SATOSHI RISK TABLES

CYRIL GRUNSPAN AND RICARDO PÉREZ-MARCO

ABSTRACT. We present Bitcoin Security Tables computing the probability of success $p(z, q, t)$ of a double spend attack by an attacker controlling a share q of the hashrate after z confirmations in time t.

1. INTRODUCTION.

The main breakthrough in [2] is the solution to the *double spend problem* of an electronic currency unit without a central authority. Bitcoin is the first form of *peer-to-peer* (P2P) electronic currency.

A double spend attack can only be attempted with a substantial fraction of the hashrate used in the *Proof-of-Work* of the Bitcoin network. The attackers will start a *double spend race* against the rest of the network to replace the last blocks of the blockchain. The last section of [2] computes the probability that the attackers catch up. Following Nakamoto, by "success of the attackers" we mean catching up the z blocks, although to replace the blocks the attackers need to validate $z + 1$. In [1] we correct Nakamoto's analysis based on an abusive approximation and give a closed-form formula for the exact probability.

Theorem 1. (C. Grunspan, R. Pérez-Marco, [1])

Let $0 < q < 1/2$*, resp.* $p = 1 - q$*, the relative hash power of the group of the attackers, resp. of honest miners. After* z *blocks have been validated by the honest miners, the probability of success of the attackers is*

$$P(z) = I_{4pq}(z, 1/2) \ ,$$

where $I_x(a, b)$ *is the Regularized Incomplete Beta Function*

$$I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1-t)^{b-1}\, dt \ .$$

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We carry out a more accurate risk analysis by considering not only the number of confirmations z but also the time t it took for the last z validated blocks, which is an information that is clearly available.

In practice, in order to avoid a double spend attack, the recipient of the bitcoin transaction waits for $z \geq 1$ confirmations. He also has the information on the time t it took to confirm the transaction z times. Obviously the probability of success of the attackers increases with t since he has more time to secretly mine his alternative blockchain. In [1] we carry out a more precise risk analysis considering this available data. The relevant dimensionless parameter introduced is the relative deviation from the expected time

$$\kappa = \frac{t}{zt_0} = \frac{pt}{z\tau_0} = \frac{1-q}{z}\frac{t}{\tau_0} ,$$

where τ_0 is the expected validation time of a new block ($\tau_0 = 10\,\mathrm{min}$ for the Bitcoin network). In [1] we give a closed-form formula for the probability $P(z, q, \kappa)$ that the attackers catch up with the current chain.

Theorem 2. (C. Grunspan, R. Pérez-Marco, [1])

We have

$$P(z, q, \kappa) = 1 - Q(z, \kappa z q/p) + \left(\frac{q}{p}\right)^z e^{\kappa z \frac{p-q}{p}} Q(z, \kappa z) .$$

After a validation z has been observed in the network and the time t being measured, and $r = \frac{t}{zt_0}$, one can compute the probabilities

$$\tilde{P}(z, q, r) = P(z, q, (1-q)r) ,$$

that give the probability of the attackers to catch up the current blockchain. We tabulate these probabilities for $z = 1, 2, \ldots, 10$.

2. USE OF THE TABLES.

The tables are useful to determine the number of confirmations needed for a given transaction. In practice, someone receiving a bitcoin transaction will check after z confirmations the corresponding table. With the measured time t, he will compute $r = \frac{t}{zt_0}$, and check the value of $\tilde{P}(z, q, r)$ and then asses the level of risk assumed accepting the payment.

3. SATOSHI RISK TABLES.

Below we give the tables for $z = 1, 2, \ldots, 9$ of $\tilde{P}(z, q, r)$ for different values of q and r in % with 2 decimal places. These are what is needed for practical applications.

z=1

$r\backslash q$	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
0.1	2.24	4.55	6.94	9.42	12	14.67	17.44	20.33	23.34	26.49	29.77	33.2	36.8
0.2	2.43	4.93	7.5	10.14	12.87	15.68	18.59	21.6	24.71	27.94	31.3	34.79	38.42
0.3	2.63	5.31	8.05	10.86	13.74	16.69	19.72	22.84	26.05	29.37	32.79	36.33	40
0.4	2.82	5.69	8.6	11.57	14.6	17.68	20.84	24.07	27.37	30.77	34.25	37.84	41.54
0.5	3.02	6.06	9.15	12.28	15.45	18.67	21.94	25.27	28.67	32.14	35.68	39.32	43.04
0.6	3.21	6.44	9.69	12.97	16.29	19.64	23.02	26.46	29.94	33.48	37.08	40.76	44.5
0.7	3.4	6.81	10.23	13.67	17.12	20.59	24.09	27.62	31.19	34.8	38.45	42.16	45.93
0.8	3.6	7.18	10.77	14.36	17.95	21.54	25.15	28.77	32.42	36.09	39.79	43.53	47.32
0.9	3.79	7.56	11.3	15.04	18.76	22.48	26.19	29.9	33.62	37.35	41.1	44.87	48.67
1	3.98	7.92	11.83	15.72	19.57	23.4	27.22	31.02	34.81	38.6	42.38	46.18	49.99
1.1	4.17	8.29	12.36	16.39	20.37	24.32	28.23	32.11	35.97	39.81	43.64	47.45	51.27
1.2	4.36	8.66	12.89	17.05	21.16	25.22	29.23	33.19	37.11	41	44.86	48.7	52.52
1.3	4.55	9.02	13.41	17.71	21.95	26.11	30.21	34.25	38.24	42.17	46.06	49.92	53.74
1.4	4.75	9.39	13.93	18.37	22.72	26.99	31.18	35.29	39.34	43.32	47.24	51.1	54.93
1.5	4.94	9.75	14.44	19.02	23.49	27.86	32.14	36.32	40.42	44.44	48.38	52.26	56.08
1.6	5.13	10.11	14.95	19.67	24.25	28.72	33.08	37.33	41.48	45.54	49.51	53.4	57.21
1.7	5.32	10.47	15.46	20.31	25.01	29.57	34.01	38.33	42.53	46.62	50.61	54.5	58.31
1.8	5.5	10.82	15.97	20.94	25.75	30.41	34.93	39.31	43.55	47.67	51.68	55.58	59.38
1.9	5.69	11.18	16.47	21.57	26.49	31.24	35.83	40.27	44.56	48.71	52.73	56.63	60.42
2	5.88	11.53	16.97	22.2	27.22	32.06	36.73	41.22	45.55	49.73	53.76	57.66	61.44
2.1	6.07	11.89	17.47	22.82	27.95	32.87	37.6	42.15	46.52	50.72	54.77	58.67	62.43
2.2	6.26	12.24	17.96	23.43	28.66	33.68	38.47	43.07	47.47	51.7	55.75	59.65	63.39
2.3	6.44	12.59	18.45	24.04	29.37	34.47	39.33	43.97	48.41	52.65	56.71	60.6	64.33
2.4	6.63	12.94	18.94	24.65	30.08	35.25	40.17	44.86	49.33	53.59	57.66	61.54	65.25
2.5	6.82	13.29	19.42	25.25	30.77	36.02	41	45.74	50.23	54.51	58.58	62.45	66.14
2.6	7	13.63	19.91	25.84	31.46	36.78	41.82	46.6	51.12	55.41	59.48	63.34	67.01
2.7	7.19	13.98	20.38	26.43	32.14	37.54	42.63	47.44	51.99	56.29	60.36	64.21	67.85
2.8	7.38	14.32	20.86	27.02	32.82	38.28	43.43	48.28	52.85	57.16	61.22	65.06	68.68
2.9	7.56	14.66	21.33	27.6	33.49	39.02	44.22	49.1	53.69	58.01	62.07	65.89	69.48
3	7.75	15	21.8	28.18	34.15	39.75	44.99	49.91	54.52	58.84	62.89	66.7	70.27
3.1	7.93	15.34	22.27	28.75	34.8	40.46	45.76	50.7	55.33	59.65	63.7	67.49	71.03
3.2	8.11	15.68	22.74	29.32	35.45	41.18	46.51	51.49	56.13	60.45	64.49	68.26	71.77
3.3	8.3	16.02	23.2	29.88	36.1	41.88	47.25	52.26	56.91	61.24	65.26	69.01	72.5
3.4	8.48	16.35	23.66	30.44	36.73	42.57	47.99	53.01	57.68	62	66.02	69.74	73.2
3.5	8.66	16.69	24.12	30.99	37.36	43.26	48.71	53.76	58.43	62.76	66.76	70.46	73.89

$\boxed{z=2}$

$r\backslash q$	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
0.1	0.05	0.21	0.49	0.89	1.45	2.16	3.06	4.15	5.47	7.04	8.89	11.06	13.58
0.2	0.06	0.25	0.58	1.05	1.69	2.51	3.52	4.74	6.2	7.92	9.92	12.25	14.92
0.3	0.07	0.3	0.68	1.23	1.96	2.89	4.03	5.39	7	8.87	11.03	13.51	16.35
0.4	0.09	0.35	0.79	1.43	2.26	3.31	4.58	6.09	7.85	9.89	12.21	14.86	17.85
0.5	0.1	0.4	0.92	1.64	2.59	3.77	5.18	6.84	8.76	10.96	13.46	16.26	19.41
0.6	0.12	0.47	1.05	1.88	2.94	4.25	5.81	7.64	9.73	12.09	14.76	17.73	21.02
0.7	0.13	0.54	1.2	2.13	3.32	4.77	6.49	8.47	10.73	13.27	16.1	19.24	22.68
0.8	0.15	0.61	1.36	2.4	3.72	5.32	7.2	9.35	11.78	14.5	17.49	20.78	24.37
0.9	0.17	0.69	1.53	2.68	4.14	5.89	7.94	10.26	12.87	15.76	18.92	22.36	26.09
1	0.2	0.77	1.71	2.98	4.58	6.5	8.71	11.21	13.99	17.05	20.37	23.97	27.83
1.1	0.22	0.86	1.89	3.3	5.05	7.12	9.51	12.18	15.14	18.37	21.86	25.6	29.59
1.2	0.24	0.95	2.09	3.63	5.53	7.77	10.33	13.18	16.32	19.71	23.36	27.24	31.35
1.3	0.27	1.05	2.3	3.97	6.03	8.44	11.18	14.21	17.52	21.08	24.87	28.89	33.11
1.4	0.3	1.15	2.51	4.33	6.54	9.13	12.04	15.26	18.74	22.46	26.4	30.54	34.87
1.5	0.33	1.26	2.74	4.69	7.08	9.84	12.93	16.32	19.97	23.85	27.94	32.2	36.63
1.6	0.36	1.37	2.97	5.08	7.63	10.56	13.84	17.41	21.22	25.26	29.48	33.85	38.37
1.7	0.39	1.49	3.21	5.47	8.19	11.31	14.76	18.5	22.48	26.67	31.02	35.5	40.1
1.8	0.42	1.61	3.46	5.88	8.77	12.06	15.7	19.61	23.76	28.08	32.55	37.14	41.81
1.9	0.46	1.74	3.72	6.29	9.36	12.84	16.65	20.73	25.03	29.5	34.09	38.77	43.5
2	0.49	1.87	3.98	6.72	9.96	13.62	17.61	21.87	26.32	30.92	35.62	40.38	45.17
2.1	0.53	2	4.26	7.15	10.58	14.42	18.59	23	27.6	32.33	37.13	41.97	46.82
2.2	0.57	2.14	4.54	7.6	11.2	15.22	19.57	24.15	28.89	33.74	38.64	43.55	48.44
2.3	0.61	2.28	4.82	8.06	11.84	16.04	20.56	25.29	30.18	35.14	40.13	45.11	50.03
2.4	0.65	2.43	5.11	8.52	12.48	16.87	21.55	26.45	31.46	36.54	41.61	46.64	51.6
2.5	0.69	2.58	5.41	8.99	13.14	17.7	22.56	27.6	32.74	37.92	43.07	48.15	53.13
2.6	0.73	2.73	5.72	9.47	13.8	18.54	23.56	28.75	34.02	39.29	44.52	49.64	54.63
2.7	0.78	2.89	6.03	9.96	14.47	19.39	24.57	29.9	35.29	40.66	45.94	51.1	56.1
2.8	0.82	3.05	6.35	10.46	15.15	20.24	25.58	31.05	36.55	42	47.35	52.54	57.54
2.9	0.87	3.21	6.67	10.96	15.84	21.1	26.6	32.2	37.81	43.34	48.73	53.95	58.95
3	0.92	3.38	7	11.47	16.53	21.96	27.61	33.34	39.05	44.66	50.1	55.33	60.32
3.1	0.97	3.55	7.34	11.99	17.22	22.83	28.63	34.48	40.29	45.96	51.44	56.68	61.65
3.2	1.02	3.73	7.68	12.51	17.92	23.7	29.64	35.62	41.51	47.24	52.75	58	62.96
3.3	1.07	3.91	8.02	13.03	18.63	24.57	30.65	36.74	42.72	48.51	54.05	59.3	64.23
3.4	1.12	4.09	8.37	13.57	19.34	25.44	31.66	37.86	43.92	49.76	55.32	60.56	65.47
3.5	1.18	4.27	8.73	14.1	20.06	26.31	32.67	38.97	45.11	50.99	56.56	61.8	66.67

$\boxed{z = 3}$

$r\backslash q$	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
0.1	0	0.01	0.03	0.08	0.17	0.32	0.54	0.85	1.28	1.87	2.65	3.68	5.01
0.2	0	0.01	0.04	0.11	0.22	0.4	0.66	1.04	1.55	2.24	3.14	4.3	5.78
0.3	0	0.02	0.06	0.14	0.28	0.5	0.82	1.27	1.87	2.67	3.7	5.01	6.66
0.4	0	0.02	0.07	0.18	0.35	0.62	1	1.54	2.25	3.17	4.34	5.82	7.65
0.5	0	0.03	0.09	0.22	0.44	0.76	1.22	1.85	2.68	3.74	5.07	6.72	8.75
0.6	0	0.03	0.12	0.27	0.54	0.93	1.48	2.22	3.17	4.39	5.89	7.73	9.95
0.7	0.01	0.04	0.14	0.34	0.65	1.12	1.77	2.63	3.73	5.1	6.79	8.83	11.26
0.8	0.01	0.05	0.18	0.41	0.79	1.34	2.09	3.09	4.34	5.9	7.78	10.03	12.67
0.9	0.01	0.06	0.21	0.49	0.94	1.58	2.46	3.6	5.02	6.76	8.85	11.31	14.17
1	0.01	0.08	0.25	0.58	1.11	1.86	2.86	4.15	5.76	7.7	9.99	12.67	15.76
1.1	0.01	0.09	0.3	0.69	1.3	2.16	3.3	4.76	6.56	8.7	11.21	14.12	17.43
1.2	0.01	0.11	0.35	0.8	1.5	2.49	3.79	5.42	7.41	9.77	12.5	15.63	19.16
1.3	0.02	0.13	0.41	0.93	1.73	2.85	4.31	6.13	8.32	10.9	13.86	17.21	20.96
1.4	0.02	0.15	0.48	1.07	1.98	3.23	4.86	6.88	9.28	12.08	15.27	18.85	22.81
1.5	0.02	0.17	0.55	1.22	2.25	3.65	5.46	7.67	10.3	13.32	16.74	20.54	24.7
1.6	0.03	0.2	0.63	1.39	2.54	4.1	6.09	8.51	11.36	14.62	18.26	22.27	26.63
1.7	0.03	0.23	0.71	1.57	2.84	4.57	6.76	9.39	12.47	15.95	19.82	24.05	28.59
1.8	0.03	0.26	0.8	1.76	3.17	5.07	7.46	10.31	13.62	17.33	21.42	25.85	30.57
1.9	0.04	0.29	0.9	1.96	3.52	5.6	8.19	11.27	14.81	18.75	23.06	27.68	32.57
2	0.04	0.33	1	2.18	3.89	6.16	8.96	12.27	16.03	20.2	24.72	29.53	34.58
2.1	0.05	0.36	1.12	2.41	4.28	6.74	9.76	13.3	17.29	21.68	26.41	31.39	36.59
2.2	0.06	0.4	1.23	2.65	4.69	7.35	10.59	14.36	18.58	23.19	28.11	33.27	38.59
2.3	0.06	0.45	1.36	2.9	5.12	7.98	11.44	15.44	19.9	24.72	29.83	35.14	40.58
2.4	0.07	0.49	1.49	3.17	5.56	8.64	12.33	16.56	21.24	26.27	31.56	37.01	42.56
2.5	0.08	0.54	1.63	3.46	6.03	9.31	13.24	17.7	22.6	27.83	33.29	38.88	44.53
2.6	0.08	0.59	1.78	3.75	6.51	10.02	14.17	18.87	23.98	29.41	35.02	40.74	46.46
2.7	0.09	0.65	1.94	4.06	7.01	10.74	15.13	20.05	25.38	30.99	36.76	42.58	48.38
2.8	0.1	0.71	2.1	4.38	7.53	11.48	16.1	21.25	26.79	32.57	38.48	44.41	50.26
2.9	0.11	0.77	2.27	4.71	8.07	12.25	17.1	22.47	28.21	34.16	40.2	46.22	52.11
3	0.12	0.83	2.45	5.06	8.62	13.03	18.11	23.71	29.64	35.75	41.91	48	53.92
3.1	0.13	0.9	2.63	5.41	9.19	13.83	19.14	24.95	31.07	37.33	43.6	49.75	55.7
3.2	0.14	0.97	2.82	5.78	9.78	14.64	20.19	26.21	32.51	38.91	45.28	51.48	57.43
3.3	0.15	1.04	3.02	6.16	10.38	15.48	21.25	27.47	33.95	40.48	46.93	53.18	59.13
3.4	0.16	1.12	3.23	6.56	10.99	16.32	22.32	28.75	35.38	42.04	48.57	54.84	60.78
3.5	0.18	1.2	3.44	6.96	11.62	17.18	23.41	30.03	36.82	43.59	50.18	56.47	62.39

$\boxed{z=4}$

$r\backslash q$	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
0.1	0	0	0	0.01	0.02	0.05	0.09	0.17	0.3	0.5	0.79	1.22	1.85
0.2	0	0	0	0.01	0.03	0.06	0.12	0.23	0.39	0.63	0.99	1.51	2.24
0.3	0	0	0	0.02	0.04	0.09	0.17	0.3	0.5	0.8	1.24	1.85	2.71
0.4	0	0	0.01	0.02	0.05	0.12	0.22	0.39	0.64	1.01	1.54	2.27	3.27
0.5	0	0	0.01	0.03	0.07	0.15	0.29	0.5	0.82	1.27	1.91	2.77	3.93
0.6	0	0	0.01	0.04	0.1	0.2	0.37	0.64	1.03	1.59	2.35	3.36	4.7
0.7	0	0	0.02	0.05	0.13	0.26	0.48	0.81	1.3	1.96	2.87	4.05	5.59
0.8	0	0	0.02	0.07	0.17	0.34	0.61	1.02	1.61	2.41	3.47	4.84	6.6
0.9	0	0.01	0.03	0.09	0.21	0.43	0.77	1.27	1.97	2.92	4.16	5.74	7.73
1	0	0.01	0.04	0.12	0.27	0.54	0.95	1.55	2.39	3.5	4.94	6.74	8.97
1.1	0	0.01	0.05	0.15	0.34	0.66	1.16	1.89	2.87	4.16	5.81	7.85	10.34
1.2	0	0.01	0.06	0.18	0.42	0.81	1.41	2.26	3.41	4.9	6.77	9.07	11.83
1.3	0	0.02	0.08	0.22	0.51	0.98	1.69	2.69	4.02	5.72	7.83	10.39	13.42
1.4	0	0.02	0.09	0.27	0.61	1.17	2	3.16	4.68	6.61	8.98	11.81	15.12
1.5	0	0.02	0.11	0.33	0.73	1.39	2.35	3.68	5.42	7.58	10.21	13.32	16.91
1.6	0	0.03	0.14	0.39	0.86	1.63	2.74	4.26	6.21	8.63	11.53	14.92	18.79
1.7	0	0.04	0.16	0.46	1.01	1.9	3.17	4.88	7.07	9.75	12.93	16.6	20.75
1.8	0	0.04	0.19	0.54	1.18	2.19	3.64	5.56	7.99	10.94	14.4	18.35	22.77
1.9	0	0.05	0.22	0.63	1.37	2.52	4.14	6.29	8.98	12.2	15.95	20.18	24.86
2	0	0.06	0.26	0.73	1.57	2.87	4.69	7.07	10.02	13.53	17.55	22.06	27
2.1	0	0.07	0.3	0.84	1.79	3.25	5.28	7.9	11.12	14.91	19.22	24	29.17
2.2	0.01	0.08	0.35	0.96	2.03	3.66	5.9	8.78	12.28	16.35	20.94	25.98	31.38
2.3	0.01	0.09	0.4	1.08	2.29	4.1	6.57	9.71	13.48	17.84	22.71	28	33.61
2.4	0.01	0.1	0.45	1.22	2.56	4.57	7.27	10.68	14.74	19.38	24.52	30.04	35.85
2.5	0.01	0.12	0.51	1.38	2.86	5.07	8.02	11.69	16.04	20.97	26.36	32.11	38.09
2.6	0.01	0.13	0.58	1.54	3.18	5.59	8.79	12.75	17.39	22.59	28.23	34.19	40.34
2.7	0.01	0.15	0.65	1.71	3.52	6.15	9.61	13.85	18.77	24.24	30.13	36.28	42.57
2.8	0.01	0.17	0.72	1.9	3.88	6.74	10.46	14.99	20.19	25.93	32.04	38.37	44.78
2.9	0.01	0.19	0.8	2.1	4.26	7.35	11.35	16.16	21.64	27.63	33.96	40.46	46.97
3	0.02	0.21	0.89	2.31	4.66	7.99	12.26	17.36	23.12	29.36	35.89	42.53	49.13
3.1	0.02	0.24	0.98	2.54	5.08	8.66	13.21	18.6	24.63	31.1	37.82	44.59	51.26
3.2	0.02	0.26	1.08	2.78	5.53	9.36	14.19	19.86	26.16	32.86	39.75	46.63	53.35
3.3	0.02	0.29	1.18	3.03	5.99	10.08	15.2	21.15	27.71	34.62	41.67	48.65	55.39
3.4	0.02	0.32	1.3	3.29	6.47	10.83	16.24	22.47	29.27	36.38	43.57	50.63	57.39
3.5	0.03	0.35	1.41	3.57	6.98	11.61	17.3	23.8	30.85	38.15	45.46	52.58	59.34

$\boxed{z = 5}$

$r\backslash q$	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
0.1	0	0	0	0	0	0.01	0.02	0.04	0.07	0.13	0.24	0.41	0.68
0.2	0	0	0	0	0	0.01	0.02	0.05	0.1	0.18	0.31	0.53	0.87
0.3	0	0	0	0	0.01	0.01	0.03	0.07	0.13	0.24	0.41	0.68	1.1
0.4	0	0	0	0	0.01	0.02	0.05	0.1	0.18	0.32	0.54	0.89	1.39
0.5	0	0	0	0	0.01	0.03	0.07	0.13	0.25	0.43	0.72	1.14	1.76
0.6	0	0	0	0.01	0.02	0.04	0.09	0.19	0.34	0.57	0.93	1.46	2.22
0.7	0	0	0	0.01	0.03	0.06	0.13	0.25	0.45	0.75	1.21	1.86	2.77
0.8	0	0	0	0.01	0.04	0.09	0.18	0.34	0.59	0.98	1.55	2.34	3.43
0.9	0	0	0	0.02	0.05	0.12	0.24	0.45	0.77	1.26	1.96	2.92	4.21
1	0	0	0.01	0.02	0.07	0.16	0.32	0.58	1	1.6	2.45	3.6	5.12
1.1	0	0	0.01	0.03	0.09	0.21	0.41	0.75	1.26	2	3.02	4.39	6.16
1.2	0	0	0.01	0.04	0.12	0.27	0.53	0.95	1.58	2.48	3.69	5.29	7.34
1.3	0	0	0.01	0.05	0.15	0.34	0.67	1.19	1.96	3.03	4.46	6.31	8.65
1.4	0	0	0.02	0.07	0.19	0.43	0.83	1.47	2.39	3.65	5.32	7.46	10.1
1.5	0	0	0.02	0.09	0.24	0.53	1.03	1.79	2.88	4.36	6.29	8.72	11.68
1.6	0	0	0.03	0.11	0.3	0.66	1.25	2.16	3.44	5.16	7.36	10.1	13.39
1.7	0	0.01	0.04	0.14	0.37	0.8	1.51	2.58	4.07	6.04	8.53	11.59	15.22
1.8	0	0.01	0.05	0.17	0.45	0.96	1.8	3.05	4.76	7	9.81	13.19	17.16
1.9	0	0.01	0.06	0.21	0.54	1.15	2.13	3.57	5.53	8.05	11.18	14.9	19.2
2	0	0.01	0.07	0.25	0.64	1.36	2.5	4.14	6.36	9.19	12.64	16.7	21.34
2.1	0	0.01	0.08	0.3	0.76	1.59	2.9	4.78	7.27	10.41	14.19	18.59	23.55
2.2	0	0.02	0.1	0.35	0.89	1.86	3.35	5.46	8.25	11.71	15.83	20.57	25.84
2.3	0	0.02	0.12	0.41	1.04	2.15	3.84	6.21	9.29	13.08	17.55	22.61	28.19
2.4	0	0.02	0.14	0.48	1.21	2.46	4.37	7.01	10.4	14.53	19.33	24.72	30.58
2.5	0	0.03	0.16	0.56	1.39	2.81	4.95	7.86	11.58	16.05	21.19	26.88	33.01
2.6	0	0.03	0.19	0.65	1.59	3.19	5.56	8.78	12.82	17.63	23.1	29.09	35.46
2.7	0	0.04	0.22	0.74	1.81	3.59	6.22	9.74	14.12	19.27	25.06	31.33	37.92
2.8	0	0.04	0.25	0.84	2.04	4.03	6.93	10.76	15.47	20.96	27.06	33.6	40.39
2.9	0	0.05	0.29	0.96	2.3	4.5	7.68	11.83	16.88	22.7	29.1	35.89	42.85
3	0	0.06	0.33	1.08	2.58	5	8.46	12.95	18.34	24.49	31.17	38.18	45.29
3.1	0	0.06	0.37	1.22	2.87	5.54	9.3	14.11	19.85	26.31	33.26	40.47	47.7
3.2	0	0.07	0.42	1.36	3.19	6.1	10.17	15.32	21.39	28.16	35.37	42.75	50.09
3.3	0	0.08	0.48	1.52	3.53	6.7	11.08	16.57	22.98	30.04	37.48	45.02	52.43
3.4	0	0.09	0.53	1.69	3.89	7.33	12.03	17.87	24.6	31.94	39.59	47.27	54.72
3.5	0	0.11	0.59	1.87	4.28	8	13.02	19.19	26.25	33.86	41.7	49.49	56.97

$$\boxed{z = 6}$$

$r\backslash q$	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
0.1	0	0	0	0	0	0	0	0.01	0.02	0.03	0.07	0.14	0.25
0.2	0	0	0	0	0	0	0	0.01	0.02	0.05	0.1	0.19	0.33
0.3	0	0	0	0	0	0	0.01	0.02	0.04	0.07	0.14	0.25	0.45
0.4	0	0	0	0	0	0	0.01	0.02	0.05	0.1	0.19	0.34	0.59
0.5	0	0	0	0	0	0.01	0.02	0.04	0.08	0.15	0.27	0.47	0.79
0.6	0	0	0	0	0	0.01	0.02	0.05	0.11	0.21	0.37	0.63	1.04
0.7	0	0	0	0	0	0.01	0.04	0.08	0.16	0.29	0.51	0.85	1.37
0.8	0	0	0	0	0.01	0.02	0.05	0.11	0.22	0.4	0.69	1.13	1.78
0.9	0	0	0	0	0.01	0.03	0.08	0.16	0.3	0.55	0.92	1.48	2.3
1	0	0	0	0	0.02	0.05	0.11	0.22	0.42	0.73	1.21	1.92	2.93
1.1	0	0	0	0.01	0.02	0.06	0.15	0.3	0.56	0.97	1.58	2.46	3.68
1.2	0	0	0	0.01	0.03	0.09	0.2	0.4	0.74	1.26	2.02	3.1	4.57
1.3	0	0	0	0.01	0.04	0.12	0.27	0.53	0.96	1.61	2.55	3.86	5.6
1.4	0	0	0	0.02	0.06	0.16	0.35	0.69	1.23	2.03	3.18	4.74	6.78
1.5	0	0	0.01	0.02	0.08	0.21	0.45	0.88	1.54	2.53	3.9	5.74	8.11
1.6	0	0	0.01	0.03	0.1	0.27	0.58	1.1	1.92	3.11	4.74	6.88	9.6
1.7	0	0	0.01	0.04	0.14	0.34	0.73	1.37	2.36	3.77	5.68	8.15	11.24
1.8	0	0	0.01	0.05	0.17	0.43	0.9	1.69	2.86	4.52	6.74	9.56	13.02
1.9	0	0	0.01	0.07	0.22	0.53	1.11	2.05	3.44	5.37	7.91	11.09	14.95
2	0	0	0.02	0.09	0.27	0.65	1.35	2.46	4.08	6.31	9.19	12.75	17
2.1	0	0	0.02	0.11	0.33	0.79	1.62	2.92	4.8	7.34	10.58	14.54	19.18
2.2	0	0	0.03	0.13	0.4	0.95	1.93	3.44	5.6	8.47	12.09	16.43	21.47
2.3	0	0	0.04	0.16	0.48	1.14	2.27	4.02	6.47	9.7	13.69	18.44	23.85
2.4	0	0	0.04	0.19	0.58	1.35	2.66	4.66	7.42	11.01	15.4	20.54	26.31
2.5	0	0.01	0.05	0.23	0.68	1.58	3.09	5.35	8.45	12.42	17.2	22.72	28.85
2.6	0	0.01	0.06	0.27	0.8	1.84	3.56	6.11	9.56	13.91	19.09	24.98	31.44
2.7	0	0.01	0.08	0.32	0.94	2.13	4.08	6.93	10.74	15.48	21.05	27.31	34.07
2.8	0	0.01	0.09	0.38	1.09	2.45	4.65	7.82	12	17.13	23.09	29.69	36.73
2.9	0	0.01	0.11	0.44	1.26	2.8	5.26	8.76	13.32	18.85	25.18	32.11	39.4
3	0	0.01	0.12	0.51	1.44	3.18	5.92	9.77	14.72	20.64	27.33	34.56	42.07
3.1	0	0.02	0.15	0.59	1.65	3.59	6.62	10.83	16.17	22.48	29.53	37.03	44.72
3.2	0	0.02	0.17	0.68	1.87	4.04	7.38	11.96	17.69	24.38	31.76	39.51	47.36
3.3	0	0.02	0.19	0.78	2.11	4.52	8.18	13.14	19.27	26.33	34.01	41.99	49.96
3.4	0	0.03	0.22	0.88	2.38	5.03	9.03	14.37	20.89	28.31	36.29	44.46	52.51
3.5	0	0.03	0.25	1	2.66	5.58	9.92	15.66	22.57	30.33	38.57	46.91	55.02

$\boxed{z = 7}$

$r\backslash q$	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
0.1	0	0	0	0	0	0	0	0	0	0.01	0.02	0.05	0.09
0.2	0	0	0	0	0	0	0	0	0.01	0.01	0.03	0.06	0.13
0.3	0	0	0	0	0	0	0	0	0.01	0.02	0.05	0.09	0.18
0.4	0	0	0	0	0	0	0	0.01	0.01	0.03	0.07	0.13	0.25
0.5	0	0	0	0	0	0	0	0.01	0.02	0.05	0.1	0.19	0.35
0.6	0	0	0	0	0	0	0.01	0.02	0.04	0.07	0.15	0.27	0.49
0.7	0	0	0	0	0	0	0.01	0.02	0.05	0.11	0.21	0.39	0.68
0.8	0	0	0	0	0	0.01	0.02	0.04	0.08	0.16	0.31	0.55	0.93
0.9	0	0	0	0	0	0.01	0.02	0.06	0.12	0.24	0.43	0.75	1.25
1	0	0	0	0	0	0.01	0.04	0.08	0.17	0.34	0.6	1.03	1.67
1.1	0	0	0	0	0.01	0.02	0.05	0.12	0.25	0.47	0.83	1.38	2.2
1.2	0	0	0	0	0.01	0.03	0.08	0.17	0.34	0.64	1.11	1.82	2.85
1.3	0	0	0	0	0.01	0.04	0.11	0.24	0.47	0.86	1.47	2.36	3.63
1.4	0	0	0	0	0.02	0.06	0.15	0.32	0.63	1.14	1.9	3.02	4.57
1.5	0	0	0	0.01	0.03	0.08	0.2	0.43	0.83	1.47	2.43	3.8	5.66
1.6	0	0	0	0.01	0.04	0.11	0.27	0.57	1.08	1.88	3.06	4.71	6.92
1.7	0	0	0	0.01	0.05	0.15	0.35	0.74	1.38	2.37	3.8	5.77	8.35
1.8	0	0	0	0.02	0.07	0.19	0.46	0.94	1.73	2.94	4.66	6.97	9.94
1.9	0	0	0	0.02	0.09	0.25	0.58	1.18	2.15	3.6	5.63	8.31	11.71
2	0	0	0.01	0.03	0.11	0.32	0.73	1.47	2.64	4.36	6.73	9.81	13.64
2.1	0	0	0.01	0.04	0.14	0.4	0.91	1.8	3.2	5.22	7.95	11.44	15.72
2.2	0	0	0.01	0.05	0.18	0.49	1.12	2.18	3.83	6.18	9.3	13.22	17.95
2.3	0	0	0.01	0.06	0.22	0.61	1.36	2.62	4.55	7.24	10.77	15.14	20.31
2.4	0	0	0.01	0.08	0.28	0.74	1.63	3.12	5.34	8.41	12.36	17.18	22.79
2.5	0	0	0.02	0.1	0.34	0.9	1.95	3.68	6.22	9.68	14.07	19.34	25.38
2.6	0	0	0.02	0.12	0.41	1.07	2.3	4.3	7.19	11.06	15.89	21.6	28.05
2.7	0	0	0.03	0.14	0.49	1.27	2.7	4.98	8.24	12.53	17.82	23.96	30.8
2.8	0	0	0.03	0.17	0.59	1.5	3.15	5.73	9.38	14.11	19.84	26.41	33.59
2.9	0	0	0.04	0.21	0.7	1.75	3.64	6.55	10.6	15.77	21.95	28.92	36.43
3	0	0	0.05	0.25	0.82	2.04	4.18	7.44	11.9	17.52	24.13	31.49	39.29
3.1	0	0	0.06	0.29	0.95	2.35	4.76	8.39	13.29	19.36	26.39	34.1	42.15
3.2	0	0.01	0.07	0.34	1.11	2.69	5.4	9.41	14.75	21.26	28.7	36.73	45
3.3	0	0.01	0.08	0.4	1.28	3.07	6.09	10.5	16.28	23.24	31.06	39.39	47.83
3.4	0	0.01	0.09	0.46	1.46	3.49	6.84	11.66	17.88	25.27	33.46	42.04	50.62
3.5	0	0.01	0.11	0.54	1.67	3.93	7.63	12.88	19.55	27.36	35.88	44.69	53.36

$\boxed{z = 8}$

$r\backslash q$	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
0.1	0	0	0	0	0	0	0	0	0	0	0.01	0.01	0.03
0.2	0	0	0	0	0	0	0	0	0	0	0.01	0.02	0.05
0.3	0	0	0	0	0	0	0	0	0	0.01	0.02	0.03	0.07
0.4	0	0	0	0	0	0	0	0	0	0.01	0.02	0.05	0.11
0.5	0	0	0	0	0	0	0	0	0.01	0.02	0.04	0.08	0.16
0.6	0	0	0	0	0	0	0	0	0.01	0.03	0.06	0.12	0.23
0.7	0	0	0	0	0	0	0	0.01	0.02	0.04	0.09	0.18	0.33
0.8	0	0	0	0	0	0	0	0.01	0.03	0.07	0.14	0.26	0.48
0.9	0	0	0	0	0	0	0.01	0.02	0.05	0.1	0.2	0.38	0.68
1	0	0	0	0	0	0	0.01	0.03	0.07	0.15	0.3	0.55	0.96
1.1	0	0	0	0	0	0.01	0.02	0.05	0.11	0.23	0.43	0.77	1.32
1.2	0	0	0	0	0	0.01	0.03	0.07	0.16	0.33	0.61	1.07	1.78
1.3	0	0	0	0	0	0.01	0.04	0.11	0.23	0.46	0.84	1.45	2.36
1.4	0	0	0	0	0.01	0.02	0.06	0.15	0.33	0.64	1.14	1.93	3.09
1.5	0	0	0	0	0.01	0.03	0.09	0.21	0.45	0.86	1.52	2.52	3.96
1.6	0	0	0	0	0.01	0.05	0.13	0.29	0.61	1.15	1.99	3.24	5
1.7	0	0	0	0	0.02	0.06	0.17	0.4	0.81	1.5	2.56	4.1	6.22
1.8	0	0	0	0.01	0.03	0.09	0.23	0.53	1.06	1.92	3.24	5.1	7.62
1.9	0	0	0	0.01	0.04	0.12	0.31	0.69	1.36	2.43	4.03	6.26	9.21
2	0	0	0	0.01	0.05	0.15	0.4	0.88	1.72	3.03	4.95	7.58	10.99
2.1	0	0	0	0.01	0.06	0.2	0.51	1.12	2.14	3.73	6	9.06	12.94
2.2	0	0	0	0.02	0.08	0.26	0.65	1.4	2.64	4.53	7.19	10.7	15.08
2.3	0	0	0	0.02	0.11	0.33	0.82	1.72	3.22	5.44	8.51	12.49	17.38
2.4	0	0	0	0.03	0.13	0.41	1.01	2.1	3.87	6.46	9.98	14.45	19.84
2.5	0	0	0.01	0.04	0.17	0.51	1.24	2.54	4.61	7.6	11.57	16.54	22.43
2.6	0	0	0.01	0.05	0.21	0.63	1.5	3.04	5.44	8.85	13.3	18.78	25.15
2.7	0	0	0.01	0.06	0.26	0.77	1.8	3.6	6.36	10.21	15.16	21.14	27.97
2.8	0	0	0.01	0.08	0.32	0.92	2.15	4.23	7.38	11.69	17.14	23.61	30.87
2.9	0	0	0.01	0.1	0.39	1.11	2.53	4.93	8.49	13.27	19.23	26.17	33.84
3	0	0	0.02	0.12	0.47	1.31	2.97	5.7	9.69	14.97	21.42	28.82	36.85
3.1	0	0	0.02	0.14	0.56	1.55	3.45	6.54	10.98	16.76	23.71	31.54	39.89
3.2	0	0	0.03	0.17	0.66	1.81	3.98	7.46	12.37	18.64	26.07	34.3	42.93
3.3	0	0	0.03	0.21	0.78	2.11	4.57	8.45	13.84	20.62	28.51	37.1	45.96
3.4	0	0	0.04	0.25	0.91	2.43	5.21	9.52	15.39	22.67	31	39.92	48.96
3.5	0	0	0.05	0.29	1.06	2.79	5.91	10.66	17.03	24.8	33.53	42.74	51.92

$\boxed{z = 9}$

$r\backslash q$	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
0.1	0	0	0	0	0	0	0	0	0	0	0	0	0.01
0.2	0	0	0	0	0	0	0	0	0	0	0	0.01	0.02
0.3	0	0	0	0	0	0	0	0	0	0	0.01	0.01	0.03
0.4	0	0	0	0	0	0	0	0	0	0	0.01	0.02	0.05
0.5	0	0	0	0	0	0	0	0	0	0.01	0.01	0.03	0.07
0.6	0	0	0	0	0	0	0	0	0	0.01	0.02	0.05	0.11
0.7	0	0	0	0	0	0	0	0	0.01	0.02	0.04	0.08	0.17
0.8	0	0	0	0	0	0	0	0	0.01	0.03	0.06	0.13	0.25
0.9	0	0	0	0	0	0	0	0.01	0.02	0.04	0.1	0.19	0.37
1	0	0	0	0	0	0	0	0.01	0.03	0.07	0.15	0.29	0.55
1.1	0	0	0	0	0	0	0.01	0.02	0.05	0.11	0.23	0.43	0.79
1.2	0	0	0	0	0	0	0.01	0.03	0.08	0.17	0.34	0.63	1.11
1.3	0	0	0	0	0	0.01	0.02	0.05	0.11	0.25	0.49	0.89	1.54
1.4	0	0	0	0	0	0.01	0.03	0.07	0.17	0.36	0.69	1.24	2.09
1.5	0	0	0	0	0	0.01	0.04	0.11	0.24	0.51	0.96	1.68	2.78
1.6	0	0	0	0	0	0.02	0.06	0.15	0.35	0.7	1.3	2.24	3.63
1.7	0	0	0	0	0.01	0.03	0.08	0.21	0.48	0.95	1.73	2.92	4.65
1.8	0	0	0	0	0.01	0.04	0.12	0.3	0.65	1.26	2.26	3.75	5.87
1.9	0	0	0	0	0.01	0.05	0.16	0.4	0.86	1.65	2.9	4.73	7.27
2	0	0	0	0	0.02	0.08	0.22	0.53	1.12	2.12	3.66	5.88	8.89
2.1	0	0	0	0.01	0.03	0.1	0.29	0.7	1.44	2.68	4.55	7.2	10.7
2.2	0	0	0	0.01	0.04	0.14	0.38	0.9	1.83	3.34	5.59	8.69	12.72
2.3	0	0	0	0.01	0.05	0.18	0.49	1.14	2.28	4.11	6.76	10.36	14.93
2.4	0	0	0	0.01	0.07	0.23	0.63	1.43	2.82	4.99	8.09	12.2	17.33
2.5	0	0	0	0.02	0.09	0.29	0.79	1.77	3.44	5.99	9.56	14.22	19.91
2.6	0	0	0	0.02	0.11	0.37	0.98	2.16	4.14	7.11	11.19	16.39	22.63
2.7	0	0	0	0.03	0.14	0.46	1.21	2.62	4.94	8.36	12.96	18.72	25.5
2.8	0	0	0	0.04	0.17	0.57	1.47	3.14	5.83	9.73	14.87	21.19	28.47
2.9	0	0	0.01	0.05	0.22	0.7	1.77	3.73	6.83	11.22	16.92	23.78	31.54
3	0	0	0.01	0.06	0.27	0.85	2.12	4.39	7.92	12.84	19.09	26.48	34.68
3.1	0	0	0.01	0.07	0.33	1.03	2.51	5.13	9.12	14.57	21.38	29.27	37.87
3.2	0	0	0.01	0.09	0.4	1.23	2.95	5.94	10.42	16.42	23.77	32.14	41.08
3.3	0	0	0.01	0.11	0.48	1.45	3.45	6.83	11.82	18.37	26.26	35.06	44.29
3.4	0	0	0.02	0.13	0.57	1.71	4	7.81	13.31	20.43	28.82	38.02	47.48
3.5	0	0	0.02	0.16	0.67	1.99	4.6	8.86	14.9	22.57	31.45	40.99	50.64

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