A Mysterious Cluster Expansion Associated to the Expectation Value of the Permanent of 0-1 Matrices

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Abstract

We consider two ensembles of 0-1 $n \times n$ matrices. The first is the set of all $n \times n$ matrices with entries zeroes and ones such that all column sums and all row sums equal r, uniformly weighted. The second is the set of $n \times n$ matrices with zero and one entries where the probability that any given entry is one is r/n, the probabilities of the set of individual entries being i.i.d.'s. Calling the two expectation values E and E_B respectively, we develop a formal relation

$$E(\text{perm}(A)) = E_B(\text{perm}(A))e^{\sum_2 T_i}. \tag{A1}$$

We use two well-known approximating ensembles to E, E_1 and E_2. Replacing E by either E_1 or E_2 we can evaluate all terms in (A1). The equality of (A1) holds for either E_1 or E_2, and the terms T_i have amazing properties. We conjecture (A1) is true for E, and all these properties hold also for E.

We happily inform the reader that no knowledge of cluster expansions is necessary to read and understand this paper, but for those interested a general exposition is given in [1]. The development of this paper is a short sequence of computational steps. We will clearly state which steps are rigorous and which are formal. There will be no theorems, but the formalism has intrinsic beauty. Our results will be computer computations (in integer arithmetic) of a large number of terms in the developed expansion, that again have a beauty and compelling force for theorems to be proved.

We begin with the definitions of the objects we will deal with. For an $n \times n$ matrix A, in addition to the permanent of A, perm(A), we will need information about the permanents of submatrices of A. Given a set of i of the rows of A, and a set of i of the columns of A, an $i \times i$ submatrix is determined. The sum of the permanents of all such $i \times i$ submatrices of A we denote by $\text{perm}_i(A)$.

We will consider a number of ensembles of $n \times n$ matrices, each with an associated expectation:

The first ensemble is the set of all 0-1 $n \times n$ matrices with all row and column sums equal to r. We let E denote the expectation determined by a uniform weighting in this ensemble.

The second is the Bernoulli random matrix ensemble where each entry independently has a probability $p = r/n$ of being one, and is zero otherwise. We denote the associated expectation by E_B.

The third ensemble is the set of nonnegative integer matrices determined by the first measure in Section 4 of [2]. It is in fact determined as a uniformly weighted sum of r independent random permutation matrices. Note that this is not necessarily a set of 0-1 matrices, but all row and column sums equal r. We denote the expectation here by E_1.

The fourth ensemble is again a set of nonnegative integer matrices determined by the second measure in Section 4 of [2]. We denote the associated expectation by E_2.

We will later work with a matrix A from the first ensemble and B from the second and then use a product expectation

$$E_\times(f(A,B)) = EE_B(f(A,B)). \tag{1}$$

Our initial object of study is the expectation of the permanent in our first ensemble

$$E(\mathrm{perm}(A)). \tag{2}$$

We write A as a sum

$$A = B + (A - B) \tag{3}$$

where B lies in our second ensemble, and we get

$$E(\mathrm{perm}(A)) = E_\times(\mathrm{perm}(B + (A - B))). \tag{4}$$

We write

$$A - B \equiv C \tag{5}$$

and note that

$$E_\times(\mathrm{perm}(B + C)) = \sum_\alpha E_B(\mathrm{perm}(B_{\bar\alpha}))E_\times(\mathrm{perm}(C_\alpha)) \tag{6}$$

where C_α denotes some submatrix of C (obtained by a selection of a set of the columns and an equal-sized set of the rows) and $B_{\bar\alpha}$ is the dual submatrix of B (obtained using the complementary set of rows and columns). The sum over α is over all such submatrices. Equation (6) follows from the definition of the permanent, and a little thinking

We now use the very special properties here that the random variables in $B_{\bar\alpha}$ and C_α are statistically independent, and that the expectations on the right side of (6) each depend only on the size of the respective submatrices. It follows, from these two very special features, that from (6) one gets

$$E(\mathrm{perm}(A)) = \sum_{i=0}^{n} f_i\, E_B(\mathrm{perm}_{n-i}(B))E_\times(\mathrm{perm}_i(C)). \tag{7}$$

Here we have set $\mathrm{perm}_n(A) = \mathrm{perm}(A)$ (for $n \times n$ matrices) and $\mathrm{perm}_0(A) = 1$. f_i is 1 over the number of distinct $i \times i$ submatrices:

$$f_i = \frac{1}{\binom{n}{i}^2}. \tag{8}$$

One should check that (6) and (7) trivially are equal, using the special properties above.

We now study $E_\times(\mathrm{perm}_i(C))$. Here we get

$$E_\times(\mathrm{perm}_i(C)) = \sum_{k=0}^{i} f(i,k)\, E(\mathrm{perm}_{i-k}(A))(-1)^k E_B(\mathrm{perm}_k(B)) \tag{9}$$

where

$$f(i,k) = \frac{\binom{n}{i}^2\binom{i}{k}^2}{\binom{n}{k}^2\binom{n}{i-k}^2}. \tag{10}$$

We note

$$f_i = f(n,i). \tag{11}$$

The development of equation (9) is as the development of (7), but requiring more careful attention to the nitty-gritty.

We do a little calculation from the easy formula

$$E_B(\mathrm{perm}_i(B)) = \binom{n}{i}^2 i!\,(r/n)^i \tag{12}$$

to get

$$\frac{E_B(\mathrm{perm}_{n-i}(B))}{E_B(\mathrm{perm}(B))} = \frac{1}{r^i}\frac{n^i(n-i)!}{n!}\cdot\frac{1}{f_i}. \tag{13}$$

We put together the above formulas to get our expression for $E(\mathrm{perm}(A))$:

$$E(\mathrm{perm}(A)) = E_B(\mathrm{perm}(B))\cdot\left(1+\sum_{i=2}^{n} C_i\right) \tag{14}$$

with

$$C_i = \left(\frac{1}{r^i}\frac{n^i(n-i)!}{n!}\right)\cdot\sum_{k=0}^{i} f(i,k)\,E(\mathrm{perm}_{i-k}(A))(-1)^k E_B(\mathrm{perm}_k(B)). \tag{15}$$

We have used the fact that $C_1 = 0$ (and $C_0 = 1$). We emphasize so far that everything is “rigorous”, formulae (14) and (15) give a neat expression for $E(\mathrm{perm}(A))$. And also note that all the expectations involving B are easily known by (12).

We now construct our cluster expansion

$$\left(1+\sum_{i=2}^{n} C_i\right) \text{ “=” } e^{\sum_{i=2}^{\infty} T_i}. \tag{16}$$

This is our first “formal” step. The T_i are selected so that if one expands the two sides in powers of the matrix entries, the two sides are equal power by power. We present the first few T_i so that one may see the pattern:

$$T_2 = C_2 \tag{17a}$$
$$T_3 = C_3 \tag{17b}$$
$$T_4 = C_4 - \tfrac{1}{2}T_2^2 \tag{17c}$$
$$T_5 = C_5 - T_3T_2 \tag{17d}$$
$$T_6 = C_6 - T_4T_2 - \tfrac{1}{6}T_2^3 - \tfrac{1}{2}T_3^2. \tag{17e}$$

I hope we have not insulted your intelligence by listing so many of the terms.

As we mentioned, the terms in (14), (15) in the E_B expectations are all known by (12). Of those in the E expectations the only term we know exactly is (a well known result)

$$E(\mathrm{perm}_2(A)) = \tfrac{1}{2}nr(rn-2r+1). \tag{18}$$

We are thus motivated to consider the formulas of (14), (15) as developed using E_1 or E_2 instead of E. The formal procedure we followed would have given the same form for (14), (15) with E replaced by either E_1 or E_2. All the expectations using E_1 or E_2 are known from Section 4 of [2] as follows

$$E_1(\mathrm{perm}_m(A)) = \frac{1}{(n!)^r}\binom{n}{m}^2 \sum_{m_1,\ldots,m_r\in\mathbb{Z}_+,\, m_1+\cdots+m_r=m} \frac{m!(n-m_1)!\cdots(n-m_r)!}{m_1!\cdots m_r!} \tag{19}$$

$$E_2(\mathrm{perm}_m(A)) = \binom{n}{m}^2 \frac{r^{2m} m!(rn-m)!}{(rn)!} \tag{20}$$

We callect the information we have obtained from computer computations as support for a few amazing conjectures.

Conjecture 1 *Using E, E_1, or E_2 we have that the following limit exists:*

$$\lim_{n\to\infty} \frac{1}{n} T_i(n). \tag{21}$$

For E we can only test this for T_2, where it is true. For E_1 and E_2 we have verified this for $i \leq 25$. (It would be easy to test this for much higher i. Someone should prove this theorem for E_1 or E_2.) One will realize that the existence of the limit in (21) is amazing if one considers that the $C_i(n)$ may behave proportional to $n^{i/2}$ for large n (by computer computation), so much cancellation must take place for the limit in (21) to exist.

Conjecture 2 *Using E, E_1, or E_2*

$$\lim_{n\to\infty} \frac{1}{n} T_i(n) = \frac{a_i}{r^{h(i)}} + \frac{b_i}{r^{h(i)+1}} \tag{22}$$

where $h(i) = i/2$ if i is even and $= (i+1)/2$ if i is odd.

This has been verified for the same cases as discussed under Conjecture 1. Again someone should prove it for E_1 or E_2.

It appears that E_1 and E_2 have the same values of a_i and b_i. We collect the first few values of a_i and b_i for the different expectations. For E one has

$$a_2 = \tfrac{1}{2}, \quad b_2 = 0. \tag{23}$$

For E_1 and E_2 one has

$$a_2 = \tfrac{1}{2}, \qquad b_2 = 0 \tag{24a}$$
$$a_3 = \tfrac{2}{3}, \qquad b_3 = 0 \tag{24b}$$
$$a_4 = -\tfrac{1}{2}, \qquad b_4 = \tfrac{5}{4} \tag{24c}$$
$$a_5 = -2, \qquad b_5 = \tfrac{14}{5} \tag{24d}$$
$$a_6 = \tfrac{5}{6}, \qquad b_6 = -\tfrac{42}{6} \tag{24e}$$
$$a_7 = 6, \qquad b_7 = -\tfrac{168}{7} \tag{24f}$$
$$a_8 = -\tfrac{14}{8}, \qquad b_8 = \tfrac{252}{8}. \tag{24g}$$

If we assume

$$\frac{1}{n} \lim \sum T_i = \sum \lim \frac{1}{n} T_i, \tag{25}$$

our second formal step, one has for either E_1 or E_2 (using (22) and (24a)–(24g))

$$\frac{1}{n} \lim \sum T_i = \frac{1}{2}\frac{1}{r} + \frac{1}{6}\frac{1}{r^2} + \frac{1}{12}\frac{1}{r^3} + \frac{1}{20}\frac{1}{r^4} \cdots. \tag{26}$$

As is usual in studying cluster expansions one studies the $\lim_{n\to\infty} \frac{1}{n}(\ \ )$. It is easy to show for $s = 1, 2$ that

$$\lim_{n\to\infty} \frac{1}{n} E_s(\mathrm{perm}(A)) = \lim_{n\to\infty} \frac{1}{n} E_B(\mathrm{perm}(A)) + 1 + (r-1)\ln\left(1 - \frac{1}{r}\right) \tag{27}$$

and this by (26) is the same as

$$\lim_{n\to\infty}\frac{1}{n}E_s(\mathrm{perm}(A)) \simeq \lim_{n\to\infty}E_B(\mathrm{perm}(A)) + \frac{1}{n}\lim\sum T_i \tag{28}$$

where the two sides are asymptotically equal to as high a power of $1/r$ as determined by the 25 first terms in the sum of T_i. One certainly has equality of the two sides of (28).

Our most daring conjecture is that

$$\lim_{n\to\infty}\frac{1}{n}E(\mathrm{perm}(A)) = \lim_{n\to\infty}\frac{1}{n}E_s(\mathrm{perm}(A)), \quad s=1,2 \tag{29}$$

but in fact this is pretty broadly believed!?

At the bottom there is a mystery. What theoretical mechanism gives rise to the structure of this cluster expansion?

References

[1] Brydges, D., A short course on cluster expansions, *Phénomènes critiques, systèmes aléatoires, théories de gauge*, Parts I, II (Les Houches, 1984), North-Holland, Amsterdam, 1986, pp. 129–183,

[2] Friedland, S. and Krop, E. and Markström, K., *On the number of matchings in regular graphs*, Electron. J. Combin., **15** (2008), pp. 1–28.