), and these intervals can be used to support functions arbitrarily close to a soliton of mass μ.

Of particular relevance is the case where the *second inequality is strict.*

Theorem 6.2 (Existence of ground states). *If $\mathcal{G}$ is non–compact and*

$$\inf_{v\in H^1_\mu(\mathcal{G})} E(v,\mathcal{G}) \quad < \quad E(\phi_\mu,\mathbb{R}),$$

then the infimum in (5) *is attained, i.e. $\mathcal{G}$ supports a ground state.*

Observe that the non–strict inequality "$\leq$" is always satisfied, due to the level–pinching inequality (Thm. 6.1).

The proof (see [9]) is quite delicate, and is based on the following *dichotomy principle* for minimizing sequences (relative to the infimum in (5)). It turns out that, in general, any minimizing sequence $\{u_n\}$ is *either*

(i) weakly convergent to zero, *or*

(ii) strongly convergent to a ground state.

But it can be proved that (i) is (in this case) *incompatible* with the assumption of the theorem, because u_n would then "escape to ∞" along a halfline of $\mathcal{G}$, approaching the shape of a soliton, and its energy level would then be *equal* to (and *not* less than) the energy level of the soliton, in the limit.

The previous result is quite abstract, but it has the following consequence, which is of quite practical use in the applications.

Corollary 6.1 (Operative version of the existence theorem). *If there exists a competitor $u\in H^1_\mu(\mathcal{G})$ such that $E(u,\mathcal{G})\leq E(\phi_\mu,\mathbb{R})$, then $\mathcal{G}$ admits a ground state.*

A sketch of the proof is as follows. Let u be a competitor satisfying the assumption of the theorem: if, by any chance, u is a ground state, then there is nothing to prove. Otherwise u is not optimal, which amounts to

$$\inf_{v\in H^1_\mu(\mathcal{G})} E(v,\mathcal{G}) < E(u,\mathcal{G}) \leq E(\phi_\mu,\mathbb{R}),$$

but in this case a ground state still exists (other than u) by the previous Theorem.

This corollary is quite useful in several concrete cases, where one can try to obtain estimates (on the ground state energy level) by graph surgery: starting from a soliton ϕ_μ on $\mathbb{R}$, one can try to "fit it to $\mathcal{G}$", without increasing its energy. Whenever this can be done, the Theorem guarantees that $\mathcal{G}$ admits a ground state.

A simple example where this can be done is the real line with a pendant, that is, the graph in Fig. 21.

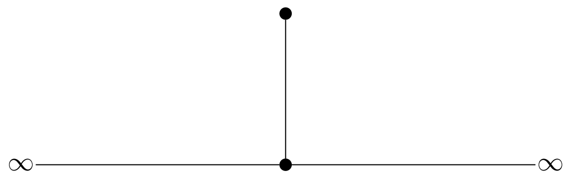

Figure 21: A line with one pendant (bounded edge) attached to it

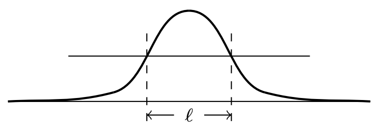

Figure 22: First step: cut the head of the soliton

Theorem 6.3. *Let $\mathcal{G}$ be the real line with a pendant of length ℓ. Then*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) < E(\phi_\mu, \mathbb{R}),$$

so that $\mathcal{G}$ admits a ground state.

The idea of the proof goes as follows. Due to the previous corollary, it suffices to construct a function $u \in H^1_\mu(\mathcal{G})$ such that $E(u, \mathcal{G}) < E(\phi_\mu, \mathbb{R})$, and this can be done by *graph surgery* combined with *rearrangements*, as follows.

(1) Take the soliton ϕ_μ centred at zero and "cut it" at a width ℓ (Fig. 22,23).

(2) Join the *two* resulting soliton *tails* at their maximum, and place them on the line in $\mathcal{G}$, with the maximum at the vertex (Fig. 24).

(3) *Rearrange* the head of the soliton to a monotone function on the interval $[0, \ell]$ (Fig. 25).

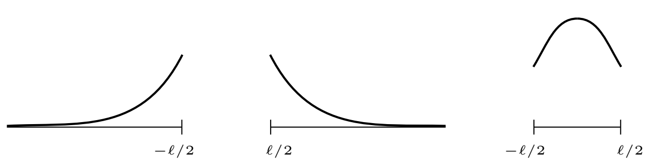

Figure 23: One is left with one head and two tails

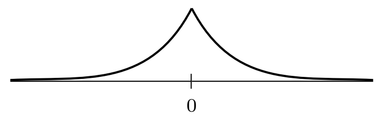

Figure 24: Second step: glue the two tails together

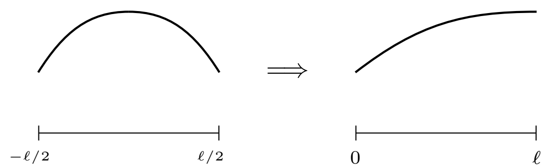

Figure 25: Third step: rearrange monotonically the head of the soliton

This monotone rearrangement lowers the energy level of this portion of function.

(4) The function on the interval can be attached to the function on the line, thus building a function on $\mathcal{G}$ (Fig. 26):

In this way, one produces a function $u \in H^1_\mu(\mathcal{G})$ such that

$$E(u, \mathcal{G}) < E(\phi_\mu, \mathbb{R})$$

(the strict inequality is due to the rearrangement performed on the interval, "from symmetric to monotone", in step (3)). By the existence theorem, then, $\mathcal{G}$ admits a ground state.

We point out that we did *not* construct the ground state, but just a *competitor* u, with an energy level lower than the soliton's.

Other examples of graphs where the corollary can be successfully applied are shown in Fig. 27, 28.

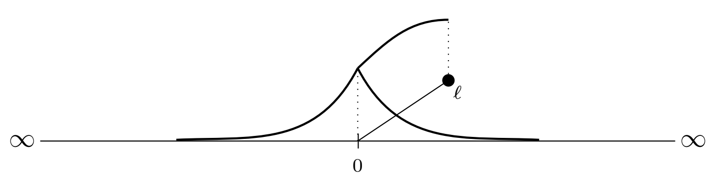

Figure 26: Last step: mount the function on $\mathcal{G}$

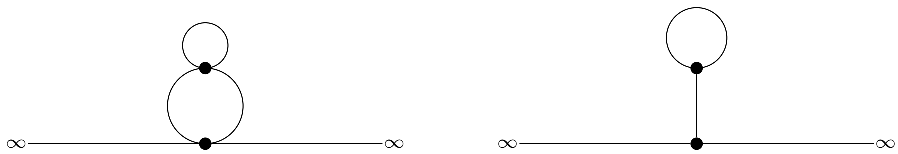

Figure 27: Left: A line with a tower of bubbles. Right: A signpost graph.

Figure 28: Left: A tadpole graph. Right: A 3-fork graph

For each of these graphs, let us shortly see how one can build a function $u \in H^1_\mu(\mathcal{G})$ such that $E(u, \mathcal{G}) \leq E(\phi_\mu, \mathbb{R})$, and thus prove the existence of a ground state.

The first case, the so called "bubble towers", are graph of the kind portrayed in Fig. 29 (as already seen in Theorem 5.1):

Each of them is obtained from $\mathbb{R}$, with the identification of some pairs of opposite points:

$$x_j \sim -x_j, \quad j = 1, \ldots, n \quad (n \text{ bubbles})$$

The symmetry of these graphs enables them to support a soliton ϕ_μ, explot-ing the even symmetry of the soliton:

$$\phi_\mu(x_j) = \phi_\mu(-x_j), \qquad j = 1, \ldots, n.$$

As Fig. 30 shows, a soliton ϕ_μ can indeed be *folded* and placed, *isometrically*, on the line with two bubbles:

Thus, in a sense, $\mathcal{G}$ "supports" a soliton ϕ_μ and this fact, combined with the level-pinching inequality, shows that

$$\inf_{v \in H^1_\mu(\mathcal{G})} E(v, \mathcal{G}) = E(\phi_\mu, \mathbb{R}).$$

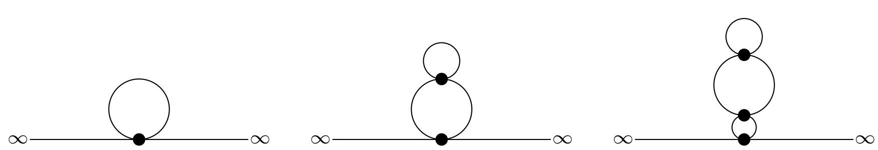

Figure 29: Some examples of bubble towers

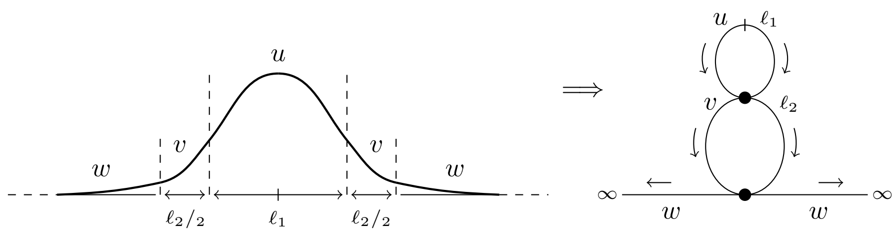

Figure 30: How to cut a soliton to fix it on a given bubble tower

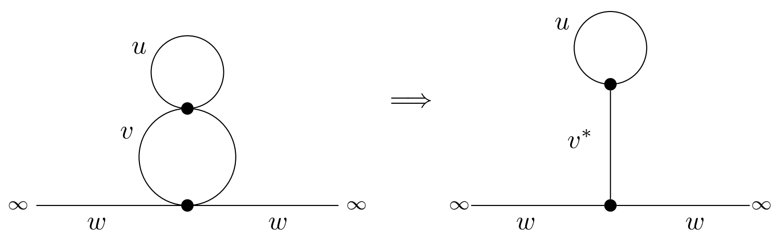

Figure 31: From a bubble tower to a signpost

This "folded soliton" is therefore not just a competitor, but precisely the *ground state*.

In a similar way, one can see that any tower of bubbles supports a (suitably folded) soliton, hence any tower of bubbles has a ground state, and it is not difficult to show that the ground state is unique, up to multiplication by a phase.

Also in the second example, the "signpost graph", there is a ground state. Indeed, a soliton ϕ_μ, initially folded on a "double bubble", can be *partially rearranged* and fitted to the signpost (see Fig. 31)

In the above picture, v^* denotes the monotone rearrangement of v (from the circle to an interval of the same length, that is, "from symmetric to monotone"). The loss of preimages in passing from v (regarded as an even function) to v^* (regarded as a decreasing function) makes the energy decrease and go *below* $E(\phi_\mu, \mathbb{R})$. As before, we did not build a ground state, just a good competitor.

Also in the case of the "tadpole graph" we can *partially rearrange* the competitor alredy built on the double bubble (see Fig. 32)

Here, in addition to the rarrangement of v, we also rearrange w (from the real line) to w^* (to the half-line), which further decreases energy.

Finally, a similar procedure applies to the case where the graph is a "3–fork".

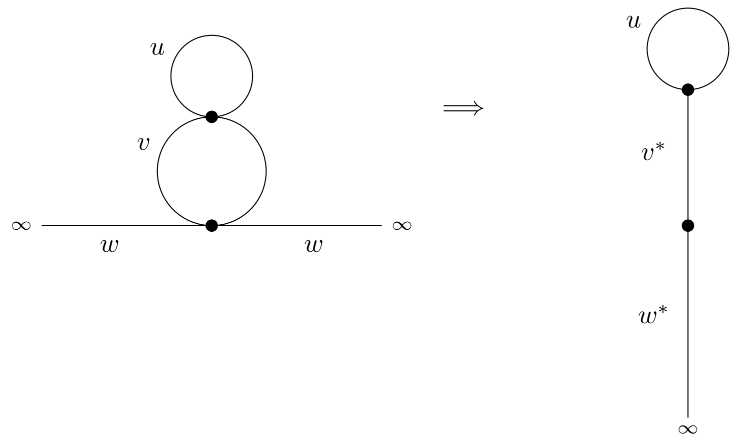

Figure 32: From bubble tower to tadpole

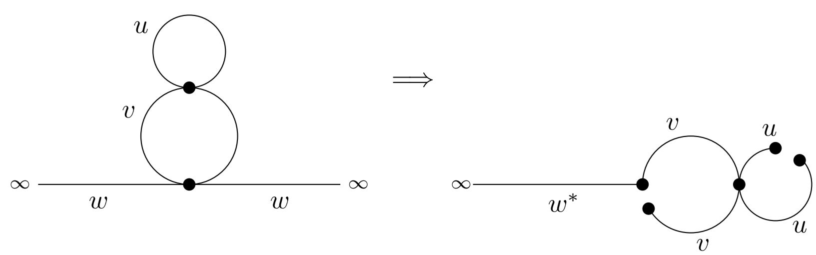

Figure 33: Open the arcs in the bubble tower

Starting from the competitor on a double bubble, we first "open up" the two circles corresponding to the two bubbles, and we rearrange w to w^* (on the half-line), as illustrated in Fig. 33

Now, the arc of circle with the free endpoint from the lower bubble, and the two arcs of circle from the upper bubble, can be seen as the three bounded edges forming the fork (see Fig. 34). Of course, by a proper choice of the size of the bubbles and the cut-points, a 3-fork with edges of any size (not necessarily equal) can be handled.

In the examples we have seen, the *topology* of $\mathcal{G}$ was enough to guarantee a ground state, while the metric of $\mathcal{G}$ (i.e. the lengths of its edges) was irrelevant.

However, in general, things are more complicated, and also the *metric* of $\mathcal{G}$ may play a role.

We will consider two examples where this is the case:

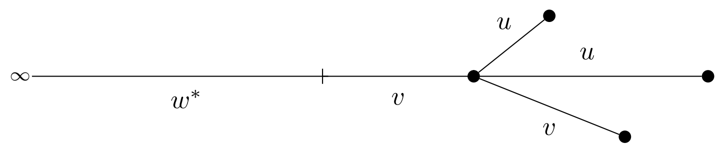

Figure 34: Reconstruct the 3-fork

Figure 35: A graph with one halfline

- graphs with just one half-line;
- graphs where *phase transitions* occur, from existence to nonexistence of ground states, if we vary the length of just *one* edge.

Let $\mathcal{G}$ be a graph with just *one half-line* (Fig. 35).

The main question is, of course, whether $\mathcal{G}$ admits a ground state for every value of the prescribed mass μ. This is nontrivial, since one can check that such a graph (due to the presence of *just one* half-line) does not satisfy assumption (H), and hence the existence of ground states cannot be a priori ruled out.

As we have seen, several examples of graphs with just one half-line (tadpole, 2–fork, 3–fork) indeed admit a ground state for every μ.

However, this is not true in general, and *counterexamples* can be constructed.

Let $\mathcal{K}$ be any compact graph, and let $\mathcal{G}$ be the graph obtained by attaching *one half-line* to $\mathcal{K}$ (Fig. 36):

Theorem 6.4. *There exists $\varepsilon > 0$ such that if*

$$\mu^\beta \operatorname{diam}(\mathcal{K}) + \frac{1}{\mu^\beta \operatorname{length}(\mathcal{K})} < \varepsilon, \qquad \beta = \frac{p-2}{6-p},$$

then $\mathcal{G}$ has no ground state with mass μ.

Figure 36: A graph obtained by attaching a half-line to a compact graph $\mathcal{K}$.

Figure 37: The n-fork graph

The proof is quite involved and requires several sharp estimates, the interested reader is referred to [9]. The main idea, however, is simple: a *small diameter* of the compact core $\mathcal{K}$, combined with a *long total length*, rules out ground states, because *tangled compact parts* are not energetically convenient. Any competitor u, due to the structure of $\mathcal{K}$, is indeed forced to either *oscillate* (and thus have many preimages) or, on the contrary, to be *almost constant*, and neither behaviour is energetically convenient if u is compared to a soliton of the same mass.

A concrete case where this result applies is when $\mathcal{G}$ has the shape of an *n–fork*, namely n terminal edges of length ℓ, attached to a half-line (Fig. 37).

In this case, we clearly have $\operatorname{diam}(\mathcal{K}) = 2\,\ell$ and $\operatorname{length}(\mathcal{K}) = n\,\ell$. If we fix the value of the mass $\mu > 0$, and then we take ℓ *small enough* (depending on μ) and n *large enough* (depending on μ and ℓ), then the theorem applies, and the resulting $\mathcal{G}$ has no ground state.

Explicit computations show that, at least when $p = 4$, any $n \geq 5$ is *sufficient* for the counterexample, while on the other hand $n > 3$ is *necessary*, because we know that any 3–fork has a ground state.

Finally, it is not known whether one can build the counterexample with $n = 4$, that is, it is not known whether a 4-fork always has a ground state.

Now we discuss an example of a metric graph $\mathcal{G}$ such that varying the length of just one edge (without affecting the topology of $\mathcal{G}$) may lead from existence to nonexistence of a ground state.

Let $\mathcal{G}_\ell$ consist of three half-lines and one *terminal edge* of length ℓ, all emanating from a common vertex (Fig. 38)

Clearly, as long as $\ell > 0$, the topology of $\mathcal{G}_\ell$ is *independent* of the length ℓ. Nevertheless, we have the following

Theorem 6.5 (phase transition). *There exists a critical length $\ell^* > 0$ such that:*

$$\mathcal{G}_\ell \text{ has a ground state} \quad \Longleftrightarrow \quad \ell \geq \ell^*.$$

The idea of the proof is that, once the mass μ has been fixed, if ℓ is long enough then $\mathcal{G}_\ell$ has a ground state (this is true in general, as soon as a graph

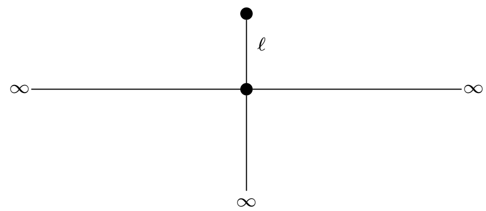

Figure 38: A graph made of three half-lines and a terminal edge

$\mathcal{G}$ has a long enough terminal edge), which resembles a half-soliton with the head at the tip of the bounded edge. On the other hand, if $\mathcal{G}_\ell$ had a ground state for every $\ell > 0$, then by a compactness argument also the 3-star graph $\mathcal{G}_0$ would inherit a ground state, which is a contradiction since $\mathcal{G}_0$ is known to have no ground state. This shows that at least one transition (from existence to nonexistence of a ground state) must occur, as ℓ is decreased: then, the fact that *exactly one* transition occurs requires a more careful analysis, based on a monotonicity argument.

This example shows that, in general, the *topology* of $\mathcal{G}$ *is not enough, alone,* to establish whether $\mathcal{G}$ has a ground state of a given mass, and also the *metric properties* of $\mathcal{G}$ (together with its topology) should be considered.

7 - The critical case: $p = 6$

In this section we describe some results concerning the existence of ground states for the *critical* NLS energy functional

$$E(u, \mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{6}\int_{\mathcal{G}} |u|^6\,dx$$

on the space $H^1_\mu(\mathcal{G})$, where $\mathcal{G}$ is a noncompact metric graph (see Fig. 1). The content of this section refers to [10].

According to (6), the solutions to (5) are solutions of the L^2–critical stationary NLS equation

$$u'' + u^5 = \omega u \qquad \text{on } \mathcal{G},$$

with Kirchhoff conditions (7) at each vertex v of the graph.

This problem is much more delicate than the subcritical one, where the exponent in the nonlinearity lies in the interval $(2, 6)$. One of the reasons is that, as discussed in Sec. 2.1 under the formal mass-preserving transformation

$$u(x) \mapsto u_\lambda(x) = \sqrt{\lambda}u(\lambda x),$$

the kinetic and the potential terms in E scale in the same way:

$$E(u_\lambda, \lambda^{-1}\mathcal{G}) = \lambda^2 E(u, \mathcal{G}), \tag{11}$$

which is typical of problems with *serious* loss of compactness.

In the critical case the problem depends very strongly on the mass μ and on the ground state energy function $\mathcal{E}_{\mathcal{G}}(\mu)$ defined in (5), which will play a central role in all of our results.

Example 7.1. *The real line ($\mathcal{G} = \mathbb{R}$). The situation is very different from the one encountered in Sec. 3.1 for the subcritical case.*
Indeed, it is known that there exists a number $\mu_{\mathbb{R}} > 0$, the critical mass, *such that*

$$\mathcal{E}_{\mathbb{R}}(\mu) = \begin{cases} 0 & \text{if } \mu \le \mu_{\mathbb{R}} \\ -\infty & \text{if } \mu > \mu_{\mathbb{R}} \end{cases} \qquad \Big(\mu_{\mathbb{R}} = \pi\sqrt{3}/2\Big).$$

Moreover $\mathcal{E}_{\mathbb{R}}(\mu)$ is attained if and only if $\mu = \mu_{\mathbb{R}}$. *The* ground states, *called solitons, form a quite large family: up to phase and translations, they can be written as*

$$\phi_\lambda(x) = \sqrt{\lambda}\phi(\lambda x), \qquad \lambda > 0,$$

where $\phi(x) = \operatorname{sech}^{1/2}(\frac{2}{\sqrt{3}}x)$.

Example 7.2. *The half-line ($\mathcal{G} = \mathbb{R}^+$). Again, there exists a number $\mu_{\mathbb{R}^+} = \mu_{\mathbb{R}}/2$, such that*

$$\mathcal{E}_{\mathbb{R}^+}(\mu) = \begin{cases} 0 & \text{if } \mu \le \mu_{\mathbb{R}^+} \\ -\infty & \text{if } \mu > \mu_{\mathbb{R}^+} \end{cases} \qquad \Big(\mu_{\mathbb{R}^+} = \pi\sqrt{3}/4\Big).$$

Moreover $\mathcal{E}_{\mathbb{R}}(\mu)$ is attained if and only if $\mu = \mu_{\mathbb{R}^+}$. *The ground states (half-solitons) are the restrictions to $\mathbb{R}^+$ of the family ϕ_λ.*

Thus on the standard domains $\mathbb{R}$ and $\mathbb{R}^+$ the minimization process (5) is *extremely unstable*, with solutions existing for a *single value* of the mass.

This behavior is due to the same homogeneity of the kinetic and potential terms under mass-preserving scalings and the invariance of $\mathbb{R}$ and $\mathbb{R}^+$ under dilations.

On a generic noncompact graph $\mathcal{G}$ however, the problem can be highly nontrivial and entirely new phenomena may arise, depending on the topology of the graph.

Here we describe these new phenomena, essentially by classifying all graphs from the point of view of existence of ground states.

7.1 - *The critical mass*

The first thing to do is to understand the appearance of the critical mass $\mu_{\mathbb{R}}$ (or $\mu_{\mathbb{R}^+}$) in the problems on classical domains and to identify the same notion for general graphs. This is carried out by analyzing the Gagliardo-Nirenberg inequality, a fundamental tool in all the existence proofs.

The Gagliardo–Nirenberg inequality on $\mathbb{R}$ reads

$$\|u\|_6^6 \le C\|u\|_2^4 \cdot \|u'\|_2^2 \qquad \forall u \in H^1(\mathbb{R}).$$

The *best constant* (the smallest C) is

$$K_{\mathbb{R}} = \sup_{\substack{u\in H^1(\mathbb{R})\\ u\not\equiv 0}} \frac{\|u\|_6^6}{\|u\|_2^4 \cdot \|u'\|_2^2} = \sup_{u\in H^1_\mu(\mathbb{R})} \frac{\|u\|_6^6}{\mu^2 \cdot \|u'\|_2^2}.$$

Therefore

$$\|u\|_6^6 \le K_{\mathbb{R}}\mu^2\|u'\|_2^2 \qquad \forall u \in H^1_\mu(\mathbb{R}).$$

Now for every $u \in H^1_\mu(\mathbb{R})$,

$$\begin{aligned} 6E(u,\mathbb{R}) &= 3\|u'\|_2^2 - \|u\|_6^6 \ge 3\|u'\|_2^2 - K_{\mathbb{R}}\mu^2\|u'\|_2^2 \\ &= \|u'\|_2^2\left(3 - K_{\mathbb{R}}\mu^2\right) \end{aligned}$$

so that

$$\mu^2 \le 3/K_{\mathbb{R}} \implies E(u,\mathbb{R}) \ge 0 \qquad \text{for all } u \in H^1_\mu(\mathbb{R}).$$

On the other hand, if $\mu^2 > 3/K_{\mathbb{R}}$, and u is close to optimality in the Gagliardo–Nirenberg inequality, i.e. $\|u\|_6^6 > (K_{\mathbb{R}} - \varepsilon)\mu^2\|u'\|_2^2$, then

$$6E(u,\mathbb{R}) = 3\|u'\|_2^2 - \|u\|_6^6 \le \|u'\|_2^2\left(3 - (K_{\mathbb{R}} - \varepsilon)\mu^2\right) < 0$$

for small ε, and therefore

$$\mu^2 > 3/K_{\mathbb{R}} \implies E(u,\mathbb{R}) < 0 \qquad \text{for some } u \in H^1_\mu(\mathbb{R}).$$

By mass–preserving scalings (11) it is then easy to see that

$$\mu^2 \le 3/K_{\mathbb{R}} \implies \mathcal{E}_{\mathbb{R}}(\mu) = 0,$$

$$\mu^2 > 3/K_{\mathbb{R}} \implies \mathcal{E}_{\mathbb{R}}(\mu) = -\infty.$$

Therefore,

$$\mu_{\mathbb{R}}^2 = \frac{3}{K_{\mathbb{R}}}.$$

This motivates the following definition.

Definition 7.1. The critical mass for a noncompact metric graph $\mathcal{G}$ is the number

$$\mu_{\mathcal{G}} = \sqrt{\frac{3}{K_{\mathcal{G}}}},$$

where $K_{\mathcal{G}}$ is the best constant for the Gagliardo–Nirenberg inequality on $\mathcal{G}$.

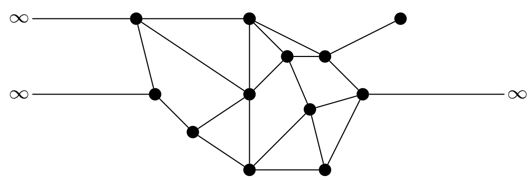

Figure 39: Case 1: a graph with a terminal point

R e m a r k 7.1. It is not difficult to see that for every noncompact $\mathcal{G}$,

$$K_{\mathbb{R}} \leq K_{\mathcal{G}} \leq K_{\mathbb{R}^+}$$

so that

$$\mu_{\mathbb{R}^+} \leq \mu_{\mathcal{G}} \leq \mu_{\mathbb{R}}.$$

Thus every noncompact graph is in this sense intermediate between $\mathbb{R}^+$ and $\mathbb{R}$. In view of the preceding discussion it is easy to prove the following statements

P r o p o s i t i o n 7.1. *Let $\mathcal{G}$ be a noncompact metric graph.*

- *If $\mu \leq \mu_{\mathcal{G}}$, then $\mathcal{E}_{\mathcal{G}}(\mu) = 0$, and is not attained when $\mu < \mu_{\mathcal{G}}$*
- *If $\mu > \mu_{\mathcal{G}}$, then $\mathcal{E}_{\mathcal{G}}(\mu) < 0$ (possibly $-\infty$)*
- *If $\mu > \mu_{\mathbb{R}}$, then $\mathcal{E}_{\mathcal{G}}(\mu) = -\infty$*

C o r o l l a r y 7.1. *A necessary condition for the existence of a ground state of mass μ is that*

$$\mu \in [\mu_{\mathcal{G}}, \mu_{\mathbb{R}}].$$

7.2 - *The results*

The necessary condition of Corollary 7.1 is far from being sufficient. The existence of ground states depends mainly on the topology of the graph $\mathcal{G}$, according to the following four mutually exclusive cases:

1. $\mathcal{G}$ has a terminal point (Fig. 39)
2. $\mathcal{G}$ satisfies Assumption (H) introduced in Sec. 4 (Fig. 40, 41).
3. $\mathcal{G}$ has exactly one half-line and no terminal point (Fig. 42)
4. $\mathcal{G}$ has none of the above properties (Fig. 43).

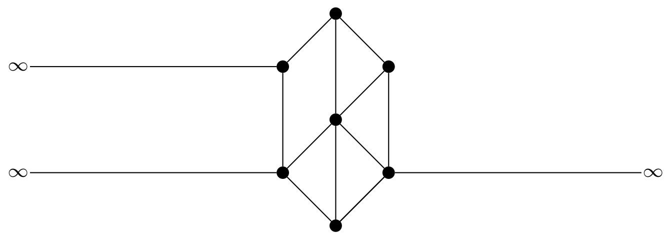

Figure 40: Case 2: a graph satisfying assumption (H)

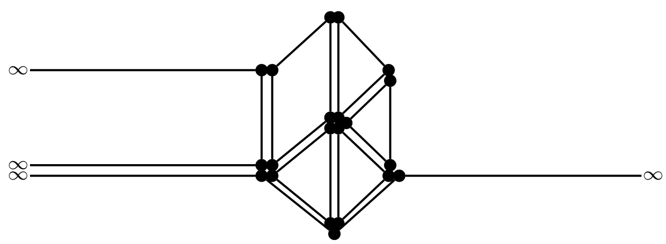

Figure 41: The graph portrayed in Fig. 40 and a covering made of 7 cycles

We now list the main results, case by case.

Theorem 7.1 (Case 1). *Assume that $\mathcal{G}$ has at least one terminal point. Then*

- $\mu_{\mathcal{G}} = \mu_{\mathbb{R}^+}$
- *when* $\mu \in (\mu_{\mathbb{R}^+}, \mu_{\mathbb{R}}]$, $\mathcal{E}_{\mathcal{G}}(\mu) = -\infty$
- *when* $\mu = \mu_{\mathbb{R}^+}$, $\mathcal{E}_{\mathcal{G}}(\mu) = 0$ *but is attained if and only if $\mathcal{G}$ is a half-line.*

The result shows that in the presence of a terminal point ground states do not exist (except when $\mathcal{G} = \mathbb{R}^+$).

The terminal edge behaves like $\mathbb{R}^+$, almost supporting a half-soliton. The "almost" however cannot be eliminated, resulting in nonexistence of ground states.

It is easy to check the second statement in the theorem. Indeed, take u compactly supported on $\mathbb{R}^+$, with $\|u\|_2^2 > \mu_{\mathbb{R}^+}$, and such that $E(u, \mathbb{R}^+) < 0$. The last condition can be fulfilled because the mass of u is strictly larger than $\mu_{\mathbb{R}^+}$, the critical mass for the half-line. Now scale u by introducing $u_\lambda(x) = \sqrt{\lambda} u(\lambda x)$, with λ so large that the support of u_λ is contained in an interval shorter than the terminal edge of $\mathcal{G}$. Place u_λ on the terminal edge of $\mathcal{G}$ and extend it to zero elsewhere on $\mathcal{G}$. Then

$$E(u_\lambda, \mathcal{G}) = E(u_\lambda, \mathbb{R}^+) < 0,$$

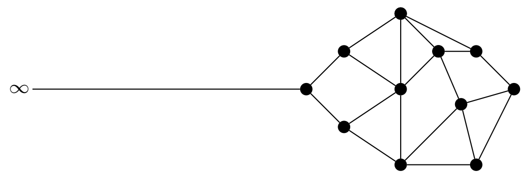

Figure 42: Case 3: a graph with exactly one half-line.

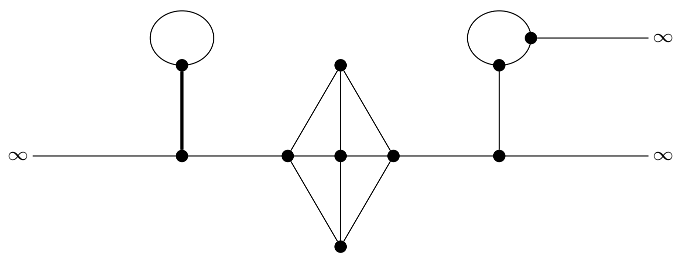

Figure 43: Case 4: a graph without terminal points, without cycle covering and with more than one half-line. No cycle can cover the thick edge.

so that

$$\lim_{\lambda\to\infty} E(u_\lambda, \mathcal{G}) = \lim_{\lambda\to\infty} E(u_\lambda, \mathbb{R}^+) = \lim_{\lambda\to\infty} \lambda^2 E(u, \mathbb{R}^+) = -\infty.$$

Theorem 7.2 (Case 2). *Assume that $\mathcal{G}$ satisfies Assumption (H), so that it has a cycle covering. Then*

- $\mu_\mathcal{G} = \mu_\mathbb{R}$
- $\mathcal{E}_\mathcal{G}(\mu_\mathbb{R}) = 0$ *and is attained if and only if* $\mathcal{G}$ *is* $\mathbb{R}$ *or a tower of bubbles.*

This result shows that in the presence of a cycle covering ground states do not exist, except when $\mathcal{G}$ is $\mathbb{R}$ or a tower of bubbles (Fig. 29).

Theorem 7.3 (Case 3). *Assume that $\mathcal{G}$ has exactly one half-line and no terminal point. Then*

- $\mu_\mathcal{G} = \mu_{\mathbb{R}^+}$
- $\mathcal{E}_\mathcal{G}(\mu) < 0$ *(and finite) for every* $\mu \in (\mu_{\mathbb{R}^+}, \mu_\mathbb{R}]$
- $\mathcal{E}_\mathcal{G}(\mu)$ *is attained if and only if* $\mu \in (\mu_{\mathbb{R}^+}, \mu_\mathbb{R}]$

This result unveils totally new phenomena: first of all, ground states exist for a *whole interval* of masses, a feature that is completely absent on the standard

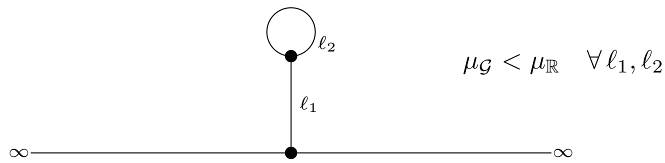

Figure 44: A signpost graph: for every value of ℓ_1, ℓ_2, this graph satisfies the hypotheses of Thm. 7.4

domains $\mathbb{R}$ and $\mathbb{R}^+$. Secondly, ground states have *negative energy*, which is normal for subcritical problems, but highly unexpected in the L^2–critical case. The ultimate reason for this is the nontrivial topology of certain graphs with respect to that of $\mathbb{R}$ or $\mathbb{R}^+$.

The proof of Theorem 7.3, is *very* involved. A sketch of some key steps will be given in Section 7.3.

We conclude with the last result, whose structure is a bit different from that of the preceding Theorems.

Theorem 7.4 (Case 4). *Assume that $\mathcal{G}$ has no terminal point, no cycle covering and more than one half-line. If, in addition,*

$$\mu_{\mathcal{G}} < \mu_{\mathbb{R}},$$

then

- $\mathcal{E}_{\mathcal{G}}(\mu) < 0$ *(and finite) for every* $\mu \in (\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$
- $\mathcal{E}_{\mathcal{G}}(\mu)$ *is attained if and only if* $\mu \in [\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$

The same comments of Theorem 7.3 apply: again ground states exist for a whole interval of masses, and again ground states have negative energy. This time however a new feature appears: ground states exist also for $\mu = \mu_{\mathcal{G}}$. This fact is particularly interesting from the functional analytic point of view. Indeed, since $\mathcal{E}_{\mathcal{G}}(\mu_{\mathcal{G}}) = 0$, any sequence such that $\|u_n'\|_{L^2(\mathcal{G})} \to 0$ is a minimizing sequence and clearly compactness is lost at this level: there exist minimizing sequences at level zero that are *not* precompact. However, a minimizer exists. To obtain a ground state it is therefore necessary to select accurately a particular minimizing sequence, in order to avoid falling onto a bad sequence.

Finally, some comments on the assumption $\mu_{\mathcal{G}} < \mu_{\mathbb{R}}$ are in order.

There are graphs where it is automatically satisfied, for example the *signpost* graph (44), independently of the lengths ℓ_1, ℓ_2:

The existence of graphs of the fourth kind where $\mu_{\mathcal{G}} = \mu_{\mathbb{R}}$ is an open problem. We conjecture that in this case the sole topology of the graph is not enough to guarantee the existence of ground states. Most likely the metric properties of the graph play a role too in this case.

7.3 - *Some key steps of the existence proofs*

The main ingredient of the existence proofs in Theorems 7.3 and 7.4 is the following modified Gagliardo-Nirenberg inequality, whose proof is technically very involved.

Lemma 7.1 (Modified Gagliardo–Nirenberg inequality). *Assume that $\mathcal{G}$ has no terminal point and let $\mu \leq \mu_{\mathbb{R}}$. For every $u \in H^1_\mu(\mathcal{G})$ there exists $\theta \in [0, \mu]$ such that*

$$\|u\|^6_{L^6(\mathcal{G})} \leq K_{\mathbb{R}}(\mu - \theta)^2 \|u'\|^2_{L^2(\mathcal{G})} + C\theta^{1/2}, \tag{12}$$

with C depending only on $\mathcal{G}$.

We recall that the standard Gagliardo-Nirenberg inequality (with best constant) reads

$$\|u\|^6_{L^6(\mathcal{G})} \leq K_{\mathcal{G}} \mu^2 \|u'\|^2_{L^2(\mathcal{G})} \qquad \forall u \in H^1_\mu(\mathcal{G}).$$

In (12) the constant θ depends on u. However, the inequality holds with a *smaller* constant ($K_{\mathbb{R}} \leq \mathcal{K}_{\mathcal{G}}$), a *smaller* mass ($\mu - \theta \leq \mu$) and the price is reasonable: $C\theta^{1/2} \leq C\mu^{1/2}$.

With this inequality, it is simple to show that minimizing sequences are bounded, which is the first (and in this case more delicate) step towards an existence result.

Indeed, take $\mu \in (\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$. Then, by Proposition 7.1, $\mathcal{E}_{\mathcal{G}}(\mu) < 0$, say $\mathcal{E}_{\mathcal{G}}(\mu) < -\alpha < 0$.

Let now u_n be a minimizing sequence for E and let θ_n be the constant in (12) associated to u_n. Then, for n large,

$$\begin{aligned}
-6\alpha \geq 6E(u_n, \mathcal{G}) &= 3\|u_n'\|_2^2 - \|u_n\|_6^6 \geq \text{(by (12))} \\
&\geq 3\|u_n'\|_2^2 - K_{\mathbb{R}}(\mu - \theta_n)^2 \ \|u_n'\|_2^2 - C\theta_n^{1/2} \\
&\geq \|u_n'\|_2^2 \ \left(3 - K_{\mathbb{R}}(\mu - \theta_n)^2\right) - C\theta_n^{1/2} \\
&\geq -C\theta_n^{1/2},
\end{aligned}$$

since

$$3 - K_{\mathbb{R}}(\mu - \theta_n)^2 \geq 3 - K_{\mathbb{R}}\mu^2 \geq 3 - K_{\mathbb{R}}\mu_{\mathbb{R}}^2 = 0.$$

This shows that

$$C\theta_n^{1/2} \geq 6\alpha,$$

uniformly in n.

For this reason, $3 - K_{\mathbb{R}}(\mu - \theta_n)^2 \geq \delta$, so that

$$-6\alpha \geq \frac{\delta}{6}\|u_n'\|_2^2 - C\theta_n^{1/2}$$

from which we see that $\|u_n'\|_2$ is uniformly bounded. Once this is established, it is also very easy to see that $\|u_n\|_{L^p(\mathcal{G})}$ is uniformly bounded too. Then one can extract suitable subsequences and the proof of existence follows easily (in most cases). Only in Theorem 7.4 (when $\mu = \mu_{\mathcal{G}}$) a supplementary analysis is needed in order to construct a particular minimizing sequence.

8 - Conclusions and perspectives

Inspired by physical applications, the problem of the existence of ground states for the focusing NLS on branched structures has proved challenging from the mathematical point of view too, giving rise to a new chapter in the Calculus of Variations, in which established techniques mix with graph theoretical notions and results. In particular, topology and metric of a graph interact in a highly nontrivial way, so that investigating the existence of a ground state can involve either topological consideration or hard estimates.

The results we presented here focus on graphs with a finite number of edges and vertices, that include at least one halfline. This means that, on the large spatial scale, all these graphs look as star graphs, and the compact core plays the role of a vertex, possibly with an internal structure. This large-scale point of view has never been seriously considered, but it could be effective in order to describe ground states at low masses.

However, in all these examples the large-scale structure is still the same of a network, while it would be interesting to consider examples in which such a structure becomes genuinely two-dimensional, reconstructing for instance a stripe in the plane or even the entire plane. For the first example, one could consider the case of a graph made of two parallel halflines joined together through infinitely many parallel edges, in such a way that the distance between two consecutive edges is constant (infinite ladder graph); for the second case, the most immediate example is surely given by the square grid. We are currently investigating this case, and we found that the two-dimensional large-scale structure plays a very important role resulting in a substantial change in the kind of results we are proving. Furthermore, periodicity avoids the presence of halflines and then of quasi-solitons, so that lack of compactness in minimizing sequences can be due either by spreading or by concentration only.

Beyond the problem of ground states, the issue of the existence and the shape of generic standing waves is very topical and will be addressed in forthcoming papers too.

Far beyond these investigations, one could also think of the possibility of approximating regular domains in more dimensions with metric graphs becoming more and more dense. This research line is very likely for a future long-term project.

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