# PROOF OF CONVERGENCE FOR THE LATTICE MONOMER-DIMER CLUSTER EXPANSION I, A SIMPLIFIED MODEL

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Abstract. In a set of papers we address the problem of making completely rigorous the development of our expression for $\lambda_d(p)$ of the monomer-dimer problem on a $d$-dimensional hypercubic lattice

$$\lambda_d(p) = \frac{1}{2}\Big(p\ln(2d) - p\ln(p) - 2(1-p)\ln(1-p) - p\Big) + \sum_{k=2} a_k(d)p^k \tag{A}$$

where $a_k(d)$ is a sum of powers $(1/d)^r$ for

$$k-1 \geq r \geq k/2 \tag{B}$$

In fact as we will point out one has already rigorously established the convergence of the sum in A for small $p$. It is the $d$ dependence of $a_k(d)$ that has yet to be rigorously shown. The study of these papers establishes the convergence of our cluster expansions, resulting in the said $d$ dependence. In this paper we study a cute little problem, claiming the general case can be dealt with as a dressing up of this skeleton problem.

## 1. The Model Problem

We start by presenting the model problem. It is interesting in its own right, and is after all the study of this paper. $r > 0$ is given and real $J_i$ are given, $i \geq 2$ satisfying

$$|J_i| \leq r^i \tag{1}$$

We define

$$Z = Z(N,p) = \sum_{\alpha_i} \prod_i \left( (J_i p^i N)^{\alpha_i} \cdot \frac{1}{\alpha_i!} \right) \tag{2}$$

The $\alpha_i, i \geq 2$ are non-negative integers, and the sum over the $a_i$ in (2) is over all values of the $\alpha_i$, restricted by

$$\sum_{i=2}^{\infty} i\alpha_i \leq \frac{pN}{2} \tag{3}$$

We will prove

**Theorem.** *There is a $p_0 > 0$ such that for $0 \leq p \leq p_0$*

$$\lim_{N\to\infty} \frac{\ln Z}{N} = \sum_2^{\infty} p^i J_i \tag{4}$$

The easiest way to prove this theorem (probably) must be to estimate the corresponding sum to that in (2) with the $\alpha_i$ satisfying the complementary inequality

$$\sum_{i=2}^{\infty} i\alpha_i > \frac{pN}{2} \tag{5}$$

If this part is "small enough" then $Z$ becomes approximately equal

$$Z \cong \prod_i \left( \sum_{\alpha_i=0}^{\infty} (J_i p^i N)^{\alpha_i} \cdot \frac{1}{\alpha_i!} \right) \tag{6}$$

and the theorem is easily proved

We follow a much more devious route to the proof, in this paper. A route whose steps can all be paralleled in the actual problem.

## 2. Introduction

The cluster expansion approach to $\lambda_d$ of the dimer problem on a hypercubic lattice was presented in [1]. A formal argument was given for the expansion

$$\lambda_d \sim \frac{1}{2}\ln(2d) - \frac{1}{2} + \sum_i c_i \frac{1}{d^i} \tag{7}$$

At present the status of this putative asymptotic expansion is still not clear. In [4], working with Friedland, a simple extension of the expansion formalism of [1] was made to treat the monomer-dimer problem. This yielded the expresion

$$\lambda_d(p) = \frac{1}{2}\Big(p\ln(2d) - p\ln p - 2(1-p)\ln(1-p) - p\Big) + \sum_{k=2} a_k(d)p^k \tag{8}$$

where $a_k(d)$ is a sum of powers of $(1/d), (1/d)^r$, with

$$k-1 \geq r \geq k/2. \tag{9}$$

The situation is a little complicated. The cluster expansion formalism yields an expression for $a_k(d)$ as a function of the Mayer Series coefficients of the dimer gas. In [4], repeated in eq (12)–(20) of [3], another route was obtained to derive the $a_k(d)$ from the Mayer Series coefficients. Although the two routes certainly give the same answer, this has not been rigorously established. The inscrutable algebraic identity that must be proved is detailed in [5]. It is child's play to see that the second route leads rigorously to an expression (8) for $\lambda_d(p)$ where the sum converges. But it is the expression in (8) derived by the first route (still not proved rigorously) for which the $a_k(d)$ have the expression in powers of $1/d$ described before (9). Proof of the convergence of the cluster expansion, which we enter into in this paper, will show that the two expressions for all $a_k(d)$ are equal, that the sum in (8) converges for small $p$, and that the $a_k(d)$ have the indicated dependence on $d$.

The limit that must be evaluated to rigorously establish the cluster expansion development is presented in Section 3. The relation of the limit of the model problem, eq (4), to this limit will be clear. In succeeding sections the theorem of eq (4) is proven, taking care continuously to carry out steps as they can easily be applied to the general limit of Section 3. The basic strategy is to arrange $\ln Z$ as the sum of terms, chunks. Within the chunks some sums are replaced by contour integrals (in many complex variables). A single stationary point of the integrand dominates in the limit of these integrals.

## 3. The Object of Study

We must analyze $Z^*$

$$Z^* = \sum_{\alpha_i} \beta(N, \textstyle\sum i\alpha_i) \prod \bar{J}_i^{\alpha_i} \dfrac{N^{\sum \alpha_i}}{\prod(\alpha_i!)} \tag{10}$$

This is eq (5.24) of [2]. Here the $\alpha_i, i \geq 2$, are non-negative integers and are restricted by eq (3)

$$\beta(N, jN) = e^{NH(p,j)} \tag{11}$$

with

$$\sum i\alpha_i = jN \tag{12}$$

and

$$H(p,j) = j \ln p + (1-2j)\ln(1-2j) + j - \frac{p}{2}\left(1 - \frac{2j}{p}\right) \cdot \ln\left(1 - \frac{2j}{p}\right) \tag{13}$$

$$\equiv j \ln p + \tilde{H}(p,j) \tag{14}$$

Eq (11) is eq (5.16) of [2], eq (13) is eq (5.17) of [2]. Eq (14) defines $\tilde{H}$. We also define $\tilde{\beta}$

$$\beta(N, \textstyle\sum i\alpha_i) \equiv p^{\sum i\alpha_i} \tilde{\beta}(N, \textstyle\sum i\alpha_i) \equiv p^{\sum i\alpha_i} e^{N\tilde{H}(p,j)} \tag{15}$$

$Z^*$ becomes

$$Z^* = \sum_{\alpha_i} \tilde{\beta}(N, \textstyle\sum i\alpha_i) \prod (\bar{J}_i p^i N)^{\alpha_i} \dfrac{1}{\prod(\alpha_i!)} \tag{16}$$

Sums are again restricted by (3).

One wants to study

$$\lim_{N\to\infty} \frac{\ln Z^*}{N} \tag{17}$$

The similarity between eq (16) and eq (17) and the pair of equations, eq (2) and eq (4), is obvious. The $\bar{J}_i$ have a weak dependence on $N$. (They are asymptotically constant.)

The desired limit of (17) we do not detail now. This limit might be found in (5.31) and (5.32) of [2], or in an entirely different form in the discussion surrounding (24)–(28) in [3], where another reference is given.

## 4. From Sums to Contour Integrals

In the next section $Z$ or $Z^*$ from (2) or (16) will be arranged into a sum of terms called chunks. In some of these chunks there will be a designated set of indices, $\mathcal{S}$, such that a portion of the chunk is of the form

$$\prod_{i\in\mathcal{S}} \left( \sum_{i=0}^{m_i} \frac{(J_i p^i N)^{\alpha_i}}{\alpha_i!} \right) \beta \tag{18}$$

Here if it is $Z$ we are working with $\beta = 1$, if it is $Z^*$, $J_i$ becomes $\bar{J}_i$; from now on such trivial differences will not be commented on. In (18) the other $\alpha_i$ are not summed, having been assigned certain values and $\beta$ may depend on $\alpha_i$. The $J_i$ for $i$ in $\mathcal{S}$ will be negative, it will be important to control cancellations between positive and negative terms in evaluating (18) accurately enough. This motivated the use of contour integrals. In fact dealing with $Z$ a simpler treatment is possible as will be pointed out later, but we want to use a method that applies to both $Z$ and $Z^*$.

We set $-a \equiv J_i N p^i$ and note

$$\sum_{\alpha=0}^{n} \frac{(-a)^\alpha}{\alpha!} f(\alpha) = \frac{1}{2\pi i} \oint_C dz \frac{\pi}{\sin \pi z} \frac{a^z}{z!} f(z) \tag{19}$$

where the contour $C$ is counterclockwise and contains $\{0, 1, \ldots, n\}$ and no other singularities of the integrand. Employing the identity (19) in (18) for all the $\alpha_i$ with $i$ in $\mathcal{S}$ we have converted all the sums in our chunk to a single multivariable contour integral. Analytic properties of $\Gamma(z)$ and $\beta$ will be dealt with later.

The division of $Z$ and $Z^*$ into chunks is to convert sums from having limits as given by (3) to limits as in (19)

$$0 \le \alpha_i \le m_i \tag{20}$$

In the space of allowed $\alpha_i$ we are fitting hyper-rectangles. This messy procedure is now illucidated. It is we feel the central idea of the proof.

## 5. Disection into Chunks

We write this section in the language of $Z^*$ of (16), changes for $Z$ of (2) trivial. We first introduce a number of parameters. There is $\tilde{M}$

$$\tilde{M} = \frac{pN}{4} \tag{21}$$

and $\mu_i$, for $i \ge 2$,

$$\mu_i = \frac{1}{i^{2+\epsilon_1}} \tag{22}$$

Each chunk is assigned a "level", a non-negative integer, and is either "free" or "boxed". It requires patience to develop the construction of these chunks.

We let $\mathcal{P}$ be the subset of indices for which $\bar{J}_i \ge 0$ if $i \in \mathcal{P}$ and $\mathcal{N}$ be the subset for which $\bar{J}_i < 0$. Each chunk has a unique $\alpha_i$ assigned to the $i \in \mathcal{P}$, say $\alpha_i = t_i$. Thus in a chunk some of the $\alpha_i$ may be summed over, but not the $\alpha_i$ with $i \in \mathcal{P}$. For any chunk we define

$$R_0 = \sum_{i \in \mathcal{P}} i t_i \tag{23}$$

**Level-zero free chunks.** A level zero chunk is uniquely specified by the set of $t_i$, $i \in \mathcal{P}$. If

$$R_0 \ge \tilde{M} \tag{24}$$

then it is a free level-zero chunk. Its precise definition is

$$\prod_{i \in \mathcal{P}} (\mathrm{subs}(\alpha_i = t_i)) \sum_{\substack{\alpha_i, i \in \mathcal{N} \\ \sum_{i \in \mathcal{N}} i\alpha_i \le \frac{pN}{2} - R_0}} \tilde{\beta}(N, \sum i\alpha_i) \left( \prod_i (\bar{J}_i p^i N)^{\alpha_i} \frac{1}{\prod(\alpha_i!)} \right) \tag{25}$$

We are using a Maple-like notation. In (25) the $\alpha_i$ for $i \in \mathcal{P}$ are set equal to $t_i$, and the remaining $\alpha_i$ are summed over subject to the restriction from (3). As with all the chunks, this chunk is some subsum of the terms in (16). Different chunks are disjoint, the union of all the chunks giving all terms in (16).

**Level-zero boxed chunks.** Here

$$R_0 < \tilde{M} \tag{26}$$

We define $C_0$ by

$$C_0 = \sup_x \left\{ x \mid \sum_{i\in\mathcal{N}} i \cdot \mathrm{floor}(x\mu_i) \le \frac{pN}{2} - R_0 \right\} \tag{27}$$

where floor$(\alpha)$ is the largest integer $\le \alpha$. We then set

$$m_0(i) = \mathrm{floor}(C_0\mu_i) \tag{28}$$

The level-zero boxed chunk defined by the $t_i$, $i \in \mathcal{P}$ is then

$$\prod_{i\in\mathcal{P}} (\mathrm{subs}(\alpha_i = t_i)) \sum_{\alpha_{a_1}=0}^{m_0(a_i)} \cdots \sum_{\alpha_{a_s}=0}^{m_0(a_s)} M \tag{29}$$

Here $M$ indicates everything after the sums in (16). $a_1, \ldots, a_s$ are the indices labelling elements of $\mathcal{N}$. By using (27) we have found the biggest box, of a certain shape, we can insert in the sum. That is we are picking the upper limits in (29) as large as possible, with a certain fixed ratio between them

The $R_0 \ge \tilde{M}$ in the free chunk will lead to enough smallness in estimates later that one will not have to study cancellations between signed terms via the contour integrals of (19). For the boxed chunks one will need to do so.

**Level-one chunks.** The level-zero chunks fail to exhaust all terms In (16) because of terms containing some $\alpha_i$ for some $i \in \mathcal{N}$ exceeding the upper limits in (29). We give a subset $\mathcal{B}_1$ of $\mathcal{N}$ and to each $i$ in $\mathcal{B}_1$ we associate a $t_i$ with

$$t_i > m_0(i) \tag{30}$$

We call the augmented set $\mathcal{B}_1$ of indices and associated $t_i$, $\bar{\mathcal{B}}_1$. We set

$$R_1 = \sum_{i\in\mathcal{B}_1} i t_i \tag{31}$$

If $R_0 + R_1 \ge \tilde{M}$ we have the level-one free chunk given as

$$\prod_{i\in\mathcal{P}\cup\mathcal{B}_1} (\mathrm{subs}(\alpha_i = t_i)) \sum_{\substack{a_i,\, i\in\mathcal{N}_1 \\ \sum_{i\in\mathcal{N}_1} i\alpha_i \le \frac{pN}{2} - R_0 - R_1}} M \tag{32}$$

We have defined $\mathcal{N}_1 = \mathcal{N} - \mathcal{B}_1$. To define the boxed chunk we define

$$C_1 = \sup_x \left\{ x \mid \sum_{i\in\mathcal{N}_1} i\, \mathrm{floor}(x\mu_i) \le \frac{pN}{2} - R_0 - R_1 \right\} \tag{33}$$

and

$$m_1(i) = \mathrm{floor}(C_1\mu_i) \tag{34}$$

Then the level-one boxed chunk is given as

$$\prod_{i\in\mathcal{P}\cup\mathcal{B}_1} (\mathrm{subs}(\alpha_i = t_i)) \sum_{\alpha_{a_1}=0}^{m_1(a_1)} \cdots \sum_{\alpha_{a_s}=0}^{m_1(a_s)} M \tag{35}$$

and here the $a_1, \ldots, a_s$ label elements of $\mathcal{N}_1$.

The set of $t_i$ associated to the $i$ in $\mathcal{P} \cup \mathcal{B}_1$ uniquely label the level-one chunks.

**General level chunks.** We assume we have defined chunks of level-zero through level-$n$, and we will derive expressions for the level-$(n+1)$ chunks. Thus we have

$$\mathcal{N} = \mathcal{N}_0 \supset \mathcal{N}_1 \supset \mathcal{N}_2 \cdots \supset \mathcal{N}_n \tag{36}$$

$$\mathcal{B}_i \subset \mathcal{N}_{i-1}, \qquad i = 1, 2, \ldots, n \tag{37}$$

$$\mathcal{N}_i = \mathcal{N}_{i-1} - \mathcal{B}_i, \qquad i = 1, 2, \ldots, n \tag{38}$$

$$\bar{\mathcal{B}}_i, \qquad i = 1, 2, \ldots, n \tag{39}$$

That is, we have an assignment $\alpha_i = t_i$ for $i$ in each $\mathcal{B}_i$. We set

$$R_k = \sum_{i \in \mathcal{B}_k} i t_i, \qquad k = 1, 2, \ldots, n \tag{40}$$

and have $C_0, C_1, \ldots, C_n$ with

$$C_k = \sup_x \left\{ x \mid \sum_{i \in \mathcal{N}_k} i \operatorname{floor}(x\mu_i) \leq \frac{pN}{2} - \sum_0^k R_k \right\}, \qquad k = 0, 1, \ldots, n \tag{41}$$

We set

$$m_k(i) = \operatorname{floor}(C_k \mu_i), \qquad k = 0, 1, \ldots, n \tag{42}$$

and for $i$ in $\mathcal{B}_k$, $k = 2, \ldots, n$ one has

$$m_{k-1}(i) < t_i \leq m_{k-2}(i) \tag{43}$$

For $k = 1$ the analogous condition is given by (30); there is in this case no upper bound here imposed. It is not difficult to see the $C_i$ from a decreasing sequence.

To go to the next level, select the set $\mathcal{B}_{n+1} \subset \mathcal{N}_n$ and for $i \in \mathcal{B}_{n+1}$ require

$$m_n(i) < t_i \leq m_{n-1}(i) \tag{44}$$

The rest follows immediately.

Let us specify free and boxed chunks of level-$(n+1)$. The free level $n+1$ chunk arises if

$$\sum_{i=0}^{n+1} R_i \geq \tilde{M} \tag{45}$$

and then is given as

$$\prod_{i \in \mathcal{P} \cup (\mathcal{N} - \mathcal{N}_{n+1})} (\operatorname{subs}(\alpha_i = t_i)) \sum_{\substack{\alpha_i, i \in \mathcal{N}_{n+1} \\ \sum_{i \in \mathcal{N}_{n+1}} i\alpha_i \leq \frac{pN}{2} - \sum_{i=0}^{n+1} R_i}} M \tag{46}$$

If (45) is not satisfied the boxed chunk is given as

$$\prod_{i \in \mathcal{P} \cup (\mathcal{N} - \mathcal{N}_{n+1})} (\operatorname{subs}(\alpha_i = t_i)) \sum_{\alpha_{a_1}=0}^{m_{n+1}(a_1)} \cdots \sum_{\alpha_{a_s}=0}^{m_{n+1}(a_s)} M \tag{47}$$

where $a_1, \ldots, a_s$ are the indices of $\mathcal{N}_{n+1}$.

## 6. Smallness from high occupation

In this section we find an upper bound on

$$\prod_i \left( \frac{(|\bar{J}_i| p^i N)^{\alpha_i}}{\alpha_i!} \right) \tag{48}$$

subject to the restriction

$$\sum_i i\alpha_i \geq \tilde{M} \tag{49}$$

We use (3)

$$|\bar{J}_i| \leq r^i \tag{50}$$

and define

$$Q = \prod_i \left( \frac{((pr)^i N)^{\alpha_i}}{\alpha_i!} \right) \tag{51}$$

One then has

$$\ln Q \leq F = N \sum_i [ix_i \ln(pr) - x_i \ln x_i + x_i] \tag{52}$$

using the well known inequality $n\ln(n/e) + 1 \leq \ln n!$, where we have set

$$\alpha_i = x_i N \tag{53}$$

$$\tilde{M} = \tilde{m} N \tag{54}$$

Then we have

$$f = \frac{F}{N} = \sum_i [ix_i \ln(pr) - x_i \ln x_i + x_i] \tag{55}$$

We ignore the restriction that $\alpha_i$ must be an integer. We note that our upper bound will apply if we restrict the set of indices in the products of (48) or (51), or require the $\alpha_i$ to be integers, each of these would lead to a lower upper bound.

Now we want to maximize $f$ subject to

$$h = \sum_i ix_i = \tilde{m} \tag{56}$$

We apply Lagrange multipliers

$$\frac{d}{dx_i}(f - \lambda h) = 0 \tag{57}$$

getting

$$x_i = (pre^{-\lambda})^i \tag{58}$$

and then, from (56)

$$\sum_{i=2}^{\infty} i(pre^{-\lambda})^i = \tilde{m} \tag{59}$$

one gets for $\tilde{m} = cp$ as $p$ goes to zero

$$F \cong -N\tilde{m}|\ln(p/\sqrt{\tilde{m}})| \tag{60}$$

## 7. The Contour Integrals, distorting the contours

We have displayed the expressions for boxed chunks of level-zero, level-one, and general level-$(n+1)$ in eqs. (29), (35), (47). We isolate the portion of the expression shown in eq. (18), and relabelling indices look at our present object of study

$$A = \sum_{\alpha_1=0}^{m_1} \cdots \sum_{\alpha_s=0}^{m_s} \prod_{i=1}^{s} \left( \frac{(J_{d_i} p^{d_i} N)^{\alpha_i}}{\alpha_i!} \right) \beta \tag{61}$$

Here all the $J_i$ are negative. We set

$$a_i = -J_{d_i} p^{d_i} N \tag{62}$$

and rewrite $A$ as

$$A = \frac{1}{(2\pi i)^s} \oint_{C_1} dz_1 \cdots \oint_{C_s} dz_s \prod_{i=1}^{s} \left( \frac{\pi}{\sin \pi z_i} \frac{a_i^{z_i}}{\Gamma(z_i+1)} \right) \beta \tag{63}$$

where we take $C_i$ as hugging the interval $[-1/2, m_i + 1/2]$, encircling it counterclockwise.

We let $g$ be the integrand of (63) so that

$$A = \frac{1}{(2\pi i)^s} \oint_{C_1} dz_1 \cdots \oint_{C_s} dz_s g \tag{64}$$

and seek a stationary point of $g$, so we solve together

$$\frac{\partial}{\partial z_i} g = 0, \qquad i = 1, \ldots, s \tag{65}$$

We now use

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{66}$$

to write $g$ as

$$g = \prod_{i=1}^{s} \left( -a_i^{z_i} \Gamma(-z_i) \right) \beta \tag{67}$$

We look at the equations (65) for a stationary point of $Z$, where then $\beta = 1$, getting

$$\frac{\partial}{\partial z_i} \left( -a_i^{z_i} \Gamma(-z_i) \right) = 0 \tag{68}$$

or taking exponentials and setting $w_i = -z_i$

$$\frac{d}{dw_i} \left[ -w_i \ln a + \ln \Gamma(w) \right] = 0 \tag{69}$$

We now use the standard large $w$ approximation

$$\ln n! \cong n \ln n - n \tag{70}$$

to get

$$w_i \cong a_i + 1 \tag{71}$$

or

$$z_i = z_{i0} \cong -a_i - 1 \tag{72}$$

at the stationary point.

In general let the coordinates of the stationary point be $z_i = z_{i0}$. We then first shift the contour $C_i$ to $C_i'$ where $C_i'$ hugs the interval $[z_{i0}, m_i + 1/2]$ counterclockwise. One hits no singularities of $g$ to distort the contour so. Next we stretch the contour to infinity, plus and minus, in the imaginary directions, leaving contours $C_i''$. $C_i''$ consists of two line segments parallel to the imaginary axis

$$\begin{aligned} C_i'' = &[-\infty i + z_{i0}, \infty i + z_{i0}] \\ &\text{union} \\ &[-\infty i + m_i + 1/2, \infty i + m_i + 1/2] \end{aligned} \tag{73}$$

Again it will not be hard to show that no singularities of the integrand are crossed in the distortion, and the segments at $\infty$ to close the $C_i''$ contours contribute nothing.

In treating $Z^*$ and studying

$$A = \frac{1}{(2\pi i)^s} \oint_{C_i''} dz_1 \cdots \oint_{C_s''} dz_s g \tag{74}$$

one will have to show the limit in eq (17) picks out just the value of the integrand at the center of the first line segment in (73). In studying $Z$ we will use a simpler route to study (61). But one easily sees that the contour integral method would yield similar estimates in this case.

## 8. Completion of the Proof for $Z$

In this section we carry through the proof of the theorem, see eq. (4), for $Z$, much in the way we plan to complete the proof for $Z^*$, see eq. (17). An important difference is that we do not need use of contour integration. We divide $Z$ into a sum of three terms

$$Z = Z(N, p) = T_1 + T_2 + T_3 \tag{75}$$

$T_1(N, p)$ is the sum of level-zero boxed chunks, $T_2(N, p)$ is the sum of the rest of the boxed chunks, and $T_3(N, p)$ is the sum of the free chunks. We will arrive at the theorem by proving, for small enough $p$, that

$$\lim_{N\to\infty} \frac{\ln T_1}{N} = \sum_2 p^i J_i \tag{76}$$

$$\lim_{N\to\infty} \frac{\ln T_2}{N} < \sum_2 p^i J_i \tag{77}$$

$$\lim_{N\to\infty} \frac{\ln T_3}{N} < \sum_2 p^i J_1 \tag{78}$$

**Study of $T_3$.** We may overestimate $T_3$ as follows

$$T_3 \le \sum_{\alpha_i} Q \tag{79}$$

where $Q$ is from (51) and the $\alpha_i$ are restricted by

$$\sum i\alpha_i \ge \frac{pN}{4} \tag{80}$$

Then we write

$$\begin{aligned} \sum_{\alpha_i} Q &= \sum_{\alpha_i} \prod_i \left( \frac{((pr)^i N)^{\alpha_i}}{\alpha_i!} \right) \\ &\le \sum_{\alpha_i} \prod_i \left( \frac{((pr)^i N)^{\alpha_i/2}}{(\alpha_i/2)!} \right) \prod_i \left( \frac{((pr)^i N)^{\alpha_i/2}}{(\alpha_i/2)!} \right) \end{aligned} \tag{81}$$

all subject to the restriction (80). This yields

$$\sum_{\alpha_i} Q \le AB \tag{82}$$

with

$$A = \sup \prod_i \left( \frac{((pr)^i N)^{\alpha_i/2}}{(\alpha_i/2)!} \right) \tag{83}$$

and

$$B = \sum_{\alpha_i} \prod_i \left( \frac{((pr)^i N)^{\alpha_i/2}}{(\alpha_i/2)!} \right) \tag{84}$$

In (82) and (83) restriction (80) is enforced, but getting a weaker inequality do not impose it in (84). For $A$ from (60) we get

$$A \cong e^{-N(p/8)|ln(p)/\sqrt{p/8}|} \tag{85}$$

Looking at $B$ we note

$$\begin{aligned} \sum_i \frac{a^{i/2}}{(i/2)!} &= \sum_i \frac{a^i}{i!} + \sum_i \frac{a^{1/2+i}}{(1/2+i)!} \\ &= \sum_i \frac{a^i}{i!} + a^{1/2} \sum_i \frac{a^i}{(1/2+i)!} \\ &\le (1 + ca^{1/2}) \sum \frac{a^i}{i!} \le (1 + ca^{1/2}) e^a \end{aligned} \tag{86}$$

Equation (78) follows for $p$ small enough with little work.

## 9. Dealing with $Z_1$ and $Z_2$

In this section we study the methods used to treat $T_1$ and $T_2$ to derive (76) and (77). We discuss the proof of (76) alone, since (77) requires no new ideas. We write $T_1$, the sum of the level zero boxed chunks as

$$T_1 = \sum_\beta A^\beta B^\beta \tag{87}$$

with

$$A^\beta = \prod_{i \in \mathcal{P}} \frac{(J_i p^i N)^{t_i}}{t_i!} \tag{88}$$

where $t_i = t_i(\beta)$ satisfy

$$R_0 = R_0(\beta) = \sum i t_i < \frac{pN}{4} \tag{89}$$

$C_0(\beta)$ is given by (27), and $m_0(i)$ by (28), using which we define

$$B^\beta = \prod_{i \in \mathcal{N}} \left( \sum_{\alpha_i=0}^{m_0(i)} \frac{(J_i p^i N)^{\alpha_i}}{\alpha_i!} \right) \tag{90}$$

We now set

$$B_0 = \prod_{i \in \mathcal{N}} (e^{J_i p^i N}) \tag{91}$$

We write

$$A^\beta B^\beta = A^\beta B_0 + A^\beta E^\beta \tag{92}$$

and

$$\sum A^\beta B^\beta = B_0 \sum A^\beta + \sum A^\beta E^\beta \tag{93}$$

We study the second term in the right side of (93) first

$$\left| \sum A^\beta E^\beta \right| \leq \left( \sum A^\beta \right) \sup_\beta |E^\beta| \tag{94}$$

If we were studying $Z^*$ instead of $Z$ we would have used contour integral techniques, but in the present case we use simpler methods

$$\sum A^\beta \leq e^{\sum_{i \in \mathcal{P}} J_i p^i N} \tag{95}$$

To study $|E^\beta|$ we first introduce

$$g(a, n) = \sum_{i=n+1}^{\infty} \left( \frac{|a|^i}{i!} \right) e^{+|a|} \tag{96}$$

and

$$g_i = g(J_i p_i N, m_0(i)) \tag{97}$$

In terms of the quantities we have

$$|E^\beta| \leq e^{\sum_{i \in \mathcal{N}} J_i p^i N} \cdot \left( \prod_{i \in \mathcal{N}} (1 + g_i) - 1 \right) \tag{98}$$

The dependence on $\beta$ comes from the dependence of $g_i$ on the $m_0(i)$ which depend on $\beta$.

If $x_i \geq 0$ and $\sum x_i \leq 1$ one has the inequality

$$\prod (1 + x_i) - 1 \leq e \sum x_i \tag{99}$$

Once we know $\sum g_i \leq 1$ we now have

$$\left| \sum A^\beta E^\beta \right| \leq e^{\sum J_i p^i N} \cdot e \cdot \sup_\beta \left( \sum_{i \in \mathcal{N}} g_i \right) \tag{100}$$

Limit (76) follows from

$$\lim_{N \to \infty} \frac{1}{N} \ln \left( B_0 \sum A^\beta \right) = \sum J_i p^i \tag{101}$$

and

$$\lim_{N \to \infty} \frac{\left| \sum A^\beta E^\beta \right|}{B_0 \sum A^\beta} = 0 \tag{102}$$

We choose to see (101) by approximating $\sum A^\beta$ by its largest term, a justifiable procedure in this case. Note that all terms in sum are positive This is the method used in [1] to treat (27) there, and again in [2] to treat (5.24) therein. In both these cases the method is used "formally" where it does not really apply since there are positive and negative terms. It is the work of the present paper and its sequel, part 2, to show the results in [1] and [2] are none the less correct.

We turn to understanding limit (102). From (100) we have

$$\frac{\left| \sum A^\beta E^\beta \right|}{B_0 \sum A^\beta} \leq \frac{e^{\sum J_i p^i N}}{B_0 \sum A^\beta} \cdot e \cdot \sup_\beta \left( \sum_{i \in \mathcal{N}} g_i \right) \tag{103}$$

From (101), now presumed true this becomes

$$\frac{\left|\sum A^\beta E^\beta\right|}{B_0 \sum A^\beta} \leq e^{N\varepsilon(N)} \cdot e \cdot \sup_\beta \left( \sum_{i \in \mathcal{N}} g_i \right) \tag{104}$$

for some $\varepsilon(N)$ that goes to zero with $N$. We may deduce an upper bound on $\sum_{i\in\mathcal{N}} g_i$ as $\sum_{i\in\mathcal{N}} \bar{g}_i$, with $\bar{g}_i$, $g_i$ computed with $R_0 = \frac{pN}{4}$.

We may get an upper bound for the $\sum_{i\in\mathcal{N}} \bar{g}_i$ by looking at

$$h(r^2p^2N, \gamma pN) \tag{105}$$

with

$$h(a,n) = \frac{a^n/n!}{e^{-a}} \tag{106}$$

$$\ln h(r^2p^2N, \gamma pN) \leq -(\gamma pN) \ln\left(\frac{\gamma pN}{r^2p^2N}\right) + (\gamma pN) - r^2p^2N \tag{107}$$

implying

$$h \leq e^{-\tilde{\gamma}N} \tag{108}$$

with $\tilde{\gamma} > 0$. The first term in the sum for $\bar{g}_2$ is of the form $h$, and the sum of all the remaining terms in $\sum_{i\in\mathcal{N}} \bar{g}_i$ are majorized by this term if p is small enough. This and (104) yield (102) and finally (76).

## References


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