A Note on the Dirac-like Equation in Non-commutative Geometry

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Abstract

We postulate the non-commutativity of 4-momenta and we derive the mass splitting in the Dirac equation. The applications are discussed.

The non-commutativity [1, 2] manifests interesting peculiarities in the Dirac case. Recently, we analized Sakurai-van der Waerden method of derivations of the Dirac (and higher-spins too) equation [3]. We can start from

$$(EI^{(2)} - \boldsymbol{\sigma}\cdot\mathbf{p})(EI^{(2)} + \boldsymbol{\sigma}\cdot\mathbf{p})\Psi_{(2)} = m^2\Psi_{(2)}\,, \tag{1}$$

or

$$(EI^{(4)} + \boldsymbol{\alpha}\cdot\mathbf{p} + m\beta)(EI^{(4)} - \boldsymbol{\alpha}\cdot\mathbf{p} - m\beta)\Psi_{(4)} = 0. \tag{2}$$

As in the original Dirac work, we have

$$\beta^2 = 1\,,\quad \alpha^i\beta + \beta\alpha^i = 0\,,\quad \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}\,. \tag{3}$$

For instance, their explicite forms can be chosen

$$\alpha^i = \begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}\,,\quad \beta = \begin{pmatrix}0 & 1_{2\times 2}\\ 1_{2\times 2} & 0\end{pmatrix}\,, \tag{4}$$

where σ^i are the ordinary Pauli 2×2 matrices.

We also postulate the non-commutativity relations for the components of 4-momenta:

$$[E, \mathbf{p}^i]_- = \Theta^{0i} = \theta^i \,, \tag{5}$$

as usual. Therefore the equation (2) will *not* lead to the well-known equation $E^2 - \mathbf{p}^2 = m^2$. Instead, we have

$$\left\{E^2 - E(\boldsymbol{\alpha}\cdot\mathbf{p}) + (\boldsymbol{\alpha}\cdot\mathbf{p})E - \mathbf{p}^2 - m^2 - i(\boldsymbol{\sigma}\otimes I_{(2)})[\mathbf{p}\times\mathbf{p}]\right\}\Psi_{(4)} = 0 \tag{6}$$

For the sake of simplicity, we may assume the last term to be zero. Thus, we come to

$$\left\{E^2 - \mathbf{p}^2 - m^2 - (\boldsymbol{\alpha}\cdot\boldsymbol{\theta})\right\}\Psi_{(4)} = 0\,. \tag{7}$$

However, let us apply the unitary transformation. It is known [4, 5] that one can[1]

$$U_1(\boldsymbol{\sigma}\cdot\mathbf{a})U_1^{-1} = \sigma_3|\mathbf{a}|\,. \tag{8}$$

For $\boldsymbol{\alpha}$ matrices we re-write (8) to

$$\mathcal{U}_1(\boldsymbol{\alpha}\cdot\boldsymbol{\theta})\mathcal{U}_1^{-1} = |\boldsymbol{\theta}|\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \alpha_3|\boldsymbol{\theta}|\,. \tag{9}$$

The explicit form of the U_1 matrix is ($a_{r,l} = a_1 \pm ia_2$):

$$\begin{aligned} U_1 &= \frac{1}{\sqrt{2a(a+a_3)}}\begin{pmatrix}a+a_3 & a_l\\ -a_r & a+a_3\end{pmatrix} = \frac{1}{\sqrt{2a(a+a_3)}}[a + a_3 + i\sigma_2 a_1 - i\sigma_1 a_2]\,,\\ \mathcal{U}_1 &= \begin{pmatrix}U_1 & 0\\ 0 & U_1\end{pmatrix}\,. \end{aligned} \tag{10}$$

Let us apply the second unitary transformation:

$$\mathcal{U}_2\alpha_3\mathcal{U}_2^\dagger = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\alpha_3\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\,. \tag{11}$$

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[1] Some relations for the components $\mathbf{a}$ should be assumed. Moreover, in our case $\boldsymbol{\theta}$ should not depend on E and $\mathbf{p}$. Otherwise, we must take the non-commutativity $[E, \mathbf{p}^i]_-$ into account again.

The final equation is

$$[E^2 - \mathbf{p}^2 - m^2 - \gamma^5_{chiral}|\boldsymbol{\theta}|]\Psi'_{(4)} = 0\,. \qquad (12)$$

In the physical sense this implies the mass splitting for a Dirac particle over the non-commutative space, $m_{1,2} = \pm\sqrt{m^2 \pm \theta}$. This procedure may be attractive for explanation of the mass creation and the mass splitting for fermions.

References

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