A POINT INTERACTION FOR THE DISCRETE SCHRÖDINGER OPERATOR AND GENERALIZED CHEBYSHEV POLYNOMIALS

D. R. YAFAEV

Abstract. We consider semi-infinite Jacobi matrices corresponding to a point interaction for the discrete Schrödinger operator. Our goal is to find explicit expressions for the spectral measure, the resolvent and other spectral characteristics of such Jacobi matrices. It turns out that their spectral analysis leads to a new class of orthogonal polynomials generalizing the classical Chebyshev polynomials.

1. Introduction

1.1. As is well known, the theories of Jacobi operators given by three-diagonal matrices (see Subsection 2.1, for the precise definitions) and of differential operators $Da(x)D + b(x)$, $D = -id/dx$, are to a large extent similar. This is true for Jacobi operators acting in the space $\ell^2(\mathbb{Z})$ and differential operators acting in the space $L^2(\mathbb{R})$ as well as for the corresponding operators acting in the spaces $\ell^2(\mathbb{Z}_+)$ and $L^2(\mathbb{R}_+)$, respectively. We refer to the book [11] where this analogy is described in a very detailed way. Both classes of the operators are very important in applications. For example, Jacobi operators play a substantial role in solid state physics (see, e.g., §1.5 of [11]) while the Schrödinger operator is the basic object of quantum mechanics. Moreover, Jacobi operators in the space $\ell^2(\mathbb{Z}_+)$ are intimately related (see, e.g., the classical book [1]) to the theory of orthogonal polynomials. We refer to the books [9, 10] for all necessary information on orthogonal polynomials.

We study Jacobi operators given in the space $\ell^2(\mathbb{Z}_+)$ by matrices

$$H_a = \frac{1}{2}\begin{pmatrix} 0 & a & 0 & 0 & 0 & \cdots \\ a & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{pmatrix}. \tag{1.1}$$

The parameter a here is an arbitrary positive number. The operator H_1 plays the role of the "free" differential operator D^2 in the space $L^2(\mathbb{R}_+)$ with the boundary condition $f(0) = 0$. We consider H_a as a perturbation of the operator H_1 which is easy to analyze directly. Clearly, the perturbation $H_a - H_1$ has rank two that allows us to

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find the resolvent of the operator H_a. The operator H_a is a rare example where all spectral quantities, such as the spectral measure, eigenfunctions (of the continuous spectrum), the wave operators, the scattering matrix, the spectral shift function, etc., can be calculated explicitly.

We show that eigenfunctions of the operator H_a are constructed in terms of a class of orthogonal polynomials $\mathrm{Ch}_n(z;a)$, $n = 0, 1, \ldots$, generalizing the classical Chebyshev polynomials. It is very well-known that $\mathrm{Ch}_n(z;1)$ are the Chebyshev polynomials of the second kind. It was also noted in the book [2] that $\mathrm{Ch}_n(z;\sqrt{2})$ are the Chebyshev polynomials of the first kind. We are studing Jacobi matrices H_a for all $a > 0$. This naturally leads to a class of polynomials $\mathrm{Ch}_n(z;a)$ parametrized by an arbitrary $a > 0$.

1.2. The relation between Jacobi matrices and orthogonal polynomials is the classical fact. In principle, it can be used in both ways. Let us mention the paper [8] devoted to the inverse problem (of reconstruction of the Jacobi matrix by its spectral measure) where this is thoroughly discussed. The paper [8] contains also numerous references to earlier works on this subject.

We proceed from analysis of the Jacobi operator H_a. In principle our method applies to arbitrary finite rank perturbations of the operator H_1. In this context, we mention paper [4] where the diagonal rank one perturbations of H_1 were considered by the methods of the orthogonal polynomials theory.

We perform the spectral analysis of the operator H_a in a very explicit way. This leads to some formulas for the polynomials $\mathrm{Ch}_n(z;a)$. A part of these formulas is perhaps new even for the classical Chebyshev polynomials.

1.3. Compared to the continuous case, the operator H_a plays the role of the self-adjoint realization A_α of the differential operator D^2 in the space $L^2(\mathbb{R}_+)$ with a boundary condition $f'(0) = \alpha f(0)$ where $\alpha \in \mathbb{R}$ or of the operator $A^{(0)}$ with the boundary condition $f(0) = 0$. The operator $A^{(0)}$ is usually taken for the "free" operator, and the operators A_α are interpreted as Hamiltonians of a point interaction of two quantum particles. Of course the essential spectra of the operators A_α and $A^{(0)}$ coincide with the half-axis $[0, \infty)$, and they coincide with the interval $[-1, 1]$ for all operators H_a. The study of the operators H_a is more difficult and the formulas obtained are less trivial than for the operators A_α. Actually, the operators H_a are more close to self-adjoint realizations of the operator D^2 in the space $\mathbb{C} \oplus L^2(\mathbb{R}_+)$ (see the paper [13] or §4.7 of the book [15]).

Let us discuss the analogy between the operators A_α and H_a in more details. The role of $x \in \mathbb{R}_+$ is played by the variable $n \in \mathbb{Z}_+$, and the role of a function $f(x)$ is played by a sequence $f = (f_0, f_1, \ldots)$. The operator H_1 acts by the formula $(H_1 f)_n = 2^{-1}(f_{n-1} + f_{n+1})$ for all $n \in \mathbb{Z}_+$ if one imposes an artificial "boundary condition" $f_{-1} = 0$. So, the operator H_1 plays the role of $A^{(0)}$. The operators $A_\alpha \geq 0$ for $\alpha \geq 0$, and they have the eigenvalues $-\alpha^2$ for all $\alpha < 0$. The case $\alpha = 0$ is critical: the operator $A_0 \geq 0$, but its resolvent $(A_0 - zI)^{-1}$ has a singularity at the bottom of its spectrum (one says that the operator A_0 has a zero-energy resonance). The operators H_a have discrete eigenvalues (one below -1 and another one above 1) if and only if

$a > \sqrt{2}$. The resolvent $(H_{\sqrt{2}} - zI)^{-1}$ of $H_{\sqrt{2}}$ has singularities at both edge points ± 1 of its spectrum. Thus the operator $H_{\sqrt{2}}$ plays the role of A_0, and the values $a < \sqrt{2}$ (resp. $a > \sqrt{2}$) correspond to the values $\alpha > 0$ (resp. $\alpha < 0$).

To a some extent, this paper was motivated by looking for a discrete analogue of the point interaction very well studied in the continuous case. However, at a technical level, the analogy between the discrete and continuous Schrödinger operators is not used in this paper. Our approach relies on a direct analysis of the resolvent of the operator H_a.

1.4. The paper is organized as follows. Section 2 is of a preliminary nature. First, we briefly recall necessary facts about Jacobi matrices and their link to orthogonal polynomials. Then we study the operator H_1. The results presented here are probably well-known, but we have not found them in the literature in an explicit and convenient for us form. Our approach relies on comparing H_1 with an auxiliary operator $\mathbf{H}$ acting in the space $\ell^2(\mathbb{Z})$ by the formula $(\mathbf{H}f)_n = 2^{-1}(f_{n-1} + f_{n+1})$. The operator $\mathbf{H}$ can be easily diagonalized by the discrete Fourier transform.

Section 3 plays the central role. Here we find the resolvent $R_a(z) = (H_a - zI)^{-1}$ of the operator H_a and its spectral family. We develop scattering theory for the pair H_1, H_a in Section 4. In particular, we calculate the corresponding scattering matrix $S_a(\lambda)$ and the spectral shift function $\xi_a(\lambda)$. We also establish a link between eigenfunctions of the operator H_a and the wave operators for the pair H_1, H_a. The traces $\operatorname{Tr}\big(H_a - H_1\big)$ and the moments of the spectral measure of the operator H_a are calculated in Section 5. In Section 6, we discuss a relation of our results on the operators H_a with results on the corresponding Hankel operators. We also study the limits $a \to 0$ and $a \to \infty$. These limits turn out to be very singular.

We use the letter I for the identity operator and do not distinguish the scalar products in various spaces in notation.

2. Jacobi matrices. The discrete Schrödinger operator

2.1. Let us briefly recall some basic facts about Jacobi operators in the space $\ell^2(\mathbb{Z}_+)$. We denote by e_n, $n \in \mathbb{Z}_+$, the canonical basis in this space, that is, all components of the vector e_n are zeros, except the n-th component which equals 1. Let $a_0, a_1, \ldots$, $b_0, b_1, \ldots$ be some real sequences and $a_n > 0$ for all $n = 0, 1, \ldots$. Then the Jacobi operator H is defined by the formula

$$He_n = a_{n-1}e_{n-1} + b_n e_n + a_n e_{n+1}, \quad n \in \mathbb{Z}_+, \tag{2.1}$$

where we accept that $e_{-1} = 0$. The operator H is obviously symmetric. We assume that the sequences $\{a_n\}$ and $\{b_n\}$ are bounded, and hence H is a bounded operator in the space $\ell^2(\mathbb{Z}_+)$. Let us denote by $dE(\lambda)$ its spectral measure and put $d\rho(\lambda) = d(E(\lambda)e_0, e_0)$. Since the support of the measure $d\rho$ is bounded, the set of all polynomials is dense in the space $L^2(\mathbb{R}; d\rho)$.

Let us define polynomials $P_n(z)$ by the recurrent relations

$$zP_n(z) = a_{n-1}P_{n-1}(z) + b_nP_n(z) + a_nP_{n+1}(z), \quad n \in \mathbb{Z}_+, \quad z \in \mathbb{C}. \tag{2.2}$$

We accept that $P_{-1}(z) = 0$ and put $P_0(z) = 1$. Then $P_n(z)$ is a polynomial of degree n and its coefficient at z^n equals $(a_0 a_1 \cdots a_{n-1})^{-1}$. Clearly, the linear sets spanned by $\{1, z, \dots, z^n\}$ and by $\{P_0(z), P_1(z), \dots, P_n(z)\}$ coincide.

Comparing relations (2.1) and (2.2) and using recursion arguments, we see that

$$e_n = P_n(H) e_0 \tag{2.3}$$

for all $n \in \mathbb{Z}_+$. Thus the set of all vectors $H^n e_0$, $n \in \mathbb{Z}_+$, is dense in $\ell^2(\mathbb{Z}_+)$, and hence the spectrum of the operator H is simple with e_0 being the generating vector. It also follows from (2.3) that

$$d(E(\lambda)e_n, e_m) = P_n(\lambda) P_m(\lambda) d\rho(\lambda) \tag{2.4}$$

whence

$$\int_{-\infty}^{\infty} P_n(\lambda) P_m(\lambda) d\rho(\lambda) = \delta_{n,m} \tag{2.5}$$

where $\delta_{n,n} = 1$ and $\delta_{n,m} = 0$ for $n \neq m$. Of course $\{P_0(\lambda), P_1(\lambda), \dots, P_n(\lambda)\}$ are obtained by the Gram-Schmidt orthogonalization of the set $\{1, \lambda, \dots, \lambda^n\}$ in the space $L^2(\mathbb{R}_+; d\rho)$.

Let us now define a mapping $U : \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}; d\rho)$ by the formula

$$(Ue_n)(\lambda) = P_n(\lambda). \tag{2.6}$$

It is isometric according to (2.5). It is also unitary because the set of all polynomials $P_n(\lambda)$, $n \in \mathbb{Z}_+$, is dense in $L^2(\mathbb{R}; d\rho)$. Finally, the intertwining property

$$(UHf)(\lambda) = \lambda (Uf)(\lambda) \tag{2.7}$$

holds. Indeed, it suffices to check it for $f = e_n$ when according to definition (2.1) $(UHe_n)(\lambda)$ coincides with the right-hand side of (2.1) while $\lambda(Ue_n)(\lambda)$ equals its left-hand side.

We note also that (2.4) ensures the formula

$$\int_{-\infty}^{\infty} P_n(\lambda) P_m(\lambda) (\lambda - z)^{-1} d\rho(\lambda) = (R(z)e_n, e_m), \quad \forall n, m \in \mathbb{Z}_+, \tag{2.8}$$

which yields an expression for the integrals in the left-hand side provided the resolvent $R(z) = (H - zI)^{-1}$ is known.

Thus, for all sequence $\{a_n\}$, $\{b_n\}$, one can construct the measure $d\rho(\lambda)$ such that the polynomials $P_n(\lambda)$ defined by (2.2) are orthogonal in $L^2(\mathbb{R}; d\rho)$. This assertion is known as the Favard theorem. Its standard proof presented here relies on the spectral theorem for self-adjoint operators and does not of course give an explicit expression for the measure $d\rho(\lambda)$. On the contrary, given a measure $d\rho(\lambda)$, one can reconstruct $\{a_n\}$, $\{b_n\}$, that is, the Jacobi matrix (2.1) with the spectral measure $d\rho(\lambda)$. The solution of this (inverse) problem is discussed from various points of view in the article [8].

In this paper we study the case when $a_0 = a/2$, $a_n = 1/2$ for $n \geq 1$ and $b_n = 0$ for all $n \geq 0$.

2.2. Let us first consider the "free" discrete Schrödinger operator (the infinite Jacobi matrix) $\mathbf{H}$ in the space $\ell^2(\mathbb{Z})$ given by the formula

$$\mathbf{H}e_n = \frac{1}{2}(e_{n-1} + e_{n+1}) \tag{2.9}$$

where e_n, $n \in \mathbb{Z}$, is the canonical basis in the space $\ell^2(\mathbb{Z})$. Evidently, the operator $\mathbf{H}$ can be explicitly diagonalized. Indeed, let $\mathcal{F}$,

$$(\mathcal{F}f)(\mu) = \sum_{n\in\mathbb{Z}} f_n \mu^n, \quad \mu \in \mathbb{T}, \quad f = (\dots, f_{-1}, f_0, f_1, \dots), \quad f_n = (f, e_n), \tag{2.10}$$

be the discrete Fourier transform. Let the unit circle $\mathbb{T}$ be endowed with the normalized Lebesgue measure

$$d\mathbf{m}(\mu) = (2\pi i\mu)^{-1} d\mu, \quad \mu \in \mathbb{T}. \tag{2.11}$$

Then the operator $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ is unitary. Since

$$(\mathcal{F}\mathbf{H}f)(\mu) = \frac{\mu + \mu^{-1}}{2} (\mathcal{F}f)(\mu), \tag{2.12}$$

the spectrum of the operator $\mathbf{H}$ is absolutely continuous, has multiplicity 2 and coincides with the interval $[-1, 1]$.

Now it is easy to calculate the resolvent $\mathbf{R}(z) = (\mathbf{H} - zI)^{-1}$ of the operator $\mathbf{H}$. Below we fix the branch of the analytic function $\sqrt{z^2 - 1}$ of $z \in \mathbb{C} \setminus [-1, 1]$ by the condition $\sqrt{z^2 - 1} > 0$ for $z > 1$. Then it equals $\pm i\sqrt{1 - \lambda^2}$ for $z = \lambda \pm i0$, $\lambda \in (-1, 1)$, and $\sqrt{z^2 - 1} < 0$ for $z < -1$.

Lemma 2.1. *For all $n, m \in \mathbb{Z}$, we have*

$$(\mathbf{R}(z)e_n, e_m) = -\frac{(z - \sqrt{z^2 - 1})^{|n-m|}}{\sqrt{z^2 - 1}}. \tag{2.13}$$

Proof. It follows from (2.10) – (2.12) that

$$\begin{aligned} (\mathbf{R}(z)e_n, e_m) &= (2\pi i)^{-1} \int_{\mathbb{T}} \Big(\frac{\mu + \mu^{-1}}{2} - z\Big)^{-1} \mu^{n-m-1} d\mu \\ &= (\pi i)^{-1} \int_{\mathbb{T}} \big(\mu^2 - 2z\mu + 1\big)^{-1} \mu^{n-m} d\mu, \quad z \in \mathbb{C} \setminus [-1, 1]. \end{aligned} \tag{2.14}$$

The equation $\mu^2 - 2z\mu + 1 = 0$ has the roots

$$\mu_\pm = z \pm \sqrt{z^2 - 1}. \tag{2.15}$$

By the proof of (2.13), we may suppose that $z > 1$. Then $\mu_+ > 1$ and $\mu_+\mu_- = 1$. Since $\mathbf{H}$ commutes with the complex conjugation, we have

$$(\mathbf{R}(z)e_n, e_m) = (\mathbf{R}(z)e_m, e_n),$$

so that we can suppose $n \geq m$ in (2.14). Then μ_- in the only singular point of the integrand in (2.14) inside the unit circle. Calculating the residue at this point, we find that

$$(\mathbf{R}(z)e_n, e_m) = 2(\mu_- - \mu_+)^{-1} \mu_-^{n-m}.$$

Substituting here expressions (2.15), we arrive at (2.13). $\square$

2.3. Now we are in a position to study the discrete "free" Schrödinger operator (the semi-infinite Jacobi matrix) H_+ in the space $\ell^2(\mathbb{Z}_+)$ with elements $f = (f_0, f_1, \dots)$. It is defined by the formula

$$H_+ e_0 = \frac{1}{2} e_1, \quad H_+ e_n = \frac{1}{2}(e_{n-1} + e_{n+1}), \quad n \geq 1. \tag{2.16}$$

Of course the operator H_+ is given by matrix (1.1) where $a = 1$.

Our first goal is to calculate the resolvent $R_+(z) = (H_+ - zI)^{-1}$ of the operator H_+. To that end, we introduce an auxiliary operator H_- in the space $\ell^2(\mathbb{Z}_-)$ where $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{Z}_+$ by the formula

$$H_- e_{-1} = \frac{1}{2} e_{-2}, \quad H_- e_{-n} = \frac{1}{2}(e_{-n-1} + e_{-n+1}), \quad n \geq 2. \tag{2.17}$$

The operators H_+ and H_- are of course unitarily equivalent, but we do not need this fact. Let us consider $H = H_+ \oplus H_-$ as an operator in the space $\ell^2(\mathbb{Z}) = \ell^2(\mathbb{Z}_+) \oplus \ell^2(\mathbb{Z}_-)$. Comparing formulas (2.9), (2.16) and (2.17), we see that the difference

$$V = H - \mathbf{H} = -\frac{1}{2}(\cdot, e_0)e_{-1} - \frac{1}{2}(\cdot, e_{-1})e_0 \tag{2.18}$$

is an operator in $\ell^2(\mathbb{Z})$ of rank 2. This allows us to easily obtain an explicit expression for the resolvent $R(z) = (H - zI)^{-1}$ of the operator H.

Let us set

$$T(z) = V - VR(z)V.$$

In terms of the operator $T(z)$, the resolvent equation for the pair $\mathbf{H}$, H can be written as

$$T(z) = V - V\mathbf{R}(z)T(z). \tag{2.19}$$

Denote

$$\omega(z) = z - \sqrt{z^2 - 1} = (z + \sqrt{z^2 - 1})^{-1}. \tag{2.20}$$

Below we often use the obvious identities:

$$1 - \omega(z)^2 = 2\sqrt{z^2 - 1}\,\omega(z), \quad 1 + \omega(z)^2 = 2z\omega(z). \tag{2.21}$$

Lemma 2.2. *The solution $T(z)$ of the equation* (2.19) *is given by the formula*

$$2T(z)f = (-f_0 + f_{-1}\omega(z))e_{-1} + (-f_{-1} + f_0\omega(z))e_0, \quad f_n = (f, e_n). \tag{2.22}$$

Proof. In view of expression (2.18), equation (2.19) implies that

$$2T(z)f = -f_0 e_{-1} - f_{-1} e_0 + (\mathbf{R}(z)T(z)f, e_0)e_{-1} + (\mathbf{R}(z)T(z)f, e_{-1})e_0 \tag{2.23}$$

where $f_n = (f, e_n)$ for all $f \in \ell^2(\mathbb{Z})$. Let us take the scalar products of this equation with $\mathbf{R}(\bar{z})e_{-1}$ and $\mathbf{R}(\bar{z})e_0$. This leads to a system of two equations for $u_{-1}(z) =$

$(\mathbf{R}(z)T(z)f, e_{-1})$ and $u_0(z) = (\mathbf{R}(z)T(z)f, e_0)$:

$$\left.\begin{aligned} 2u_{-1}(z) =& -(\mathbf{R}(z)e_{-1}, e_{-1})f_0 - (\mathbf{R}(z)e_0, e_{-1})f_{-1} \\ & + (\mathbf{R}(z)e_{-1}, e_{-1})u_0(z) + (\mathbf{R}(z)e_0, e_{-1})u_{-1}(z) \\ 2u_0(z) =& -(\mathbf{R}(z)e_{-1}, e_0)f_0 - (\mathbf{R}(z)e_0, e_0)f_{-1} \\ & + (\mathbf{R}(z)e_{-1}, e_0)u_0(z) + (\mathbf{R}(z)e_0, e_0)u_{-1}(z) \end{aligned}\right\}. \tag{2.24}$$

To solve system (2.24), we take into account expression (2.13). Then (2.24) can be rewritten as

$$\left.\begin{aligned} (2\sqrt{z^2-1} + \omega(z))u_{-1}(z) + u_0(z) =& f_0 + f_{-1}\omega(z) \\ u_{-1}(z) + (2\sqrt{z^2-1} + \omega(z))u_0(z) =& f_0\omega(z) + f_{-1} \end{aligned}\right\}$$

which yields

$$u_{-1}(z) = f_0\omega(z), \quad u_0(z) = f_{-1}\omega(z).$$

Therefore the representation (2.22) is a direct consequence of equation (2.23). □

Now we are in a position to calculate matrix elements of the resolvent

$$R(z) = \mathbf{R}(z) - \mathbf{R}(z)T(z)\mathbf{R}(z). \tag{2.25}$$

Proposition 2.3. *Let the operator H_+ be defined in the space $\ell^2(\mathbb{Z}_+)$ by relations (2.16). Then the matrix elements of its resolvent $R_+(z) = (H_+ - zI)^{-1}$ are given by the formula*

$$(R_+(z)e_n, e_m) = \frac{(z-\sqrt{z^2-1})^{n+m+2} - (z-\sqrt{z^2-1})^{|n-m|}}{\sqrt{z^2-1}} =: r_{n,m}(z) \tag{2.26}$$

for all $n, m \in \mathbb{Z}_+$. In particular,

$$(R_+(z)e_0, e_0) = 2(\sqrt{z^2-1} - z). \tag{2.27}$$

Proof. It follows from formula (2.13) and Lemma 2.2 that, for all $n, m \geq 0$,

$$(\mathbf{R}(z)T(z)\mathbf{R}(z)e_n, e_m) = \frac{\omega(z)^{n+m}}{z^2-1}(T(z)(\omega(z)e_{-1} + e_0), \omega(\bar{z})e_{-1} + e_0). \tag{2.28}$$

According to (2.22) and the first identity (2.21), we have

$$2(T(z)(\omega(z)e_{-1} + e_0), \omega(\bar{z})e_{-1} + e_0) = \omega(z)^3 - \omega(z) = -2\sqrt{z^2-1}\,\omega(z)^2$$

and therefore (2.28) implies

$$(\mathbf{R}(z)T(z)\mathbf{R}(z)e_n, e_m) = -\frac{\omega(z)^{n+m+2}}{\sqrt{z^2-1}}. \tag{2.29}$$

Let us now take into account representation (2.25) and observe that $(R_+(z)e_n, e_m) = (R(z)e_n, e_m)$ if $n \geq 0$ and $m \geq 0$. Thus formula (2.26) is an immediate corollary of (2.13) and (2.29). □

The following result is a direct consequence of the standard relation

$$2\pi i \frac{d(E_+(\lambda)e_0, e_0)}{d\lambda} = (R_+(\lambda + i0)e_0, e_0) - (R_+(\lambda - i0)e_0, e_0)$$

between the spectral measure $dE_+(\lambda)$ of the self-adjoint operator H_+ and its resolvent.

Corollary 2.4. *For all $\lambda \in (-1, 1)$, we have*

$$d(E_+(\lambda)e_0, e_0) = 2\pi^{-1}\sqrt{1-\lambda^2}d\lambda. \tag{2.30}$$

3. Point interaction. The spectral measure

3.1. Here we consider a generalization H_a of the operator (2.16) given in the space $\ell^2(\mathbb{Z}_+)$ by the matrix (1.1). To put it differently, this operator is defined by the formulas

$$H_a e_0 = \frac{a}{2} e_1, \quad H_a e_1 = \frac{1}{2}(a e_0 + e_2), \quad H_a e_n = \frac{1}{2}(e_{n-1} + e_{n+1}), \quad n \geq 2, \tag{3.1}$$

where $a > 0$. Obviously, $H_1 = H_+$. Of course, the operators H_a are particular cases of the Jacobi operators discussed in Subsection 2.1 corresponding to the case $a_0 = a/2$, $a_n = 1/2$ for $n \geq 1$ and $b_n = 0$ for all $n \geq 0$. So, all the results exposed there are automatically true now.

Our first goal here is to give explicit expressions for the objects introduced in Subsection 2.1. For the operator H_a, they will be denoted $dE_a(\lambda)$, $d\rho_a(\lambda)$, $R_a(z) = (H_a - zI)^{-1}$, U_a, etc. Let us start with the polynomials $P_n(z)$. For the case of the operator H_a, they will be denoted $\mathrm{Ch}_n(z; a)$. Explicitly, these polynomials are defined by the recurrent relations below.

Definition 3.1. Set $\mathrm{Ch}_0(z; a) = 1$, $\mathrm{Ch}_1(z; a) = 2a^{-1}z$ and

$$\mathrm{Ch}_{n-1}(z; a) + \mathrm{Ch}_{n+1}(z; a) = 2z\,\mathrm{Ch}_n(z; a), \quad n \geq 1.$$

Note that $\mathrm{Ch}_n(z; 1) = \mathsf{U}_n(z)$ (the Chebyshev polynomials of the second kind) and $\mathrm{Ch}_n(z; \sqrt{2}) = \mathsf{T}_n(z)$ (the Chebyshev polynomials of the first kind) for all $n \in \mathbb{Z}_+$. For an arbitrary $a > 0$, we use the term "generalized Chebyshev polynomials" for $\mathrm{Ch}_n(z; a)$. It is possible to give an explicit expression for these polynomials.

Proposition 3.2. *Let us set*

$$\gamma_\pm(z; a) = \frac{a}{2} \pm \frac{a^2 - 2}{2a} \frac{z}{\sqrt{z^2 - 1}}. \tag{3.2}$$

Then

$$\mathrm{Ch}_n(z; a) = \gamma_+(z; a)\omega(z)^n + \gamma_-(z; a)\omega(z)^{-n}, \quad \forall n \geq 1, \tag{3.3}$$

where the function $\omega(z)$ is defined by formula (2.20).

Proof. Since

$$\omega(z) + \omega(z)^{-1} = 2z,$$

both sequences $f_n^{(+)}(z) = \omega(z)^n$ and $f_n^{(-)}(z) = \omega(z)^{-n}$ satisfy the equations

$$f_{n-1}(z) + f_{n+1}(z) = 2z f_n(z), \quad n \geq 2.$$

Therefore their arbitrary linear combination (3.3) also solves these equations. To find the constants $\gamma_+(z;a)$ and $\gamma_-(z;a)$, it remains to satisfy the equations $\mathrm{Ch}_1(z;a) = 2a^{-1}z$ and $a\,\mathrm{Ch}_0(z;a) + \mathrm{Ch}_2(z;a) = 2z\,\mathrm{Ch}_1(z;a)$ where $\mathrm{Ch}_0(z;a) = 1$. This yields the system

$$\left.\begin{aligned} \gamma_+(z;a)\omega(z) + \gamma_-(z;a)\omega(z)^{-1} &= 2a^{-1}z \\ a + \big(\gamma_+(z;a)\omega(z)^2 + \gamma_-(z;a)\omega(z)^{-2}\big) &= 4a^{-1}z^2 \end{aligned}\right\}.$$

It is easy to see that its solution is given by formula (3.2). □

Remark 3.3. All functions $\gamma_\pm(z;a)$ and $\omega(z)$ are holomorphic on the set $\mathbb{C}\setminus[-1,1]$ only. Nevertheless their combination in the right-hand side of (3.3) is a polynomial.

Remark 3.4. Representation (3.3) remains true for $z = \pm 1$. In this case

$$\mathrm{Ch}_n(\pm 1;a) = (\pm 1)^n\big(2na^{-1} - (n-1)a\big), \quad n \geq 1. \tag{3.4}$$

Indeed, let $z \to \pm 1$ in (3.3). Observe that $\omega(\pm 1) = \pm 1$,

$$\omega(z)^n - \omega(z)^{-n} = -2nz^{n-1}\sqrt{z^2-1} + O(z^2-1)$$

and

$$\gamma_+(z) = \frac{a^2-2}{2a}\frac{z}{\sqrt{z^2-1}} + O(1) \quad \text{as} \quad z \to \pm 1.$$

Since $\gamma_+(z;a) + \gamma_-(z;a) = a$, formula (3.4) follows from (3.3) in the limit $z \to \pm 1$.

Let us now use (3.3) for $z = \lambda + i0$, $\lambda \in (-1,1)$, and observe that according to (2.20) and (3.2)

$$\omega(\lambda \pm i0) = \lambda \mp i\sqrt{1-\lambda^2} \tag{3.5}$$

and

$$\gamma_\pm(\lambda + i0;a) = \frac{a}{2} \mp i\frac{a^2-2}{2a}\frac{\lambda}{\sqrt{1-\lambda^2}}. \tag{3.6}$$

Note that $\mathrm{Ch}_n(-\lambda;a) = (-1)^n\,\mathrm{Ch}_n(\lambda;a)$ because $\omega(-\lambda+i0) = -\overline{\omega(\lambda+i0)}$ and $\gamma_+(-\lambda+i0;a) = \gamma_-(\lambda+i0;a)$.

Proposition 3.2 implies the following assertion. We recall that the values of the function arccot belong to the interval $(0,\pi)$.

Corollary 3.5. *Let* $\lambda \in (-1,1)$. *Put*

$$\varphi(\lambda) = \operatorname{arccot}\frac{-\lambda}{\sqrt{1-\lambda^2}}, \quad \delta(\lambda;a) = \operatorname{arccot}\frac{a^2-2}{a^2}\frac{\lambda}{\sqrt{1-\lambda^2}} \tag{3.7}$$

and

$$\kappa(\lambda;a) = \frac{\sqrt{a^4 + 4(1-a^2)\lambda^2}}{a\sqrt{1-\lambda^2}}.$$

Then

$$\mathrm{Ch}_n(\lambda;a) = \kappa(\lambda;a)\sin\Big(n\varphi(\lambda) + \delta(\lambda;a)\Big), \quad n \geq 1. \tag{3.8}$$

Proof. Notice that $|\omega(\lambda \pm i0)| = 1$, $|\gamma_\pm(\lambda + i0; a)| = \kappa(\lambda; a)/2$ and with definitions (3.7)

$$-\omega(\lambda + i0) = e^{i\varphi(\lambda)}, \quad i\gamma_+(\lambda + i0; a) = 2^{-1}\kappa(\lambda; a)e^{i\delta(\lambda;a)}.$$

Therefore (3.8) is a direct consequence of (3.3) for $z = \lambda + i0$. $\square$

Representation (3.8) extends the well-known formula (10.11.2) in the book [6] for the classical Chebyshev polynomials to arbitrary $a > 0$.

Of course passing in (3.8) to the limit $\lambda \to \pm 1$, we recover relation (3.4).

3.2. Now we perform an explicit spectral analysis of the operators H_a for all $a > 0$. In particular, we find the orthogonality measure $d\rho_a(\lambda) = d(E_a(\lambda)e_0, e_0)$ for the polynomials $\mathrm{Ch}_n(\lambda; a)$. Let us consider the operator H_a as a perturbation of H_1. Obviously, the perturbation

$$V_a = H_a - H_1 = \alpha(\cdot, e_0)e_1 + \alpha(\cdot, e_1)e_0, \quad \alpha = \frac{a-1}{2}, \tag{3.9}$$

has rank 2. Of course, the essential spectrum of H_a is the same as that of H_1, i.e., it coincides with the interval $[-1, 1]$.

We need a particular case of Proposition 2.3. It can be easily deduced from formula (2.26) if the identities (2.21) are taken into account.

Lemma 3.6. *Let the function $\omega(z)$ be defined by relation* (2.20)*, and let*

$$r_{n,m}(z) = (R_1(z)e_n, e_m). \tag{3.10}$$

Then

$$\begin{aligned} r_{n,0}(z) &= -2\omega^{n+1}(z), \quad n \geq 0, \\ r_{n,1}(z) &= -4z\omega^{n+1}(z), \quad n \geq 1. \end{aligned} \tag{3.11}$$

Our first goal is to calculate the perturbation determinant

$$D_a(z) = \mathrm{Det}\big(I + V_a R_1(z)\big) \tag{3.12}$$

for the pair H_1, H_a.

Proposition 3.7. *For all $a > 0$, the perturbation determinant is given by the formula*

$$D_a(z) = 1 + (1 - a^2)\omega(z)^2. \tag{3.13}$$

In particular,

$$D_{\sqrt{2}}(z) = 2\sqrt{z^2 - 1}\,\omega(z).$$

Proof. It follows from definitions (3.9) and (3.10) that

$$\begin{aligned} D_a(z) = \mathrm{Det}\begin{pmatrix} 1 + \alpha r_{1,0}(z) & \alpha r_{0,0}(z) \\ \alpha r_{1,1}(z) & 1 + \alpha r_{0,1}(z) \end{pmatrix} \\ = (1 + \alpha r_{1,0}(z))(1 + \alpha r_{0,1}(z)) - \alpha^2 r_{0,0}(z) r_{1,1}(z). \end{aligned}$$

Substituting here expressions (3.11), we get (3.13). $\square$

Next, we calculate eigenvalues of the operator H_a.

Proposition 3.8. *If $a \leq \sqrt{2}$, then the set $(-\infty, -1) \cup (1, \infty)$ consists of regular points of the operator H_a. If $a > \sqrt{2}$, then the operator H_a has exactly two isolated eigenvalues*

$$\lambda_{\pm}(a) = \pm \frac{a^2}{2\sqrt{a^2 - 1}}. \tag{3.14}$$

The points 1 and -1 are not eigenvalues of the operators H_a.

Proof. Suppose first that $|\lambda| > 1$. Recall that λ is an eigenvalue of H_a if and only if $D_a(\lambda) = 0$. In view of relation (3.13) and the second identity (2.21) this equation is equivalent to $2\lambda = a^2\omega(\lambda)$ or, by definition (2.20),

$$(a^2 - 2)\lambda = \pm a^2\sqrt{\lambda^2 - 1} \quad \text{if} \quad \pm\lambda > 1.$$

Obviously, this equation has solutions if and only if $a^2 > 2$. In this case these solutions are given by formula (3.14).

Let now $\lambda = 1$ or $\lambda = -1$ and $(H_1 + V_a)\psi = \lambda\psi$ for some $\psi \in \ell^2(\mathbb{Z}_+)$. Put $\psi_n = (\psi, e_n)$ and

$$\varphi = (H_1 - \lambda)\psi = -V_a\psi.$$

It follows from formula (3.9) that

$$\varphi = -\alpha\psi_0 e_1 - \alpha\psi_1 e_0$$

and hence

$$\psi = R_1(\lambda)\varphi = -\alpha\psi_0 R_1(\lambda)e_1 - \alpha\psi_1 R_1(\lambda)e_0.$$

Using formulas (3.11) where $\omega(\lambda) = \lambda$, we find that

$$\psi_n = 2\alpha\lambda^{n+1}(2\lambda\psi_0 + \psi_1), \quad n \geq 0.$$

Comparing these two equations for $n = 0$ and $n = 1$, we see that $\psi_1 = \lambda\psi_0$ whence $\psi_n = 6\alpha\lambda^n\psi_0$. Since $\lambda = \pm 1$, the sequence $\psi \in \ell^2(\mathbb{Z}_+)$ for $\psi = 0$ only. $\square$

3.3. Now we are in a position to calculate the resolvent $R_a(z) = (H_a - zI)^{-1}$ of the operator H_a. Similarly to Subsection 2.3, it is more convenient to work with the operator

$$T_a(z) = V_a - V_a R_a(z) V_a. \tag{3.15}$$

Then

$$R_a(z) = R_1(z) - R_1(z)T_a(z)R_1(z) \tag{3.16}$$

and the resolvent equation for the pair H_1, H_a can be written as

$$T_a(z) = V_a - V_a R_1(z) T_a(z). \tag{3.17}$$

Proposition 3.9. *Let the operator H_a be defined by formulas (3.1). Then for all $f \in \ell^2(\mathbb{Z}_+)$, we have*

$$2D_a(z)T_a(z)f = (b_{0,0}(z)f_0 + b_{0,1}(z)f_1)e_0 + (b_{1,0}(z)f_0 + b_{1,1}(z)f_1)e_1 \tag{3.18}$$

where the perturbation determinant $D_a(z)$ is given by formula (3.13), $f_n = (f, e_n)$ *and*

$$\begin{aligned} b_{0,0}(z) &= 2(a-1)^2 z\omega(z)^2, \quad b_{1,1}(z) = (a-1)^2\omega(z), \\ b_{0,1}(z) &= b_{1,0}(z) = a-1-(a-1)^2\omega(z)^2. \end{aligned} \tag{3.19}$$

Proof. In view of expression (3.9) for V_a, equation (3.17) can be rewritten as

$$T_a(z)f = \alpha f_1 e_0 + \alpha f_0 e_1 - \alpha(R_1(z)T_a(z)f, e_1)e_0 - \alpha(R_1(z)T_a(z)f, e_0)e_1. \tag{3.20}$$

Let us take the scalar products of this equation with $R_1(\bar z)e_0$ and $R_1(\bar z)e_1$. This leads to a system of two equations for $u_0(z) = (R_1(z)T_a(z)f, e_0)$ and $u_1(z) = (R_1(z)T_a(z)f, e_1)$:

$$\left.\begin{aligned} u_0(z) =& \alpha(R_1(z)e_1, e_0)f_0 + \alpha(R_1(z)e_0, e_0)f_1 \\ &- \alpha(R_1(z)e_1, e_0)u_0(z) - \alpha(R_1(z)e_0, e_0)u_1(z) \\ u_1(z) =& \alpha(R_1(z)e_1, e_1)f_0 + \alpha(R_1(z)e_0, e_1)f_1 \\ &- \alpha(R_1(z)e_1, e_1)u_0(z) - \alpha(R_1(z)e_0, e_1)u_1(z) \end{aligned}\right\}.$$

Let us solve this system. Using notation (2.26), we can rewrite it as

$$\left.\begin{aligned} (1+\alpha r_{1,0}(z))u_0(z) + \alpha r_{0,0}(z)u_1(z) =& \alpha r_{1,0}(z)f_0 + \alpha r_{0,0}(z)f_1 \\ \alpha r_{1,1}(z)u_0(z) + (1+\alpha r_{0,1}(z))u_1(z) =& \alpha r_{1,1}(z)f_0 + \alpha r_{0,1}(z)f_1 \end{aligned}\right\}. \tag{3.21}$$

It can be easily checked that the solution of system (3.21) is given by the formulas

$$\begin{aligned} D_a(z)u_0(z) =& -a(a-1)\omega(z)^2 f_0 - (a-1)\omega(z)f_1, \\ D_a(z)u_1(z) =& -2(a-1)z\omega(z)^2 f_0 - a(a-1)\omega(z)^2 f_1. \end{aligned} \tag{3.22}$$

According to (3.20) we have

$$T_a(z)f = \alpha(f_1 - u_1(z))e_0 + \alpha(f_0 - u_0(z))e_1,$$

which in view of (3.22) yields the representation (3.18), (3.19). $\square$

To find the spectral measure of the operator H_a, we calculate the matrix element $(R_a(z)e_0, e_0)$ of the resolvent; it is also known as the Weyl m-function. Other matrix elements of $R_a(z)$ will be found in Subsection 5.1.

Theorem 3.10. *The representation*

$$(R_a(z)e_0, e_0) = -\frac{2\omega(z)}{D_a(z)} = \frac{2}{(a^2-2)z - a^2\sqrt{z^2-1}} \tag{3.23}$$

holds.

Proof. In view of the identity (2.21), it follows from formula (2.26) that $(R_1(z)e_0, e_0) = -2\omega(z)$ and $(R_1(z)e_0, e_1) = -2\omega(z)^2$. Therefore representation (3.18) implies that

$$\begin{aligned} 2D_a(z)(R_1(z)T_a(z)R_1(z)e_0, e_0) =& 8D_a(z)\omega(z)^2(T_a(z)(e_0 + \omega(z)e_1), e_0 + \omega(\bar z)e_1) \\ =& 4\omega(z)^2(b_{0,0}(z) + 2\omega(z)b_{0,1}(z) + \omega(z)^2 b_{1,1}(z)). \end{aligned}$$

Substituting here expressions (3.19), we see that

$$2D_a(z)(R_1(z)T_a(z)R_1(z)e_0, e_0) = 4(a^2-1)\omega(z)^3.$$

In view of (3.16), this yields the representation

$$(R_a(z)e_0, e_0) - (R_1(z)e_0, e_0) = -2(a^2-1)\frac{\omega(z)^3}{D_a(z)}.$$

Using also equalities (2.27) for $(R_1(z)e_0, e_0)$ and (3.13) for $D_a(z)$, we get (3.23). $\square$

Observe now that the denominator in (3.23) is a continuous function of $z \in \mathbb{C}\setminus[-1,1]$ (up to the cut along $[-1,1]$), and it is not 0 for $z = \lambda \pm i0$, $\lambda \in (-1,1)$. Applying the formula

$$2\pi i \frac{d(E_a(\lambda)e_0, e_0)}{d\lambda} = (R_a(\lambda+i0)e_0, e_0) - (R_a(\lambda-i0)e_0, e_0),$$

we find the spectral measure $d(E_a(\lambda)e_0, e_0)$ of the self-adjoint operator H_a for $\lambda \in (-1,1)$. Finally, calculating the residues of function (3.23) at the points $\lambda_\pm(a)$, we obtain the spectral measure at these points.

Theorem 3.11. *Let the Jacobi operator H_a in the space $\ell^2(\mathbb{Z}_+)$ be defined by formula* (1.1)*. Then:*

(i) *The spectrum of the operator H_a is absolutely continuous on $[-1,1]$ and*

$$d(E_a(\lambda)e_0, e_0) = 2\pi^{-1}\frac{a^2\sqrt{1-\lambda^2}}{a^4 - 4(a^2-1)\lambda^2}d\lambda =: d\rho_a(\lambda) \tag{3.24}$$

for all $\lambda \in (-1,1)$.

(ii) *If $a \le \sqrt{2}$, then the set $(-\infty,-1)\cap(1,\infty)$ consists of regular points of the operator H_a. If $a > \sqrt{2}$, then the operator H_a has two simple eigenvalues $\lambda_\pm(a)$ given by formula* (3.14)*. In this case, we have*

$$(E_a(\{\lambda_\pm(a)\})e_0, e_0) = \frac{a^2-2}{2(a^2-1)}. \tag{3.25}$$

Note that the measure $d(E_a(\lambda)e_0, e_0)$ is invariant with respect to the reflection $\lambda \mapsto -\lambda$. Of course for $a = 1$, we recover expression (2.30) for the spectral measure of the discrete Schrödinger operator $H_+ = H_1$ and the corresponding Chebyshev polynomials $\mathsf{U}_n(\lambda)$ of the second kind. If $a = \sqrt{2}$, then (3.24) yields the standard expression for the orthogonality measure of Chebyshev polynomials $\mathsf{T}_n(\lambda)$ of the first kind. According to formula (3.24) the generalized Chebyshev polynomials $\mathrm{Ch}_n(\lambda; a)$ fit into the class of Szegö polynomials (see, for example, §10.21 of the book [6] or §2.6 of the book [10]) for all $a \in (0, \sqrt{2}\,]$ (but not for $a > \sqrt{2}$).

In view of general formula (2.4), Theorem 3.11 implies the next result.

Corollary 3.12. *For all $n, m \in \mathbb{Z}_+$, the following representations*

$$d(E_a(\lambda)e_n, e_m) = 2\pi^{-1}\frac{a^2\sqrt{1-\lambda^2}}{a^4-4(a^2-1)\lambda^2}\,\mathrm{Ch}_n(\lambda;a)\,\mathrm{Ch}_m(\lambda;a)d\lambda, \quad \lambda \in (-1,1),$$

and

$$(E_a(\{\lambda_\pm(a)\})e_n, e_m) = \frac{a^2-2}{2(a^2-1)}\,\mathrm{Ch}_n(\lambda_\pm(a);a)\,\mathrm{Ch}_m(\lambda_\pm(a);a), \quad a > \sqrt{2},$$

are true.

3.4. Following Subsection 2.1 (cf. definition (2.6)), we introduce the operators $U_a : \ell^2(\mathbb{Z}_+) \to L^2(\mathbb{R}; d\rho_a)$ by the formula $(U_a e_n)(\lambda) = \mathrm{Ch}_n(\lambda; a)$, but it will be convenient to remove the point part of $d\rho_a$ (which is non-trivial for $a > \sqrt{2}$ only). Moreover, making a change of variables we replace $L^2((-1,1); d\rho_a)$ by $L^2(-1,1)$ (L^2 with the Lebesgue measure). Thus we set

$$(V_a f)(\lambda) = \sqrt{\frac{2}{\pi}} \frac{a\sqrt[4]{1-\lambda^2}}{\sqrt{a^4 - 4(a^2-1)\lambda^2}} f(\lambda)$$

so that the operator $V_a : L^2((-1,1); d\rho_a) \to L^2(-1,1)$ is unitary and introduce the operator $F_a : \ell^2(\mathbb{Z}_+) \to L^2(-1,1)$ by the formula

$$(F_a f)(\lambda) = (V_a \chi_{(-1,1)} U_a f)\lambda), \quad \lambda \in (-1,1), \tag{3.26}$$

where $\chi_{(-1,1)}$ is the multiplication operator by the characteristic function of the interval $(-1,1)$. To put it differently, we set

$$\psi_n(\lambda; a) = \sqrt{\frac{2}{\pi}} \frac{a\sqrt[4]{1-\lambda^2}}{\sqrt{a^4 - 4(a^2-1)\lambda^2}} \mathrm{Ch}_n(\lambda; a), \quad \lambda \in (-1,1). \tag{3.27}$$

Then

$$(F_a e_n)(\lambda) = \psi_n(\lambda; a), \quad \lambda \in (-1,1). \tag{3.28}$$

The operator $F_a^* : L^2(-1,1) \to \ell^2(\mathbb{Z}_+)$ adjoint to F_a is given by the formula

$$(F_a^* g)_n = \int_{-1}^{1} \psi_n(\lambda; a) g(\lambda) d\lambda, \quad n \in \mathbb{Z}_+.$$

According to (3.26) it follows from the unitarity of the operator U_a that

$$F_a F_a^* = I, \quad F_a^* F_a = E_a(-1,1), \tag{3.29}$$

where $E_a(-1,1)$ is the spectral projection of the operator H_a corresponding to the interval $(-1,1)$; of course, $E_a(-1,1) = I$ if $a \leq \sqrt{2}$.

3.5. Finally, we note that $\omega(z)$ and $D_a(z)$ are analytic functions on a two sheets Riemann surface. The second sheet is distinguished by the condition $\sqrt{z^2-1} < 0$ for $z > 1$. With this convention, formula (3.13) for the perturbation determinant $D_a(z)$ remains true on the second sheet. Its zeros are usually interpreted as resonances (also called anti-bound states). In our case these zeros are simple.

Quite similarly to Proposition 3.8, one proves the following result.

Proposition 3.13. *If $a \in (0,1)$, then the operator H_a has two resonances at the points $\pm i 2^{-1} a^2 (1-a^2)^{-1/2}$ lying on the imaginary axis. If $1 < a \leq \sqrt{2}$, then the operator H_a has two real resonances at the points $\pm 2^{-1} a^2 (a^2-1)^{-1/2}$. If $a > \sqrt{2}$, then the operator H_a does not have resonances (but it has two eigenvalues).*

Compared to the operator A_α in the space $L^2(\mathbb{R}_+)$ the picture is of course essentially more complicated. Note that the operator H_0 has a (simple) eigenvalue at the point $\lambda = 0$. As a increases, it splits into two resonances lying on the imaginary axis and tending to $\pm i\infty$ as $a \to 1-0$. If $a \in (1, \sqrt{2}]$, these resonances belong to the real axis and tend from $\pm\infty$ to ± 1 as a increases from 1 to $\sqrt{2}$. For $a > \sqrt{2}$, the resonances become the eigenvalues.

4. Scattering theory

4.1. The wave operators $W_\pm(H_a, H_1)$ for a pair of self-adjoint operators H_1, H_a are defined as strong limits

$$W_\pm(H_a, H_1) = \operatorname*{s-lim}_{t\to\pm\infty} e^{iH_a t} e^{-iH_1 t}. \tag{4.1}$$

We refer to the book [14] for basic notions of scattering theory. Under the assumption of the existence of limits (4.1), $W_\pm(H_a, H_1)$ are isometric operators and enjoy the intertwining property $H_a W_\pm(H_a, H_1) = W_\pm(H_a, H_1) H_1$.

For the operators H_1, H_a defined by formula (3.1), the perturbation (3.9) has finite rank (V_a has rank 2). By the classical Kato theorem, in this case the wave operators $W_\pm(H_a, H_1)$ exist. Moreover, the wave operators $W_\pm(H_a, H_1)$ are complete, that is, their ranges coincide with the absolutely continuous subspace of the operator H_a (in particular, they are unitary if $a \leq \sqrt{2}$). Therefore the scattering operator

$$\mathbf{S}_a = W_+(H_a, H_1)^* W_-(H_a, H_1) \tag{4.2}$$

is unitary and commutes with H_1: $\mathbf{S}_a H_1 = H_1 \mathbf{S}_a$. All these results will also be obtained in Subsection 4.3 by a direct method relying on Theorem 3.11.

To define the corresponding scattering matrix, we use the diagonalization of H_1 by the operator F_1. Since the scattering operator $\mathbf{S}_a$ commutes with H_1 and the operator H_1 has simple spectrum, we have

$$(F_1 \mathbf{S}_a f)(\lambda) = S_a(\lambda)(F_1 f)(\lambda), \quad \lambda \in (-1, 1), \tag{4.3}$$

where $S_a(\lambda) \in \mathbb{C}$ and $|S_a(\lambda)| = 1$. The function $S_a(\lambda)$ is known as the scattering matrix for the pair H_1, H_a and the value λ of the spectral parameter. Note that the scattering matrix does not depend on the diagonalization of H_1 since all its diagonalizations have the form $\theta(\lambda) F_1$ where $\theta(\lambda) \in \mathbb{C}$ and $|\theta(\lambda)| = 1$.

The simplest way to calculate the scattering matrix is to use its abstract expression (this is a particular case of the general Birman-Kreĭn formula; see, e.g., §8.4 of [14]) via the corresponding perturbation determinant $D_a(z)$:

$$S_a(\lambda) = \frac{D_a(\lambda - i0)}{D_a(\lambda + i0)}, \quad \lambda \in (-1, 1).$$

It follows from formula (3.5) and the representation (3.13) that the limits

$$D_a(\lambda \pm i0) = 1 + (1 - a^2)(2\lambda^2 - 1 \mp 2i\lambda\sqrt{1 - \lambda^2}), \quad \lambda \in (-1, 1), \tag{4.4}$$

exist, are continuous functions of $\lambda \in (-1,1)$ and $D_a(\lambda \pm i0) \neq 0$. It is also easy to see that $|D_a(\lambda \pm i0)| = \sqrt{a^4 + 4(1-a^2)\lambda^2}$ and

$$\lim_{\lambda \to \pm 1} D_a(\lambda \pm i0) = 2 - a^2.$$

Let us state the result obtained.

Theorem 4.1. *The scattering matrix $S_a(\lambda)$ for the pair of the operators H_1, H_a is given by the formula*

$$S_a(\lambda) = \frac{1 + (1-a^2)(2\lambda^2 - 1 + 2i\lambda\sqrt{1-\lambda^2})}{1 + (1-a^2)(2\lambda^2 - 1 - 2i\lambda\sqrt{1-\lambda^2})}, \quad \lambda \in (-1,1). \tag{4.5}$$

In particular, $S_a(-\lambda) = \overline{S_a(\lambda)}$ and $S_a(0) = 1$. Moreover, $S_a(\lambda) \to 1$ as $\lambda \to \pm 1$ unless $a = \sqrt{2}$.

Corollary 4.2. *Let $a = \sqrt{2}$. Then, for all $\lambda \in (-1,1)$,*

$$D_{\sqrt{2}}(\lambda \pm i0) = 2\sqrt{1-\lambda^2}(\sqrt{1-\lambda^2} \pm i\lambda)$$

and

$$S_{\sqrt{2}}(\lambda) = \frac{\sqrt{1-\lambda^2} - i\lambda}{\sqrt{1-\lambda^2} + i\lambda}.$$

In particular, $S_a(\lambda) \to -1$ as $\lambda \to \pm 1$.

4.2. Let us now discuss the Kreĭn spectral shift function $\xi_a(\lambda)$ for the pair of the operators H_1 and H_a (see, for example, Chapter 8 of the book [14], for all necessary definitions). In terms of the perturbation determinant (3.12), this function can be introduced by the formula

$$\xi_a(\lambda) = \pi^{-1} \lim_{\varepsilon \to +0} \arg D_a(\lambda + i\varepsilon), \quad \lambda \in \mathbb{R}. \tag{4.6}$$

Recall that the perturbation determinant $D_a(z)$ is given by the explicit formula (3.13). Since $D_a(z) \to 1$ as $|z| \to \infty$ and $D_a(z) \neq 0$ for $\operatorname{Im} z \neq 0$, the branch of $\arg D_a(z)$ is correctly fixed for $\operatorname{Im} z > 0$ by the condition $\arg D_a(z) \to 0$ as $|z| \to \infty$. In the framework of abstract scattering theory, the limits in (4.6) exist for almost all $\lambda \in \mathbb{R}$, but in our particular case they exist for all $\lambda \in \mathbb{R}$ except eventually the points ± 1 and $\lambda_\pm(a)$. If $a \leq \sqrt{2}$, then $\xi(\lambda) = 0$ for all $|\lambda| > 1$. In the case $a > \sqrt{2}$, the function $\xi_a(\lambda) = 0$ for $|\lambda| > |\lambda_\pm(a)|$, $\xi_a(\lambda) = 1$ for $\lambda \in (1, \lambda_+(a))$ and $\xi_a(\lambda) = -1$ for $\lambda \in (\lambda_-(a), -1)$.

Let us now consider the function $\xi_a(\lambda)$ for $\lambda \in (-1,1)$. Since $D_a(\lambda + i0)$ is a continuous function of $\lambda \in (-1,1)$ and $D_a(\lambda + i0) \neq 0$, the spectral shift function $\xi_a(\lambda)$ depends also continuously on $\lambda \in (-1,1)$. By (2.21), (3.13), $\operatorname{Im} D_a(iy) = 0$ and thus $D_a(iy) > 0$ for all $y \geq 0$ so that $\xi_a(0) = 0$. It also follows from (4.4) that $D_a(-\lambda + i0) = \overline{D_a(\lambda + i0)}$ whence $\xi_a(-\lambda) = -\xi_a(\lambda)$. So it suffices to study $\xi_a(\lambda)$ for $\lambda \in (0,1)$.

Putting together formulas (4.4) and (4.6), we see that

$$\tan(\pi\xi_a(\lambda)) = \frac{2(a^2-1)\lambda\sqrt{1-\lambda^2}}{1+(1-a^2)(2\lambda^2-1)}. \tag{4.7}$$

Let us now consider separately the cases $a<\sqrt{2}$, $a>\sqrt{2}$ and $a=\sqrt{2}$. In the first case the denominator in (4.7) does not equal zero for $\lambda\in(0,1)$, and hence formula (4.7) can be rewritten as

$$\xi_a(\lambda) = \frac{1}{\pi}\arctan\frac{2(a^2-1)\lambda\sqrt{1-\lambda^2}}{1+(1-a^2)(2\lambda^2-1)}. \tag{4.8}$$

Obviously, $\xi_a(\lambda)\to 0$ as $\lambda\to 1$. It is also easy to see that $\xi_a(\lambda)<0$ for $a\in(0,1)$ and $\xi_a(\lambda)>0$ for $a\in(1,\sqrt{2})$. Moreover, $\xi_a(\lambda)$ has one extremum at the point $\lambda=a/\sqrt{2}$ where

$$\xi_a(a/\sqrt{2}) = \frac{1}{\pi}\arctan\frac{(a^2-1)}{a\sqrt{2-a^2}}.$$

This point is the minimum of $\xi_a(\lambda)$ for $a\in(0,1)$ and its maximum for $a\in(1,\sqrt{2})$.

If $a>\sqrt{2}$, then the denominator in (4.7) equals zero at the point $a2^{-1/2}(a^2-1)^{-1/2}\in(0,1)$. In this case it follows from (4.7) that

$$\xi_a(\lambda) = \frac{1}{\pi}\operatorname{arccot}\frac{1-(a^2-1)(2\lambda^2-1)}{2(a^2-1)\lambda\sqrt{1-\lambda^2}}. \tag{4.9}$$

In particular, we see that $\xi_a'(\lambda)>0$ and $\xi_a(\lambda)\to 1$ as $\lambda\to 1$.

Let us summarize the results obtained.

Theorem 4.3. *Let the Jacobi operator H_a in the space $\ell^2(\mathbb{Z}_+)$ be defined by formula* (1.1)*, and let $\xi_a(\lambda)$ be the spectral shift function for the pair H_1, H_a. Then $\xi_a(\lambda)$ is an odd function of $\lambda\in\mathbb{R}$ and the following results are true.*

(i) *Let $a\in(0,\sqrt{2})$. Then $\xi_a(\lambda)$ is given by formula* (4.8) *for $\lambda\in[0,1)$ and $\xi_a(\lambda)=0$ for $\lambda\geq 1$.*

(ii) *Let $a>\sqrt{2}$. Then $\xi_a(\lambda)$ is given by formula* (4.9) *for $\lambda\in[0,1)$, $\xi_a(\lambda)=1$ for $\lambda\in[1,\lambda_+(a))$ and $\xi_a(\lambda)=0$ for $\lambda\geq\lambda_+(a)$.*

(iii) *In the intermediary case $a=\sqrt{2}$, we have*

$$\xi_{\sqrt{2}}(\lambda) = \frac{1}{\pi}\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}, \quad \lambda\in[0,1),$$

$\xi_{\sqrt{2}}(\lambda)\to 1/2$ as $\lambda\to 1$ and $\xi_{\sqrt{2}}(\lambda)=0$ for $\lambda>1$.

4.3. The wave operators and the scattering matrix can be expressed via eigenfunctions (of the continuous spectrum) of the operator H_a. These eigenfunctions can be constructed in terms of the generalized Chebyshev polynomials. It is convenient to introduce the operator A of multiplication by λ in the space $L^2(-1,1)$. Let the operators F_a be defined by formula (3.26). According to the equation (2.7) the intertwining property

$$H_a F_a^* = F_a^* \mathsf{A} \tag{4.10}$$

holds.

Next, we find a relation between the operators F_a and the wave operators $W_\pm(H_a, H_1)$. We use the following elementary result.

Lemma 4.4. *Let the function $\omega(\lambda \pm i0)$ be given by formula (3.5), and let $g \in C_0^\infty(-1,1)$. Then, for an arbitrary $p \in \mathbb{Z}_+$ and $t \to \mp\infty$, we have*

$$\Big| \int_{-1}^{1} \omega(\lambda \pm i0)^n e^{-i\lambda t} g(\lambda) d\lambda \Big| \leq C_p (n + |t|)^{-p}, \quad n \in \mathbb{Z}_+, \tag{4.11}$$

with some constant C_p depending on p only.

Proof. Let us integrate by parts in (4.11) and observe that

$$\frac{d}{d\lambda}\big(n \ln \omega(\lambda \pm i0) - i\lambda t\big) = \pm i \big(\frac{n}{\sqrt{1-\lambda^2}} \mp t \big).$$

If $\mp t > 0$, the modulus of this expression is bounded from below by $n + |t|$ which yields estimate (4.11) for $p = 1$. Similarly, integrating in (4.11) p times by parts, we obtain estimate (4.11) for the same value of p. $\square$

Let us now set

$$\sigma_\pm(\lambda; a) = \frac{a^2 + 2(1-a^2)\lambda^2 \mp i2(a^2-1)\lambda\sqrt{1-\lambda^2}}{\sqrt{a^4 - 4(a^2-1)\lambda^2}}. \tag{4.12}$$

Clearly, $|\sigma_\pm(\lambda; a)| = 1$ so that the operator $\Sigma_\pm(\lambda; a)$ of multiplication by $\sigma_\pm(\lambda; a)$ is unitary in the space $L^2(-1,1)$.

Lemma 4.5. *For all $g \in L^2(-1,1)$, we have*

$$\lim_{t \to \pm\infty} \| (F_a^* \Sigma_\pm(a) - F_1^*) e^{-i\mathsf{A}t} g \| = 0 \tag{4.13}$$

Proof. It suffices to check (4.13) for $g \in C_0^\infty(-1,1)$. Let us proceed from definition (3.27) where the polynomial $\mathrm{Ch}_n(\lambda; a)$ is given by formula (3.3) with $z = \lambda + i0$ and the numbers $\omega(\lambda \pm i0)$ and $\gamma_\pm(\lambda + i0; a)$ are given by formulas (3.5) and (3.6), respectively:

$$\psi_n(\lambda; a) = \sqrt{\frac{2}{\pi}} \frac{a\sqrt[4]{1-\lambda^2}}{\sqrt{a^4 - 4(a^2-1)\lambda^2}} \big(\gamma_+(\lambda + i0; a)\omega(\lambda + i0)^n + \gamma_-(\lambda + i0; a)\omega(\lambda - i0)^n \big). \tag{4.14}$$

In particular, for $a = 1$, we have

$$\psi_n(\lambda; 1) = \sqrt{\frac{2}{\pi}} \sqrt[4]{1-\lambda^2} \big(\gamma_+(\lambda + i0; 1)\omega(\lambda + i0)^n + \gamma_-(\lambda + i0; 1)\,\omega(\lambda - i0)^n \big). \tag{4.15}$$

Lemma 4.4 implies that relation (4.13) is satisfied if the coefficient at $\omega(\lambda \pm i0)^n$ in the expression

$$\psi_n(\lambda; a)\sigma_\pm(\lambda; a) - \psi_n(\lambda; 1)$$

is zero. According to (4.14), (4.15) this yields the equation

$$\frac{a}{\sqrt{a^4 - 4(a^2-1)\lambda^2}} \gamma_\pm(\lambda + i0; a)\sigma_\pm(\lambda; a) = \gamma_\pm(\lambda + i0; 1)$$

for $\sigma_\pm(\lambda;a)$. Its solution is given by formula (4.12). $\square$

Proposition 4.6. *For all $a>0$, the strong limits* (4.1) *exist and*

$$W_\pm(H_a,H_1)=F_a^*\Sigma_\pm(a)F_1. \tag{4.16}$$

Proof. We have to check that

$$\lim_{t\to\pm\infty}\|e^{iH_at}e^{-iH_1t}f-F_a^*\Sigma_\pm(a)F_1f\|=0$$

for all $f\in\ell^2(\mathbb{Z}_+)$. In view of the intertwining property (4.10) this relation can be rewritten as

$$\lim_{t\to\pm\infty}\|e^{-iH_1t}f-F_a^*\Sigma_\pm(a)e^{-i\mathsf{A}t}F_1f\|=0. \tag{4.17}$$

Set now $g=F_1f$. Then again in view of the intertwining property (4.10) for $a=1$, we see that relations (4.13) and (4.17) are equivalent. $\square$

It follows from relations (3.29) and (4.16) that the scattering operator (4.2) is given by the equality

$$\mathbf{S}_a=F_1^*\Sigma_+(a)^*\Sigma_-(a)F_1.$$

Putting together this relation with the definition (4.3) of the scattering matrix, we see that

$$S_a(\lambda)=\frac{\sigma_-(\lambda;a)}{\sigma_+(\lambda;a)}.$$

Thus according to formula (4.12) for $\sigma_\pm(\lambda;a)$, we recover the representation (4.5) for $S_a(\lambda)$. Still another proof of (4.5) will be given in Subsection 6.3.

5. Trace identities and moment problems

5.1. Let us first find matrix elements $(R_a(z)e_n,e_m)$ for all $n,m\in\mathbb{Z}_+$. Recall that for $a=1$ they are given by formula (2.26) and denoted $r_{n,m}(z)$. As usual, $\omega(z)$ and $D_a(z)$ are the functions (2.20) and (3.13), respectively.

Proposition 5.1. *Let $n\geq 1$. Then*

$$(R_a(z)e_n,e_m)=r_{n,m}(z)-4(a^2-1)D_a(z)^{-1}z\omega(z)^{n+m+2} \tag{5.1}$$

for $m\geq 1$ and

$$(R_a(z)e_n,e_0)=r_{n,0}(z)-2(a-1)D_a(z)^{-1}\omega(z)^{n+2}(2z+a\omega(z)). \tag{5.2}$$

Proof. We follow the scheme of the proof of Theorem 3.10. If $n\geq 1$ and $m\geq 1$, then Lemma 3.6 and Proposition 3.9 imply that

$$\begin{aligned}D_a(z)&(R_1(z)T_a(z)R_1(z)e_n,e_m)\\&=4D_a(z)\omega(z)^{n+m+2}(T_a(z)(e_0+2ze_1),e_0+2\bar{z}e_1)\\&=2\omega(z)^{n+m+2}(b_{0,0}(z)+4zb_{0,1}(z)+4z^2b_{1,1}(z)).\end{aligned} \tag{5.3}$$

Substituting here expressions (3.19), we obtain that

$$D_a(z)(R_1(z)T_a(z)R_1(z)e_n,e_m)=4(a^2-1)z\omega(z)^{n+m+2}. \tag{5.4}$$

If $n\geq 1$, but $m=0$, then instead of (5.3) we have the formula

$$\begin{aligned} D_a(z)(R_1(z)T_a(z)R_1(z)e_n,e_0) \\ &= 4D_a(z)\omega(z)^{n+2}(T_a(z)(e_0+2ze_1),e_0+\omega(\bar z)e_1) \\ &= 2\omega(z)^{n+2}(b_{0,0}(z)+2zb_{1,0}(z)+\omega(z)b_{0,1}(z)+2z\omega(z)b_{1,1}(z)). \end{aligned}$$

Substituting here expressions (3.19), we obtain that

$$D_a(z)(R_1(z)T_a(z)R_1(z)e_n,e_0)=2(a-1)\omega(z)^{n+2}(2z+a\omega(z)). \tag{5.5}$$

In view of formula (3.16) relations (5.1) and (5.2) follow from (5.4) and (5.5), respectively. ☐

Corollary 5.2. *Let $n\geq 1$. Then*

$$(R_{\sqrt2}(z)e_n,e_m)=r_{n,m}(z)-2(z^2-1)^{-1/2}z\omega(z)^{n+m+1}$$

for $m\geq 1$ and

$$(R_{\sqrt2}(z)e_n,e_0)=r_{n,0}(z)-(2-\sqrt2)(z^2-1)^{-1/2}\omega(z)^{n+1}(\sqrt2 z+\omega(z)).$$

5.2. Using relation (2.8) and Theorem 3.11, we can now extend the identity (5.6) to all generalized Chebyshev polynomials $\mathrm{Ch}_n(\lambda;a)$.

Proposition 5.3. *Let the coefficients $(R_a(z)e_n,e_m)$ be defined by formulas (3.23), (5.1) and (5.2), and let the numbers $\lambda_\pm(a)$ be given by equality (3.14). Then for all $a>0$ and all $n,m\in\mathbb{Z}_+$, we have the identity*

$$\frac{2a^2}{\pi}\int_{-1}^1\frac{\mathrm{Ch}_n(\lambda;a)\mathrm{Ch}_m(\lambda;a)\sqrt{1-\lambda^2}}{(\lambda-z)(a^4-4(a^2-1)\lambda^2)}d\lambda=(R_a(z)e_n,e_m)-I_a(z;n,m)$$

where $I_a(z;n,m)=0$ if $a\leq\sqrt2$ and

$$I_a(z;n,m)=\frac{a^2-2}{2(a^2-1)}\Big(\frac{\mathrm{Ch}_n(\lambda_+(a);a)\mathrm{Ch}_m(\lambda_+(a);a)}{\lambda_+(a)-z}+\frac{\mathrm{Ch}_n(\lambda_-(a);a)\mathrm{Ch}_m(\lambda_-(a);a)}{\lambda_-(a)-z}\Big)$$

if $a>\sqrt2$.

Let us state a consequence of this result for $a=1$ and $a=\sqrt2$ when $\mathrm{Ch}_n(\lambda;1)=\mathsf{U}_n(\lambda)$ and $\mathrm{Ch}_n(\lambda;\sqrt2)=\mathsf{T}_n(\lambda)$ are the classical Chebyshev polynomials. The first assertion follows, actually, from Proposition 2.3.

Corollary 5.4. *For all $n,m\geq 0$, we have*

$$\int_{-1}^1(\lambda-z)^{-1}\mathsf{U}_n(\lambda)\mathsf{U}_m(\lambda)\sqrt{1-\lambda^2}d\lambda=\pi\frac{(z-\sqrt{z^2-1})^{n+m+2}-(z-\sqrt{z^2-1})^{|n-m|}}{2\sqrt{z^2-1}} \tag{5.6}$$

The second assertion requires Corollary 5.2 only.

Corollary 5.5. *The integral*

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\mathsf{T}_n(\lambda)\mathsf{T}_m(\lambda)}{(\lambda-z)\sqrt{1-\lambda^2}}d\lambda$$

equals $-(z^2-1)^{-1/2}$ *if* $n=m=0$. *It equals* $-\sqrt{2}\omega(z)^n$ *if* $n\geq 1$ *but* $m=0$, *and it equals*

$$r_{n,m}(z)-\frac{2\omega(z)^{n+m+1}}{\sqrt{z^2-1}}$$

if $n\geq 1$, $m\geq 1$.

5.3. To calculate the trace

$$\operatorname{Tr}\big(R_a(z)-R_1(z)\big)=\sum_{n=0}^{\infty}\big((R_a(z)e_n,e_n)-(R_1(z)e_n,e_n)\big),$$

it is convenient to use a link between the trace and the perturbation determinant (3.12):

$$\operatorname{Tr}\big(R_a(z)-R_1(z)\big)=-\frac{D_a'(z)}{D_a(z)}. \tag{5.7}$$

Differentiating expression (3.13), we see that

$$D_a'(z)=2(1-a^2)\omega(z)\omega'(z)$$

where, by definition (2.20),

$$\omega'(z)=-\frac{\omega(z)}{\sqrt{z^2-1}}.$$

Substituting these expressions into (5.7), we obtain the following result.

Proposition 5.6. *For all* $z\in\mathbb{C}\setminus[-1,1]$, *we have*

$$\operatorname{Tr}\big(R_a(z)-R_1(z)\big)=\frac{2(1-a^2)\omega(z)^2}{D_a(z)\sqrt{z^2-1}}. \tag{5.8}$$

It is now easy to obtain a generating function for the sequence $\operatorname{Tr}\big(H_a^n-H_1^n\big)$. Indeed, the left-hand side of (5.8) has the asymptotic expansion as $z\to\infty$ (for example, along the positive half-line):

$$\operatorname{Tr}\big(R_a(z)-R_1(z)\big)=-\sum_{n=1}^{\infty}z^{-n-1}\operatorname{Tr}\big(H_a^n-H_1^n\big).$$

Putting now $\zeta=z^{-1}$ and accepting that the function $\sqrt{1-\zeta^2}=1$ at $\zeta=0$, we can state a consequence of Proposition 5.6.

Corollary 5.7. *We have*

$$\sum_{n=1}^{\infty}\zeta^n\operatorname{Tr}\big(H_a^n-H_1^n\big)=\frac{2(a^2-1)}{\sqrt{1-\zeta^2}}\frac{\big(1-\sqrt{1-\zeta^2}\big)^2}{\zeta^2+(1-a^2)\big(1-\sqrt{1-\zeta^2}\big)^2}$$

where the series in the left-hand side converges for $|\zeta| < 1$ *if* $a \leq \sqrt{2}$ *and for* $|\zeta| < 2a^{-2}\sqrt{a^2-1}$ *if* $a > \sqrt{2}$. *In particular,* $\operatorname{Tr}\big(H_a^n - H_1^n\big) = 0$ *if* n *is odd.*

We note the paper [5] where the expressions for $\operatorname{Tr}\big(H_a^n - H_1^n\big)$ were obtained for general Jacobi operators H. Of course these expressions are less explicit than for the particular case of the operators $H = H_a$.

5.4. Finally, we consider the moments

$$\kappa_n(a) = \int_{-\infty}^{\infty} \lambda^n d\rho_a(\lambda) \tag{5.9}$$

of the measure $d\rho_a(\lambda) = d(E_a(\lambda)e_0, e_0)$. By Theorem 3.11, this measure is absolutely continuous on the interval $[-1, 1]$ and has two atoms at the points $\lambda_\pm(a)$ if $a > \sqrt{2}$. We recall that the points $\lambda_\pm(a)$ and the measures $\rho_a(\{\lambda_\pm(a)\})$ are given by formulas (3.14) and (3.25), respectively. Of course $\kappa_n(a) = 0$ for n odd because the measure $d\rho_a(\lambda)$ is even.

We will find an explicit expression for a generating function of the sequence $\kappa_n(a)$. It follows from Theorem 3.10 that

$$\int_{-\infty}^{\infty} (\lambda - z)^{-1} d\rho_a(\lambda) = \frac{2}{(a^2-2)z - a^2\sqrt{z^2-1}}.$$

Using the same arguments as those used for the proof of Corollary 5.7, we arrive at the next result.

Proposition 5.8. *We have*

$$\sum_{n=0}^{\infty} \kappa_n(a)\zeta^n = \frac{2}{a^2\sqrt{1-\zeta^2} + 2 - a^2}$$

where the series in the left-hand side converges for $|\zeta| < 1$ *if* $a \leq \sqrt{2}$ *and for* $|\zeta| < 2a^{-2}\sqrt{a^2-1}$ *if* $a > \sqrt{2}$.

Of course, Proposition 5.8 is consistent with the well-known expressions

$$\kappa_{2m}(1) = \frac{(2m-1)!!}{(m+1)!\,2^m}, \quad \kappa_{2m}(\sqrt{2}) = \frac{(2m-1)!!}{m!\,2^m}$$

for the classical Chebyshev polynomials. It follows from the Stirling formula (see, e.g., formula (1.18.4) in [6]) that

$$\kappa_{2m}(1) = \frac{1}{\sqrt{\pi}m^{3/2}}\big(1 + O(\frac{1}{m})\big), \quad \kappa_{2m}(\sqrt{2}) = \frac{1}{\sqrt{\pi}m^{1/2}}\big(1 + O(\frac{1}{m})\big) \tag{5.10}$$

as $m \to \infty$. Since the asymptotics as $m \to \infty$ of the integral (5.9) is determined by neighborhoods of the points $\lambda = \pm 1$, it is easy to deduce from expression (3.24) for $d\rho_a(\lambda)$ and the first relation (5.10) that for all $a < \sqrt{2}$

$$\kappa_{2m}(a) = \big(\frac{a}{a^2-2}\big)^2 \frac{1}{\sqrt{\pi}m^{3/2}}\big(1 + O(\frac{1}{m})\big), \quad m \to \infty. \tag{5.11}$$

If $a > \sqrt{2}$, then relations (3.14) and (3.25) imply that $\kappa_{2m}(a)$ tend to infinity exponentially:

$$\kappa_{2m}(a) = \frac{a^2-2}{a^2-1}\Big(\frac{a^4}{4(a^2-1)}\Big)^m + O(1), \quad m \to \infty.$$

6. Miscellaneous

6.1. Let us now briefly discuss the connection of the moment problem with Hankel operators. We denote by $\mathcal{D} \subset \ell^2(\mathbb{Z}_+)$ the set of vectors $f = (f_0, f_1, \ldots)$ with only a finite number of non-zero components f_n. The Hankel operator K corresponding to a sequence κ_n is formally defined by the relation

$$(Kf)_n = \sum_{n=0}^{\infty} \kappa_{n+m} f_m, \quad \forall f \in \mathcal{D}.$$

The classical Hamburger theorem (see, e.g., the book [1]) states that the Hankel quadratic form

$$\kappa[f,f] := \sum_{n,m=0}^{\infty} \kappa_{n+m} f_m \overline{f_n} \geq 0, \quad \forall f \in \mathcal{D},$$

if and only if

$$\kappa_n = \int_{-\infty}^{\infty} \lambda^n d\rho(\lambda) \tag{6.1}$$

with some nonnegative measure $d\rho(\lambda)$.

If the form $\kappa[f,f]$ is closable in $\ell^2(\mathbb{Z}_+)$, then using the Friedrichs construction (see, e.g., the book [3]) one can standardly associate with it a nonnegative operator $K = K(\kappa)$ in this space. The necessary and sufficient condition obtained in [16] for the form determined by sequence (6.1) to be closable is that $\operatorname{supp}\rho \subset [-1,1]$ and $\rho(\{1\}) = \rho(\{-1\}) = 0$. For the sequence $\kappa_n(a)$ defined by (6.1) with the measure $d\rho_a(\lambda)$ given by (3.24), this condition is satisfied if and only if $a \leq \sqrt{2}$.

Let $\operatorname{supp}\rho \subset [-1,1]$ and $\rho(\{1\}) = \rho(\{-1\}) = 0$. Then the Widom theorem [12] states that the operator K with matrix elements (6.1) is bounded (resp. compact) if and only if $\rho(1-\varepsilon, 1) = O(\varepsilon)$ and $\rho(-1, -1-\varepsilon) = O(\varepsilon)$ (resp, $\rho(1-\varepsilon,1) = o(\varepsilon)$ and $\rho(-1,-1-\varepsilon) = o(\varepsilon)$) as $\varepsilon \to 0$. Therefore the Hankel operator $K(a)$ with the matrix elements $\kappa_n(a)$ is unbounded if $a = \sqrt{2}$. On the contrary, the operators $K(a)$ are compact if $a < \sqrt{2}$.

6.2. Let us now discuss the behavior of the spectral measure $d\rho_a(\lambda)$, of the scattering matrix $S_a(\lambda)$ and of the spectral shift function $\xi_a(\lambda)$ in the limits $a \to \infty$ and $a \to 0$. Both these limits are very singular.

If $a \to \infty$ (the "large coupling" limit), then according to (3.25)

$$\rho_a(\{\lambda_\pm(a)\}) = \frac{a^2-2}{2(a^2-1)} \to 1/2$$

and therefore $\rho_a(-1,1)\to 0$. On the contrary, according to (4.5) the scattering matrix $S_a(\lambda)$ defined for $\lambda\in(-1,1)$ has a non-trivial limit:

$$\lim_{a\to\infty} S_a(\lambda)=\frac{2\lambda^2-1+2i\lambda\sqrt{1-\lambda^2}}{2\lambda^2-1-2i\lambda\sqrt{1-\lambda^2}}\neq 1$$

(if $\lambda\neq 1$). Similarly, according to (4.9)

$$\lim_{a\to\infty}\xi_a(\lambda)=\frac{1}{\pi}\operatorname{arccot}\frac{1-2\lambda^2}{2\lambda\sqrt{1-\lambda^2}},\quad \lambda\in(0,1).$$

Let now $a\to 0$. By (1.1), the operator H_0 can be considered as the orthogonal sum of the operators acting in the space $\mathbb{C}\oplus\ell^2(\mathbb{N})$ where $\mathbb{N}=(1,2,\dots)$. The restriction of H_0 on $\mathbb{C}$ is zero, and its restriction on $\ell^2(\mathbb{N})$ is again given by matrix (1.1) where $a=1$. According to (3.24) for every $\varepsilon\in(0,1)$ we have

$$\rho_a(-1,-1+\varepsilon)\to 0,\quad \rho_a(1-\varepsilon,1)\to 0$$

and therefore $\rho_a(-\varepsilon,\varepsilon)\to 1$ as $a\to 0$. It follows from (4.5) that the scattering matrix $S_a(\lambda)$ has a non-trivial limit:

$$\lim_{a\to 0} S_a(\lambda)=\frac{\lambda+i\sqrt{1-\lambda^2}}{\lambda-i\sqrt{1-\lambda^2}}\neq 1,\quad \lambda\in(-1,1).$$

Similarly, it follows from (4.8) that

$$\lim_{a\to 0}\xi_a(\lambda)=-\frac{1}{\pi}\arctan\frac{\sqrt{1-\lambda^2}}{\lambda},\quad \lambda\in(-1,1).$$

6.3. The expression for the scattering matrix can also be obtained in terms of the operator $T_a(z)$ defined by formula (3.15). Let $t_a(\lambda,\mu;z)$ be the integral kernel of the operator $F_1T_a(z)F_1^*$ where F_1 is given by (3.28) with

$$\psi_n(\lambda;1)=\sqrt{\frac{2}{\pi}}\sqrt[4]{1-\lambda^4}\,\mathsf{U}_n(\lambda)$$

(recall that $\mathsf{U}_n(\lambda)=\mathrm{Ch}_n(\lambda;1)$). Then

$$S_a(\lambda)=1-2\pi i\,t_a(\lambda,\lambda;\lambda+i0),\quad \lambda\in(-1,1). \tag{6.2}$$

This formula has been obtained in the paper [7] in the framework of the Friedrichs-Faddeev model and discussed in the book [14] in a more general setting.

It follows from formula (3.18) that

$$\pi D_a(z)t_a(\lambda,\mu;z)=(1-\lambda^2)^{1/4}(1-\mu^2)^{1/4}\Big(b_{0,0}(z)\mathsf{U}_0(\lambda)\mathsf{U}_0(\mu)+b_{0,1}(z)\mathsf{U}_0(\lambda)\mathsf{U}_1(\mu)$$
$$+(b_{1,0}(z)\mathsf{U}_1(\lambda)\mathsf{U}_0(\mu)+b_{1,1}(z)\mathsf{U}_1(\lambda)\mathsf{U}_1(\mu)\Big).$$

Since $\mathsf{U}_0(\lambda)=1$, $\mathsf{U}_1(\lambda)=2\lambda$, representation (6.2) implies that

$$S_a(\lambda)=1-\frac{2i\sqrt{1-\lambda^2}}{D_a(\lambda+i0)}\big(b_{0,0}(\lambda+i0)+2(b_{0,1}(\lambda+i0)+b_{1,0}(\lambda+i0))\lambda+4b_{1,1}(\lambda+i0)\lambda^2\big). \tag{6.3}$$

Let us now use formulas (3.19) and the second identity (2.21):

$$b_{0,0}(\lambda+i0)+2(b_{0,1}(\lambda+i0)+b_{1,0}(\lambda+i0))\lambda+4b_{1,1}(\lambda+i0)\lambda^2$$
$$=2(a-1)\lambda\big(2-(a-1)\omega(\lambda+i0)^2+2(a-1)\omega(\lambda+i0)\lambda\big)=2(a^2-1)\lambda.$$

Substituting this expression into formula (6.3), we recover expression (4.5).

References

[1] N. Akhiezer, *The classical moment problem and some related questions in analysis*, Oliver and Boyd, Edinburgh and London, 1965.

[2] Yu. M. Berezanskii, *Expansion in eigenfunctions of selfadjoint operators*, Amer. Math. Soc., Providence, R.I., 1968.

[3] M. Sh. Birman and M. Z. Solomyak, *Spectral theory of selfadjoint operators in Hilbert space*, Reidel, Doldrecht, 1987.

[4] V. V. Borzov, E. V. Damaskinsky, *On the spectrum of discrete Schrödinger equation with one dimensional perturbation*, arXiv:1609.05527, Days of Diffraction (2016), 73-78.

[5] K. M. Case, *Orthogonal polynomials.* II, Journal of Math. Phys., **16** (1975), 1435-1440.

[6] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions*, Vol. 1, 2, McGraw-Hill, New York-Toronto-London, 1953.

[7] L. D. Faddeev, *On the Friedrichs model in the theory of perturbations of the continuous spectrum*, Amer. Math. Soc. Transl. (Ser. 2) **62** (1967), 177-203.

[8] F. Gesztesy, B. Simon, *m-Functions and inverse spectral analysis for finite and semi-infinite Jacobi matrices*, Journal d'Analyse Mathém., **73** (1997), 267-297.

[9] M. E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Cambridge University Press, Cambridge, 2005.

[10] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc., Providence, R. I., 1978.

[11] G. Teschl, *Jacobi operators and completely integrable nonlinear lattices*, Amer. Math. Soc., Providence, R. I., 2000.

[12] H. Widom, *Hankel matrices*, Trans. Amer. Math. Soc. **121** (1966), 1-35.

[13] D. R. Yafaev, *On a zero-range interaction of a quantum particle with the vacuum*, J. Phys. A, **25** (1992), 963-978.

[14] D. R. Yafaev, *Mathematical scattering theory: General theory*, Amer. Math. Soc., Providence, R. I., 1992.

[15] D. R. Yafaev, *Mathematical scattering theory: Analytic theory*, Amer. Math. Soc., Providence, R. I., 2010.

[16] D. R. Yafaev, *Unbounded Hankel operator and moment problems*, Integral Eq. Oper. Theory, **85** (2016), 289-300.

IRMAR, Université de Rennes I, Campus de Beaulieu, Rennes, 35042 FRANCE and SPGU, Univ. Nab. 7/9, Saint Petersburg, 199034 RUSSIA

E-mail address: `yafaev@univ-rennes1.fr`