Stable standing waves for a NLS on star graphs as local minimizers of the constrained energy

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Abstract

On a star graph made of $N \geq 3$ halflines (edges) we consider a Schrödinger equation with a subcritical power-type nonlinearity and an attractive delta interaction located at the vertex. From previous works it is known that there exists a family of standing waves, symmetric with respect to the exchange of edges, that can be parametrized by the mass (or L^2-norm) of its elements. Furthermore, if the mass is small enough, then the corresponding symmetric standing wave is a ground state and, consequently, it is orbitally stable. On the other hand, if the mass is above a threshold value, then the system has no ground state.
Here we prove that orbital stability holds for every value of the mass, even if the corresponding symmetric standing wave is not a ground state, since it is anyway a *local* minimizer of the energy among functions with the same mass.
The proof is based on a new technique that allows to restrict the analysis to functions made of pieces of soliton, reducing the problem to a finite-dimensional one. In such a way, we do not need to use direct methods of Calculus of Variations, nor linearization procedures.

1 Introduction

The subject of nonlinear dynamics on quantum graphs (or *networks*, see [12, 22] for an exhaustive introduction) dates back to the seminal papers by Ali Mehmeti (see [10] and references therein), where the focus was on dispersive properties, and by von Below [11], who first set variational problems on a network. Since then, the interest increased at a growing rate up to the flourishing of results of the last decade, motivated by the need for simple models in contexts where two features coexist: a basic environment endowed with branches, junctions, ramifications, and the presence of non-negligible nonlinear effects. Such models range from quantum optics (see, e.g., [19]) to Bose-Einstein condensation (see, e.g., [27] and, for a more comprehensive introduction to physical applications, [23]). An important part of recent results concerns the study of the Nonlinear Schrödinger Equation (NLS), see e.g. [1, 2, 4, 5, 8, 9, 13, 14, 15, 21, 25, 26], and many of them are concerned with the seek for *standing waves*, i.e. solutions to the NLS that preserve the spatial shape and harmonically oscillate in time (namely, solutions of the form $\Phi(t) = e^{i\omega t}\Phi_\omega$, where Φ_ω is the space profile, called *stationary* or *bound state*), or even for *ground states*, i.e., stationary states that minimize the NLS energy among all functions with the same L^2-norm or *mass*.

To this regard, in spite of the quasi one-dimensional nature of networks, the structure of the family of standing waves, as well as the problem of the existence of a ground state, is far richer and more complicated than for the NLS on the line (for recent developments in this direction, see [8, 9]).

In the present paper we consider a star graph $\mathcal{G}$ made of N (≥ 3) halflines that meet one another at the unique vertex V (see Fig.1), and study the dynamics generated on it by the NLS with an attractive δ-interaction of strength $-\alpha$, $\alpha > 0$, placed at V (for the precise definitions see formulas (2.1), (2.2)). It

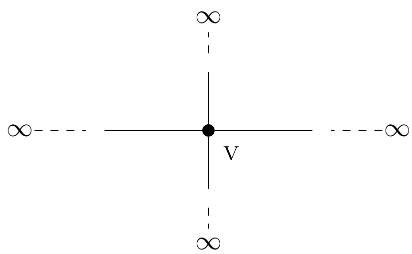

Figure 1: A star graph made of four halfline and a vertex.

is already known (see [4, 5]) that for any frequency $\omega \in (\alpha^2/N^2, +\infty)$ there exists a unique, real stationary state Ψ_ω whose associated solution to the NLS oscillates at the frequency ω and is symmetric under exchange of edges, namely, the restriction of Ψ_ω to any halfline gives the same function (see Fig.2). The main result of this paper is the orbital stability of Ψ_ω and can be expressed as follows:

Theorem 1. *On the star graph $\mathcal{G}$ made of $N \geq 3$ halflines intersecting one another at the vertex* V, *consider the Schrödinger Equation* (2.1) *with a focusing*

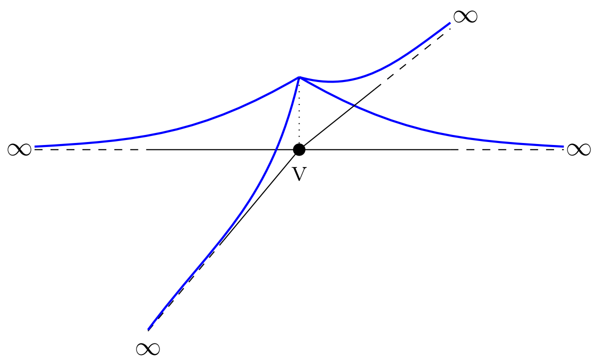

Figure 2: A representation of a symmetric stationary state Ψ_ω on the star graph made of four halflines.

nonlinearity of power $2\mu+1$ *and a delta interaction at* V *with strength* $-\alpha$*, with* $0<\mu<2$ *and* $\alpha>0$*.*
Then, given $\omega>\alpha^2/N^2$*, the unique standing wave* $e^{i\omega t}\Psi_\omega$ *symmetric under exchange of edges, is orbitally stable.*

We recall here that orbital stability is Lyapunov stability for orbits instead of states. Indeed, in order to hold for the dynamics generated by the NLS, that enjoys phase invariance, Lyapunov stability must be weakened: it cannot hold for states, but it may hold for orbits. In other words, Theorem 1 establishes that a solution remains arbitrarily close *to the orbit* of Ψ_ω, provided that the initial data had been chosen as suitably close *to the same orbit*.

With Theorem 1 we complete the analysis carried out in [4], where the existence of a ground state at a fixed mass has been established assuming that the mass is smaller than a critical value. In that case, orbital stability follows from the fact that Ψ_ω is a ground state, so the celebrated general result by Cazenave and Lions [16] applies. On the other hand, if its mass is larger than the critical threshold, then Ψ_ω is *not* a ground state (see point 4. in Section 2), therefore one is forced to use a criterion for orbital stability for which it is not necessary to assume that Ψ_ω is a ground state. In fact, Theorem 3 in [20] establishes that a stationary state is orbitally stable if and only if it is a *local* minimum for the energy functional among the functions with the same mass. Even though, in general, proving that a function is a local minimum of a functional is a difficult task, in our case it is possible to exploit the particular structure of the star graph made of N halflines, and Th. 4.1 in [8] which states that the minimizer of the NLS energy on the halfline under the mass constraint and a nonhomegeneous Dirichlet condition at the origin is given by a unique *soliton branch* (we recall its explicit expression in Sec. 2). This fact allows to introduce the so called *multi-soliton transformation*, that maps almost every function on the graph into a function (*multi-soliton*) made of N pieces of solitons, one for each halfline, in such a way that the mass is preserved and the energy is lowered. The space of multi-solitons with the same mass, denoted by $\mathcal{M}$ (see Def. 2.1), is a *finite-dimensional* manifold that contains all the stationary states. Thus, proving that a stationary state is a local minimum in

$\mathcal{M}$ requires a finite-dimensional analysis only. Nevertheless, this turns out to be not an immediate issue, because the study of the sign of the Hessian of the energy requires a certain degree of explicitness. In order to get it, one has to further reduce the problem from N halflines to two, and use the fact that on the real line the orbital stability has been already proven by Fukuizumi, Ohta and Ozawa in [18]. Finally, owing to the continuity of the multi-soliton transform, one shows that the symmetric stationary state is not only a local minimum of the energy among the multi-soliton states, but also among all states with the same mass.

The statement of Theorem 1 emphasizes the dynamical content of the result, i.e. the orbital stability. It emerges from the preceding discussion that one can give a variational version of the same result, that highlights the fact that the examined standing wave corresponds in fact to a *local* minimizer of the energy in the appropriate space, which is in general a highly non-trivial goal in the Calculus of Variations. For this reason we remark that Theorem 1 can be stated also in the following variational version:

Theorem 1'. (Variational version) *On the star graph $\mathcal{G}$ made of $N \geq 3$ halflines intersecting one another at the vertex* V*, consider the Schrödinger Equation* (2.1) *with a focusing nonlinearity of power $2\mu + 1$ and a delta interaction at* V *with strength $-\alpha$, where $0 < \mu < 2$ and $\alpha > 0$.*
Then, given $\omega > \alpha^2/N^2$, the unique positive bound state Ψ_ω symmetric under exchange of edges, is a strict (up to phase invariance) local minimizer for the energy functional associated to the considered evolution equation, constrained to the manifold of constant L^2-norm.

In [5] we treated the dual problem, namely the minimization of the *action* functional on the associated Nehari manifold. It was proved that in order to have a minimizer, the delta interaction must be strong enough. The link with the small mass condition given in [4] can be reconstructed by noting that, given a frequency ω, the mass of Ψ_ω is a monotonically decreasing function of the interaction strength α, so that the assumption of large α can be thought of as a small mass hypothesis. Through Theorem 1 one can get rid of every such assumptions: Ψ_ω is always orbitally stable, irrespective of the mass and of the interaction strentgh. Notice that in [5] it was proven that the energy constrained at constant mass has also other stationary points than the symmetric states above discussed and so the NLS on a star graph admits several families of non symmetric standing waves. In [4] it is shown that they are excited states, in the sense that their energy is above the energy of the symmetric state. They are believed to be saddle points of the constrained energy, which is consistent with the fact that the local minimum represented by the symmetric state is above the infimum of the energy for large mass.

Theorem 1 is already known to hold for $N = 2$ too, i.e. for the NLS with an attractive delta interaction on the line, since in that case Ψ_ω (which is the unique bound state at its frequency and at its mass) is always a ground state (see [7, 17, 18, 24]).

The structure of the paper is the following: in Section 2 we introduce the notation, recall the state of the art (in particular we explain the result of Th. 4.1 in [8]), give some preliminary results, introduce and describe the *multi-soliton manifold* $\mathcal{M}$. In Section 3 we introduce the *multi-soliton transformation*, that allows to reduce the problem to a finite-dimensional setting. In Section 4 we prove that Ψ_ω is actually a local minimum for the restriction of the NLS energy to $\mathcal{M}$. Finally, in Section 5 we extend the minimality property of Ψ_ω from the multi-soliton manifold to the natural space of the functions with finite energy and constant mass, so accomplishing the proof of Theorem 1.

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2 Setting, notation, previous results

We follow in notation the previous papers [2, 3, 4, 5]. The basic environment is the metric graph $\mathcal{G}$, defined as the star graph made of N halflines intersecting at their origins, where the unique vertex V is located. This convention fixes a coordinate system on $\mathcal{G}$, namely, each edge is identified with the real nonnegative halfline $[0,+\infty)$ and is endowed with its own coordinate (denoted by x_j for the j.th edge), while $x_j = 0$ is the coordinate of V, regardless of j. Functions $\Psi : \mathcal{G} \longrightarrow \mathbb{C}$ can be represented as vectors where the component on the j.th halfline is a scalar function $\psi_j : [0,+\infty) \longrightarrow \mathbb{C}$. For the sake of clarity, we shall often use the column vector representation

$$\Psi \;=\; \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}.$$

Besides, the component of Ψ on the j.th edge is denoted by ψ_j or by $(\Psi)_j$. In general, we use uppercase greek letters for functions defined on $\mathcal{G}$, and lowercase greek letters for functions acting on the halfline or on the line.

We denote by Ψ' and Ψ'' the vector valued functions with components ψ_j' and ψ_j'' respectively, the derivatives taken along the coordinate x_j.

Spaces $L^p(\mathcal{G})$ are naturally defined as the direct sum of N copies of $L^p(0,+\infty)$, while $H^1(\mathcal{G})$ is the direct sum of N copies of $H^1(0,+\infty)$ with the additional condition of *continuity at the vertex*

$$\psi_1(0) = \psi_2(0) = \cdots = \psi_N(0) = \Psi(\mathrm{V}).$$

On $\mathcal{G}$ we consider the dynamics generated by the *Schrödinger equation with power nonlinearity and an attractive point interaction at the vertex*, formally

$$i\partial_t \Psi(t) \;=\; -\Delta\Psi(t) - |\Psi(t)|^{2\mu}\Psi(t) - \alpha\delta_{\mathrm{V}}\Psi(t),$$

where $\mu > 0$ and the notation $|\Psi|^{2\mu}\Psi$ is understood in vector representation as

$$|\Psi|^{2\mu}\Psi \;=\; \begin{pmatrix} |\psi_1|^{2\mu}\psi_1 \\ |\psi_2|^{2\mu}\psi_2 \\ \vdots \\ |\psi_N|^{2\mu}\psi_N \end{pmatrix},$$

moreover, $\alpha > 0$ and δ_{V} denotes a delta potential located at the vertex. More precisely,

$$i\partial_t \Psi(t) \;=\; H\Psi(t) - |\Psi(t)|^{2\mu}\Psi(t), \tag{2.1}$$

where H is the linear operator defined as

$$\begin{aligned} D(H) \;&= \{\Psi \in H^1(\mathcal{G}),\ \text{s.t. } \psi_j \in H^2(0,+\infty),\ \sum_{j=1}^{N} \psi_j'(0) = -\alpha\Psi(\mathrm{v})\} \\ (H\Psi)_j \;&=\; -\,\psi_j''\ . \end{aligned} \tag{2.2}$$

Global well-posedness of (2.1) (for $0 < \mu < 2$) has been proved first in [1] for the cubic case, then in [5] for the case of a more general power, and for the case $N = 2$ (real line) with a general point interaction in [6]. Furthermore, it has been also proved that the L^2-norm, or *mass*

$$Q(\Psi) \;=\; \|\Psi\|^2_{L^2(\mathcal{G})}$$

and the energy

$$E(\Psi, \mathcal{G}) \;=\; \frac{1}{2}\|\Psi'\|^2_{L^2(\mathcal{G})} - \frac{1}{2\mu+2}\|\Psi\|^{2\mu+2}_{L^{2\mu+2}(\mathcal{G})} - \frac{\alpha}{2}|\Psi(0)|^2 \tag{2.3}$$

are conserved by the flow.

We shall occasionally make use of the functionals $E(\cdot, \mathbb{R}^+)$ and $E(\cdot, \mathbb{R})$, that share the formal expression of (2.3) but are evaluated on functions on the halfline and on the line, respectively.

As explained in Sec.1, we aim at proving that Ψ_ω is a local minimizer for $E(\cdot, \mathcal{G})$ with the mass constraint, so, chosen $M > 0$, our reference space is

$$H^1_M(\mathcal{G}) \;:=\; H^1(\mathcal{G}) \cap \{Q(\Psi) = M\}.$$

Let us now recall some preliminary notions, together with some further basic definitions and notation.

1. For every $\omega > 0$, the *soliton*

$$\phi_\omega(x) = [(\mu+1)\omega]^{\frac{1}{2\mu}} \operatorname{sech}^{\frac{1}{\mu}}(\mu\sqrt{\omega}x)$$

is the unique positive square-integrable solution to the stationary NLS equation

$$\varphi''(x) + \varphi^{2\mu+1}(x) \;=\; \omega\varphi(x).$$

As a consequence, the function $e^{i\omega t}\phi_\omega$ is a standing wave for the NLS on the line with power nonlinearity $2\mu + 1$. Besides, for any $\omega > 0$ the *mass* of the soliton is given by

$$\|\phi_\omega\|^2_{L^2(\mathbb{R})} = 2\frac{(\mu+1)^{\frac{1}{\mu}}}{\mu}\omega^{\frac{1}{\mu}-\frac{1}{2}} \int_0^1 (1-t^2)^{\frac{1}{\mu}-1} dt, \tag{2.4}$$

and is a monotonically increasing function of ω.

2. As proved in [5], for every $\alpha > 0$, the unique positive stationary solution to Eq. (2.1) with frequency $\omega > \alpha^2/N^2$, symmetric under exchange of edges, is given by

$$(\Psi_\omega)_i\,(x_i) = \phi_\omega(x_i + \zeta), \quad i = 1, \ldots, N$$

with

$$\zeta = \frac{1}{\mu\sqrt{\omega}} \operatorname{arctanh}\left(\frac{\alpha}{N\sqrt{\omega}}\right). \tag{2.5}$$

Notice that $\zeta > 0$, so that every halfline hosts a *soliton tail*, which is a monotonically decreasing function (see Fig.2).

3. The *mass function*

$$M(\omega) \;:=\; Q(\Psi_\omega) \;=\; N\frac{(\mu+1)^{\frac{1}{\mu}}}{\mu}\omega^{\frac{1}{\mu}-\frac{1}{2}} \int_{\frac{\alpha}{N\sqrt{\omega}}}^{1} (1-t^2)^{\frac{1}{\mu}-1} dt, \tag{2.6}$$

(see formula (5.1) in [4]) is strictly monotonically increasing, and ranges from 0 (excluded) to $+\infty$ as ω goes from α^2/N^2 (excluded) to $+\infty$. Then, in the same way as for the soliton on the real line, ω can be interpreted as a relabelling of the mass, and for every positive M there exists exactly one symmetric stationary state Ψ_ω with mass M (see [4]).

4. As mentioned in Sec.1, there are some values of M such that the corresponding stationary state Ψ_ω is *not* a ground state. Indeed, in [5] we exhibited a sequence $\Phi_n \in H^1_M(\mathcal{G})$ s.t. $E(\Phi_n, \mathcal{G}) \to E(\phi_{\omega_\mathbb{R}}, \mathbb{R})$ as n goes to infinity (see formula (3.12) in [5]), where $\omega_\mathbb{R}$ is the unique value of ω such that $\|\phi_\omega\|^2_{L^2(\mathbb{R})} = M$, see Eq. (2.4). Roughly speaking, such a sequence is supported on a single edge and asymptotically reconstructs a soliton at infinity. Therefore, in order for Ψ_ω to be a ground state, it must be $E(\Psi_\omega, \mathcal{G}) \leq E(\phi_{\omega_\mathbb{R}}, \mathbb{R})$. By explicitly computing the involved energies, see also formulas (4.10) and (4.12) in [4], such inequality can be rewritten as

$$(2-\mu)\omega_\mathbb{R} M \;\leq\; (2-\mu)\omega M + \alpha\mu(\mu+1)^{\frac{1}{\mu}}\left(\omega - \frac{\alpha^2}{N^2}\right). \tag{2.7}$$

Furthermore, since $\|\Psi_\omega\|^2_{L^2(\mathcal{G})} = M = \|\phi_{\omega_\mathbb{R}}\|^2_{L^2(\mathbb{R})}$, and from Eqs. (2.4) and (2.6), one has the identities

$$\begin{aligned} M \;&= 2\frac{(\mu+1)^{\frac{1}{\mu}}}{\mu}\omega_\mathbb{R}^{\frac{1}{\mu}-\frac{1}{2}} \int_0^1 (1-t^2)^{\frac{1}{\mu}-1} dt \\ &= N\frac{(\mu+1)^{\frac{1}{\mu}}}{\mu}\omega^{\frac{1}{\mu}-\frac{1}{2}} \int_{\frac{\alpha}{N\sqrt{\omega}}}^{1} (1-t^2)^{\frac{1}{\mu}-1} dt. \end{aligned} \tag{2.8}$$

Hence, for M large, the l.h.s. of (2.7) is of order $M^{\frac{\mu+2}{2-\mu}}$, as the first term in the r.h.s., while the second term in the r.h.s. is of order $M^{\frac{2}{2-\mu}}$, then it can be neglected. Expliciting $\omega_\mathbb{R}$ and ω as functions of M in (2.8), one has that inequality (2.7) amounts to

$$\int_0^1 (1-t^2)^{\frac{1}{\mu}-1} dt \;\geq\; \frac{N}{2}\int_{\frac{\alpha}{N\sqrt{\omega}}}^{1} (1-t^2)^{\frac{1}{\mu}-1} dt,$$

that is violated for M large, as ω becomes large too. Then, large mass implies that Ψ_ω is not a ground state, and, since by Lemma 5.2 in [4], Ψ_ω is the minimizer of the energy among all stationary states, one concludes that there is no ground state if the mass exceeds a critical threshold.

The last part of the present section is devoted to the introduction of the finite-dimensional manifold to which we shall reduce the problem.

We preliminarily recall Theorem 4.1 of [8], which establishes that, given $a, m > 0$, there exists a *unique* couple $\omega > 0, \xi \in \mathbb{R}$ such that

$$\int_0^{+\infty} \phi_\omega^2(x+\xi)\, dx = m, \qquad \phi_\omega(\xi) = a. \tag{2.9}$$

Furthermore, the function $\phi_\omega(\cdot + \xi)$ minimizes the energy

$$\frac{1}{2}\|\phi'\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{2\mu+2}\|\phi\|^{2\mu+2}_{L^{2\mu+2}(\mathbb{R}^+)}$$

among the functions in $H^1(\mathbb{R}^+)$ with mass m and whose value at zero equals a (Dirichlet constraint), hence, within such class of functions, it is also the minimizer of $E(\cdot, \mathbb{R}^+)$.
By this result one can introduce two functions $\omega = \omega(m, a), \xi = \xi(m, a)$, that give the value of the soliton parameters ω and ξ such that $\phi_\omega(\cdot + \xi)$ minimizes the functional $E(\cdot, \mathbb{R}^+)$ with both mass and Dirichlet constraints given by Eq. (2.9).

As we will show in Sec.3, the existence of such functions is the starting point for the reduction of the problem to a finite-dimensional manifold, whose definition is:

Definition 2.1. *Fixed $M > 0$, we call* multi-soliton manifold *of mass M, and denote by $\mathcal{M}$, the subset of $H^1_M(\mathcal{G})$ made of functions whose restriction at every halfline of $\mathcal{G}$ gives a piece of soliton.*

Remark 2.2. Notice that all functions in $\mathcal{M}$ are positive.

Remark 2.3. Every element of $\mathcal{M}$ has the following form:

$$\Phi_{\underline{m},a} = \begin{pmatrix} \phi_{\omega(m_1,a)}(\cdot + \xi(m_1,a)) \\ \phi_{\omega(m_2,a)}(\cdot + \xi(m_2,a)) \\ \vdots \\ \phi_{\omega(M - \sum_{j=1}^{N-1} m_j, a)}(\cdot + \xi(M - \sum_{j=1}^{N-1} m_j, a)) \end{pmatrix} \tag{2.10}$$

where

- we used the notation $\underline{m} := (m_1, m_2, \ldots m_{N-1})$, with $m_i > 0$ and $\sum_{i=1}^{N-1} m_i < M$;
- by the definition of the functions ω and ξ, m_i is the mass located on the i.th edge, i.e.,
$$\int_0^\infty \phi^2_{\omega(m_i,a)}(x + \xi(m_i,a))\, dx = m_i, \tag{2.11}$$
and a is the value attained by $\Phi_{\underline{m},a}$ at V, in particular, for all $i = (1, \ldots, N-1)$
$$\phi_{\omega(m_i,a)}(\xi(m_i,a)) = a. \tag{2.12}$$

Notice that the global mass constraint $M(\Phi_{\underline{m},a}) = M$ is guaranteed by the last compoment of the vector in the r.h.s. of (2.10).

Since the function $\Phi_{\underline{m},a}$ depends on N parameters $m_1, \ldots, m_{N-1}, a$, $\mathcal{M}$ is a N-dimensional submanifold of $H^1_M(\mathcal{G})$.

Remark 2.4. The real symmetric stationary state Ψ_ω with $Q(\Psi_\omega) = M$, belongs to $\mathcal{M}$. Indeed, it is immediately seen that $\Psi_\omega = \Phi_{\underline{\widetilde{m}},\widetilde{a}}$, where the vector $\underline{\widetilde{m}}$ reads $\widetilde{m}_1 = \cdots = \widetilde{m}_{N-1} = \frac{M}{N}$ and $\widetilde{a}$ is defined as the unique solution to the transcendental equation

$$\widetilde{a} = \phi_{\omega(M/N,\widetilde{a})}(\xi(M/N,\widetilde{a})) \ = \ [(\mu+1)\omega(M/N,\widetilde{a})]^{\frac{1}{2\mu}} \left(1 - \frac{\alpha^2}{N^2\omega(M/N,\widetilde{a})}\right)^{\frac{1}{2\mu}},$$

where we observed that $\xi(M/N,\widetilde{a}) = \zeta$ and then used (2.5).

When dealing with functions in the manifold $\mathcal{M}$, the energy functional becomes a real-valued function of N real, positive variables. We emphasize this change of point of view by introducing the *reduced energy* E_r as

$$\begin{aligned} E_r \ &: \ \{(\underline{m}, a) \in (0,+\infty)^N, \ \text{s.t.} \ \sum_{i=1}^{N-1} m_i < M\} \to \mathbb{R} \\ E_r(\underline{m}, a) \ &:= E(\Phi_{\underline{m},a}, \mathcal{G}). \end{aligned} \tag{2.13}$$

Let us define the function $F : [0, M] \times \mathbb{R}^+ \to \mathbb{R}$ as

$$\begin{aligned} F(m,a) \ =& \ \frac{1}{2}\|\phi'_{\omega(m,a)}(\cdot + \xi(m,a))\|^2_{L^2(\mathbb{R}^+)} \\ & - \frac{1}{2\mu+2}\|\phi_{\omega(m,a)}(\cdot + \xi(m,a))\|^{2\mu+2}_{L^{2\mu+2}(\mathbb{R}^+)} - \frac{\alpha}{2N}a^2 \end{aligned} \tag{2.14}$$

so that the function E_r decomposes as follows

$$E_r(\underline{m}, a) \ = \ \sum_{i=1}^{N-1} F(m_i, a) + F(M - \sum_{j=1}^{N-1} m_j, a). \tag{2.15}$$

3 Multi-soliton transformation

The first step of the proof of Theorem 1 consists in reducing the problem to a finite-dimensional manifold, made of functions obtained by gluing together pieces of soliton. To this aim, we start by transforming every function in $H^1_M(\mathcal{G})$ that does not vanish at V into a function of $\mathcal{M}$, in such a way that the value at V and the mass at any edge are preserved.

Owing to Theorem 4.1 in [8] and to the definition of the functions ω and ξ, introduced in Sec.2, it is possible to give the following

Definition 3.1. *The* soliton transformation $S\eta \in H^1(\mathbb{R}^+)$ *of a function* $\eta \in H^1(\mathbb{R}^+)$ *such that* $\eta(0) \neq 0$, *is defined as*

$$S\eta \ := \ \phi_{\omega(m,a)}(\cdot + \xi(m,a)),$$

where we denoted $m = \int_{\mathbb{R}^+} |\eta(x)|^2\,dx$, $a = |\eta(0)|$.

By Theorem 4.1 in [8], the function $S\eta$ is unique and

$$\frac{1}{2}\|(S\eta)'\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{2\mu+2}\|S\eta\|^{2\mu+2}_{L^{2\mu+2}(\mathbb{R}^+)} \leq \frac{1}{2}\|\eta'\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{2\mu+2}\|\eta\|^{2\mu+2}_{L^{2\mu+2}(\mathbb{R}^+)},$$

hence, since $(S\eta)(0) = |\eta(0)|$, Theorem 4.1 in [8] implies

$$E(S\eta, \mathbb{R}^+) \leq E(\eta, \mathbb{R}^+),$$

where equality holds if and only if $\eta = e^{i\theta}\phi_\omega(\cdot + \xi)$, for some values of θ, ω, and ξ.

Furthermore, notice that the soliton transformation acts trivially on pieces of soliton, namely $S\phi_\omega(\cdot + \xi) = \phi_\omega(\cdot + \xi)$.

Finally, the soliton transformation is defined for $a > 0$ only. However, we do not need to cover the case $a = 0$. The only property we shall use is the continuity of S at the stationary state Ψ_ω.

Proposition 3.2. *The soliton transformation S is continuous from the space $H^1_M(\mathcal{G})$ to itself.*

Proof. We decompose S in three steps and show continuity at every step. Schematically,

$$\eta \overset{S_1}{\mapsto} (a, m) \overset{S_2}{\mapsto} (\omega, \xi) \overset{S_3}{\mapsto} \phi_\omega(\cdot + \xi),$$

where $a = |\eta(0)|$, $m = \int_0^{+\infty} |\eta(x)|^2 dx$.

First, S_1 is continuous because the pointwise value and the L^2-norm are continuous in $H^1(\mathbb{R}^+)$.
As already stated, the fact that the map S_2 is well-defined follows from Theorem 4.1 in [8]. The proof that it is also continuous can be made by closely following the proof of Theorem 4.1 in [8]. We sketch the procedure since, due to differences in the notation, this step could not be straightforward.

By using the scaling property

$$\phi_\omega(x) = \omega^{\frac{1}{2\mu}} \phi_1(\sqrt{\omega}x)$$

in the identities (2.11) and (2.12) we obtain

$$\int_0^\infty \phi_1^2(x + \sqrt{\omega}\xi) dx = m\omega^{-\frac{2-\mu}{2\mu}} \qquad \text{and} \qquad \phi_1(\sqrt{\omega}\xi) = a\omega^{-\frac{1}{2\mu}}. \tag{3.1}$$

Putting them together one obtains the identity

$$g(\sqrt{\omega}\xi) = \frac{m}{a^{2-\mu}}, \tag{3.2}$$

with

$$g(z) = \big(\phi_1(z)\big)^{-(2-\mu)} \int_0^\infty \phi_1^2(x + z) dx.$$

The function g is continuous and strictly monotonically decreasing (for the proof of this statement we refer to the proof of Th. 4.1 in [8] again). We remark that the definition of the function ϕ_1 in [8] is slightly different from ours, basically the two definitions involve different scalings of the hyperbolic secant. This is due to the fact that in [8] the authors parametrize solitons through the mass instead of the frequency. In spite of that, the argument in Th. 4.1 still applies, being based only the asymptotic, monotonicity, and log-concavity properties of sech.

By (3.2), we infer that the quantity $\sqrt{\omega}\xi$ is a continuous function of m and a. Moreover, for fixed m and a there exists a unique value of $\sqrt{\omega}\xi$ such that identity (3.2) is satisfied. Furthermore, fixed the quantity $\sqrt{\omega}\xi$, there exists a unique ω such that the second identity in (3.1) is satisfied, and such a ω is a continuous function of a and m. As a consequence ξ is a continuous function of a and m too, and this proves the continuity of S_2.

In order to prove the continuity of S_3, let us fix the couple (ω, ξ) and prove that

$$\lim_{\omega_1\to\omega,\xi_1\to\xi} \|\phi_\omega(\cdot+\xi) - \phi_{\omega_1}(\cdot+\xi_1)\|_{H^1(\mathbb{R}^+)} \ = \ 0. \tag{3.3}$$

Preliminarily, we use the triangular inequality

$$\begin{aligned}
&\|\phi_\omega(\cdot+\xi) - \phi_{\omega_1}(\cdot+\xi_1)\|^2_{H^1(\mathbb{R}^+)} \\
&\leq 2\|\phi_\omega(\cdot+\xi) - \phi_\omega(\cdot+\xi_1)\|^2_{H^1(\mathbb{R}^+)} + 2\|\phi_\omega(\cdot+\xi_1) - \phi_{\omega_1}(\cdot+\xi_1)\|^2_{H^1(\mathbb{R}^+)} \\
&\leq 2\|\phi_\omega(\cdot+\xi) - \phi_\omega(\cdot+\xi_1)\|^2_{H^1(\mathbb{R}^+)} + 2\int_0^{+\infty} |\phi_\omega(x+\xi_1) - \phi_{\omega_1}(x+\xi_1)|^2 dx \\
&\quad + 2\int_0^{+\infty} |\phi'_\omega(x+\xi_1) - \phi'_{\omega_1}(x+\xi_1)|^2 dx \\
&\leq 2\|\phi_\omega(\cdot+\xi_1) - \phi_\omega(\cdot+\xi)\|^2_{H^1(\mathbb{R}^+)} + 2\int_{-\infty}^{+\infty} |\phi_\omega(x) - \phi_{\omega_1}(x)|^2 dx \\
&\quad + 2\int_{-\infty}^{+\infty} |\phi'_\omega(x) - \phi'_{\omega_1}(x)|^2 dx
\end{aligned} \tag{3.4}$$

Concerning the first term in the r.h.s., a straightforward computation gives

$$\|\phi_\omega(\cdot+\xi) - \phi_\omega(\cdot+\xi_1)\|^2_{L^2(\mathbb{R}^+)} \ \leq \ |\xi-\xi_1|^2 \|\phi'_\omega\|^2_{L^2(\mathbb{R}^+)},$$

and, repeating the same computation for the derivatives, one gets

$$\|\phi_\omega(\cdot+\xi) - \phi_\omega(\cdot+\xi_1)\|^2_{H^1(\mathbb{R}^+)} \ \leq \ |\xi-\xi_1|^2 \|\phi_\omega\|^2_{H^2(\mathbb{R}^+)}. \tag{3.5}$$

So the first summand in the r.h.s vanishes as ξ_1 approaches ξ. For the second term in the r.h.s. of (3.4), first observe that, assuming $\omega/2 \leq \omega_1 \leq 2\omega$,

$$\begin{aligned}
&|\phi_\omega(x) - \phi_{\omega_1}(x)|^2 \ \leq \ 2|\phi_\omega(x)|^2 + 2|\phi_{\omega_1}(x)|^2 \\
&\leq C\left(\omega^{\frac{1}{\mu}} e^{-2\sqrt{\omega}|x|} + {\omega_1}^{\frac{1}{\mu}} e^{-2\sqrt{\omega_1}|x|}\right) \ \leq \ C\omega^{\frac{1}{\mu}} e^{-\sqrt{\omega}|x|},
\end{aligned}$$

and the last quantity is an integrable function in the variable x. Analogously, for the last term in the r.h.s. of inequality (3.4), one gets

$$\begin{aligned}
&|\phi'_\omega(x) - \phi'_{\omega_1}(x)|^2 \ \leq \ 2|\phi'_\omega(x)|^2 + 2|\phi'_{\omega_1}(x)|^2 \\
&\leq C\left(\omega^{1+\frac{1}{\mu}} e^{-2\sqrt{\omega}|x|} + \omega_1^{1+\frac{1}{\mu}} e^{-2\sqrt{\omega_1}|x|}\right) \ \leq \ C\omega^{1+\frac{1}{\mu}} e^{-\sqrt{\omega}|x|}.
\end{aligned}$$

Then, by dominated convergence theorem, one has that the last two terms in (3.4) vanish as ω_1 goes to ω, thus (3.3) is proved and the proof is complete.

□

We can now introduce the multi-soliton trasformation Σ as the natural generalization of the soliton transformation S to the star graph $\mathcal{G}$.

Definition 3.3. *Given a function $\Phi \in H^1_M(\mathcal{G})$ such that $\Phi(\mathrm{V}) \neq 0$, the* multi-soliton *transformation $\Sigma\Phi$ of Φ is the function defined on $\mathcal{M}$ as*

$$(\Sigma\Phi)_j \ := \ S\Phi_j,$$

where S is the soliton transformation introduced in Definition 3.1.

Remark 3.4. The multi-soliton transformation Σ inherits the following properties from the soliton transformation S:

1. Σ is continuous from the space of the functions in $H^1_M(\mathcal{G})$ that do not vanish at the vertex, to $H^1_M(\mathcal{G})$.
2. Σ preserves the mass distribution on the edges of the star graph, i.e. $\|(\Sigma\Phi)_j\|^2_{L^2(\mathbb{R}^+)} = \|(\Phi)_j\|^2_{L^2(\mathbb{R}^+)}$, and the absolute value of the function in the vertex, namely $\Sigma\Phi(\mathrm{V}) = |\Phi(\mathrm{V})|$.
3. For every Φ in the domain of Σ,
$$E(\Sigma\Phi, \mathcal{G}) \leq E(\Phi, \mathcal{G}), \tag{3.6}$$
where equality holds if and only if $\Phi \in \mathcal{M}$, up to a constant phase factor.
4. The multi-soliton transformation Σ acts trivially on $\mathcal{M}$. In particular, $\Sigma\Psi_\omega = \Psi_\omega$.

4 Local minimality of Ψ_ω in $\mathcal{M}$

Here we treat the finite-dimensional problem of the local minimality of Ψ_ω in the manifold $\mathcal{M}$. In this section we always refer to the reduced energy E_r (see Definition 2.13) and then, when possible, avoid any reference to functions, that can be replaced by points of $\mathbb{R}^N$. In particular, according to Remark 2.3, the bound state Ψ_ω corresponds to the point $\widetilde{P} := (\underline{\widetilde{m}}, \widetilde{a}) = (M/N, \ldots, M/N, \widetilde{a})$. However, in order to prove even this finite-dimensional minimality, along the proof we must come back to function representation and make use of a well-known result of the Grillakis-Shatah-Strauss theory on stability of standing waves (see Theorem 3.4 in [20]), that we straightforwardly apply in order to get inequality (4.5).

Proposition 4.1. *Fixed $M > 0$, the point $\widetilde{P}$ is a strict local minimum for the function $E_r(\underline{m}, a)$ defined in* (2.13).

Proof. First, notice that $\widetilde{P}$ is an internal point for the domain of E_r, therefore it suffices to show that $\widetilde{P}$ is a stationary point for E_r and that the Hessian matrix of E_r evaluated at $\widetilde{P}$ is positive definite.

The fact that Ψ_ω is a stationary point immediately gives that $\widetilde{P}$ is a stationary point for E_r, so we turn to the study of the sign of the Hessian matrix. By straightforward calculations,

$$\begin{aligned} \frac{\partial^2 E_r}{\partial m_i \partial m_j}(\widetilde{P}) &= (1+\delta_{ij}) \frac{\partial^2 F}{\partial m^2}\left(\frac{M}{N}, \widetilde{a}\right) \\ \frac{\partial^2 E_r}{\partial a^2}(\widetilde{P}) &= N \frac{\partial^2 F}{\partial a^2}\left(\frac{M}{N}, \widetilde{a}\right) \\ \frac{\partial^2 E_r}{\partial a \partial m_i}(\widetilde{P}) &= 0, \end{aligned} \tag{4.1}$$

with $1 \le i, j \le N-1$.
Consequently, the Hessian matrix of E_r computed at $\widetilde{P}$ is a $N \times N$ block matrix. The high $(N-1)\times(N-1)$ left block, that we call $\mathbb{H}_1$, reads

$$\mathbb{H}_1 := \frac{\partial^2 F}{\partial m^2}\left(\frac{M}{N}, \widetilde{a}\right)(\mathbb{J}+\mathbb{I})$$

where $\mathbb{J}$ is the matrix with all elements equal to one. By elementary linear algebra, one immediatly finds that $\mathbb{H}_1$ has two eigenvalues: $N\frac{\partial^2 F}{\partial m^2}\left(\frac{M}{N}, \widetilde{a}\right)$ with multiplicity 1, and $\frac{\partial^2 F}{\partial m^2}\left(\frac{M}{N}, \widetilde{a}\right)$ with multiplicity $N-2$. Therefore, $\mathbb{H}_1$ is positive definite if and only if

$$\frac{\partial^2 F}{\partial m^2}\left(\frac{M}{N}, \widetilde{a}\right) > 0. \tag{4.2}$$

The second diagonal block, denoted by $\mathbb{H}_2$, is 1×1 and reduces to $N\frac{\partial^2 F}{\partial a^2}\left(\frac{M}{N}, \widetilde{a}\right)$, so it is positive defined if and only if

$$\frac{\partial^2 F}{\partial a^2}\left(\frac{M}{N}, \widetilde{a}\right) > 0. \tag{4.3}$$

Thus, in order to prove that $\widetilde{P}$ is actually a local minimum of the function E_r, one has to prove (4.2) and (4.3).

To prove (4.2), consider the curve in the manifold $\mathcal{M}$ given by $\Phi_{\underline{m}(t),\widetilde{a}}$, with

$$m_1(t) = \frac{M}{N} + t,\ m_2(t) = \cdots = m_{N-1}(t) = \frac{M}{N}, \qquad t \in (-\varepsilon, \varepsilon)$$

and define the function

$$f(t) := E_r(\underline{m}(t), \widetilde{a}).$$

By definition of partial derivative and using (4.1), one has

$$f''(0) \ = \ \frac{\partial^2 E_r}{\partial m_1^2}(\widetilde{P}) \ = \ 2\frac{\partial^2 F}{\partial m^2}\left(\frac{M}{N}, \widetilde{a}\right).$$

So condition (4.2) reduces to $f''(0) > 0$. Now, let us write the curve in terms of elements of $\mathcal{M}$:

$$\Phi_{\underline{m}(t),\widetilde{a}} = \begin{pmatrix} \phi_{\omega(M/N+t,\widetilde{a})}(\cdot + \xi(M/N+t,\widetilde{a})) \\ \phi_{\omega(M/N,\widetilde{a})}(\cdot + \xi(M/N,\widetilde{a})) \\ \vdots \\ \phi_{\omega(M/N-t,\widetilde{a})}(\cdot + \xi(M/N-t,\widetilde{a})) \end{pmatrix}$$

It transpires that $\Phi_{\underline{m}(t),\widetilde{a}}$ varies with t in the first and the N.th components only. Since the total mass is conserved, as t changes *there is a transfer of mass from one edge to the other without changing the value at the vertex.* We define the operator

$$\tau : L^2(\mathcal{G}) \longrightarrow L^2(\mathbb{R})$$

acting as $\tau\Xi \ = \ \eta$, with

$$\eta(x) \ = \ \chi_{\mathbb{R}^+}(x)(\Xi)_1(x) + \chi_{\mathbb{R}^-}(x)(\Xi)_N(-x),$$

where, as usual, we denoted by $(\Xi)_j$ the component of the wave function Ξ on the j.th edge. In other words, η is the function on the line obtained from Ξ by matching together edges 1 and N and neglecting all the others. Therefore,

$$\begin{aligned}
& f(t) - E_r(\widetilde{P}) \;=\; E_r(\underline{m}(t), a) - E_r(\widetilde{P}) \\
= \;& \sum_{i=1}^{N-1} F(m_i(t), \widetilde{a}) + F\left(\frac{M}{N} - t, \widetilde{a}\right) - NF\left(\frac{M}{N}, \widetilde{a}\right) \\
= \;& F\left(\frac{M}{N} + t, \widetilde{a}\right) + F\left(\frac{M}{N} - t, \widetilde{a}\right) - 2F\left(\frac{M}{N}, \widetilde{a}\right) \\
= \;& E_{2\alpha/N}(\tau\Phi_{\underline{m}(t),\widetilde{a}}, \mathbb{R}) - E_{2\alpha/N}(\tau\Psi_\omega, \mathbb{R}),
\end{aligned} \tag{4.4}$$

where we exploited the definition (2.14) of the function F and introduced the functional

$$E_{2\alpha/N}(u, \mathbb{R}) \;:=\; \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} - \frac{1}{2\mu+2}\|u\|^{2\mu+2}_{L^{2\mu+2}(\mathbb{R})} - \frac{\alpha}{N}|u(0)|^2$$

acting on $H^1(\mathbb{R})$. Notice that $E_{2\alpha/N}(\cdot, \mathbb{R})$ is the functional representing the energy associated to the focusing Schrödinger equation with power nonlinearity $2\mu+1$ and a delta interaction placed at the origin with strength $2\alpha/N$ (see [18] and [6]).

Furthermore,

$$(\tau\Psi_\omega)(x) \;=\; [(\mu+1)\omega]^{\frac{1}{2\mu}} \cosh^{-\frac{1}{\mu}}\left(\mu\sqrt{\omega}(|x| + \zeta)\right),$$

with ζ given by Eq. (2.5), which is the ground state of the NLS on the line with a delta interaction located at the origin, of strength $\alpha' = \frac{2\alpha}{N}$. In that case (see [18] and [7]), such a state is known to be a stable global minimizer of the constrained problem, so it falls into the scope of Theorem 3.4 of [20]. In the notation of that theorem, $T(s(u)) = 1$ since we are dealing with real functions. Therefore, for ε sufficiently small, the theorem yields

$$\begin{aligned}
f(t) - E_r(\widetilde{P}) \;&=\; E_{2\alpha/N}(\tau\Phi_{\underline{m}(t),\widetilde{a}}, \mathbb{R}) - E_{2\alpha/N}(\tau\Psi_\omega, \mathbb{R}) \\
&\geq\; c\|\tau\Phi_{\underline{m}(t),\widetilde{a}} - \tau\Psi_\omega\|^2_{L^2(\mathbb{R})},
\end{aligned} \tag{4.5}$$

for any $t \in (-\varepsilon, \varepsilon)$. We claim that there exists a positive constant c such that, for ε small enough,

$$\|\tau\Phi_{\underline{m}(t),\widetilde{a}} - \tau\Psi_\omega\|^2_{L^2(\mathbb{R})} \geq c\,t^2. \tag{4.6}$$

This gives the bound $f(t) - E_r(\widetilde{P}) \geq c\,t^2$, which, together with $f(0) - E_r(\widetilde{P}) = 0$ and $f'(0) = 0$, implies $f''(0) \geq c > 0$, and concludes the proof of (4.2).
It remains to prove the claim (4.6). To this aim, we write

$$\begin{aligned}
& \|\tau\Phi_{\underline{m}(t),\widetilde{a}} - \tau\Psi_\omega\|^2_{L^2(\mathbb{R})} \\
= & \int_0^\infty \left(\phi_{\omega(M/N+t,\widetilde{a})}(x + \xi(M/N+t, \widetilde{a})) - \phi_{\omega(M/N,\widetilde{a})}(x + \xi(M/N, \widetilde{a}))\right)^2 dx \\
& + \int_0^\infty \left(\phi_{\omega(M/N-t,\widetilde{a})}(x + \xi(M/N-t, \widetilde{a})) - \phi_{\omega(M/N,\widetilde{a})}(x + \xi(M/N, \widetilde{a}))\right)^2 dx
\end{aligned}$$

and note that

$$\|\tau\Phi_{\underline{m}(t),\widetilde{a}}-\tau\Psi_\omega\|^2_{L^2(\mathbb{R})}\Big|_{t=0}=0 \qquad \text{and} \qquad \frac{d}{dt}\|\tau\Phi_{\underline{m}(t),\widetilde{a}}-\tau\Psi_\omega\|^2_{L^2(\mathbb{R})}\Big|_{t=0}=0.$$

To conclude the proof of (4.6) it is enough to show that

$$\frac{d^2}{dt^2}\|\tau\Phi_{\underline{m}(t),\widetilde{a}}-\tau\Psi_{\omega,0}\|^2_{L^2(\mathbb{R})}\Big|_{t=0}\geq c.$$

We compute the second derivative at $t=0$ and obtain (we omit the dependence of ω and ξ on m and a)

$$\begin{aligned}&\frac{d^2}{dt^2}\|\tau\Phi_{\underline{m}(t),\widetilde{a}}-\tau\Psi_\omega\|^2_{L^2(\mathbb{R})}\Big|_{t=0}\\ &=4\int_0^\infty\left(\frac{\partial\phi_\omega}{\partial\omega}(x+\xi)\,\frac{\partial\omega}{\partial m}+\ \frac{\partial\phi_\omega}{\partial x}(x+\xi)\,\frac{\partial\xi}{\partial m}\right)^2 dx\Big|_{(m,a)=(M/N,\widetilde{a})}.\end{aligned}\tag{4.7}$$

Next we prove that the integrand is not identically equal to zero, which in turn implies that the second derivative in $t=0$ is strictly positive.

In Eq. (3.2) we set $z=\sqrt{\omega}\xi$ and take the derivative with respect to m, this gives the identity

$$\frac{\partial g}{\partial z}\,\frac{\partial z}{\partial m}=\frac{1}{a^{2-\mu}},$$

which tells us that $\frac{\partial z}{\partial m}\neq 0$. Since

$$\frac{\partial z}{\partial m}=\frac{\xi}{2\sqrt{\omega}}\frac{\partial\omega}{\partial m}+\sqrt{\omega}\frac{\partial\xi}{\partial m},$$

we conclude that $\frac{\partial\omega}{\partial m}$ and $\frac{\partial\xi}{\partial m}$ cannot be both equal to zero (recall that $\xi\neq 0$ whenever $\alpha\neq 0$). Since the functions $\frac{\partial\phi_\omega}{\partial\omega}(x)$ and $\frac{\partial\phi_\omega}{\partial x}(x)$ are linearly independent, the integrand in Eq. (4.7) does not vanish identically.

In order to prove (4.3) one proceeds analogously. First define a curve $\Phi_{\underline{\widetilde{m}},a(t)}$ in the manifold $\mathcal{M}$, with

$$a(t)=\widetilde{a}+t,\qquad t\in(-\varepsilon,\varepsilon).$$

Then, introduced the function $h(t):=E_r(\underline{\widetilde{m}},a(t))$, one immediately has

$$h''(0)=\frac{\partial^2E_r}{\partial a^2}(\widetilde{P})\ =\ N\frac{\partial^2F}{\partial a^2}\left(\frac{M}{N},\widetilde{a}\right).$$

Hence, to prove (4.3) it is enough to show that $h''(0)>0$. Notice that, for any t, the function $\Phi_{\underline{\widetilde{m}},a(t)}$ is changing symmetrically on any edge, so that, using Theorem 3.4 of [20] again,

$$\begin{aligned}h(t)-E_r(\widetilde{P})\ &=\ NF\left(\frac{M}{N},a(t)\right)-NF\left(\frac{M}{N},\widetilde{a}\right)\\ =\ &\ \frac{N}{2}\big(E_{2\alpha/N}(\tau\Phi_{\underline{\widetilde{m}},a(t)},\mathbb{R})-E_{2\alpha/N}(\tau\Psi_\omega,\mathbb{R}),\big)\\ \geq\ &\ \frac{N}{2}c\|\tau\Phi_{\underline{\widetilde{m}},a(t)}-\tau\Psi_\omega\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

To prove the inequality $h''(0) > 0$ it is enough to show that $\|\tau\Phi_{\underline{\widetilde{m}},a(t)} - \tau\Psi_\omega\|^2_{L^2(\mathbb{R})} \geq c\,t^2$ for t small enough. Arguing as in the proof of (4.6), we conclude that this is certainly true if $\frac{\partial\omega}{\partial a}$ and $\frac{\partial\xi}{\partial a}$ are not both equal to zero. To see that this is actually the case, we take the derivative of Eq. (3.2) with respect to a and obtain an identity which is not compatible with $\frac{\partial z}{\partial a} = 0$. This in turns implies that or $\frac{\partial\omega}{\partial a} \neq 0$ or $\frac{\partial\xi}{\partial a} \neq 0$.

$\square$

Remark 4.2. As an alternative formulation of Proposition 4.1, we have that Ψ_ω is a *strict* local minimizer for the restriction of $E(\cdot,\mathcal{G})$ to the manifold $\mathcal{M}$.

5 Orbital stability

The next step consists in passing from the local minimality of Ψ_ω in $\mathcal{M}$ to the local minimality of Ψ_ω on $H^1_M(\mathcal{G})$. This step is immediate once one considers that: first, the reduction from $H^1_M(\mathcal{G})$ to $\mathcal{M}$ through the transformation Σ lowers the energy level; second, Ψ_ω is invariant under Σ; third, Σ is continuous at Ψ_ω. Some care must be dedicated to the fact that multiplying by a constant phase factor does not lower the energy. Anyway, one has

Proposition 5.1. *Ψ_ω is a strict (up to multiplication by phase) local minimizer of $E(\cdot,\mathcal{G})$ in $H^1_M(\mathcal{G})$.*

Proof. First we prove that Ψ_ω is a *strict* local minimizer among *real* functions in $H^1_M(\mathcal{G})$. According to Remark 4.2, Ψ_ω is a strict local minimum of $E(\cdot,\mathcal{G})$ restricted to $\mathcal{M}$. This means that there exists $\varepsilon > 0$ such that, if $\Phi \neq \Psi_\omega$ is a real element of $H^1_M(\mathcal{G})$ with $\Phi(\mathrm{v}) \neq 0$, and $\|\Sigma\Phi - \Psi_\omega\|_{H^1(\mathcal{G})} < \varepsilon$, then

$$E(\Psi_\omega,\mathcal{G}) \leq E(\Sigma\Phi,\mathcal{G}), \tag{5.1}$$

where equality holds if and only if $\Sigma\Phi = \Psi_\omega$.

Moreover, by Rermark 3.4 we know that Σ is continuous at Ψ_ω, then there exists $\delta > 0$ such that if $0 < \|\Phi - \Psi_\omega\|_{H^1(\mathcal{G})} < \delta$, then

$$\varepsilon \;>\; \|\Sigma\Phi - \Sigma\Psi_\omega\|_{H^1(\mathcal{G})} \;=\; \|\Sigma\Phi - \Psi_\omega\|_{H^1(\mathcal{G})},$$

where we used the invariance of Ψ_ω under the action of Σ. But then, using inequalities (5.1) and (3.6),

$$E(\Psi_\omega,\mathcal{G}) \;\leq\; E(\Sigma\Phi,\mathcal{G}) \;\leq\; E(\Phi,\mathcal{G}),$$

where the first inequality becomes an equality if and only if $\Sigma\Phi = \Psi_\omega$, while the second one becomes an equality if and only if $\Phi = \Sigma\Phi$. Then, we proved that there exists a $\delta > 0$ such that if $0 < \|\Phi - \Psi_\omega\|_{H^1(\mathcal{G})} < \delta$, then $E(\Phi,\mathcal{G}) < E(\Psi,\mathcal{G})$, so we have that Ψ_ω strictly minimizes $E(\cdot,\mathcal{G})$ locally among the real functions in $H^1_M(\mathcal{G})$. Of course, extending the analysis to non-real functions, Ψ_ω cannot be a strict minimizer due to phase invariance of the energy functional: $E(e^{i\theta}\Psi_\omega,\mathcal{G}) = E(\Psi_\omega,\mathcal{G})$. However, for any Φ outside the phase orbit of Ψ_ω whose distance from the orbit is less than δ, one has $E(\Phi,\mathcal{G}) > E(\Psi_\omega,\mathcal{G})$. Indeed, there exists $\theta \in [0,2\pi)$ s.t.

$$\delta > \|\Phi - e^{i\theta}\Psi_\omega\|_{H^1_M(\mathcal{G})} = \|e^{-i\theta}\Phi - \Psi_\omega\|_{H^1_M(\mathcal{G})} \geq \||e^{-i\theta}\Phi| - \Psi_\omega\|_{H^1_M(\mathcal{G})},$$

so that, since $|e^{-i\theta}\Phi|$ is real and different from Ψ_ω,

$$E(\Psi_\omega, \mathbb{R}) \ < \ E(|e^{-i\theta}\Phi|, \mathcal{G}). \tag{5.2}$$

On the other hand, a straigthforward computation gives

$$E(|e^{-i\theta}\Phi|, \mathcal{G}) \ \leq \ E(e^{-i\theta}\Phi, \mathcal{G}),$$

that, together with (5.2), concludes the proof.

□

So we can conclude by invoking Grillakis-Shatah-Strauss theory.

Proof of Theorem 1. Owing to Theorem 3 in [20], the local minimality property in $H^1_M(\mathcal{G})$ is equivalent to orbital stability. The proof is complete. □

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