# ON STABLE SOLUTIONS OF BOUNDARY REACTION-DIFFUSION EQUATIONS AND APPLICATIONS TO NONLOCAL PROBLEMS WITH NEUMANN DATA

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Abstract. We study reaction-diffusion equations in cylinders with possibly nonlinear diffusion and possibly nonlinear Neumann boundary conditions. We provide a geometric Poincaré-type inequality and classification results for stable solutions, and we apply them to the study of an associated nonlocal problem. We also establish a counterexample in the corresponding framework for the fractional Laplacian.



Contents



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## 1. Introduction

1.1. **Boundary reactions and stable solutions.** In this paper, we will study reaction-diffusion equations, i.e. mathematical models in which the diffusion process is in balance with a nonlinear reaction. The diffusion is modeled by a (possibly nonlinear) operator of elliptic type, and the reaction may occur on the domain as well as on the boundary, via a Neumann condition. A typical example of reaction-diffusion model is given by the Peierls-Nabarro model for atom dislocations in crystals, in which the elastic force acting on the dislocation function is balanced through a potential acting on the slip plane and thus producing a boundary reaction (see e.g. [17] or Section 2 of [8] for a physical derivation of such model).

Other models which naturally produce reaction-diffusion equations concern the distribution of chemical substances, such as in the case of the so called Fisher-KPP equation (see [13] and [18]).

The domain that we will consider in this paper is a cylinder that is infinite in one direction, namely the Cartesian product of a smooth domain $\Omega$ and $(0, +\infty)$. Homogeneous Neumann conditions are prescribed along the lateral boundary $\partial\Omega \times \{y\}$, for any $y > 0$, and possibly nonhomogeneous and weighted Neumann data are given on the bottom of the domain $\Omega \times \{0\}$. The interest for this Neumann type conditions in cylinder is also related to the representation of the powers of the Laplacian in the spectral sense, see [19, 28, 29].

The main problem that we address here is the classification of stable solutions, i.e. solutions of the equation which correspond to a nonnegative second variation of the associated energy functional (notice that, in particular, minimal solutions fall into this category). The classification of stable solutions of elliptic equations with homogeneous Neumann data goes back at least to the celebrated results in [7], which show that the only stable solutions of semilinear equations in a domain with homogeneous Neumann conditions are the constants, under suitable convexity assumptions either on the domain or on the nonlinearity.

Our main results concern the extension of these type of classifications for reaction-diffusion equations on cylindrical domains with reactive boundary conditions (in this circumstances, as we will see, the stable solutions are not necessarily constant, but will depend only on the "vertical" variable).

Related, but rather different in spirit, classification results for reaction-diffusions in low-dimensional halfspaces have been obtained in [6, 24, 25, 5, 3, 22, 4] (in this case, the stable solutions only depend on one "horizontal" variable).

Also, we will provide a geometric Poincaré-type formula, which can be seen as the counterpart of an inequality obtained in [27] for elliptic equations.

Since the results obtained are related to fractional equations, we will firstly apply our main results to a Neumann boundary value problem for the spectral Neumann Laplacian. Afterwards, we provide a counterexample that prevents classification in a related, but different, nonlocal setting.

Now we introduce the model under consideration in further details and give precise statements of the results obtained.

1.2. **The mathematical setting.** The problem under investigation in this paper is the following:

$$\begin{cases} \operatorname{div}(a(y,|\nabla u|)\nabla u) = g(y,u) & \text{in } \Omega\times(0,+\infty) =: \mathcal{C}, \\ \partial_\nu u = 0 & \text{on } \partial\Omega\times(0,+\infty) =: \partial_L\mathcal{C}, \\ -a(y,|\nabla u|)\partial_y u = f(u) & \text{on } \Omega\times\{0\} =: \partial_B\mathcal{C}. \end{cases} \tag{1.1}$$

Here and in the rest of the paper, the set $\Omega \subset \mathbb{R}^n$ is a bounded and sufficiently regular (say of class $C^{4,\alpha}$) domain. As for the forcing terms $g$ and $f$, we suppose that $g$ is continuous with respect to the first variable and locally Lipschitz with respect to the second variable, and that $f \in C^{2,\alpha}(\mathbb{R})$, for some $\alpha \in (0,1)$.

The variables in the cylinder $\mathcal{C}$ are denoted by $x \in \Omega \subset \mathbb{R}^n$, and $y \in (0,+\infty)$. In some cases, we will use the notation $X := (x,y) \in \mathcal{C}$.

Moreover, in the whole of the paper we will assume the following structural conditions on the function $a$: we assume that

$$a \in C((0,+\infty)\times[0,+\infty)) \cap C^1((0,+\infty)\times(0,+\infty)),$$

that

$$a(y,t) > 0 \quad \text{and} \quad a(y,t) + t a_t(y,t) > 0 \tag{1.2}$$

for any $y > 0$ and $t \geqslant 0$, that there exists $C > 0$ such that

$$t\,|a_t(y,t)| \leqslant C\, a(y,t) \tag{1.3}$$

for any $y > 0$ and $t \geqslant 0$, and that

$$\lim_{t\to 0} t\, a_t(y,t) = 0 \tag{1.4}$$

for any $y > 0$.

Here and in what follows the subscript $t$ stays for the derivative of $a$ with respect to the second variable. From the analytical point of view, condition (1.2) may be seen as a rather general form of ellipticity (this will be detailed in Lemma 2.3). Some examples of $a(y,t)$ that we take into account are

$$\begin{aligned} a(y,t) &= y^\vartheta, \text{ with } \vartheta \in (-1,1), \\ a(y,t) &= y^\vartheta(1+t^2)^{p/2}, \text{ with } \vartheta \in (-1,1) \text{ and } p > 1, \\ a(y,t) &= \frac{y^\vartheta}{\sqrt{1+t^2}}, \text{ with } \vartheta \in (-1,1) \text{ and } |\nabla u| \in L^\infty(\mathcal{C}). \end{aligned}$$

In particular, our assumptions comprise the quasilinear equations of $p$-Laplace type and mean curvature type, possibly weighted by Muckenhoupt weights. The case $a(y,t) = y^\vartheta$ naturally arises in some extension problems for the spectral fractional Laplacian with Neumann boundary condition, see [19, 28].

We now clarify the type of solutions that we are going to consider. We always suppose that

$$
\begin{aligned}
&u \in C(\overline{\mathcal{C}}) \cap C^2(\overline{\Omega} \times (0,+\infty)),\\
&\nabla_x u,\ D^2_x u \in L^2(\Omega \times \{0\}),\\
&\partial_\nu u(x,y) = 0 \quad \text{for all } (x,y) \in \partial\Omega \times [0,+\infty) \text{ and}\\
&a(y,|\nabla u|)\left(|\nabla u|^2 + |D^2_x u|^2 + |\nabla_x u_y|^2 + |D^3_x u|^2 + |D^2_x u_y|^2\right) \in L^1(\Omega \times (0,R))\\
&\text{for all } R>0,
\end{aligned}
\tag{1.5}
$$

where $\nu = (\tilde\nu, 0) \in \mathbb{R}^n \times \mathbb{R}$ and $\tilde\nu$ denotes the outer unit vector field on $\partial\Omega$.

Here and in the rest of the paper, the notation $\nabla_x$ stands for the gradient only in the $x$ variable (in particular, $\nabla_x u$ is an $n$-dimensional vector field). The second condition in (1.5) is intended in the sense of trace. Notice also that the second and the third conditions in (1.5) do not require $u$ to be of class $C^1$ near $\partial\Omega \times \{0\}$, since only the derivatives of $u$ with respect to $x$ are taken into account. We shall see that the previous assumptions are naturally satisfied when (1.1) is seen as extension of a nonlocal boundary value problem. Moreover, they can be directly checked in many concrete cases using the classical regularity theory for elliptic equation (up to the boundary), for which we refer to [1, 2].

Concerning the last equation in (1.1), under reasonable assumptions on $a$, $g$ and $f$ it can be interpreted in the classical case as

$$
\lim_{y\to 0} f(u(x,y)) + a\big(y, |\nabla u(x,y)|\big)\,\partial_y u(x,y) = 0 \quad \text{for any } x \in \Omega. \tag{1.6}
$$

In general we will not need such a regularity. On the contrary, we call solution of (1.1) any function $u$ satisfying (1.5) and such that (1.1) holds in the following weak sense:

$$
\int_{\mathcal{C}} a(y,|\nabla u|)\nabla u\cdot\nabla\varphi + \int_{\mathcal{C}} g(y,u)\varphi = \int_{\partial_B \mathcal{C}} f(u)\varphi \tag{1.7}
$$

for any $\varphi \in \mathcal{A}$, where

$$
\mathcal{A} := \left\{ \varphi \in W^{1,1}_{\mathrm{loc}}(\mathcal{C}) \;\middle|\; \begin{array}{l} \varphi \text{ has bounded support in } y,\ a(y,|\nabla u|)\,|\nabla\varphi|^2 \in L^1(\mathcal{C}) \\ \text{and } \varphi|_{\Omega\times\{0\}} \in L^2(\Omega) \end{array} \right\}. \tag{1.8}
$$

It is clear that any classical solution is also a weak one, in the sense specified above.

Let us consider now the symmetric matrix

$$
\mathcal{B}(y,\eta)_{ij} := a(y,|\eta|)\delta_{ij} + \frac{a_t(y,|\eta|)}{|\eta|}\eta_i\eta_j \qquad \text{for all } i,j = 1,\dots,n+1, \tag{1.9}
$$

where $\eta = (\eta_1,\dots,\eta_{n+1})$, and we mean that the latter term is zero if $\eta$ is zero.

The matrix $\mathcal{B}$ plays a role in the linearized equation (in a sense that will be clarified in Lemma 2.7).

We write that $u$ is a *stable solution* of (1.1) if it is a solution (in the sense of (1.7)) and if

$$
I(\varphi) := \int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\varphi, \nabla\varphi\rangle + \int_{\mathcal{C}} g_u(y,u)\varphi^2 - \int_{\partial_B\mathcal{C}} f'(u)\varphi^2 \geqslant 0 \tag{1.10}
$$

for any $\varphi \in \mathcal{A}$.

Having introduced the main definitions and notation, we are in position to present our main results, which are: a geometric Poincaré-type formula, the classification of stable solutions when $\Omega$ is convex, the classification of bounded stable solutions in case of convex/concave boundary reaction $f$, and the application of these results to nonlocal problems in $\Omega$ related to the spectral Neumann Laplacian. Finally, we also present a counterexample for the fractional Laplacian with point-wise Neumann boundary condition.

1.3. **A Poincaré-type formula.** The first result that we present is a weighted Poincaré-type inequality. A weighted $L^2(\mathcal{C})$-norm of any test function will be bounded by a weighted $L^2(\mathcal{C})$-norm of its gradient. The weights are non-negative and possess a neat geometric interpretation. This type of Poincaré-type formulas are indeed an extension of a celebrated result obtained in [27] for classical elliptic equations. The precise statement in our framework is the following:

**Theorem 1.1.** *Let $u$ be a stable solution of* (1.1). *Then, for any $\psi \in C^1(\overline{\mathcal{C}})$ with bounded support in $y$ and such that $\psi_{x_j} \in \mathcal{A}$ for any $j = 1, \ldots, n$, we have*

$$\begin{aligned} &\int_{\mathcal{C}} \left[ \sum_{j=1}^{n} \langle \mathcal{B}(y, \nabla u)\, \nabla u_{x_j}, \nabla u_{x_j} \rangle - \langle \mathcal{B}(y, \nabla u) \nabla |\nabla_x u|,\, \nabla |\nabla_x u| \rangle \right] \psi^2 \\ &\qquad - \int_{\partial_L \mathcal{C}} a(y, |\nabla u|)\, \big(\nabla u \cdot \partial_\nu(\nabla u)\big)\, \psi^2 \leqslant \int_{\mathcal{C}} \langle \mathcal{B}(y, \nabla u) \nabla \psi,\, \nabla \psi \rangle\, |\nabla_x u|^2. \end{aligned} \tag{1.11}$$

We remark that the weights in (1.11) have a simple, concrete interpretation in terms of the level sets of the solution $u$. As a matter of fact, fixed $y > 0$, if $(x, y) \in \{u = c\} \cap \{\nabla_x u \neq 0\}$, then the $c$-level set of $u(\cdot, y)$ in the vicinity of $(x, y)$ is a smooth $(n-1)$-dimensional manifold $\mathcal{S}_y$ in $\Omega \times \{y\}$, and we can therefore consider the tangential gradient $\nabla_{\mathcal{S}_y}$ along $\mathcal{S}_y$ and the principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$. In this way, one can consider the norm of the second fundamental form, i.e.

$$\mathcal{K} := \sqrt{\sum_{i=1}^{n-1} \kappa_i}$$

and bound the weight on the left hand side of (1.11) in terms of these quantities.

More explicitly (see formula (1.20) in [25]), one has that on $\mathcal{C} \cap \{\nabla_x u \neq 0\}$ it results

$$\begin{aligned} & \sum_{j=1}^{n} \langle \mathcal{B}(y, \nabla u)\, \nabla u_{x_j}, \nabla u_{x_j} \rangle - \langle \mathcal{B}(y, \nabla u) \nabla |\nabla_x u|,\, \nabla |\nabla_x u| \rangle \\ = \;& a(y, |\nabla u|)\, \mathcal{K}_0 + \frac{a_t(y, |\nabla u|)}{|\nabla u|}\, \mathcal{K}_\sharp, \end{aligned}$$

where

$$\mathcal{K}_0 := \sum_{j=1}^{n} u_{x_j\, y}^2 - (\partial_y |\nabla_x u|)^2 + \mathcal{K}^2\, |\nabla_x u|^2 + \big|\nabla_{\mathcal{S}_y} |\nabla_x u|\big|^2$$

$$\text{and} \qquad \mathcal{K}_\sharp := \sum_{j=1}^{n} (\nabla u \cdot \nabla u_{x_j})^2 - (\nabla u \cdot \nabla |\nabla_x u|)^2.$$

Since $\nabla u_{x_j} = 0 = \nabla|\nabla_x u|$ for almost every point in $\{\nabla_x u = 0\}$, this type of inequalities has also a deep relevance for rigidity and symmetry results, as pointed out by [11], see also [12, 24, 25].

1.4. **Classification of stable solutions in convex domains.** One of the main goal of this paper is to classify stable solutions of (1.1) under suitable assumptions on either the domain or on the nonlinearities. In this spirit, the first result that we present concerns classification in convex domains.

**Theorem 1.2.** *Let $\Omega \subset \mathbb{R}^n$ be convex, and let $u$ be a stable solution of* (1.1) *satisfying the energy bound*

$$\int_{\Omega\times(0,R)} a(y,|\nabla u|)\,|\nabla_x u|^2 \leqslant CR^2, \tag{1.12}$$

*for some constant $C>0$ independent of $R$.*

*Then there exist $\omega : (0,+\infty) \to \mathbb{S}^{N-1}$ and $u_0 : \mathbb{R}\times[0,+\infty) \to \mathbb{R}$ such that*

$$u(x,y) = u_0(\omega(y)\cdot x, y)$$

*for any $(x,y) \in \mathcal{C}$. In particular, the level sets of the function $u(\cdot,y)$ are parallel $(n-1)$-dimensional planes in $\mathbb{R}^n \times \{y\}$ for any fixed $y>0$.*

*In addition, if $\{\nabla_x u = 0\} = \varnothing$ then the function $\omega(\cdot)$ is constant.*

*Moreover, if $n \geqslant 2$ and the principal curvatures along $\partial\Omega$ are strictly positive, then $u$ depends only on $y$.*

We remark that assumption (1.12) is satisfied, for instance, if $a$ grows in $y$ at infinity as $y^\vartheta$ (with $\vartheta \in (-1,1)$) and $|\nabla_x u|$ has growth bounded by $y^{\frac{1-\vartheta}{2}}$; in particular, unbounded solutions may be also taken into account.

We also stress that, in general (and differently from the setting in [7]), it is not possible to deduce, in the setting of Theorem 1.2, that the solution $u$ is constant (as a counterexample, one may consider the case in which $u := y$, $a := 1$, $f := -1$, $g := 0$; clearly, $u$ is stable being a harmonic function).

Notice that Theorem 1.2 yields 1-dimensional symmetry of possibly unbounded stable solutions to (1.1) provided that $\Omega$ is convex and $\partial\Omega$ has strictly positive principal curvature. It is possible to obtain the same result removing this last assumption, but adding an integrability condition, as stated in the following result:

**Theorem 1.3.** *Let $\Omega \subset \mathbb{R}^n$ be convex, and let $u$ be a stable solution of* (1.1), *satisfying the integrability assumption*

$$a(y,|\nabla u|)\left(|\nabla_x u|^2 + |D_x^2 u|^2 + |\nabla_x u_y|^2\right) + \left|g_u(y,u)\right|\,|\nabla_x u|^2 \in L^1(\mathcal{C}). \tag{1.13}$$

*Then $u$ depends only on $y$.*

It is worth to observe that (1.13) can be considerably weakened when $a(y,t)$ is independent of $t$ and $g \equiv 0$. This case is particularly interesting, as we will discuss in the forthcoming Section 1.6. To this goal, we have the following result:

**Proposition 1.4.** *Let $\Omega \subset \mathbb{R}^n$ be convex. If $a$ is independent of $t$ and $g \equiv 0$, then* (1.13) *is satisfied provided that*

$$a(y)\,|\nabla u|^2 \in L^1(\mathcal{C}). \tag{1.14}$$

Notice that condition (1.14) is naturally satisfied when searching for solutions to (1.1) by means of global variational arguments.

In the case of classical elliptic equations, a classification of stable solutions in convex domains with homogeneous Neumann boundary data was given in [7]. Indeed, our Theorem 1.3 may be seen as the extension of Theorem 2 of [7] to the case of boundary reaction-diffusion equations.

In many concrete cases, once one knows that the solution only depends on $y$ (as given for instance by Theorem 1.3), then (1.1) simplifies and can be often explicitly integrated. For instance, if $u = u(y)$ and $a = a(y)$ only depend on $y$, and $g$ vanishes identically, then (1.1) reduces to an ordinary differential equation which provides the family of solutions

$$u(y) = c - f(c) \int_0^y \frac{d\zeta}{a(\zeta)}, \tag{1.15}$$

for $c \in \mathbb{R}$. We also remark that in the model case in which $a(y) = y^\vartheta$, with $\vartheta \in (-1, 1)$, the functions $u$ of the form (1.15) that satisfy $au_y^2 \in L^1(\mathcal{C})$ are the constants (see also Lemma 4.10 in [12] for classification results of ordinary differential equations).

Moreover, we stress that, in general, stable solutions of (1.1) are *not necessarily constant*. As an example, one can consider $u(x, y) := e^{-y}$, $a(y, t) := e^y$, $f := 1$ and $g := 0$ (notice that in this case (1.1), (1.5) and (1.13) are all satisfied; moreover, since $g \equiv 0$ and $f' \leqslant 0$, and recalling Lemma 2.3, then we see that $u$ is stable, in the sense of (1.10)). In this sense, Theorems 1.2 and 1.3 are optimal.

Though the statements of Theorems 1.2 and 1.3 are somehow similar, we prove them by different methods. Indeed, the proof of Theorem 1.2 relies on the Poincaré-type geometric inequality stated in Theorem 1.1 (by the choice of an appropriate test function), while the proof of Theorem 1.3 is based on the relation between maximum principles and stability conditions in view of a suitable spectral analysis.

1.5. **Classification of bounded stable solutions for convex/concave nonlinearities.** Now we address the problem of classifying stable solutions if the nonlinearity $f$ is either convex or concave. To this aim, we shall make the assumption that $g \equiv 0$. The precise result obtained is the following:

**Theorem 1.5.** *Assume that*

$$a_t(y, t) \leqslant 0 \text{ and } g(y, t) = 0 \text{ for any } t,\ y > 0. \tag{1.16}$$

*Let $u$ be a bounded and stable solution of (1.1), such that*

$$\begin{aligned} & a(y, |\nabla u|)|\nabla u|^2 \in L^1(\mathcal{C}), \\ & \lim_{R \to +\infty} \frac{1}{R^2} \int_{\Omega \times (R, 2R)} a(y, |\nabla u|) = 0 \\ \text{and} \qquad & a(y, |\nabla u|)\, \partial_y u \in C(\overline{\Omega} \times [0, +\infty)). \end{aligned} \tag{1.17}$$

*If either $f$ is strictly convex, or $f$ is strictly concave, then $u$ is constant in $\mathcal{C}$.*

We remark that Theorem 1.5 is proved here under the additional assumption in (1.16), stating that $a$ is nonincreasing with respect to the variable $t$. This assumption is of course satisfied in all the cases in which $a$ is independent of $t$, that is, if one is considering semilinear reaction-diffusion equation. Nevertheless, we remark that condition (1.16) is satisfied also in the case of important quasilinear reaction-diffusion equations, such as the one driven by mean curvature-type operators, in which

$$a(y,t) = \frac{y^{\vartheta}}{\sqrt{1+t^2}},$$

with $\vartheta \in (-1,1)$. We think that it is an interesting open problem to decide for which type of quasilinear reaction-diffusion equations similar statements hold true.

In the case of elliptic equations with inner reaction, the classification of stable solutions with Neumann data under suitable convexity or concavity assumptions on the nonlinear term was obtained in [7]. In this sense, our Theorems 1.5 is the extension of Theorem 3 of [7] to the boundary reaction-diffusion equation in (1.1).

1.6. **Application to nonlocal Neumann problems.** Now we discuss the classification of stable solutions in a problem driven by the square root of the Laplacian in the spectral sense. For this scope, let $\{\varphi_k : k \in \mathbb{N} \cup \{0\}\}$ and $\{\lambda_k : k \in \mathbb{N} \cup \{0\}\}$ be the eigenfunctions and the eigenvalues of $-\Delta$ in $\Omega$ with homogeneous Neumann conditions on $\partial\Omega$. We normalize the sequence of eigenfunctions in such a way that they form an orthonormal basis of $L^2(\Omega)$.

The Neumann Laplacian $-\Delta_N$ is the operator acting on an $L^2(\Omega)$-function

$$w(x) = \sum_{k=0}^{\infty} w_k\, \varphi_k(x),$$

where

$$w_k := \int_\Omega w(x)\, \varphi_k(x)\, dx,$$

as

$$-\Delta_N w(x) := \sum_{k=0}^{\infty} \lambda_k\, w_k\, \varphi_k(x).$$

Then, for $s \in (0,1)$, the $s$-Neumann Laplacian is given by

$$(-\Delta_N)^s w(x) := \sum_{k=0}^{\infty} \lambda_k^s\, w_k\, \varphi_k(x). \tag{1.18}$$

With the language of the semigroups introduced in [28], it is possible to show that $(-\Delta_N)^s$ is a nonlocal operator.

From now on we focus on the case $s = 1/2$ and, given $f \in C^{2,\alpha}(\mathbb{R})$, we consider the semilinear equation

$$\begin{cases} (-\Delta_N)^{1/2} v = f(v) & \text{in } \Omega \\ \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.19}$$

The problem can be considered in weak sense, namely we consider the space

$$H^{1/2}(\Omega) := \left\{ w = \sum_{k=0}^{+\infty} w_k \varphi_k \in L^2(\Omega) \text{ s.t. } \sum_{k=0}^{+\infty} \lambda_k^{1/2} |w_k|^2 < +\infty \right\}.$$

Then we say that $v$ is a solution of (1.19) if

$$v = \sum_{k=0}^{+\infty} v_k \varphi_k \in H^{1/2}(\Omega)$$

and

$$\sum_{k=0}^{+\infty} \lambda_k^{1/2}\, v_k\, \zeta_k = \int_\Omega f(v(x))\, \zeta(x)\, dx \qquad \text{for any } \zeta = \sum_{k=0}^{+\infty} \zeta_k \varphi_k \in H^{1/2}(\Omega).$$

We observe that the latter integral is finite since $f$ is bounded by our assumptions. Also, thanks to the results in [28, 29] (see also [19]), the previous nonlocal problem is related to the following local one, with boundary reaction:

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{C} \\ \partial_\nu u = 0 & \text{on } \partial_L \mathcal{C} \\ -\partial_y u = f(u) & \text{on } \partial_B \mathcal{C}. \end{cases} \tag{1.20}$$

More precisely, let us define $\mathcal{H}(\mathcal{C})$ as the completion of $H^1(\mathcal{C})$ with respect to the scalar product

$$(u_1, u_2)_{\mathcal{H}(\mathcal{C})} := \int_{\mathcal{C}} \nabla u_1 \cdot \nabla u_2 + \int_\Omega u_1 u_2,$$

where $u_i|_\Omega$ has to be understood in the sense of traces (notice that this is possible, see Section 2 in [29]). It results that $\mathcal{H}(\mathcal{C}) \supset H^1(\mathcal{C})$ (notice in particular that constant functions are in $\mathcal{H}(\mathcal{C})$ but not in $H^1(\mathcal{C})$; for this reason, $\mathcal{H}(\mathcal{C})$ is a more suitable space than $H^1(\mathcal{C})$ to set (1.20) in weak sense). Then a weak solution $v$ to (1.19) can be defined as the trace over $\Omega$ of a function $u \in \mathcal{H}(\mathcal{C})$ such that

$$\int_{\mathcal{C}} \nabla u \cdot \nabla \varphi - \int_{\Omega \times \{0\}} f(u)\varphi = 0 \qquad \text{for all } \varphi \in \mathcal{H}(\mathcal{C}), \tag{1.21}$$

see again [29].

Notice that this setting falls exactly under the general setting considered in (1.1), with $a \equiv 1$ and $g \equiv 0$ (compare (1.21) with (1.7)).

In this framework, we say that a solution $v \in H^{1/2}(\Omega)$ to (1.19) is *stable* if its extension $u \in \mathcal{H}(\mathcal{C})$ in (1.20) is stable according to (1.10) (with $\mathcal{B}$ the identity matrix and $g \equiv 0$), i.e.

$$\int_{\mathcal{C}} |\nabla \varphi|^2 - \int_{\Omega \times \{0\}} f'(u)\varphi^2 \geqslant 0 \tag{1.22}$$

for any $\varphi \in \mathcal{A}$.

With this definition, we can prove the following classification theorems for stable solutions to (1.19).

**Theorem 1.6.** *Let $\Omega \subset \mathbb{R}^n$ be convex, and let $v \in H^{1/2}(\Omega) \cap L^\infty(\Omega)$ be a stable solution to* (1.19)*. Then $v$ is constant.*

Theorem 1.6 establishes that equation (1.19) in convex domains does not admit noncostant stable solutions. The same conclusion holds, if, instead of the convexity of $\Omega$, we assume the convexity (or the concavity) of the nonlinearity $f$, according to the following result.

**Theorem 1.7.** *Let $f$ be either strictly convex, or strictly concave, and let $v \in H^{1/2}(\Omega) \cap L^\infty(\Omega)$ be a stable solution to* (1.19)*. Then $v$ is constant.*

The previous results can be considered as the counterpart of those in [7] for (1.19). Clearly, a natural question consist in finding easy and natural assumptions on $f$ or on $v$ allowing to show that $v$ is stable in the sense of (1.22). A very simple condition consists in $f' \leqslant 0$.

We also point out that, in our framework, Theorems 1.6 and 1.7 will be obtained by using Theorems 1.3 and 1.5, respectively.

As a further remark, we observe that if $v \in H^{1/2}(\Omega) \cap L^\infty(\Omega)$, then, by [29, Theorem 3.5], we have that $v \in \mathcal{C}^1(\overline{\Omega})$. Therefore the boundary condition $\partial_\nu v = 0$ on $\partial\Omega$ can be understood in classical sense.

We also mention that our focus on the case $s = 1/2$ is due to the fact that we recalled and used some results contained in [29]. Once that similar results are established for the case $s \neq 1/2$ (this is announced in [29]), our results would also hold for the general case $s \in (0,1)$. Indeed, for $s \in (0,1)$, the extension problem associated to the $s$-Neumann Laplacian will be of type (1.1) with $a(y,t) = y^\vartheta$, with $\vartheta \in (-1,1)$, and $g \equiv 0$.

1.7. **A counterexample in a different nonlocal setting.** In Section 1.6, we have considered classification results for stable solutions of spectral versions of fractional Laplacians (in the sense given by (1.18)).

In the literature, other nonlocal elliptic operators of fractional type have been widely studied. Of particular interest is the integral version of the fractional Laplacian, defined (up to normalizing constants), for any $s \in (0,1)$, as

$$(-\Delta)^s v(x) := \mathrm{pv} \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x-y|^{n+2s}}\, dy. \tag{1.23}$$

As usual, pv stays for the principal value. We stress that the operators in (1.18) and (1.23) are indeed different (see e.g. [23]).

In this setting, a natural fractional normal derivative at the boundary (see e.g. [20]) is given by

$$(\partial_\nu)^s v(x) := \lim_{t \to 0^+} \frac{v(x + t\tilde{\nu}(x)) - v(x)}{t^s},$$

where $\tilde{\nu}(x)$ denotes the outer unit normal to $\partial\Omega$ at $x \in \partial\Omega$.

With this, one may wonder whether "nice" and "stable" solutions to the equation

$$\begin{cases} (-\Delta)^s v = f(v) & \text{in } \Omega \\ (\partial_\nu)^s v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.24}$$

in convex domains or with convex nonlinearities are necessarily constant, or, at least, if they enjoy some rigid geometric properties.

While a suitable notion of stability should be introduced in this setting, the further assumption that $f \equiv 0$ would imply stability in any reasonable definition, thus the basic question boils down to determine any rigidity properties of solutions of

$$\begin{cases} (-\Delta)^s v = 0 & \text{in } \Omega \\ (\partial_\nu)^s v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.25}$$

possibly in convex domains.

Quite surprisingly, we now show that no classification (and even no rigidity) results hold true for equation (1.25). This phenomenon shows that the "right" choice of fractional operator, endowed with the appropriate boundary conditions, plays a crucial role in nonlocal problems.

In concrete, the result that we show is the following:

**Example 1.8.** Let $s \in (0,1)$, $h \in C^2(\mathbb{R})$ and $\epsilon \in (0,1)$. Then, there exist $\delta_1, \delta_2 \in \left[0, \frac{\epsilon}{2}\right]$ and $v \in C^2((-1-\epsilon, 1+\epsilon)) \cap C(\mathbb{R})$ such that

$$\begin{aligned}
&\|v - h\|_{C^2((-1,1))} \leqslant \epsilon, \\
&(-\partial_x^2)^s v(x) := \mathrm{pv} \int_{\mathbb{R}} \frac{v(x) - v(y)}{|x-y|^{1+2s}}\, dy = 0 \text{ for any } x \in (-1-\delta_1, 1+\delta_2), \\
&(\partial_\nu)^s v(x) = \lim_{\substack{y \in (-1-\delta_1, 1+\delta_2) \\ y \to x}} \frac{v(x) - v(y)}{|x-y|^s} = 0 \text{ for any } x \in \{-1-\delta_1, 1+\delta_2\}, \\
&v'(-1-\delta_1) = v'(1+\delta_2) = 0, \\
&v \text{ has bounded support.}
\end{aligned}$$

We remark that the operator $(-\partial_x^2)^s$ is simply $(-\Delta)^s$, as defined in (1.23), when the domain is one-dimensional. Also, the "fractional" boundary derivative $(\partial_\nu)^s$ has nice regularity properties and natural applications in Pohozaev-type identities and in rigidity results for overdetermined problems (see e.g. [15, 20, 21, 10, 26]); nevertheless it cannot characterize solutions $v$ of the fractional equation (1.24) in convex domains, which, as stated in Example 1.8, at least for $f \equiv 0$, can have essentially the same local qualitative properties of any prescribed function $h$.

### 1.8. **Organization of the paper.**

The rest of the paper is organized as follows. In Section 2, we collect some preliminary computations that are needed in the proofs of the main results. In Section 3, we prove the Poincaré-type geometric inequality stated in Theorem 1.1. The classification of solutions to (1.1) when $\Omega$ is convex, together with the proofs of Theorems 1.2 and 1.3, is contained in Sections 4 and 5. The proof of Theorems 1.5, with the classification of stable solutions in case of convex/concave nonlinearities is contained in Section 6. Section 7 is devoted to the study of classification results involving the spectral $s$-Neumann Laplacian $(-\Delta_N)^s$. Section 8 contains the discussion related to Example 1.8.

## 2. Toolbox

In this section we collect several intermediate statements which will be used in the proof of our main results.

### 2.1. **Some inequalities coming from the Neumann condition.**

Next result deals with the geometric analysis related to functions satisfying a Neumann condition.

**Lemma 2.1.** *Let $\Omega \subset \mathbb{R}^n$ be an open set with boundary of class $C^1$. Let $u \in C^2(\overline{\Omega} \times (0,+\infty))$, with $\partial_\nu u = 0$ on $\partial\Omega \times (0,+\infty)$.*

*Assume that $\bar{x} = (\bar{x}', \bar{x}_n) \in \partial\Omega$ and that in a neighborhood of $\bar{x}$ the domain $\Omega$ can be written in normal coordinates as the epigraph of a function $\gamma \in C^2(\mathbb{R}^{n-1})$, i.e.*

$$\Omega \cap B_r(\bar{x}) = \{x = (x', x_n) \in B_r(\bar{x}) \text{ s.t. } x_n > \gamma(x')\},$$

*for some $r>0$, with $\gamma(\bar{x}')=\bar{x}_n$ and $\nabla\gamma(\bar{x}')=0$. Then, for any $y>0$,*

$$\begin{aligned}\nabla u(\bar{x},y)\cdot\partial_\nu\left(\nabla u(\bar{x},y)\right) &= \nabla_x u(\bar{x},y)\cdot\partial_\nu\left(\nabla_x u(\bar{x},y)\right)\\ &= -\sum_{i,j=1}^{n-1}\gamma_{x_i x_j}(\bar{x}')\,u_{x_i}(\bar{x},y)\,u_{x_j}(\bar{x},y).\end{aligned} \tag{2.1}$$

*In particular, if $\Omega$ is convex, then*

$$\nabla u(\bar{x},y)\cdot\partial_\nu\left(\nabla u(\bar{x},y)\right)=\nabla_x u(\bar{x},y)\cdot\partial_\nu\left(\nabla_x u(\bar{x},y)\right)\leqslant 0. \tag{2.2}$$

*Proof.* Up to a translation, we can assume that $\bar{x}=0$. Thus, in the vicinity of the origin we can write the unit normal vector as

$$\nu(x,y)=\frac{1}{\sqrt{|\nabla\gamma(x')|^2+1}}\left(\nabla\gamma(x'),-1,0\right).$$

Therefore, the condition $\partial_\nu u(x,y)=0$, for $x\in\partial\Omega\cap B_r$, reads as

$$\sum_{i=1}^{n-1}u_{x_i}\big(x',\gamma(x'),y\big)\,\gamma_{x_i}(x')-u_{x_n}\big(x',\gamma(x'),y\big)=0.$$

So, taking the derivative with respect to $x_j$ with $j=1,\dots,n-1$, we obtain

$$\begin{aligned}&\sum_{i=1}^{n-1}u_{x_i x_j}\big(x',\gamma(x'),y\big)\gamma_{x_i}(x')\\ &\qquad+\sum_{i=1}^{n-1}u_{x_i x_n}\big(x',\gamma(x'),y\big)\,\gamma_{x_j}(x')\,\gamma_{x_i}(x')+\sum_{i=1}^{n-1}u_{x_i}\big(x',\gamma(x')\big)\,\gamma_{x_i x_j}(x')\\ &\qquad\qquad-u_{x_j x_n}\big(x',\gamma(x')\big)-u_{x_n x_n}\big(x',\gamma(x'),y\big)\,\gamma_{x_j}(x')=0.\end{aligned}$$

Hence, recalling that $\nabla\gamma(0')=0$, we infer that

$$\sum_{i=1}^{n-1}u_{x_i}(0,y)\gamma_{x_i x_j}(0,y)-u_{x_j x_n}(0,y)=0,$$

which proves that, for any $y>0$ and any $j=1,\dots,n-1$,

$$u_{x_j x_n}(\bar{x},y)=\sum_{i=1}^{n-1}u_{x_i}(\bar{x},y)\gamma_{x_i x_j}(\bar{x},y). \tag{2.3}$$

Now we observe that

$$\nu(0,y)=-e_n \text{ and so } u_{x_n}(0,y)=-\partial_\nu u(0,y)=0. \tag{2.4}$$

By differentiating this identity in $y$, we deduce that

$$u_{x_n y}(0,y)=0. \tag{2.5}$$

The considerations in (2.4) and (2.5) prove that, for any $y>0$,

$$u_{x_n}(\bar{x},y)=u_{x_n y}(\bar{x},y)=0. \tag{2.6}$$

Moreover,

$$u_y(0,y)\cdot\partial_\nu u_y(0,y)=-u_y(0,y)\,u_{x_n y}(0,y)=0. \tag{2.7}$$

Moreover, using again (2.4), we see that

$$\begin{aligned}\nabla_x u(0,y)\cdot\partial_\nu\left(\nabla_x u(0,y)\right) &= -\nabla_x u(0,y)\cdot\partial_{x_n}\left(\nabla_x u(0,y)\right)\\ &= -\sum_{i=1}^{n} u_{x_i}(0,y)u_{x_i x_n}(0,y) = -\sum_{i=1}^{n-1} u_{x_i}(0,y)u_{x_i x_n}(0,y).\end{aligned} \tag{2.8}$$

Now we plug (2.3) into (2.8), and we deduce that

$$\nabla_x u(0,y)\cdot\partial_\nu\left(\nabla_x u(0,y)\right) = -\sum_{i,j=1}^{n-1}\gamma_{x_i x_j}(0')\,u_{x_i}(0,y)\,u_{x_j}(0,y).$$

From this and (2.7), we obtain (2.1). Formula (2.2) follows from (2.1) and the convexity of $\Omega$ (which boils down to the convexity of $\gamma$). □

For completeness, we also recall the following result:

**Lemma 2.2.** *Let $n \geqslant 2$. If $\Omega$ is convex, then $\partial\Omega$ is pathwise connected.*

*Proof.* Fix $Z \in \Omega$. Given two points $A$, $B \in \partial\Omega$ we will construct a path joining $A$ to $B$ and lying on $\partial\Omega$. For this, we consider the segment $S$ that joins $A$ and $B$. Notice that $S \subseteq \overline{\Omega}$, by convexity. Given any $X \in S$, we can consider the halfline $r(X)$ that emanates from $Z$ and passes through $X$. We remark that $r(X)$ must intersect $\partial\Omega$, since $\Omega$ is bounded. This intersection point must be unique: indeed, if there are two points $Y_1$, $Y_2 \in \partial\Omega\cap r(X)$, since $B_\rho(Z)\subset\Omega$ for some $\rho>0$, we have that the convex hull of $Y_1$, $Y_2$ and $B_\rho(Z)$ lies in $\overline{\Omega}$, and this contradicts that both $Y_1$ and $Y_2$ are boundary point.

Therefore, for any $X \in S$, we can define a continuous function $\pi : S \to \partial\Omega$, as the intersection of $r(X)$ with $\partial\Omega$. Then, the image of $S$ via $\pi$ provides the desired path lying on $\partial\Omega$ and joining $A$ to $B$. □

## 2.2. **Positive definiteness of $\mathcal{B}$.**

**Lemma 2.3.** *For any $y>0$ and $\eta\in\mathbb{R}^{n+1}\setminus\{0\}$, the matrix $\mathcal{B}(y,\eta)$ is positive definite.*

*More precisely, the matrix $\mathcal{B}(y,\eta)$ has eigenvalues $a(y,|\eta|)+|\eta|\,a_t(y,|\eta|)$ (with multiplicity 1) and $a(y,|\eta|)$ (with multiplicity $n$).*

*Proof.* The second statement implies the first one, thanks to (1.2). So we focus on the proof of the second statement. For this, we fix an orthonormal basis of $\mathbb{R}^{n+1}$, say $\{E_1,\dots,E_{n+1}\}$, such that $E_1 := \eta/|\eta|$. We will use this basis to diagonalize the matrix $\mathcal{B}(y,\eta)$. Indeed, for any $k=2,\dots,n+1$, we have that $E_1\cdot E_k=0$. Thus, for any $i=1,\dots,n+1$

$$\begin{aligned}\big(\mathcal{B}(y,\eta)\,E_1\big)_i &= \sum_{j=1}^{n+1}\mathcal{B}(y,\eta)_{ij}\frac{\eta_j}{|\eta|} = a(y,|\eta|)\frac{\eta_i}{|\eta|}+\sum_{j=1}^{n+1}\frac{a_t(y,|\eta|)}{|\eta|^2}\eta_i\eta_j^2\\ &= \big(a(y,|\eta|)+|\eta|\,a_t(y,|\eta|)\big)\,(E_1)_i,\end{aligned}$$

while for any $k=2,\dots,n+1$

$$\begin{aligned}\big(\mathcal{B}(y,\eta)\,E_k\big)_i &= a(y,|\eta|)(E_k)_i+\sum_{j=1}^{n+1}\frac{a_t(y,|\eta|)}{|\eta|}\eta_i\eta_j(E_k)_j\\ &= a(y,|\eta|)(E_k)_i + a_t(y,|\eta|)\,\eta_i E_1\cdot E_k = a(y,|\eta|)(E_k)_i.\end{aligned}$$

□

2.3. **Some results available in the literature.** Here we recall some known auxiliary statements, which will be needed in the proof of our main results (these statements have been included for the facility of the reader, to make the paper more self-contained).

The following is a variant of Lemma 10 in [24]. The proof can be easily obtained modifying the argument therein, and thus is omitted.

**Lemma 2.4.** *Let $R > 0$ and $h : \Omega \times (0, R) \to \mathbb{R}$ be a nonnegative measurable function. For any $\rho \in (0, R)$, let*

$$\eta(\rho) := \int_{\Omega \times (0,R)} h.$$

*Then*

$$\int_{\Omega \times (\sqrt{R}, R)} \frac{h(X)}{y^2}\, dX \leqslant 2 \int_{\sqrt{R}}^{R} t^{-3} \eta(t)\, dt + \frac{\eta(R)}{R^2}.$$

**Lemma 2.5** (Lemma 4.2 in [25])**.** *In $\mathcal{C} \cap \{\nabla_x u \neq 0\}$, it results*

$$\sum_{j=1}^{n} \langle \mathcal{B}(y, \nabla u)\, \nabla u_{x_j},\, \nabla u_{x_j} \rangle - \langle \mathcal{B}(y, \nabla u)\, \nabla |\nabla_x u|,\, \nabla |\nabla_x u| \rangle \geqslant 0. \tag{2.9}$$

**Corollary 2.6** (Corollary 4.3 and Theorem 4.4 in [25], and Lemma 2.11 in [12])**.** *Let $U \subset \mathcal{C} \cap \{\nabla_x u \neq 0\}$ be an open set. Suppose that*

$$\sum_{j=1}^{n} \langle \mathcal{B}(y, \nabla u) \nabla u_{x_j}, \nabla u_{x_j} \rangle - \langle \mathcal{B}(y, \nabla u) \nabla |\nabla_x u|,\, \nabla |\nabla_x u| \rangle = 0$$

*almost everywhere in $U$. Then there exist $\omega : (0, +\infty) \to \mathbb{S}^{N-1}$ and $u_0 : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ such that*

$$u(x, y) = u_0(\omega(y) \cdot x, y).$$

*In particular $u_0(\cdot, y)$ is 1-dimensional for every $y > 0$ fixed.*

*In addition, if $\{\nabla_x u = 0\} = \varnothing$ then the function $\omega(\cdot)$ is constant.*

2.4. **The linearized equation.** Now we consider the so-called linearized equation, that is the equation satisfied by the derivatives of the solution in the variables $x$ (this equation is clearly related with the stability condition in (1.10)). The result needed for our purposes is the following:

**Lemma 2.7.** *Assume that $u$ is a solution of* (1.1)*. Then, for any $j = 1, \ldots, n$, we have that $u_{x_j}$ satisfies*

$$\begin{aligned} \int_{\mathcal{C}} \Big[ \langle \mathcal{B}(y, \nabla u)\, \nabla u_{x_j}, \nabla \psi \rangle + g_u(y, u)\, u_{x_j}\, \psi \Big] \\ - \int_{\partial_L \mathcal{C}} \Big[ a(y, |\nabla u|)\, (\nabla u \cdot \nabla \psi)\, \nu_j + g(y, u)\, \psi\, \nu_j \Big] \\ = \int_{\partial_B \mathcal{C}} f'(u)\, u_{x_j}\, \psi - \int_{\partial\Omega \times \{0\}} f(u)\, \psi\, \nu_j \end{aligned}$$

*for any $\psi \in \mathcal{A}$ and and such that $\psi_{x_j} \in \mathcal{A}$ for any $j = 1, \ldots, n$.*

*Proof.* First, we observe that, for any $j = 1, \dots, n$,

$$\partial_{x_j}\Big(a\big(y, |\nabla u(x,y)|\big)\Big) = \sum_{k=1}^{n+1} a_t\big(y, |\nabla u(x,y)|\big)\,\frac{u_{X_k}(x,y)\,u_{X_k x_j}(x,y)}{|\nabla u(x,y)|}.$$

As a consequence, for any fixed $j = 1, \dots, n$ and $m = 1, \dots, n+1$,

$$\begin{aligned}
& \Big(a(y,|\nabla u|)\,\nabla u_{x_j} + \partial_{x_j}\big(a(y,|\nabla u|)\big)\,\nabla u\Big)_m \\
= \;& a(y,|\nabla u|)\,u_{X_m x_j} + \partial_{x_j}\big(a(y,|\nabla u|)\big)\,u_{X_m} \\
= \;& \sum_{k=1}^{n+1} a(y,|\nabla u|)\,\delta_{km}\,u_{X_k x_j} + \sum_{k=1}^{n+1} a_t(y,|\nabla u|)\,\frac{u_{X_k}\;u_{X_m}\,u_{X_k x_j}}{|\nabla u|} \\
= \;& \sum_{k=1}^{n+1} \mathcal{B}_{km}(y,\nabla u)\,u_{X_k x_j},
\end{aligned}$$

where we have used (1.9). Therefore, for any $j = 1, \dots, n$,

$$\partial_{x_j}\big(a(y,|\nabla u|)\,\nabla u\big) = \mathcal{B}(y,\nabla u)\,\nabla u_{x_j}. \tag{2.10}$$

Using (2.10), we have the equality (in $L^1$ sense)

$$a(y,|\nabla u|)\,\nabla u\cdot\nabla\psi_{x_j} = \partial_{x_j}\Big(a(y,|\nabla u|)\,\nabla u\cdot\nabla\psi\Big) - \langle \mathcal{B}(y,\nabla u)\,\nabla u_{x_j}, \nabla\psi\rangle.$$

We wish now to use (1.7) with $\varphi = \psi_{x_j}$ (notice that this is possible since we are supposing that $\psi_{x_j} \in \mathcal{A}$). To this aim, we use the Divergence Theorem to obtain

$$\begin{aligned}
&\int_{\mathcal{C}} a(y,|\nabla u|)\,\nabla u\cdot\nabla\psi_{x_j} \\
&\qquad = \int_{\partial_L\mathcal{C}} a(y,|\nabla u|)\,(\nabla u\cdot\nabla\psi)\,\nu_j - \int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\,\nabla u_{x_j}, \nabla\psi\rangle,
\end{aligned} \tag{2.11}$$

for any $j = 1, \dots, n$. In a similar way

$$\int_{\mathcal{C}} g(y,u)\,\psi_{x_j} = \int_{\partial_L\mathcal{C}} g(y,u)\,\psi\,\nu_j - \int_{\mathcal{C}} g_u(y,u)\,u_{x_j}\,\psi. \tag{2.12}$$

Finally, using again the Divergence Theorem,

$$\int_{\partial_B\mathcal{C}} f(u)\,\psi_{x_j} = \int_{\partial\Omega\times\{0\}} f(u)\,\psi\,\nu_j - \int_{\partial_B\mathcal{C}} f'(u)\,u_{x_j}\,\psi. \tag{2.13}$$

Notice that $\nu_j$ denotes the $j$ component of $\nu = (\tilde\nu, 0)$ and $\tilde\nu$ is the outer unit normal vector of $\partial\Omega$ in $\mathbb{R}^n$.

We can now insert (2.11), (2.12) and (2.13) into (1.7) and obtain the desired conclusion. $\square$

As a consequence of Lemma 2.7, we can test the linearized equation against $\psi_j := u_{x_j}\varphi$, where $\varphi$ is a "nice" function with bounded support in $y$: in this case, the particular choice of the test function and the Neumann condition provide some additional simplifications, as stated in the following result:

**Corollary 2.8.** *Assume that $u$ is a solution of* (1.1). *Then*

$$\begin{aligned}&\sum_{j=1}^{n}\int_{\mathcal{C}}\Big[\langle \mathcal{B}(y,\nabla u)\,\nabla u_{x_j},\nabla u_{x_j}\rangle\,\varphi+\langle \mathcal{B}(y,\nabla u)\,\nabla u_{x_j},\nabla\varphi\rangle\,u_{x_j}\Big]\\&\qquad+\int_{\mathcal{C}} g_u(y,u)\,|\nabla_x u|^2\,\varphi\\&\qquad\qquad=\int_{\partial_L\mathcal{C}} a(y,|\nabla u|)\,\big(\nabla u\cdot\partial_\nu(\nabla u)\big)\,\varphi+\int_{\partial_B\mathcal{C}} f'(u)\,|\nabla_x u|^2\varphi\end{aligned}\tag{2.14}$$

*for any $\varphi\in C^1(\overline{\mathcal{C}})$ with bounded support in $y$ and such that $\varphi_{x_j}\in\mathcal{A}$, for any $j=1,\dots,n$.*

*Proof.* We use $\psi_j := u_{x_j}\varphi$ as test function in Lemma 2.7, observing that this is possible by (1.5) and the assumptions on $\varphi$. First of all, for any $x\in\partial\Omega$ and $y>0$, if $\psi_j := u_{x_j}\varphi$ then on $\partial_L\mathcal{C}$

$$\sum_{j=1}^{n}\psi_j\,\nu_j=\nabla_x u\cdot\nu\,\varphi=\nabla u\cdot\nu\,\varphi=0,$$

where we used the fact that the last component of $\nu$ is 0 on $\partial_L\mathcal{C}$. Since the normal along $\partial_L\mathcal{C}$ coincides with the one along $\partial\Omega$ by projection, we deduce from (1.5) that

$$\sum_{j=1}^{n}\psi_j\nu_j=0$$

also on $\partial\Omega\times\{0\}$. These considerations imply that

$$\sum_{j=1}^{n}\int_{\partial_L\mathcal{C}} g(y,u)\,\psi_j\,\nu_j=0\quad\text{and}\quad\sum_{j=1}^{n}\int_{\partial\Omega\times\{0\}} f(u)\,\psi_j\,\nu_j=0.\tag{2.15}$$

We also observe that

$$\begin{aligned}&a(y,|\nabla u|)\,(\nabla u\cdot\nabla\psi_j)\,\nu_j\\&\qquad=a(y,|\nabla u|)\,(\nabla u\cdot\nabla u_{x_j})\,\varphi\,\nu_j+a(y,|\nabla u|)\,(\nabla u\cdot\nabla\varphi)\,u_{x_j}\,\nu_j\end{aligned}$$

on $\partial_L\mathcal{C}$, so that, using again the homogeneous Neumann condition,

$$\sum_{j=1}^{n}\int_{\partial_L\mathcal{C}} a(y,|\nabla u|)\,(\nabla u\cdot\nabla\psi_j)\,\nu_j=\sum_{j=1}^{n}\int_{\partial_L\mathcal{C}} a(y,|\nabla u|)\,(\nabla u\cdot\nabla u_{x_j})\,\varphi\,\nu_j.\tag{2.16}$$

Plugging (2.15) and (2.16) into the formula in Lemma 2.7, the thesis follows. $\square$

A refinement of Corollary 2.8, under additional integrability assumptions, goes as follows:

**Corollary 2.9.** *Let $\Omega$ be a convex domain, and let $u$ be a solution of* (1.1)*, satisfying the integrability assumption* (1.13). *Then*

$$\begin{aligned}&\sum_{j=1}^{n}\int_{\mathcal{C}}\langle \mathcal{B}(y,\nabla u)\,\nabla u_{x_j},\nabla u_{x_j}\rangle+\int_{\mathcal{C}} g_u(y,u)\,|\nabla_x u|^2\\&\qquad=\int_{\partial_L\mathcal{C}} a(y,|\nabla u|)\,\big(\nabla u\cdot\partial_\nu(\nabla u)\big)+\int_{\partial_B\mathcal{C}} f'(u)\,|\nabla_x u|^2.\end{aligned}$$

*Proof.* We take $\phi := \phi_R(y)$ to be a smooth nonnegative function such that $\phi_R(y) = 1$ if $y \in [0,R]$, $\phi_R(y) = 0$ if $y \in [2R,+\infty)$ and $|\phi_R'| \leqslant 10/R$. We apply Corollary 2.8 and we send $R \to +\infty$, using assumption (1.13). More precisely, by (1.9) and (1.3), we can bound $|\mathcal{B}(y,\nabla u)|$ by $a(y,|\nabla u|)$, up to multiplicative constants; therefore

$$|\mathcal{B}(y,\nabla u)|\left(|\nabla_x u|^2 + |D_x^2 u|^2 + |\nabla_x u_y|^2\right) \in L^1(\mathcal{C})$$

thanks to (1.13), and this allows us to pass to the limit as $R \to +\infty$ in the first term on the left hand side in (2.14). As far as the second term, we use the fact that $g_u(y,u)|\nabla_x u|^2 \in L^1(\mathcal{C})$ and argue in a similar way. Finally, for the first term on the right hand side of (2.14) we apply the Monotone Convergence Theorem, observing that, thanks to the convexity of $\Omega$, Lemma 2.1 implies that

$$a(y,|\nabla u|)\,(\nabla \cdot \partial_\nu(\nabla u)) \leqslant 0 \quad \text{on } \partial_L \mathcal{C}. \qquad \square$$

## 3. A Poincaré-type geometric inequality: proof of Theorem 1.1

In this section we prove the geometric inequality of Poincaré type stated in Theorem 1.1.

*Completion of the proof of Theorem 1.1.* We use Corollary 2.8 with $\varphi := \psi^2$ and $\psi \in C^1(\overline{\mathcal{C}})$ with bounded support in $y$ and such that $\psi_{x_j} \in \mathcal{A}$ for any $j = 1,\dots,n$. In this way, we have that

$$\begin{aligned}
&\sum_{j=1}^n \int_{\mathcal{C}} \Big[\langle \mathcal{B}(y,\nabla u)\,\nabla u_{x_j}, \nabla u_{x_j}\rangle\,\psi^2 + 2\langle \mathcal{B}(y,\nabla u)\,\nabla u_{x_j}, \nabla\psi\rangle\, u_{x_j}\,\psi\Big] \\
&\qquad + \int_{\mathcal{C}} g_u(y,u)\,|\nabla_x u|^2\,\psi^2 \\
&\qquad\qquad = \int_{\partial_L \mathcal{C}} a(y,|\nabla u|)\,\big(\nabla u \cdot \partial_\nu(\nabla u)\big)\,\psi^2 + \int_{\partial_B \mathcal{C}} f'(u)\,|\nabla_x u|^2 \psi^2.
\end{aligned} \tag{3.1}$$

Now we use the fact that $u$ is stable, and we choose $|\nabla_x u|\psi$ as test function in the definition of stability (1.10) (we observe that this choice is admissible, thanks to (1.5)): in this way, we conclude that

$$\begin{aligned}
&\int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla|\nabla_x u|,\, \nabla|\nabla_x u|\rangle\,\psi^2 + \int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\psi,\, \nabla\psi\rangle\,|\nabla_x u|^2 \\
&\qquad + \int_{\mathcal{C}} 2\langle \mathcal{B}(y,\nabla u)\nabla|\nabla_x u|,\, \nabla\psi\rangle\,|\nabla_x u|\,\psi \\
&\qquad + \int_{\mathcal{C}} g_u(y,u)\,|\nabla_x u|^2\,\psi^2 - \int_{\partial_B \mathcal{C}} f'(u)\,|\nabla_x u|^2\,\psi^2 \geqslant 0.
\end{aligned} \tag{3.2}$$

It is convenient to observe that

$$|\nabla_x u|\,\nabla|\nabla_x u| = \frac{1}{2}\nabla|\nabla_x u|^2 = \frac{1}{2}\sum_{j=1}^n \nabla u_{x_j}^2 = \sum_{j=1}^n u_{x_j}\nabla u_{x_j},$$

and so we can rewrite (3.2) as

$$\begin{aligned}&\int_{\mathcal{C}}\langle\mathcal{B}(y,\nabla u)\nabla|\nabla_x u|,\,\nabla|\nabla_x u|\rangle\,\psi^2+\int_{\mathcal{C}}\langle\mathcal{B}(y,\nabla u)\nabla\psi,\,\nabla\psi\rangle\,|\nabla_x u|^2\\&\quad+\sum_{j=1}^n\int_{\mathcal{C}}2\langle\mathcal{B}(y,\nabla u)\nabla u_{x_j},\,\nabla\psi\rangle\,u_{x_j}\,\psi\\&\quad+\int_{\mathcal{C}}g_u(y,u)\,|\nabla_x u|^2\,\psi^2-\int_{\partial_B\mathcal{C}}f'(u)\,|\nabla_x u|^2\,\psi^2\geqslant 0.\end{aligned}$$

This expression and (3.1) have three terms in common, which can be simplified appropriately, thus establishing (1.11). □

## 4. Classification in convex domains I: proof of Theorem 1.2

We claim that there exist $\omega:(0,+\infty)\to\mathbb{S}^{n-1}$ and $u_0:\mathbb{R}\times[0,+\infty)\to\mathbb{R}$ such that

$$u(x,y)=u_0(\omega(y)\cdot x,y) \tag{4.1}$$

and that

$$\nabla u\cdot\partial_\nu(\nabla u)=0 \text{ on } \partial_L\mathcal{C}. \tag{4.2}$$

Since (4.1) and (4.2) are valid if $n=1$, we can suppose that $n\geqslant 2$.

Let $R>10$ (to be taken arbitrarily large in the sequel). We consider a smooth function $\tau_R:[0,+\infty)\to[0,1]$ supported in $[\sqrt{R},\,R]$, such that $\tau_R=1$ in $[\sqrt{R}+1,\,R-1]$ and $|\nabla\tau_R|\leqslant 10$. For any $y\geqslant 0$, we define

$$\psi_R(y):=\int_y^R\frac{\tau_R(\zeta)}{\zeta}\,d\zeta.$$

Since $\tau_R$ vanishes in $[0,\sqrt{R}]$ we have that $\psi_R$ is smooth, continuous in $[0,+\infty)$ and

$$\begin{aligned}\psi_R(y)&=\int_{\sqrt{R}}^R\frac{\tau_R(\zeta)}{\zeta}\,d\zeta\geqslant\int_{\sqrt{R}+1}^{R-1}\frac{\tau_R(\zeta)}{\zeta}\,d\zeta\\&=\int_{\sqrt{R}+1}^{R-1}\frac{d\zeta}{\zeta}=\log\frac{R-1}{\sqrt{R}+1}\geqslant\log\frac{\sqrt{R}}{2},\end{aligned} \tag{4.3}$$

for any $y\in[0,\sqrt{R}]$, as long as $R$ is sufficiently large. In addition, since $\tau_R$ vanishes also in $[R,+\infty)$, we have that $\psi_R(y)=0$ for any $y\geqslant R$, hence $\psi_R$ is compactly supported in $[0,+\infty)$. Finally, $\psi_R\in C^2(\overline{\mathcal{C}})$ by the choice of $\tau_R$. As a consequence, we can use $\psi_R$ as a test function in (1.11): this yields

$$\begin{aligned}&\int_{\mathcal{C}}\left[\sum_{j=1}^n\langle\mathcal{B}(y,\nabla u)\,\nabla u_{x_j},\nabla u_{x_j}\rangle-\langle\mathcal{B}(y,\nabla u)\nabla|\nabla_x u|,\,\nabla|\nabla_x u|\rangle\right]\,\psi_R^2\\&\quad-\int_{\partial_L\mathcal{C}}a(y,|\nabla u|)\,\big(\nabla u\cdot\partial_\nu(\nabla u)\big)\,\psi_R^2\leqslant\int_{\mathcal{C}}\langle\mathcal{B}(y,\nabla u)\nabla\psi_R,\,\nabla\psi_R\rangle\,|\nabla_x u|^2.\end{aligned} \tag{4.4}$$

Now we use (2.2), (2.9), (4.3), and the fact that $\nabla_x u$ is constant almost everywhere on $\{\nabla_x u = 0\}$, by Stampacchia's Theorem, to see that

$$\tag{4.5}
\begin{aligned}
&\int_{\mathcal{C}} \left[ \sum_{j=1}^{n} \langle \mathcal{B}(y,\nabla u)\, \nabla u_{x_j}, \nabla u_{x_j} \rangle - \langle \mathcal{B}(y,\nabla u)\nabla|\nabla_x u|,\, \nabla|\nabla_x u|\rangle \right] \psi_R^2 \geqslant \\
&\left(\log \frac{\sqrt{R}}{2}\right)^2 \int_{\Omega\times(0,\sqrt{R})} \left[ \sum_{j=1}^{n} \langle \mathcal{B}(y,\nabla u)\, \nabla u_{x_j}, \nabla u_{x_j} \rangle - \langle \mathcal{B}(y,\nabla u)\nabla|\nabla_x u|,\, \nabla|\nabla_x u|\rangle \right]
\end{aligned}$$

and

$$\tag{4.6}
\begin{aligned}
&- \int_{\partial_L \mathcal{C}} a(y,|\nabla u|)\, \big(\nabla u \cdot \partial_\nu(\nabla u)\big)\, \psi_R^2 \\
&\geqslant - \left(\log \frac{\sqrt{R}}{2}\right)^2 \int_{\partial_L \mathcal{C} \cap \{y \in (0,\sqrt{R})\}} a(y,|\nabla u|)\, \big(\nabla u \cdot \partial_\nu(\nabla u)\big).
\end{aligned}$$

In this way we can estimate the left hand side in (4.4). As far as the right hand side is concerned, by (1.3) we obtain

$$\begin{aligned}
&\int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\psi_R,\, \nabla\psi_R\rangle\, |\nabla_x u|^2 \leqslant C_1 \int_{\mathcal{C}} a(y,|\nabla u|)\, |\nabla\psi_R|^2\, |\nabla_x u|^2 \\
&\leqslant C_2 \int_{\Omega\times(\sqrt{R},R)} a(y,|\nabla u|)\, \frac{|\tau_R(y)|^2\, |\nabla_x u|^2}{y^2} \leqslant C_2 \int_{\Omega\times(\sqrt{R},R)} a(y,|\nabla u|)\, \frac{|\nabla_x u|^2}{y^2},
\end{aligned}$$

for some $C_1$, $C_2 > 0$. So, by using Lemma 2.4 with $h := a(y,|\nabla u|)\, |\nabla_x u|^2$ and recalling (1.12), we conclude that

$$\tag{4.7}
\begin{aligned}
&\int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\psi_R,\, \nabla\psi_R\rangle\, |\nabla_x u|^2 \\
&\leqslant C_3 \int_{\sqrt{R}}^{R} \left[ \int_{\Omega\times(0,t)} a(y,|\nabla u|)\, |\nabla_x u|^2 \right] \frac{dt}{t^3} + \frac{C_3}{R^2} \int_{\Omega\times(0,R)} a(y,|\nabla u|)\, |\nabla_x u|^2 \\
&\leqslant C_4 \int_{\sqrt{R}}^{R} \frac{dt}{t} + C_4 \leqslant C_5 \log\sqrt{R},
\end{aligned}$$

provided that $R$ is large enough, for some constants $C_3$, $C_4$, $C_5 > 0$. Thus we insert (4.5), (4.6) and (4.7) into (4.4), we divide by $\left(\log \frac{\sqrt{R}}{2}\right)^2$ and we deduce

$$\begin{aligned}
&\int_{\Omega\times(0,\sqrt{R})} \left[ \sum_{j=1}^{n} \langle \mathcal{B}(y,\nabla u)\, \nabla u_{x_j}, \nabla u_{x_j} \rangle - \langle \mathcal{B}(y,\nabla u)\nabla|\nabla_x u|,\, \nabla|\nabla_x u|\rangle \right] \\
&\qquad - \int_{\partial_L \mathcal{C} \cap \{y \in (0,\sqrt{R})\}} a(y,|\nabla u|)\, \big(\nabla u \cdot \partial_\nu(\nabla u)\big) \leqslant \frac{C_6 \log R}{\left(\log \frac{\sqrt{R}}{2}\right)^2}.
\end{aligned}$$

Since the latter term is infinitesimal as $R \to +\infty$, the previous estimate implies

$$\int_{\mathcal{C}} \left[ \sum_{j=1}^{n} \langle \mathcal{B}(y, \nabla u)\, \nabla u_{x_j}, \nabla u_{x_j} \rangle - \langle \mathcal{B}(y, \nabla u) \nabla |\nabla_x u|,\, \nabla |\nabla_x u| \rangle \right] - \int_{\partial_L \mathcal{C}} a(y, |\nabla u|) \big( \nabla u \cdot \partial_\nu (\nabla u) \big) \leqslant 0$$

and as a consequence

$$\left[ \sum_{j=1}^{n} \langle \mathcal{B}(y, \nabla u)\, \nabla u_{x_j}, \nabla u_{x_j} \rangle - \langle \mathcal{B}(y, \nabla u) \nabla |\nabla_x u|,\, \nabla |\nabla_x u| \rangle \right] = 0 \text{ in } \mathcal{C} \tag{4.8}$$

and

$$a(y, |\nabla u|) \big( \nabla u \cdot \partial_\nu (\nabla u) \big) = 0 \text{ on } \partial_L \mathcal{C},$$

thanks to (2.2) and (2.9). This establishes (4.2) (recall also (1.2)).

Also, by (4.8) and Corollary 2.6, we obtain that (4.1) holds.

In addition Corollary 2.6 gives that, if $\{\nabla_x u = 0\} = \varnothing$ then the function $\omega(\cdot)$ is constant, as desired.

Now we suppose that $\partial\Omega$ has positive principal curvatures and $n \geqslant 2$. We claim that

$$u \text{ is constant along } \partial\Omega \times \{\bar{y}\}, \tag{4.9}$$

for any fixed $\bar{y} > 0$. To prove it assume by contradiction that $u(p, \bar{y}) \neq u(q, \bar{y})$ for some $p$, $q \in \partial\Omega$. By Lemma 2.2, we know that we can connect $p$ to $q$ with a continuous path $\sigma : [0, 1] \to \partial\Omega$. Let $\zeta(t) := u(\sigma(t), \bar{y})$. Then $\zeta(0) \neq \zeta(1)$, and therefore there exists $\bar{t} \in (0, 1)$ such that $\dot{\zeta}(\bar{t}) \neq 0$. That is

$$0 \neq \dot{\zeta}(\bar{t}) = \nabla_x u(\sigma(\bar{t}), \bar{y}) \cdot \dot{\sigma}(\bar{t}). \tag{4.10}$$

We let $\bar{x} := \sigma(\bar{t})$. Up to a change of coordinates, we may suppose that the exterior normal of $\partial\Omega$ at $\bar{x}$ coincides with $-e_n$, hence, near $\bar{x}$ the domain $\Omega$ can be written in normal coordinates as the epigraph of a function $\gamma \in C^2(\mathbb{R}^{n-1})$. The fact that the principal curvatures of $\partial\Omega$ are positive implies that

$$\text{the Hessian of } \gamma \text{ is positive definite.} \tag{4.11}$$

On the other hand, by (4.2) and (2.1), we have that

$$0 = -\nabla u(\bar{x}, \bar{y}) \cdot \partial_\nu \left( \nabla u(\bar{x}, \bar{y}) \right) = \sum_{i,j=1}^{n-1} \gamma_{x_i x_j}(\bar{x}')\, u_{x_i}(\bar{x}, \bar{y})\, u_{x_j}(\bar{x}, \bar{y}).$$

This and (4.11) give that $u_{x_i}(\bar{x}, \bar{y}) = 0$ for any $i = 1, \ldots, n-1$. By the Neumann condition and the choice of the coordinate system, we also know that $u_{x_n} = -\partial_\nu u = 0$ in $(\bar{x}, \bar{y})$. Hence $\nabla_x u(\bar{x}, \bar{y}) = 0$, in contradiction with (4.10); this proves (4.9).

Now we show that

$$u \text{ is constant in } \Omega \times \{\bar{y}\}, \tag{4.12}$$

for any fixed $\bar{y} > 0$. For this, we let $c(\bar{y})$ be the value attained by $u$ along $\partial\Omega \times \{\bar{y}\}$, as given by (4.9). We also let $\omega(\bar{y})$ as in (4.9) and we take a vector $\varpi(\bar{y})$ orthogonal to $\omega(\bar{y})$.

Then we take a point $x_0 \in \Omega$. We consider the straight line

$$\{x_0 + \varpi(\bar{y})\, t,\ t \in \mathbb{R}\}.$$

Since the domain $\Omega$ is bounded, such a line must intersect somewhere the boundary of $\Omega$, i.e. there exists $t_0$ such that $x_0 + \varpi(\bar{y})\, t_0 \in \partial\Omega$. Therefore, by (4.9),

$$u\big(x_0 + \varpi(\bar{y})\, t_0,\, \bar{y}\big) = c(\bar{y}). \tag{4.13}$$

On the other hand, by (4.1),

$$u\big(x_0 + \varpi(\bar{y})\, t_0, \bar{y}\big) = u_0\Big(\omega(\bar{y}) \cdot \big(x_0 + \varpi(\bar{y})\, t_0\big), \bar{y}\Big) = u_0\big(\omega(\bar{y}) \cdot x_0, \bar{y}\big) = u(x_0, \bar{y}).$$

This and (4.13) give that

$$u(x_0, \bar{y}) = c(\bar{y}).$$

Since this holds for any point $x_0 \in \Omega$, we have established (4.12), and thus completed the proof of Theorem 1.2.

## 5. Classification in convex domains II: proof of Theorem 1.3

Now we address the proof of Theorem 1.3. For this goal, we need a detailed study of functions which attain the minimum of the stability functional $I$ introduced in (1.10). After this, we will use the geometric observations exposed in Section 2 and suitable test functions to complete the proof of Theorem 1.3.

5.1. **Rigidity of minimal solutions.** In this part, we study the rigidity properties of the minimizers of the stability functional $I$, as defined in (1.10). To this aim, we introduce

$$\mathcal{A}^* := \left\{ \varphi \in W^{1,1}_{\rm loc}(\mathcal{C}) \;\middle|\; \begin{array}{l} a(y, |\nabla u|)\left(\varphi^2 + |\nabla\varphi|^2\right) + |g_u(y,u)|\varphi^2 \in L^1(\mathcal{C}) \\ \text{and } \varphi|_{\Omega\times\{0\}} \in L^2(\Omega) \end{array} \right\}. \tag{5.1}$$

With respect to the space $\mathcal{A}$ defined in (1.8), we replace the requirement that $\varphi$ has bounded support in $y$ with an integrability condition. From now on, we assume that $u$ is a stable solution of (1.1), according to (1.10).



**Lemma 5.1.** *Let $I$ be defined by* (1.10). *Then $I(\varphi) \geqslant 0$ for every $\varphi \in \mathcal{A}^*$.*

*Proof.* We proceed by approximation in the following way: let $\tau_R$ be a smooth function such that

$$\tau_R(t) = \begin{cases} 1, & \text{if } 0 < t \leqslant R, \\ 0, & \text{if } t \geqslant 2R, \end{cases}$$

and $|\tau_R'| \leqslant C/R$ for some $C > 0$. Given $\varphi \in \mathcal{A}^*$, the function $\varphi_R(x,y) := \varphi(x,y)\, \tau_R(y)$ belongs to $\mathcal{A}$, and hence can be used as a test function in the stability assumption (1.10). Therefore

$$I(\varphi_R) \geqslant 0. \tag{5.2}$$

In order to obtain the desired result, we aim at passing to the limit as $R \to +\infty$. The details go as follows. First of all, we recall (1.9) and (1.3), to point out that

$$\big|\mathcal{B}(y,\eta)\big| \leqslant C_1\Big(a(y,|\eta|) + a_t(y,|\eta|)\,|\eta|\Big) \leqslant C_2\, a(y,|\eta|),$$

for some $C_1$, $C_2 > 0$. As a consequence

$$\lim_{R\to+\infty} \frac{1}{R^2} \int_{\Omega\times(R,2R)} \big|\mathcal{B}(y,\nabla u)\big|\, \varphi^2 \leqslant \lim_{R\to+\infty} \frac{C_2}{R^2} \int_{\mathcal{C}} a(y,|\nabla u|)\, \varphi^2 = 0, \tag{5.3}$$

where we used (5.1).

Now, from (5.2) we infer that

$$
\begin{aligned}
0 \leqslant & \int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\varphi,\nabla\varphi\rangle\,\tau_R^2 + \int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\tau_R,\nabla\tau_R\rangle\,\varphi^2 \\
& + 2\int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\varphi,\nabla\tau_R\rangle\,\varphi\,\tau_R + g_u(y,u)\varphi^2\tau_R^2 - \int_{\partial_B\mathcal{C}} f'(u)\varphi^2.
\end{aligned} \tag{5.4}
$$

Let $\epsilon > 0$ (to be taken arbitrarily small in the sequel); we use a weighted Hölder inequality to observe that

$$
\begin{aligned}
& \int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\varphi,\nabla\tau_R\rangle\,\varphi\,\tau_R \\
& \qquad \leqslant \epsilon \int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\varphi,\nabla\varphi\rangle\,\tau_R^2 + C_\epsilon \int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\tau_R,\nabla\tau_R\rangle\,\varphi^2
\end{aligned}
$$

for some $C_\epsilon > 0$. Plugging this into (5.4), we obtain

$$
\begin{aligned}
0 \quad \leqslant \quad & (1+\epsilon)\int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\varphi,\nabla\varphi\rangle\tau_R^2 + \frac{1+C_\epsilon}{R^2}\int_{\Omega\times(R,2R)} \big|\mathcal{B}(y,\nabla u)\big|\,\varphi^2 \\
& + \int_{\mathcal{C}} g_u(y,u)\varphi^2\tau_R^2 - \int_{\partial_B\mathcal{C}} f'(u)\varphi^2.
\end{aligned}
$$

Recalling (5.3) and the definition of $\mathcal{A}^*$, we can pass to the limit as $R \to +\infty$, concluding that

$$
0 \leqslant (1+\epsilon)\int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\varphi,\nabla\varphi\rangle + \int_{\mathcal{C}} g_u(y,u)\varphi^2 - \int_{\partial_B\mathcal{C}} f'(u)\varphi^2.
$$

Since $\epsilon > 0$ has been arbitrarily chosen, the thesis follows. □

In light of the previous result, we write that $\varphi \in \mathcal{A}^*$ is a *minimizer* for $I$ if $I(\varphi) \leqslant 0$ (equivalently, by Lemma 5.1, if $I(\varphi) = 0$). First, we show that minimizers satisfy a suitable reaction-diffusion equation, both in the weak and in the strong sense:

**Lemma 5.2.** *Assume that $I(\varphi) \leqslant 0$, for some $\varphi \in \mathcal{A}^*$. Then*

$$
\int_{\mathcal{C}} \langle \mathcal{B}(y,\nabla u)\nabla\varphi,\nabla\zeta\rangle + \int_{\mathcal{C}} g_u(y,u)\varphi\zeta - \int_{\partial_B\mathcal{C}} f'(u)\varphi\zeta = 0,
$$

*for any $\zeta \in \mathcal{A}^*$.*

*Proof.* For any $\epsilon \in \mathbb{R}$ and any test function $\zeta$, we have that $I(\varphi+\epsilon\zeta) \geqslant 0 \geqslant I(\varphi)$, and therefore, dividing by $\epsilon$ and sending $\epsilon \to 0$, we obtain the desired result. □

**Lemma 5.3.** *Assume also that $I(\varphi) \leqslant 0$, for some $\varphi \in \mathcal{A}^* \cap C^2(\mathcal{C}) \cap C^1(\overline{\Omega} \times [\alpha,\beta])$, for any $\beta > \alpha > 0$ Then, $\varphi$ is a solution of*

$$
\sum_{i,j=1}^{n+1} \mathcal{B}_{ij}(y,\nabla u)\partial^2_{X_iX_j}\varphi + \partial_{X_i}\mathcal{B}_{ij}(y,\nabla u)\partial_{X_j}\varphi - g_u(y,u)\varphi = 0 \text{ in } \mathcal{C}, \tag{5.5}
$$

$$
\text{with } \partial_\nu\varphi = 0 \text{ on } \partial_L\mathcal{C}. \tag{5.6}
$$

*Proof.* We use Lemma 5.2. Indeed, by taking $\zeta$ supported inside $\mathcal{C}$, we obtain (5.5). By taking $\zeta$ supported near any given point of $\partial_L\mathcal{C}$, we conclude that also (5.6) holds. □

With this, we are in the position of obtaining a strict sign for nonnegative minimizers, up to the boundary, in the spirit of a strict comparison principle, according to the following result:

**Corollary 5.4.** *Assume that* $I(\varphi) \leqslant 0$ *for* $\varphi \in \mathcal{A}^* \cap C^2(\mathcal{C}) \cap C^1(\overline{\Omega} \times [\alpha, \beta])$*, for any* $\beta > \alpha > 0$*. Assume in addition that* $\varphi(x, y) \geqslant 0$ *for any* $(x, y) \in \mathcal{C}$*.*

*Then either* $\varphi(x, y) > 0$ *for any* $x \in \overline{\Omega}$ *and* $y > 0$*, or* $\varphi(x, y) = 0$ *for any* $x \in \Omega$ *and any* $y > 0$*.*

*Proof.* Suppose that $\varphi(x_o, y_o) = 0$ for some $x_o \in \partial\Omega$ and $y_o > 0$. Then, we look at the equation satisfied by $\varphi$ in $\Omega_{\alpha,\beta}$, where $\Omega_{\alpha,\beta}$ is a smooth domain that contains $\Omega \times (\alpha, \beta)$ and is contained in $\Omega \times (\alpha/2, \beta)$ with $0 < \alpha < y_o < \beta$. Indeed, we define

$$\begin{aligned}
M &:= \sup_{(x,y)\in\Omega_{\alpha,\beta}} \big|g_u(y, u(x, y))\big| < +\infty,\\
a_{ij}(x, y) &:= \mathcal{B}_{ij}\big(y, \nabla u(x, y)\big),\\
b_j(x, y) &:= \sum_{i=1}^{n+1} \partial_{X_i} \mathcal{B}_{ij}\big(y, \nabla u(x, y)\big)\\
c(x, y) &:= -M - g_u(y, u(x, y)).
\end{aligned}$$

Notice that $c \leqslant 0$ and $a_{ij}$ defines an elliptic matrix on $\Omega \times (\alpha, \beta)$, for fixed $\alpha$ and $\beta$, thanks to Lemma 2.3. Moreover, by (5.5)

$$\sum_{i,j=1}^{n+1} a_{ij} \partial^2_{X_i X_j} \varphi + \sum_{j=1}^{n+1} b_j \partial_{X_j} \varphi + c\varphi = -M\varphi \leqslant 0.$$

Notice that $a_{ij}$, $b_j$, $c \in C(\overline{\Omega}_{\alpha,\beta})$, thanks to (1.5). Also, we have that $\varphi$ attains its minimum in $\overline{\Omega}_{\alpha,\beta}$ at $(x_o, y_o)$. As a consequence, by the Hopf Lemma (see e.g. Corollary 1.6 in Chapter 2 of [16]), either $\varphi$ vanishes identically in $\Omega_{\alpha,\beta}$ or $0 \neq \partial_\nu \varphi(x_o, y_o)$. The latter possibility cannot hold, in light of (5.6), and therefore $\varphi$ must, in this case, vanish identically in the domain $\Omega_{\alpha,\beta}$. Since $\alpha$ can be taken as close to 0 as we wish and $\beta$ can be taken arbitrarily large, this implies that $\varphi$ must vanish everywhere in $\Omega \times (0, +\infty)$. $\square$

As a matter of fact, we can strengthen Corollary 5.4 by removing the sign assumption on $\varphi$. Namely, we have that:

**Proposition 5.5.** *Assume that* $I(\varphi) \leqslant 0$ *for* $\varphi \in \mathcal{A}^* \cap C^2(\mathcal{C}) \cap C^1(\overline{\Omega} \times [\alpha, \beta])$*, for any* $\beta > \alpha > 0$*. Then, one and only one of these three possibilities holds true:*

- $\varphi(x, y) > 0$ *for any* $x \in \overline{\Omega}$ *and* $y > 0$*,*
- $\varphi(x, y) < 0$ *for any* $x \in \overline{\Omega}$ *and* $y > 0$*,*
- $\varphi(x, y) = 0$ *for any* $x \in \Omega$ *and any* $y > 0$*.*

*Proof.* We claim that

$$\text{either } \varphi \geqslant 0 \text{ or } \varphi \leqslant 0 \text{ in } \mathcal{C}. \tag{5.7}$$

To prove this, we consider $\varphi^+ := \max\{\varphi, 0\}$ and $\varphi^- := \max\{-\varphi, 0\}$. We observe that $\varphi^\pm \in \mathcal{A}^*$. Thus, by Lemma 5.2,

$$\begin{aligned} I(\varphi^\pm) &= \int_{\mathcal{C}} \langle \mathcal{B}(y, \nabla u)\, \nabla\varphi^\pm,\, \nabla\varphi^\pm\rangle + \int_{\mathcal{C}} g_u(y,u)\,(\varphi^\pm)^2 - \int_{\partial_B \mathcal{C}} f'(u)\,(\varphi^\pm)^2 \\ &= \int_{\mathcal{C}} \langle \mathcal{B}(y, \nabla u)\, \nabla\varphi,\, \nabla\varphi^\pm\rangle + \int_{\mathcal{C}} g_u(y,u)\,\varphi\,\varphi^\pm - \int_{\partial_B \mathcal{C}} f'(u)\,\varphi\,\varphi^\pm = 0. \end{aligned}$$

Hence, using again Lemma 5.2, we have that, for any $\zeta \in C_0^\infty(\mathcal{C})$,

$$\int_{\mathcal{C}} \langle \mathcal{B}(y, \nabla u)\, \nabla\varphi^\pm,\, \nabla\zeta\rangle + \int_{\mathcal{C}} g_u(y,u)\,\varphi^\pm\,\zeta = 0.$$

Therefore, the function $\varphi^\pm$ is a weak solution to

$$\operatorname{div}\big(\mathcal{B}(y,\nabla u)\nabla\varphi^\pm\big) = g_u(y,u)\varphi^\pm \qquad \text{in } \mathcal{C},$$

according to the notation of Chapter 8 in [14] (in particular, formula (8.5) there is a consequence of Lemma 2.3 here and formula (8.6) applies here with $b^i := c^i := 0$ and $d := g_u(\cdot, u)$). Consequently, by Theorem 8.20 in [14], for any point $X_o \in \mathcal{C}$ and any $R > 0$ such that $B_{4R}(X_o) \subset \mathcal{C}$,

$$\sup_{B_R(X_o)} \varphi^\pm \leqslant C_R \inf_{B_R(X_o)} \varphi^\pm,$$

for some $C_R > 0$. This implies that if $\varphi^\pm$ vanishes somewhere in $\mathcal{C}$, then it must vanish identically in $\mathcal{C}$, thus completing the proof of (5.7).

Thanks to (5.7) we can now exploit Corollary 5.4 (applying this to $\varphi$ if $\varphi \geqslant 0$ or to $-\varphi$ if $\varphi \leqslant 0$). From this, we obtain the desired result. $\square$

With this preparatory work, we are now in the position of finishing the proof of Theorem 1.3:

*Completion of the proof of Theorem 1.3.* By (1.5) and the integrability assumption (1.13), we have that $u_{x_j} \in \mathcal{A}^*$ for every $j = 1, \dots, n$. Thus, by Lemma 5.1, we have that $I(u_{x_j}) \geqslant 0$. On the other hand, using Corollary 2.9 and Lemma 2.1,

$$\sum_{j=1}^n I(u_{x_j}) = \int_{\partial_L \mathcal{C}} a(y, |\nabla u|)\,\big(\nabla u \cdot \partial_\nu(\nabla u)\big) \leqslant 0,$$

so that necessarily $I(u_{x_j}) = 0$ for any $j = 1, \dots, n$. Therefore, by Proposition 5.5, we deduce that either $u_{x_j}$ never vanishes in $\overline{\Omega} \times (0, +\infty)$, or $u_{x_j}$ vanishes identically in $\mathcal{C}$. But the first possibility cannot occur: to see this, let us slide a hyperplane normal to $e_j$ till it touches $\partial\Omega$ at some point $x_j^\star$.

By construction, the normal of $\partial\Omega$ at $x_j^\star$ is $e_j$, hence the homogeneous Neumann condition $\partial_\nu u = 0$ on $\partial_L \mathcal{C}$ implies that $0 = \partial_\nu u(x_j^\star, 1) = u_{x_j}(x_j^\star, 1)$. This shows that the first above-mentioned possibility cannot occur, and as a consequence $u_{x_j}$ vanishes identically in $\mathcal{C}$. Since this is valid for any $j = 1, \dots, n$, this implies that $u$ does not depend on $x$. $\square$

We conclude this section by proving Proposition 1.4.

*Proof of Proposition 1.4.* Recall that $a$ does not depend on $t$ and $g \equiv 0$. We fix $R > 3$, to be taken as large as we wish at the end of this proof. We take $\phi \in C^\infty(\mathcal{C})$ such that $\phi(x,y)=1$ if $y\in[3,R]$, $\phi(x,y)=0$ if $y\in[0,2]\cup[R+1,+\infty)$ and $|\nabla\phi|\leqslant 2$. We test (1.7) against

$$\varphi := \phi^2\Delta_x u = \phi^2\,(u_{x_1x_1}+\cdots+u_{x_nx_n}),$$

we have that

$$\int_{\mathcal{C}} 2a(y)\,\phi\,\Delta_x u\,\nabla u\cdot\nabla\phi + \int_{\mathcal{C}} a(y)\phi^2\,\nabla u\cdot\nabla\Delta_x u = 0. \tag{5.8}$$

By the Divergence Theorem,

$$\begin{aligned} &\int_{\mathcal{C}} a(y)\,\phi^2\,\nabla u\cdot\nabla\Delta_x u \\ &\quad= \sum_{i=1}^{n+1}\int_{\mathcal{C}} \mathrm{div}_x\Big(a(y)\,\phi^2\,u_{X_i}\,\nabla_x u_{X_i}\Big) - \sum_{i=1}^{n+1}\int_{\mathcal{C}} a(y)\,\nabla_x\Big(\phi^2\,u_{X_i}\Big)\cdot\nabla_x u_{X_i} \\ &\quad= \sum_{i=1}^{n+1}\int_{\partial_L\mathcal{C}} a(y)\,\phi^2\,u_{X_i}\,\partial_\nu u_{X_i} \\ &\qquad - \sum_{i=1}^{n+1}\int_{\mathcal{C}} 2a(y)\,\phi u_{X_i}\nabla_x\phi\cdot\nabla_x u_{X_i} - \sum_{i=1}^{n+1}\int_{\mathcal{C}} a(y)\,\phi^2\,|\nabla_x u_{X_i}|^2. \end{aligned} \tag{5.9}$$

Also, using the convexity of $\Omega$ and (2.2), we have that

$$\sum_{i=1}^{n+1} a(y)\,\phi^2\,u_{X_i}\,\partial_\nu u_{X_i} = a(y)\,\phi^2\nabla u\cdot\partial_\nu(\nabla u)\leqslant 0.$$

Combining this with (5.9), we obtain that

$$\begin{aligned} &\int_{\mathcal{C}} a(y)\,\phi^2\,\nabla u\cdot\nabla\Delta_x u \\ &\qquad\leqslant -\sum_{i=1}^{n+1}\int_{\mathcal{C}} 2a(y)\,\phi u_{X_i}\nabla_x\phi\cdot\nabla_x u_{X_i} - \sum_{i=1}^{n+1}\int_{\mathcal{C}} a(y)\,\phi^2\,|\nabla_x u_{X_i}|^2. \end{aligned} \tag{5.10}$$

Notice that

$$a(y)\,|\phi|\,|u_{X_i}|\,|\nabla_x\phi|\,|\nabla_x u_{X_i}| \leqslant \frac{a(y)}{8}\phi^2|\nabla_x u_{X_i}|^2 + C\,a(y)\,|u_{X_i}|^2|\nabla_x\phi|^2$$

for some $C>0$. By plugging this into (5.10), we deduce that

$$\int_{\mathcal{C}} a(y)\,\phi^2\,\nabla u\cdot\nabla\Delta_x u \leqslant C\int_{\mathcal{C}} a(y)\,|\nabla u|^2|\nabla\phi|^2 - \frac{3}{4}\int_{\mathcal{C}} a(y)\,\phi^2\,|D^2_{X\,x}u|^2.$$

This and (5.8) imply

$$\int_{\mathcal{C}} 2a(y)\,\phi\,\Delta_x u\,\nabla u\cdot\nabla\phi + C\int_{\mathcal{C}} a(y)\,|\nabla u|^2|\nabla\phi|^2 - \frac{3}{4}\int_{\mathcal{C}} a(y)\,\phi^2\,|D^2_{X\,x}u|^2 \geqslant 0. \tag{5.11}$$

Furthermore

$$a(y)\,|\phi|\,|\Delta_x u|\,|\nabla u|\,|\nabla\phi| \leqslant \frac{a(y)}{8}\phi^2\,|D^2_{X\,x}u|^2 + C\,a(y)\,|\nabla u|^2\,|\nabla\phi|^2,$$

for some $C>0$, and so (5.11) becomes

$$C\int_{\mathcal{C}} a(y)\,|\nabla u|^2|\nabla\phi|^2 - \frac{1}{2}\int_{\mathcal{C}} a(y)\,\phi^2\,|D^2_{X\,x}u|^2 \geqslant 0,$$

up to renaming $C$. By our assumptions on $\phi$, we thus obtain that

$$\int_{\Omega\times[3,R]} a(y)\,|D^2_{X\,x}u|^2 \leqslant \int_{\mathcal{C}} a(y)\,\phi^2\,|D^2_{X\,x}u|^2 \leqslant C\int_{\mathcal{C}} a(y)\,|\nabla u|^2,$$

up to renaming $C$. We stress that $C$ is independent of $R$, thus, taking $R$ as large as we wish, we conclude that

$$\int_{\Omega\times[3,+\infty)} a(y)\,|D^2_{X\,x}u|^2 \leqslant C\int_{\mathcal{C}} a(y)\,|\nabla u|^2 < +\infty, \tag{5.12}$$

where the latter step follows from (1.14).

Also, the integrability condition in (1.5) gives that

$$\int_{\Omega\times(0,3]} a(y)\,|D^2_{X\,x}u|^2 < +\infty.$$

By combining this information with (5.12), we obtain that

$$\int_{\mathcal{C}} a(y)\,|D^2_{X\,x}u|^2 < +\infty,$$

as desired. □

## 6. Classification of stable solutions for convex/concave nonlinearities: proof of Theorems 1.5

We now address the case in which $f$ satisfies suitable convexity or concavity assumption. In this setting, we need some preliminary work in order to detect the sign of the nonlinearity at the maximum or at the minimum. We stress that, in this section, we always suppose that $u$ is a bounded stable solution to (1.1), and that assumptions (1.16) and (1.17) are in force.

### 6.1. **Detecting the sign of the nonlinearity.**

A classical tool in partial differential equations is the use of various forms of maximum and comparison principles in order to check the sign of the nonlinearities at the points in which solutions of elliptic equations attain their extremal values. In our setting, we adapt these type of strategies, with the aim of detecting the sign of $f$ at the extremal values for solutions of the reaction-diffusion equation (1.1). This goal will be accomplished in Corollary 6.2. For this, we need an auxiliary result, which locates the extremal values at the bottom boundary of the domain.

**Lemma 6.1.** *Let $v(x) := u(x,0)$. Then*

$$\min_{x\in\overline{\Omega}} v(x) = \inf_{(x,y)\in\mathcal{C}} u(x,y).$$

*Proof.* We argue by contradiction and we suppose that

$$b_+ := \min_{x\in\overline{\Omega}} v(x) > \inf_{(x,y)\in\mathcal{C}} u(x,y) = b_-.$$

As a consequence, there exists

$$\ell \in (b_-,\, b_+)\,. \tag{6.1}$$

We define $w(x,y) := \big(\ell - u(x,y)\big)^+$, and we observe that

$$0 \leqslant w \leqslant \ell - \inf_{(x,y)\in\mathcal{C}} u(x,y) < b_+ - b_-,$$

so that $w \in L^\infty(\mathcal{C})$. We claim that

$$\overline{\{u<\ell\}} \subset \overline{\Omega}\times(0,+\infty) \quad \Longrightarrow \quad w=0 \text{ on } \partial_B \mathcal{C}. \tag{6.2}$$

To prove this, suppose by contradiction, that there exists a sequence $(x_k, y_k) \in \{u<\ell\}$ with $y_k \to 0$ as $k \to +\infty$. Then, up to subsequence, we have that $x_k \to x_o$, for some $x_o \in \overline{\Omega}$, and by continuity (recall (1.5))

$$\ell \geqslant \lim_{k\to+\infty} u(x_k, y_k) = u(x_o, 0) = v(x_o) \geqslant \min_{x\in\overline{\Omega}} v(x) = b_+.$$

This is in contradiction with (6.1), and hence it proves (6.2).

Now we take a smooth function $\tau_R : [0,+\infty) \to [0,1]$ such that $\tau_R = 1$ in $[0,R]$, $\tau_R = 0$ in $[2R,+\infty)$ and $|\tau_R'| \leqslant 10/R$. We use (1.7) with $\varphi := w\tau_R$ (notice that such function lies in $\mathcal{A}$, thanks to (1.5)): in this way, and recalling (1.16) and (6.2), we obtain that

$$\begin{aligned} 0 &= \int_{\mathcal{C}} a(y,|\nabla u|)\,\nabla u\cdot\nabla w\,\tau_R + \int_{\mathcal{C}} a(y,|\nabla u|)\,\nabla u\cdot\nabla\tau_R\, w \\ &= -\int_{\mathcal{C}} a(y,|\nabla u|)\,|\nabla w|^2\,\tau_R + \int_{\mathcal{C}} a(y,|\nabla u|)\,\nabla u\cdot\nabla\tau_R\, w. \end{aligned} \tag{6.3}$$

Now we use the positivity of $a$ and the boundedness of $w$ to see that

$$\begin{aligned} &\left|\int_{\mathcal{C}} a(y,|\nabla u|)\,\nabla u\cdot\nabla\tau_R\, w\right| \\ &\qquad\leqslant \|w\|_{L^\infty(\mathcal{C})}\sqrt{\int_{\mathcal{C}} a(y,|\nabla u|)\,|\nabla u|^2}\,\sqrt{\int_{\mathcal{C}} a(y,|\nabla u|)\,|\nabla\tau_R|^2} \\ &\qquad\leqslant \|w\|_{L^\infty(\mathcal{C})}\sqrt{\int_{\mathcal{C}} a(y,|\nabla u|)\,|\nabla u|^2}\,\sqrt{\frac{C}{R^2}\int_{\Omega\times(R,2R)} a(y,|\nabla u|)}, \end{aligned}$$

and this quantity is infinitesimal as $R \to +\infty$, thanks to (1.17). Using this and the Dominated Convergence Theorem, we can pass to the limit into formula (6.3), obtaining

$$0 \leqslant -\int_{\mathcal{C}} a(y,|\nabla u|)\,|\nabla w|^2.$$

This implies that $\nabla w$ vanishes identically, and so $w$ is constant. Thus, recalling (6.2), we conclude that $w$ vanishes identically, and therefore $u(x,y) \geqslant \ell$ for any $(x,y) \in \mathcal{C}$. Accordingly, we have that $b_- \geqslant \ell$, in contradiction with (6.1). □

**Corollary 6.2.** *Under the previous notation, let*

$$c := \min_{x\in\overline{\Omega}} v(x) = \inf_{(x,y)\in\mathcal{C}} u(x,y).$$

*Then $f(c) \leqslant 0$.*

*Proof.* Let $x_\star \in \overline{\Omega}$ be a minimum point for $v$, and let us assume by contradiction that $f(c) > 0$. By continuity (see (1.17)), there exists $y_\star > 0$ such that

$$a\big(y,|\nabla u(x_\star,y)|\big)\,\partial_y u(x_\star,y) \leqslant -\frac{f(c)}{2} \qquad \text{for all } y\in(0,y_\star].$$

Since $a(y,t) > 0$ for any $y > 0$ and $t \geqslant 0$, for every $y \in (0, y_\star]$

$$\partial_y u(x_\star,y) \leqslant -\frac{f(c)}{2}\,\frac{1}{a\big(y,|\nabla u(x_\star,y)|\big)}.$$

Hence, for any $y_o \in \big(0, \frac{y_\star}{2}\big]$

$$
\begin{aligned}
u(x_\star, y_\star) - u(x_\star, y_o) &= \int_{y_o}^{y_\star} \partial_y u(x_\star, y)\, dy \\
&\leqslant -\frac{f(c)}{2} \int_{y_o}^{y_\star} \frac{dy}{a\big(y, |\nabla u(x_\star, y)|\big)} \leqslant -\frac{f(c)}{2} \int_{y_\star/2}^{y_\star} \frac{dy}{a\big(y, |\nabla u(x_\star, y)|\big)} =: -b_\star,
\end{aligned}
$$

and $b_\star > 0$. Taking the limit as $y_o \to 0$, we infer that

$$
\inf_{(x,y)\in\mathcal{C}} u(x, y) - \min_{x\in\overline{\Omega}} v(x) \leqslant u(x_\star, y_\star) - v(x_\star) = u(x_\star, y_\star) - u(x_\star, 0) \leqslant -b_\star,
$$

in contradiction with Lemma 6.1. □

6.2. **Classification of stable solutions in the case of convex/concave nonlinearities and end of the proof of Theorem 1.5.** With the previous preliminary work, we are now in the position of completing the proof of Theorem 1.5. The proof will borrow an idea of [7], that is to test the equation against a vertical translation of the solution (in this way, comparing the equation with the linearized equation, the nonlinearity is compared with its derivative, hence convexity comes naturally into play).



*Completion of the proof of Theorem 1.5.* We suppose that $f$ is convex.

For any $\varphi \in \mathcal{A}$, we define

$$
J(\varphi) := -\int_{\mathcal{C}} \frac{a_t(y, |\nabla u|)}{|\nabla u|} \big(\nabla u \cdot \nabla\varphi\big)^2.
$$

By (1.9),

$$
\langle \mathcal{B}(y, \nabla u)\nabla\varphi, \nabla\varphi\rangle = a(y, |\nabla u|)\, |\nabla\varphi|^2 + \frac{a_t(y, |\nabla u|)}{|\nabla u|} \big(\nabla u \cdot \nabla\varphi\big)^2 \tag{6.4}
$$

Accordingly, and being $g \equiv 0$, the stability of $u$ implies that

$$
\int_{\mathcal{C}} a(y, |\nabla u|)|\nabla\varphi|^2 - J(\varphi) - \int_{\partial_B\mathcal{C}} f'(u)\varphi^2 = I(\varphi) \geqslant 0, \tag{6.5}
$$

for any $\varphi \in \mathcal{A}$. Let $c := \inf_{\mathcal{C}} u$, and let us consider a smooth function $\tau_R : [0, +\infty) \to [0, 1]$ such that $\tau_R = 1$ in $[0, R]$, $\tau_R = 0$ in $[2R, +\infty)$ and $|\tau_R'| \leqslant 10/R$. We exploit first (6.5) with $\varphi(x, y) := \big(u(x, y) - c\big)\, \tau_R(y)$, and we obtain that

(6.6)

$$
\begin{aligned}
0 \leqslant &\int_{\mathcal{C}} a(y, |\nabla u|)\, |\nabla u|^2\, \tau_R^2 + \int_{\mathcal{C}} a(y, |\nabla u|)\, |\nabla\tau_R|^2\, (u - c)^2 \\
&+ 2\int_{\mathcal{C}} a(y, |\nabla u|)\, (u - c)\, \tau_R \nabla u \cdot \nabla\tau_R - J\big((u - c)\, \tau_R\big) - \int_{\partial_B\mathcal{C}} f'(u)(u - c)^2.
\end{aligned}
$$

Now we use[1] (1.7) with $\varphi := \big(u(x, y) - c\big)\, \tau_R^2(y)$:

$$
\begin{aligned}
0 = &\int_{\mathcal{C}} a(y, |\nabla u|)\, |\nabla u|^2\, \tau_R^2 + 2\int_{\mathcal{C}} a(y, |\nabla u|)\, (u - c)\, \tau_R \nabla u \cdot \nabla\tau_R \\
&- \int_{\partial_B\mathcal{C}} f(u)\, (u - c).
\end{aligned} \tag{6.7}
$$

[1]As a curiosity, we stress that the choice of the test function exploited to obtain (6.6) is not the same as the one exploited to obtain (6.7).

Subtracting (6.7) from (6.6), we obtain

$$\begin{aligned} 0 \leqslant \int_{\mathcal{C}} a(y,|\nabla u|)\, |\nabla \tau_R|^2\, (u-c)^2 - J\big((u-c)\,\tau_R\big) \\ - \int_{\partial_B \mathcal{C}} \big(f'(u)(c-u)+f(u)\big)\,(c-u). \end{aligned} \tag{6.8}$$

Now we use the convexity of $f$ to see that

$$f(u)+f'(u)\,(c-u)\leqslant f(c).$$

Since $f$ is strictly convex, the inequality is strict provided $\{u\neq c\}\neq\varnothing$. Moreover, by Corollary 6.2, we know that $u\geqslant c$, therefore

$$\big(f(u)+f'(u)\,(c-u)\big)\,(c-u)\geqslant f(c)\,(c-u), \tag{6.9}$$

with strict inequality if $\{u\neq c\}\neq\varnothing$.

We also observe that, by (1.16), $-J((u-c)\tau_R)\leqslant 0$. Plugging this and (6.9) into (6.8), we conclude that

$$0\leqslant \int_{\mathcal{C}} a(y,|\nabla u|)\,|\nabla\tau_R|^2\,(u-c)^2 - \int_{\partial_B\mathcal{C}} f(c)\,(c-u), \tag{6.10}$$

with strict inequality if $\partial_B\mathcal{C}\cap\{u\neq c\}\neq\varnothing$. The previous inequality is satisfied for every $R>0$. Passing to the limit as $R\to+\infty$, we have

$$0\leqslant \lim_{R\to+\infty}\int_{\mathcal{C}} a(y,|\nabla u|)\,|\nabla\tau_R|^2\,(u-c)^2 \leqslant \lim_{R\to+\infty}\frac{C}{R^2}\int_{\Omega\times(R,2R)} a(y,|\nabla u|)=0, \tag{6.11}$$

thanks to (1.17), the boundedness of $u$, and our choice of $\tau_R$. Coming back to (6.10), this gives

$$0\leqslant \int_{\partial_B\mathcal{C}} f(c)\,(u-c)$$

with strict inequality if $\{u\neq c\}\neq\varnothing$. But recalling that $f(c)\leqslant 0$ and $u\geqslant c$ in $\mathcal{C}$, thanks to Corollary 6.2, we have that the right hand side is nonpositive, so that the previous inequality cannot be strict; hence $\{u\neq c\}\cap\partial_B C=\varnothing$, i.e. $u$ is constant on $\partial_B\mathcal{C}$.

To complete the proof, we come back to (6.7). Using the fact that $u=c$ on $\partial_B\mathcal{C}$, the last integral there vanishes, so that

$$0=\int_{\mathcal{C}} a(y,|\nabla u|)\,|\nabla u|^2\,\tau_R^2+2\int_{\mathcal{C}} a(y,|\nabla u|)\,(u-c)\,\tau_R\nabla u\cdot\nabla\tau_R \tag{6.12}$$

for every $R>0$. Moreover

$$\begin{aligned} \left|\int_{\mathcal{C}} a(y,|\nabla u|)(u-c)\tau_R\nabla u\cdot\nabla\tau_R\right| &= \left|\int_{\Omega\times(R,2R)} a(y,|\nabla u|)(u-c)\tau_R\nabla u\cdot\nabla\tau_R\right| \\ &\leqslant \frac12\int_{\Omega\times(R,2R)} a(y,|\nabla u|)|\nabla u|^2+\frac12\int_{\Omega\times(R,2R)} a(y,|\nabla u|)(u-c)^2|\nabla\tau_R|^2\to 0 \end{aligned}$$

as $R\to+\infty$, where we used the first integrability assumption in (1.17) and (6.11). Therefore, passing to the limit as $R\to+\infty$ into (6.12), we deduce by the Monotone Convergence Theorem that

$$\int_{\mathcal{C}} a(y,|\nabla u|)\,|\nabla u|^2=0,$$

i.e. $u$ is constant in the whole $\mathcal{C}$. $\square$

## 7. Application to nonlocal problems with Neumann boundary conditions

*Proof of Theorems 1.6 and 1.7.* Let $v \in H^{1/2}(\Omega) \cap L^\infty(\Omega)$ be a solution to (1.19). Then, by [29], $v$ is the trace on $\Omega \times \{0\}$ of a weak solution $u \in \mathcal{H}(\mathcal{C})$ to (1.20):

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{C} \\ \partial_\nu u = 0 & \text{on } \partial_L \mathcal{C} \\ -\partial_y u = f(u) & \text{on } \partial_B \mathcal{C}. \end{cases}$$

By Proposition 1.4, any such solution is such that

$$|D_x^2 u|^2 + |\nabla_x u_y|^2 \in L^1(\mathcal{C}),$$

so that assumption (1.13) is satisfied. Moreover, arguing as in Theorem 3.5-part 4 in [29] (recalling that $f \in C^{2,\alpha}(\mathbb{R})$ and $\Omega$ is of class $C^{4,\alpha}$, and using higher order Schauder estimates), one sees that $u \in C^{3,\alpha}(\overline{\mathcal{C}})$. In particular, we obtain that $u$ is a classical, stable solution of (1.20), and satisfies all the assumptions in (1.5).

As a consequence, if $\Omega$ is convex we are in position to apply Theorem 1.3, deducing that $u$ depends only on $y$. Therefore, $v = u(\cdot, 0)$ has to be constant.

On the other hand, if $f$ is either strictly convex, or strictly concave, Theorem 1.5 implies in the same way that $v$ is constant. $\square$

## 8. A counterexample

Now we prove the statement given in Example 1.8

*Proof of Example 1.8.* We let $\epsilon \in (0,1)$ and $h_* \in C^2(\mathbb{R})$ be such that $h_*(x) = h(x)$ for any $x \in [-1,1]$, $h_*(x) = 2x$ if $|x| \in \left[1+\frac{\epsilon}{11}, 1+\frac{2\epsilon}{11}\right]$, and $h_*(x) = -2x$ if $|x| \in \left[1+\frac{3\epsilon}{11}, 1+\frac{4\epsilon}{11}\right]$.

Then, by Theorem 1.1 in [9], there exists $v \in C^2((-2,2)) \cap C(\mathbb{R})$ which is compactly supported, such that $(-\Delta)^s v = 0$ in $(-2,2)$ and $\|v - h_*\|_{C^2((-2,2))} \leqslant \epsilon$. In particular,

$$\|v - h\|_{C^2((-1,1))} = \|v - h_*\|_{C^2((-1,1))} \leqslant \epsilon.$$

Moreover, if $x \in \left[1+\frac{\epsilon}{11}, 1+\frac{2\epsilon}{11}\right]$, we have that

$$v'(x) \geqslant h_*'(x) - \|v - h_*\|_{C^1((-2,2))} \geqslant 2 - \|v - h_*\|_{C^2((-2,2))} \geqslant 1.$$

Similarly, if $x \in \left[1+\frac{3\epsilon}{11}, 1+\frac{4\epsilon}{11}\right]$, we have that $v'(x) \leqslant -1$.

As a consequence, there exist $\delta_1, \delta_2 \in \left[\frac{\epsilon}{11}, \frac{4\epsilon}{11}\right]$ such that $v'(-1-\delta_1) = v'(1+\delta_2) = 0$. In particular, for any $x \in \{-1-\delta_1, 1+\delta_2\}$,

$$\lim_{\substack{y \in (-1-\delta_1, 1+\delta_2) \\ y \to x}} \frac{v(x) - v(y)}{|x-y|^s} = 0.$$

Then, $v$ satisfies the desired result. $\square$

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