# Asymptotic stability for standing waves of a NLS equation with concentrated nonlinearity in dimension three. II

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## Abstract

In this paper the study of asymptotic stability of standing waves for a model of Schrödinger equation with spatially concentrated nonlinearity in dimension three, begun in [3], is continued. The nonlinearity studied is a power nonlinearity concentrated at the point $x = 0$ obtained considering a contact (or $\delta$) interaction with strength $\alpha$, which consists of a singular perturbation of the Laplacian described by a selfadjoint operator $H_\alpha$, and letting the strength $\alpha$ depend on the wavefunction in a prescribed way: $i\dot{u} = H_\alpha u$, $\alpha = \alpha(u)$. For power nonlinearities in the range $(\frac{1}{\sqrt{2}}, 1)$ there exist orbitally stable standing waves $\Phi_\omega$, and the linearization around them admits two imaginary eigenvalues (absent in the range $(0, \frac{1}{\sqrt{2}})$ previously treated) which in principle could correspond to non decaying states, so preventing asymptotic relaxation towards an equilibrium orbit. This situation is usually treated requiring the validity of a *nonlinear Fermi golden rule*, which assures the presence of a dissipative term in the modulation equation ruling the complex amplitudes $z(t), \bar{z}(t)$ associated to the discrete part of the linearized spectrum. Here without the use of FGR it is proven that, in the range $(\frac{1}{\sqrt{2}}, \sigma^*)$ for a certain $\sigma^* \in (\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}]$, the dynamics near the orbit of a standing wave asymptotically relaxes in the following sense: consider an initial datum $u(0)$ near the standing wave $\Phi_{\omega_0}$, written in the form $u(0) = u_0 = e^{i\omega_0 + \gamma_0}\Phi_{\omega_0} + e^{i\omega_0+\gamma_0}[(z_0 + \overline{z_0})\Psi_1 + i(z_0 - \overline{z_0})\Psi_2] + f_0$ with $z_0$ small and $f_0$ small in energy and in a certain weighted space $L^1_w$; then the solution $u(t)$ can be asymptotically decomposed as

$$u(t) = e^{i\omega_\infty t + ib_1 \log(1+\epsilon k_\infty t)}\Phi_{\omega_\infty} + U_t * \psi_\infty + r_\infty, \quad \text{as } t \to +\infty,$$

where $\omega_\infty$, $k_\infty > 0$, $b_1 \in \mathbb{R}$, and $\psi_\infty$ and $r_\infty \in L^2(\mathbb{R}^3)$ , $U(t)$ is the free Schrödinger group and

$$\|r_\infty\|_{L^2} = O(t^{-1/4}) \quad \text{as } t \to +\infty \ .$$

We stress the fact that in the present case and contrarily to the main results in the field, the admitted nonlinearity is $L^2$-subcritical.



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## I. INTRODUCTION

We continue here the analysis of a model of nonlinear Schrödinger equation with a concentrated nonlinearity in dimension three begun in [3]. We recall that such a model is defined by the equation

$$i\frac{du}{dt} = H_{\alpha(u)}u. \tag{1}$$

where the nonlinear operator $H_{\alpha(u)}$ is a point interaction (or "delta potential") in dimension three with a strength $\alpha = \alpha(u)$ depending on the wavefunction $u$ in a prescribed way. Here the nonlinearity is of power type and focusing. More precisely, the domain of $H_{\alpha(u)}$ is given by

$$D(H_{\alpha(u)}) = \Big\{u \in L^2(\mathbb{R}^3) : \ u(x) = \phi(x) + qG_0(x) \ \text{ with } \ \phi \in H^2_{loc}(\mathbb{R}^3),\ \Delta\phi \in L^2(\mathbb{R}^3), \tag{2}$$

$$q \in \mathbb{C}, \ \ \lim_{x\to 0}(u(x) - qG_0(x)) = \phi(0) = -\nu|q|^{2\sigma}q\}, \ \ G_0(x) = \frac{1}{4\pi|x|}, \ \ \nu > 0\ .\Big\}$$

and the action is given by

$$H_\alpha u(x) = -\Delta\phi(x)\ , \quad x \in \mathbb{R}^3 \setminus \{0\}\ . \tag{3}$$

The nonlinearity, displayed in the boundary condition, means that the value at zero of the "regular part" $\phi$ of the element domain is related in a nonlinear way to the so called "charge" $q$ of the same element domain, which is the coefficent of the "singular part" $G_0$. According to (2), our choice for the function $\alpha(u)$ is

$$\alpha(u) \ = \ -\nu|q|^{2\sigma}, \qquad \nu, \sigma > 0$$

When $\sigma = 0$ one obtains the well known contact interaction (of strength $\alpha = -\nu$), which, due to the sign, is a so called attractive $\delta$ interaction (see [4]). When $\sigma \neq 0$ the nonlinearity is in some sense acting at the single point zero, coinciding with the location of the singularity of the contact interaction. This is the origin of the denomination of concentrated nonlinearity (see [1, 2, 19]). The above model can be derived from a standard nonlinear Schrödinger equationquation with a inhomogeneous nonlinearity, i.e. with a nonlinearity space dependent, and shrinking at a point in a suitable scaling limit. This derivation is rigorously treated in [10] for the one dimensional case and in the forthcoming paper [11] for the present three dimensional case. The problem (1) describes a Hamiltonian system for which global well posedness holds in the nonlinearity range $\sigma \in (0, 1)$. More precisely we endow the space $L^2(\mathbb{R}^3,\mathbb{C}) \simeq L^2(\mathbb{R}^3,\mathbb{R}) \oplus L^2(\mathbb{R}^3,\mathbb{R})$ (assuming the usual identification $z \simeq (\Re z, \Im z)$) with the symplectic form

$$\Omega(u,v) = \Im \int_{\mathbb{R}^3} u\overline{v}\ dx = \int_{\mathbb{R}^3} (\Re v \Im u - \Im v \Re u) dx = \int_{\mathbb{R}^3} (u_2 v_1 - u_1 v_2) dx \tag{4}$$

Of course $H^1(\mathbb{R}^3,\mathbb{R}) \oplus H^1(\mathbb{R}^3,\mathbb{R})$ is a symplectic submanifold, and we associate to (1) the Hamiltonian functional coinciding with the total conserved energy associated to the evolution equation (1), that is given by

$$E(u(t)) = \frac{1}{2}\|\nabla\phi\|^2_{L^2} - \frac{\nu}{2\sigma+2}|q|^{2\sigma+2}, \quad u = \phi + qG_0 \in V. \tag{5}$$

where $V$ is the domain of *finite energy states*

$$V = \{u \in L^2(\mathbb{R}^3) :\ u(x) = \phi(x) + qG_0(x), \text{ with } \phi \in L^2_{loc}(\mathbb{R}^3),\ \nabla\phi \in L^2(\mathbb{R}^3),\ q \in \mathbb{C}\}, \tag{6}$$

which is a Hilbert space endowed with the norm

$$\|u\|^2_V = \|\nabla\phi\|^2_{L^2} + |q|^2. \tag{7}$$

Note that for a generic element $u$ of the form domain the charge $q$ and its regular part $\phi$ are independent of each other.

Correspondingly, the NLSE (1) can be rephrased in the hamiltonian form

$$\frac{du}{dt} = J\ E^{'}(u(t))\ . \tag{8}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard symplectic matrix. In [3] the case of power $\sigma \in (0, \frac{1}{\sqrt{2}})$ was studied, showing existence of nonlinear bound states, orbital stability and relaxation to (relative) equilibrium asymptotically in time. For such values of the power nonlinearity $\sigma$ the asymptotic analysis is simplified by the fact that linearization around standing waves has no eigenvalues apart from the zero eigenvalue, always existing due to the gauge $U(1)$ symmetry. For $\sigma \in (\frac{1}{\sqrt{2}}, 1)$ the linearization around a standing wave admits pure imaginary eigenvalues $\pm i\xi$, which correspond to existence of neutral oscillation in the linearized dynamics. If these neutral oscillations persist as invariant tori in the phase space of the complete nonlinear system, relaxation to an asymptotic equilibrium standing wave is precluded. The analysis of so called asymptotic stability of solitons started at the beginning of the nineties with the studies of Soffer and Weinstein ([21, 22]) on the

NLS equation with an external potential in dimension three and of Buslaev and Perelman on the translation invariant NLS in dimension one ([7, 8]).

In our model the analysis goes as follows. We consider the nonlinear evolution problem

$$i\frac{du}{dt} = H_\alpha u, \qquad u(0) = u_0 \in D(H_\alpha), \tag{9}$$

In [3] it is shown the existence of a solitary wave manifold for (9)

$$\mathcal{M} = \left\{ \Phi_\omega(x) = \left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{2\sigma}} \frac{e^{-\sqrt{\omega}|x|}}{4\pi|x|} \, : \, \omega > 0 \right\} , \tag{10}$$

and it is shown that these solitary waves, which belong to $D(H_\alpha)$, are orbitally stable in the same range ($\sigma \in (0,1)$) where global well posedness of equation (9) is guaranteed.

One aims at proving that an initial datum near this solitary manifold relaxes for $t \to +\infty$ to the solitary manifold itself.

The solitary manifold turns out to be a symplectic two dimensional submanifold.

Now writing $u = e^{i\omega t}(\Phi_\omega + R)$, we obtain that $R$ satisfies the first order the linearized canonical system

$$\frac{dR}{dt} = -J \begin{bmatrix} H_{\alpha_1} + \omega & 0 \\ 0 & H_{\alpha_2} + \omega \end{bmatrix} R \; = \; \begin{bmatrix} 0 & L_2 \\ -L_1 & 0 \end{bmatrix} R \; \equiv LR \; , \tag{11}$$

and $L_j = H_{\alpha_j} + \omega$ for $j = 1, 2$, $\alpha_1 = -(2\sigma + 1)\frac{\sqrt{\omega}}{4\pi}$ and $\alpha_2 = -\frac{\sqrt{\omega}}{4\pi}$.

As recalled before (see [3], Section IV A, and Appendix VI A of this paper) in the case $\sigma \in (1/\sqrt{2}, 1)$ the discrete spectrum of $L$ consists of the eigenvalue 0 with algebraic multiplicity 2 and of two purely imaginary eigenvalues $\pm i\xi$ with

$$\xi = 2\sigma\sqrt{1-\sigma^2}\omega, \tag{12}$$

and corresponding eigenvectors $\Psi$ and $\Psi^*$ (see Appendix VI C of this paper). As a consequence, the domain of the operator $L$ can be decomposed in three symplectic subspaces, more precisely

$$D(L) = X^0 \oplus X^1 \oplus X^c,$$

where $X^0$, $X^1$, and $X^c$ are respectively the generalized kernel of $L$, the eigenspace corresponding to the $L$ eigenfunctions $\Psi$ and $\Psi^*$, and the subspace associated to the absolutely continuos spectrum of $L$. In particular $X^0$ is generated by the tangent vectors to the symplectic solitary submanifold.

The corresponding symplectic projection operators from $L^2(\mathbb{R}^3)$ onto $X^0$, $X^1$ and $X^c$ are denoted by $P^0$, $P^1$, $P^c$ respectively (see appendix C for explicit representation .) A further step in the analysis consists in decomposing the solution of equation (9) in the sum of a soliton-like part $e^{i\Theta(t)}\Phi_{\omega(t)}(x)$ and a fluctuating part $\chi(t,x)$, introducing the Ansatz

$$u(t,x) = e^{i\Theta(t)}\left(\Phi_{\omega(t)}(x) + \chi(t,x)\right), \tag{13}$$

with

$$\Theta(t) = \int_0^t \omega(s)ds + \gamma(t), \tag{14}$$

and

$$\chi(t,x) = z(t)\Psi(t,x) + \overline{z(t)}\Psi^*(t,x) + f(t,x) \equiv \psi(t,x) + f(t,x), \quad \psi \in X^1, \quad f \in X^c \tag{15}$$

with $\omega(t)$, $\gamma(t)$, $z(t)$ and $f(t,x)$ up to now unprejudiced. Notice that the parameters are now time dependent, as well as the functions $\Psi$ and $\overline{\Psi}$: this is due to the fact that $\Psi$ depends on $t$ through $L$ and then through $\omega$.

The goal now is to show that the fluctuating part is decaying ($\Psi$-$\Psi^*$ component) or dispersing ($f$-component), to provide convergence of parameters $\omega(t), \gamma(t)$ to possibly unknown asymptotic values; all of this finally gives relaxation to the solitary manifold. To this end one has to fix the above underdetermined representation, and then obtain equations for $\omega(t)$, $\gamma(t)$ and $\chi(t,x)$. This is achieved by requiring the fluctuation $\chi$ to be symplectically orthogonal to the solitary manifold $\mathcal{M}$ (or, equivalently, orthogonal to the generalized kernel $N_g(L)$) for every time $t \geq 0$. This procedure yields the so called "modulation equations" for $\omega(t)$, $\gamma(t)$ and $\chi(t,x)$. Moreover, exploiting further orthogonality relations between $\Psi, \Psi^*, \Phi_\omega$, one also gets equations for the coefficient $z$ and the dispersive term $f$ in $\chi$ (See Theorem II.1 in Section II for more details). At this point two problems occur. The first is that the modulation equation for the fluctuating component $\chi$, due to the introduction of the time dependent Ansatz, contains a non autonomous linear part. So, the use of dispersive estimates to show the wanted decay of this component requires to preliminarily "freeze" the dynamics at a certain fixed time $T$; the subsequent step is to show the uniform character of the estimates in $T$. The frozen equations are written in Section II. In the same section the leading terms in the modulation equations are identified, and the estimates on the remainder terms are displayed. This is however not yet sufficient to get rid of the complex oscillation described by $z$ and $z^*$. So the modulation equations are rewritten in a canonical way by means of a Poincaré normal form pushed to the third order which preserves estimates on the remainders. In particular, the transformed oscillating component $z_1 = z_1(z, \bar{z})$ satisfies the equation

$$\dot{z_1} = i\xi z_1 + iK|z_1|^2 z_1 + \widehat{Z}_R \ . \tag{16}$$

The asymptotic behavior of the solution of the $z_1$ component depends on the coefficient $iK$ and more precisely on the sign of its real part.

This is the point where nonlinear Fermi golden rule (FGR) enters the game. It turns out in relevant examples that if a resonance condition between a higher harmonic (a multiple) of $i\xi$ and the continuous spectrum of linearization is satisfied (the FGR), then $\Re(iK)$ is strictly negative; this gives decay of the oscillating modes of the linearization (see the seminal paper cited above and moreover [24], [25] and [9]). The decay is ultimately due to coupling of oscillating modes with the continuous spectrum assured by FGR, and consequent drift of energy from the discrete component to the continuous one; so the mechanism is dissipation by dispersion. Furthermore, exploiting the hamiltonian structure of the system it can be shown that the above situation is in some sense generic ([13, 14],[5] and reference therein).

In the present model things go as follows. In the first place the second harmonic $2i\xi$ of the discrete eigenvalue $i\xi$ lies in the interior of the continuous spectrum if the nonlinearity power $\sigma \in (\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}})$. Let us then denote by $N_2(q,q)$ the quadratic terms coming from the Taylor expansion of the nonlinearity (see formula (20)), and by $\Psi(\omega_0) = \begin{pmatrix} \Psi_1(\omega_0) \\ \Psi_2(\omega_0) \end{pmatrix}$ the eigenfunction of the linearized operator associated to $i\xi_0$ (given explicitely in Appendix VI C, Proposition VI 4). The nonlinear Fermi golden rule (FGR) adapted to our case is the following non-degeneracy condition:

$$JN_2(q_{\Psi(\omega_0)}, q_{\Psi(\omega_0)})\overline{q_{\Psi_+(2i\xi_0)}} \neq 0, \tag{17}$$

where $\Psi_+(2i\xi_0)$ is the generalized eigenfunction associated to $+2i\xi_0$ (for the explicit representation see Appendix D).

This nondegeneracy condition should imply $\Re(iK) < 0$ in (16).

As a matter of fact, thanks to the explicit character of our model, we are able to directly verify, without use of the nondegeneracy condition, that $\Re(iK) < 0$ (that gives dissipation in (16)) holds for any $\sigma$ in the range $\left(\frac{1}{\sqrt{2}}, \sigma^*\right)$, for a certain $\sigma^* \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$ (see Section III D). Moreover, we have numerical evidence that this is true on the whole interval $\left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$. With these premises, we eventually prove the following result.

**Theorem (Asymptotic stability in the case of purely imaginary eigenvalues)** Assume that $u \in C(\mathbb{R}^+, V)$ is a solution to equation (9) with a power nonlinearity (see (2)) given by $\sigma \in (\frac{1}{\sqrt{2}}, \sigma^*)$, for a certain $\sigma^* \in (\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}]$. Moreover, suppose that the initial datum is close to a standing wave of (9), in the sense that

$$u(0) = u_0 = e^{i\omega_0 + \gamma_0}\Phi_{\omega_0} + e^{i\omega_0+\gamma_0}[(z_0 + \overline{z_0})\Psi_1 + i(z_0 - \overline{z_0})\Psi_2] + f_0 \in V \cap L^1_w(\mathbb{R}^3),$$

with $\omega_0 > 0$, $\gamma_0$, $z_0 \in \mathbb{R}$, and $f_0 \in L^2(\mathbb{R}^3) \cap L^1_w(\mathbb{R}^3)$ and

$$|z_0| \leq \epsilon^{1/2} \qquad \text{and} \qquad \|f_0\|_{L^1_w} \leq c\epsilon^{3/2},$$

where $c$, $\epsilon > 0$.

Then, provided $\epsilon$ is sufficiently small, the solution $u(t)$ can be asymptotically decomposed as

$$u(t) = e^{i\omega_\infty t + ib_1 \log(1+\epsilon k_\infty t)}\Phi_{\omega_\infty} + U_t * \psi_\infty + r_\infty, \quad \text{as } t \to +\infty,$$

where $\omega_\infty$, $\epsilon k_\infty > 0$, $b_1 \in \mathbb{R}$, and $\psi_\infty$, $r_\infty \in L^2(\mathbb{R}^3)$ such that

$$\|r_\infty\|_{L^2} = O(t^{-1/4}) \quad \text{as } t \to +\infty,$$

in $L^2(\mathbb{R}^3)$.

Some remarks are in order.

We stress again that the range of the admitted nonlinearities $\sigma$ implies that $\pm 2i\xi$ is in the essential spectrum of the linearized operator. Notice that $\frac{\sqrt{3}+1}{2\sqrt{2}} \approx 0.96$, and that when $\sigma \to 1$ the discrete eigenvalues of linearization, $\pm i\xi$, collide at zero. So, in order to cover a larger range of values of $\sigma$ one has to consider harmonics higher than the second and consider normal form pushed to order higher than the third. As shown in [3] the standing waves in the case $\sigma = 1$ are unstable by blow-up. We recall that $\sigma = \frac{1}{\sqrt{2}}$ is a threshold resonance for the linearization. Its presence does not allow enough decay of the linearized dynamics to obtain relaxation to solitary manifold.

The asymptotic stability result is achieved following the outline of [9] and [18] and using the machinery already developed in [3]. In particular, in [18] the same problem for the analogous one-dimensional model is studied.

Nevertheless, the three-dimensional case presents some differences. The first one is that the concentrated nonlinearity imposes to develop the analysis at the form level. This means that the estimates on the evolution of the initial data are more delicate.

The second main difference is the faster decay of the propagator of the free Laplacian. This allows to develop the the analysis using just the structural weight $w = 1 + \frac{1}{|x|}$ which arises from the dispersive estimate instead of introducing new weighted spaces as done in the one-dimensional case treated in [6, 18].

Finally, the eigenfunctions associated to the purely imaginary eigenvalues do not have oscillating terms occurring in the one-dimensional case but they exponentially decrease as $|x| \to +\infty$. This fact will be useful in order to get the decay in time of the radiation term.

A last comment of general nature is in order. As in the one dimensional case studied by Buslaev, Komech, Kopylova, and Stuart in [6] and Komech, Kopylova, and Stuart in [18], and the three dimensional model analyzed in [3], the analysis of a specific model allows to obtain asymptotic stability of standing waves without a priori assumptions. In particular the nonlinearity is fixed, of power type and subcritical, in the sense that it falls in the range of global well posedness of the equation (see [17] for a different example where asymptotic stability is proven in subcritical regime); no spectral assumptions are needed; no smallness of initial data is required, in the sense that we give results for every standing wave of the model and initial data near the family of standing waves.

As a final remark, while Komech, Kopylova, and Stuart in [18] find a link between the Fermi Golden Rule and the decay of normal modes of linearization, here such decay is directly verified. This fact seems to indicate that some of these assumptions or hypotheses are in fact unnecessary when enough information about the model is known.

For the sake of completeness, in the course of the paper we will repeat proofs requiring some modifications because of the facts mentioned above; on the contrary, where the arguments hold unchanged, just a reference will be given. Moreover in the appendices we give information about linearization operator recalling useful material from [3] and giving detailed properties of discrete and continuous eigenfunctions.

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## II. MODULATION EQUATIONS

We recall that the operators we are dealing with have different domains while the forms associated share the common domain $V$, introduced in (6). It proves convenient to describe $V$ in an alternative way: fixed an arbitrary $\lambda > 0$ and denoted $G_\lambda(x) := e^{-\lambda|x|}/4\pi$, one finds

$$V = \left\{u = \phi_\lambda + qG_\lambda, \text{ with}\, \phi_\lambda \in H^1(\mathbb{R}^3),\ q \in \mathbb{C}\right\}.$$

The arbitrary positive parameter $\lambda$ is customary in the description of a point interaction and moreover allows the regular part of the element's domain to be in the Sobolev space $H^1$ instead of the corresponding homogeneous counterpart, which is often useful. Everything is in fact independent on the choice of $\lambda$ .

Correspondingly, we will perform computations at the form level. In order to do that let us recall that the variational formulation of equation (1) is

$$\left(i\frac{du}{dt}(t), v\right) = Q_\alpha(u(t), v) \quad \forall v \in V. \tag{18}$$

where, given $u, v \in V$ with $u = \phi_{u,\lambda} + q_u G_\lambda$, $v = \phi_{v,\lambda} + q_v G_\lambda$,

$$Q_\alpha(u, v) = (\nabla\phi_{u,\lambda}, \nabla\phi_{v,\lambda})_{L^2} + \bar{q}_u q_v \left(\alpha + \frac{\sqrt{\lambda}}{8\pi}\right) - \lambda\bar{q}_u(G_\lambda, \phi_{v,\lambda})_{L^2} - \lambda q_v(\phi_{u,\lambda}, G_\lambda)_{L^2}$$

Note that the equation (18) makes sense because $V$ is independent of the positive parameter $\lambda$ and it is a Hilbert space with the norm

$$\|u\|_V^2 = \|\nabla\phi_\lambda\|_{L^2}^2 + |q|^2, \quad \forall u \in V.$$

In order to inspect the asymptotic stability of equation (1) it is useful to represent the solution $u$ by the aid of the Ansatz (13) together with defintions (14) and (15). From now on we always refer to such formulas.

Hence, we want to construct a solution of equation (1) close at each time to a solitary wave. Let us notice that the solitary wave does not need to be the same at every time, which means that the parameters $\omega(t)$ and $\Theta(t)$ are free to vary in time.

Exactly as in the case $\sigma \in \left(0, \frac{1}{\sqrt{2}}\right)$ (see [3], Section V) the function $\chi$ solves

$$\left(i\frac{d\chi}{dt}(t), v\right)_{L^2} = Q_{\alpha,Lin}(\chi(t), v) + \dot{\gamma}(t)(\Phi_{\omega(t)} + \chi(t), v)_{L^2} + \tag{19}$$

$$+\dot{\omega}(t)\left(-i\frac{d\Phi_{\omega(t)}}{d\omega}, v\right)_{L^2} + N(q_\chi(t), q_v),$$

for all $v \in V$. I

Here $Q_{\alpha,Lin}$ is the quadratic form of the linearization operator acting as

$$Q_{\alpha,Lin}(\chi, v) = (\nabla\phi_\chi, \nabla\phi_v)_{L^2} - \frac{\sqrt{\omega}}{4\pi}\Re(q_\chi\overline{q_v}) - \sigma\frac{\sqrt{\omega}}{2\pi}\Re q_\chi \Re q_v + \omega(\chi, v)_{L^2} \ ,$$

and the nonlinear remainder $N(q_\chi, q_v)$ is given by

$$N(q_\chi, q_v) = -\nu|q_\chi + q_\omega|^{2\sigma}\Re((q_\chi + q_\omega)\overline{q_v}) + \nu(2\sigma+1)|q_\omega|^{2\sigma}\Re q_\chi\Re q_v + \nu|q_\omega|^{2\sigma}\Im q_\chi\Im q_v + \nu|q_\omega|^{2\sigma}\Re(q_\omega\overline{q_v}), \tag{20}$$

where (see section II B in [3]), $q_\omega = \left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{\frac{1}{2\sigma}}$.

Since $\omega(t)$, $\gamma(t)$, and $\chi(x,t)$ are unknown and the propagator grows in time along the directions of the generalized kernel of the operator $L$, the idea is to get a determined system requiring the function $\chi(t)$ to be orthogonal to the generalized kernel of $L$ at any time $t \geq 0$. Hence, one obtains that $\omega$, $\gamma$, $z$, and $f$ must solve the following system of equations.

**Theorem II.1. (Modulation equations)** *If $\chi(t)$ is a solution of equation* (19) *such that $P_0\chi(t) = 0$ for all $t \geq 0$ and $\omega(t)$ and $\gamma(t)$ are continuously differentiable in time, then $\omega$ and $\gamma$ are solutions of*

$$\dot{\omega} = \frac{\Re\ (JN(q_\chi)\overline{q_{P_0^*(\Phi_\omega+\chi)}})}{\left(\varphi_\omega - \frac{dP_0}{d\omega}\chi, \Phi_\omega + \chi\right)_{L^2}}, \tag{21}$$

$$\dot{\gamma} = \frac{\Re\ (JN(q_\chi)\overline{q_{J(\varphi_\omega - \frac{dP_0}{d\omega}\chi)}})}{\left(\varphi_\omega - \frac{dP_0}{d\omega}\chi, \Phi_\omega + \chi\right)_{L^2}}. \tag{22}$$

*Furthermore, $z$ and $f$ satisfy*

$$(\Psi, J\Psi)_{L^2}(\dot{z} - i\xi z) = \Re(JN(q_\chi)\overline{q_{J\Psi}}) + \dot{\omega}\left[\left(f, J\frac{d\Psi}{d\omega}\right)_{L^2} - \left(\frac{d\psi}{d\omega}, J\Psi\right)_{L^2}\right] + \dot{\gamma}(\chi, \Psi)_{L^2}, \tag{23}$$

$$\left(\frac{df}{dt}, v\right)_{L^2} = Q_L(f, v) + \left(-\dot{\omega}\left(zP^c\frac{d\Psi}{d\omega} + \overline{z}P^c\frac{d\Psi^*}{d\omega}\right) + \dot{\gamma}P^cJ\chi, v\right)_{L^2} + \tag{24}$$

$$+(8\pi\sqrt{\lambda}P^cJN(q_\chi)G_\lambda, q_vG_\lambda)_{L^2},$$

*for all $v \in V$.*

*Proof.* Equations (21) and (22) can be proved with the same argument exploited in the case $\sigma \in \left(0, \frac{1}{\sqrt{2}}\right)$ (see [3], Theorems V.3 and V.6).
Equation (23) can be obtained taking $v = J\Psi$ as test function and noting that

- $\frac{d\chi}{dt} = \dot{z}\Psi + \dot{\overline{z}}\Psi^* + \dot{\omega}\left(z\frac{d\Psi}{d\omega} + \overline{z}\frac{d\Psi^*}{d\omega}\right) + \frac{df}{dt}$,
- $(\Psi^*, J\Psi)_{L^2} = 0$,
- $\left(\frac{d\Phi_\omega}{d\omega}, J\Psi\right)_{L^2} = 0$,
- $\left(\frac{df}{dt}, J\Psi\right)_{L^2} = -\dot{\omega}\left(f, J\frac{d\Psi}{dt}\right)_{L^2}$, and
- $\dot{\omega}\left(\frac{d\Psi^*}{d\omega}, J\Psi\right)_{L^2} = -\left(\Psi^*, J\frac{d\Psi}{dt}\right)_{L^2}$.

Finally, equation (24) follows taking the projection onto the continuous spectrum $P^c$ of both sides of equation (19) and recalling that $f \in X^c$. □

### A. Frozen spectral decomposition

The goal of this subsection is to get an autonomous linearized equation for the component $f$.
According to (11), having fixed $T > 0$ we define

$$L_T = L(\omega_T) := -J \begin{bmatrix} H_{\alpha_1(T)} + \omega_T & 0 \\ 0 & H_{\alpha_2(T)} + \omega_T \end{bmatrix},$$

where $\alpha_1(T) = -(2\sigma + 1)\frac{\sqrt{\omega_T}}{4\pi}$, $\alpha_2 = -\frac{\sqrt{\omega_T}}{4\pi}$. and $\omega_T = \omega(T)$. Then for any $t \in [0, T]$ one can decompose $f(t) \in X^c = X^c(t)$ as

$$f = g + h \qquad \text{with} \quad g \in X_T^d = X_T^0 \oplus X_T^1, \ h \in X_T^c,$$

where the subscript $T$ means that the time is fixed at $t = T$. We recall that $P^0$ and $P^1$ are the symplectic projections on the kernel and nonzero eigenvalues of the linearization (see Appendix C). Correspondingly we denote by $P_T^0$ and $P_T^1$ the projections on the analogous subspaces frozen at time $T$. Finally denote $P_T^d = P_T^0 + P_T^1$.
Then for the quadratic form

$$Q_L(u, v) - Q_{L_T}(u, v) = \frac{\sqrt{\omega} - \sqrt{\omega_T}}{4\pi} \Re(\mathbb{T} q_u \overline{q_v}) - (\omega_T - \omega)(Ju, v)_{L^2},$$

for all $u$, $v \in V$, where

$$\mathbb{T} = \begin{bmatrix} 0 & -1 \\ 2\sigma+1 & 0 \end{bmatrix}.$$

Hence, observing that $P^c\Psi = 0$, the equation (24) for $f$ is equivalent to

$$\left(\frac{df}{dt}, v\right)_{L^2} = Q_{L_T}(f,v) + \left((\omega-\omega_T)Jf + \dot{\omega}\frac{dP^c}{d\omega}\psi + \dot{\gamma}P^cJ\chi, v\right)_{L^2} + \tag{25}$$

$$+\left(8\pi\sqrt{\lambda}\left(\frac{\sqrt{\omega}-\sqrt{\omega_T}}{4\pi}\mathbb{T}q_f + P^cJN(q_\chi)\right)G_\lambda, q_vG_\lambda\right)_{L^2},$$

for all $v \in V$.

Since our dispersive estimate holds only on the continuous spectral subspace, we need to prove that it is enough to estimate the symplectic projection of $\chi(t)$ onto that subspace. This is stated in the following lemma where we denote by $\mathcal{R}(a)$ a bounded continuous real valued functions vanishing as $a \to 0$, and

$$\mathcal{R}_1(\omega) = \mathcal{R}(\|\omega-\omega_0\|_{C^0([0,T])}).$$

Along the paper, with a slight abuse, we shall use the symbol $\mathcal{R}(a,b)$ to denote bounded continuous real valued functions vanishing as $a$, $b \to 0$.

**Lemma II.2.** *If $|\omega-\omega_T|$ is small enough, then the function $g$ can be estimated in terms of $h$ as follows:*

$$\|g\|_{L^\infty_{w^{-1}}} \le \mathcal{R}_1(\omega)|\omega-\omega_T|\|h\|_{L^\infty_{w^{-1}}}.$$

The last lemma can be proved following the proof of Lemma 3.2 in [18].

As a consequence, one can apply the operator $P^c_T$ to both sides of the equation for $f$ and obtain

$$\left(\frac{dh}{dt}, v\right)_{L^2} = Q_{L_T}(h,v) + \left(P^c_T\left[(\omega-\omega_T)Jf + \dot{\omega}\frac{dP^c}{d\omega}\psi + \dot{\gamma}P^cJ\chi\right], v\right)_{L^2} + \tag{26}$$

$$+\left(8\pi\sqrt{\lambda}P^c_T\left(\frac{\sqrt{\omega}-\sqrt{\omega_T}}{4\pi}\mathbb{T}q_f + P^cJN(q_\chi)\right)G_\lambda, q_vG_\lambda\right)_{L^2},$$

for any $v \in V$.

## B. Asymptotic expansion of dynamics

In order to prove the asymptotic stability of the ground state we need to show that for large times $z$ and $h$ are small. For this purpose, in this section we expand the inhomogeneous terms in the modulation equations, then rewrite the equations of $\omega$, $\gamma$, $z$ and $h$ as in (29), (30), (31) and (36), and for each equation we estimate the error terms.

In what follows we denote

$$(q,p) = q_1 p_1 + q_2 p_2, \qquad \forall p, q \in \mathbb{C}^2.$$

With an abuse of notation we denote by $q_\omega = \begin{pmatrix} \left(\frac{\sqrt{\omega}}{4\pi\nu}\right)^{1/(2\sigma)} \\ 0 \end{pmatrix}$ the charge of the function $\begin{pmatrix} \Phi_\omega \\ 0 \end{pmatrix}$.

As a preliminary step, we expand the nonlinear part of the equation (19) $N(q_\chi)$ as

$$N(q_\chi) = N_2(q_\chi) + N_3(q_\chi) + N_R(q_\chi), \tag{27}$$

where $N_2$ and $N_3$ are the quadratic and cubic terms in $q_\chi$ respectively, while $N_R$ is the remainder. Exploiting the Taylor expansion of the function $F(t) = t^\sigma$ around $|q_\omega|^2$, one gets

$$\Re(N_2(q_\chi)\overline{q_v}) = \Re((\sigma|q_\omega|^{2(\sigma-1)}|q_\chi|^2 q_\omega + 2\sigma|q_\omega|^{2(\sigma-1)}(q_\omega, q_\chi)q_\chi + 2(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)}(q_\omega, q_\chi)^2 q_\omega)\overline{q_v}),$$

and

$$\Re(N_3(q_\chi)\overline{q_v}) = \Re((\sigma|q_\omega|^{2(\sigma-1)}|q_\chi|^2 q_\chi + 2(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)}(q_\omega, q_\chi)^2 q_\chi +$$

$$+2(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)}(q_\omega, q_\chi)|q_\chi|^2 q_\omega + \frac{4}{3}(\sigma-2)(\sigma-1)\sigma|q_\omega|^{2(\sigma-3)}(q_\omega, q_\chi)^3 q_\omega)\overline{q_v}),$$

for any $q_v \in \mathbb{C}$. For later convenience, let us define the following symmetric forms

$$N_2(q_1, q_2) = \sigma|q_\omega|^{2(\sigma-1)}(q_1, q_2)q_\omega + \sigma|q_\omega|^{2(\sigma-1)}[(q_\omega, q_1)q_2 + (q_\omega, q_2)q_1] +$$

$$+2(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)}(q_\omega, q_1)(q_\omega, q_2)q_\omega,$$

and

$$N_3(q_1, q_2, q_3) = \frac{1}{6}\sigma|q_\omega|^{2(\sigma-1)}\sum_{i,j,k=1}^{3}(q_i, q_j)q_k + \frac{1}{3}(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)}\sum_{i,j,k=1}^{3}(q_\omega, q_i)(q_\omega, q_j)q_k +$$

$$+\frac{1}{3}(\sigma-1)\sigma|q_\omega|^{2(\sigma-2)}\sum_{i,j,k=1}^{3}(q_\omega,q_i)(q_j,q_k)q_\omega+\frac{4}{3}(\sigma-2)(\sigma-1)\sigma|q_\omega|^{2(\sigma-3)}(q_\omega,q_1)(q_\omega,q_2)(q_\omega,q_3)q_\omega.$$

In order to prove the asymptotic stability result, we shall prove in Section 2.4, the following asymptotics

$$\|f(t)\|_{L^\infty_{w^{-1}}}\sim t^{-1},\qquad z(t)\sim t^{-\frac{1}{2}},\qquad \|\psi(t)\|_V\sim t^{-\frac{1}{2}}, \tag{28}$$

as $t\to+\infty$.

**Remark II.3.** *As in [18], the first step in proving these expected asymptotics is to separate leading terms and remainders in the right hand sides of the modulation equations* (21) - (23), (26)*. Basically, in the next subsections, we will expand the expression for $\dot\omega$, $\dot\gamma$, and $\dot z$ up to and including the terms of order $t^{-3/2}$, and for $\dot h$ up to and including $t^{-1}$.*

**Remark II.4.** *Note that since the nonlinearity depends only on the charges the same holds for its Taylor expansion.*

### *1. Equation for $\omega$*

Substituting the expansion for the nonlinear part $N$ given in (27) in equation (21) and considering the asymptotics (28) one gets

$$\dot\omega=\frac{1}{\Delta}\Re((JN_2(q_\psi)+2JN_2(q_\psi,q_f)+JN_3(q_\psi))q_\omega)+\frac{1}{\Delta^2}\left(\psi,\frac{d\Phi_\omega}{d\omega}\right)_{L^2}\Re(JN_2(q_\psi)q_\omega)+\Omega_R,$$

where $\Delta=\frac{1}{2}\frac{d}{d\omega}\|\Phi_\omega\|^2_{L^2}$ and the remainder $\Omega_R$ is estimated by

$$|\Omega_R|\le\mathcal{R}(\omega,|z|+\|f\|_{L^\infty_{w^{-1}}})(|z|^2+\|f\|_{L^\infty_{w^{-1}}})^2.$$

Recalling that $\psi=z\Psi+\overline{z}\Psi^*$, one can rewrite the previous equation for $\dot\omega$ as

$$\dot\omega=\Omega_{20}z^2+\Omega_{11}z\overline{z}+\Omega_{02}\overline{z}^2+\Omega_{30}z^3+\Omega_{21}z^2\overline{z}+\Omega_{12}z\overline{z}^2+\Omega_{03}\overline{z}^3+z(q_f,\Omega'_{10})+\overline{z}(q_f,\Omega'_{01})+\Omega_R, \tag{29}$$

where the $\Omega_{ij}$'s and the $\Omega'_{ij}$'s are suitable coefficients, while $\Omega_R$ is a remainder term.

**Remark II.5.** *Since the second component of the vector $q_\omega$ equals* 0*, one has*

$$\Omega_{11}=2\frac{q_\omega}{\Delta}Re(JN_2(q_\Psi)\overline{q_{\Psi^*}})=0.$$

*This fact will turn out to be useful in writing the canonical form of the modulation equations.*

*2. Equation for $\gamma$*

As in the previous subsection the equation for $\dot\gamma$ (22) can expanded as

$$\dot\gamma = \frac{1}{\Delta}\Re((JN_2(q_\psi) + 2JN_2(q_\psi, q_f) + JN_3(q_\psi))\overline{q_{J\frac{d\Phi_\omega}{d\omega}}}) + \frac{1}{\Delta^2}\left(\psi, J\frac{d^2\Phi_\omega}{d^2\omega}\right)_{L^2}\Re(JN_2(q_\psi)q_\omega) + \Gamma_R,$$

where the remainder $\Gamma_R$ is estimated by

$$|\Gamma_R| \le \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2.$$

As before, the equation for $\dot\gamma$ shall be written in the form

$$\dot\gamma = \Gamma_{20}z^2 + \Gamma_{11}z\overline{z}\Gamma_{02}\overline{z}^2 + \Gamma_{30}z^3 + \Gamma_{21}z^2\overline{z} + \Gamma_{12}z\overline{z}^2 + \Gamma_{03}\overline{z}^3 + z(q_f, \Gamma'_{10}) + \overline{z}(q_f, \Gamma'_{01}) + \Gamma_R. \quad (30)$$

**Remark II.6.** *In this case $\Gamma_{11}$ does not vanish as in equation* (29).

*3. Equation for $z$*

Exploiting the results of the previous subsections, equation (23) can be expanded as

$$\begin{aligned}\dot z - i\xi z &= \tfrac{2}{\kappa}\Re(JN_2(q_\psi)\overline{q_f}) + \tfrac{1}{\kappa}\Re((JN_2(q_\psi) + JN_3(q_\psi))\overline{q_{J\Psi}}) + \\ &\quad -\tfrac{1}{\Delta\kappa}\left(\tfrac{d\psi}{d\omega}, J\Psi\right)_{L^2}\Re(JN_2(q_\psi)q_\omega) + \tfrac{1}{\Delta\kappa}(\psi,\Psi)_{L^2}\Re(JN_2(q_\psi)\overline{q_{J\frac{d\Phi_\omega}{d\omega}}}) + Z_R,\end{aligned}$$

where $\kappa = -(\Psi, J\Psi)_{L^2}$ and

$$|Z_R| \le \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2.$$

Mimicking the notation employed in (29) and (30), the previous equation can be written in the form

$$\dot z = i\xi z + Z_{20}z^2 + Z_{11}z\overline{z} + Z_{02}\overline{z}^2 + Z_{30}z^3 + Z_{21}z^2\overline{z} + Z_{12}z\overline{z}^2 + Z_{03}\overline{z}^3 + z\Re(q_f\overline{Z'_{10}}) + \overline{z}\Re(q_f\overline{Z'_{01}}) + Z_R, \quad (31)$$

and it turns out that

$$\begin{aligned} Z_{11} &= \tfrac{2}{\kappa}\Re(JN_2(q_\Psi, q_{\Psi^*})\overline{q_\Psi}), \quad Z_{20} = \tfrac{1}{\kappa}\Re(JN_2(q_\Psi)\overline{q_\Psi}), \quad Z_{02} = \tfrac{1}{\kappa}\Re(JN_2(q_{\Psi^*})\overline{q_\Psi}), \\ Z_{21} &= \tfrac{3}{\kappa}\Re(JN_3(q_{\Psi^*}, q_\Psi, q_\Psi)\overline{q_\Psi}) + \tfrac{1}{\Delta\kappa}\Big[\left(\tfrac{d\Psi^*}{d\omega}, j\Psi\right)_{L^2}\Re(JN_2(q_\Psi)\overline{q_{J\Phi_\omega}}) + \\ &\quad -(\Psi^*,\Psi)_{L^2}\Re(JN_2(q_\Psi)\overline{q_{\frac{d\Phi_\omega}{d\omega}}}) - 2\|\Psi\|^2_{L^2}\Re(JN_2(q_{\Psi^*}, q_\Psi)\overline{q_{\frac{d\Phi_\omega}{d\omega}}})\Big], \\ Z'_{10} &= 2\tfrac{JN_2(q_{\Psi^*}, q_\Psi)}{\overline{\kappa}}, \quad Z'_{01} = 2\tfrac{JN_2(q_\Psi)}{\overline{\kappa}}. \end{aligned} \quad (32)$$

*4. Equation for h*

In order to expand asymptotically the equation (26) for $h$, the following remark will be useful.

**Remark II.7.** *For any $f \in L^2(\mathbb{R}^3)$ the following holds*

$$P^c_T P^c f = P^c_T (I - P^d) f = P^c_T (P^c_T + P^d_T - P^d) f = P^c_T f + P^c_T (P^d_T - P^d) f.$$

Let us denote

$$\rho(t) = \omega(t) - \omega_T + \dot{\gamma}(t), \tag{33}$$

then equation (26) can be rewritten as

$$\left(\frac{dh}{dt}, v\right)_{L^2} = Q_{L_T}(h, v) + (\rho P^c_T J h, v)_{L^2} + (8\pi P^c_T J N_2(q_\psi) G_\lambda, q_v G_\lambda)_{L^2} +$$

$$+ \left( P^c_T \left[ \dot{\omega} \frac{dP^c}{d\omega} \psi + \dot{\gamma} P^c J \psi + \rho J g + \dot{\gamma} (P^d_T - P^d) J f \right], v \right)_{L^2} +$$

$$+ \left( 8\pi \sqrt{\lambda} P^c_T \left( \frac{\sqrt{\omega} - \sqrt{\omega_T}}{4\pi} \mathbb{T} q_f + P^c J N(q_\chi) - J N_2(q_\psi) \right) G_\lambda, q_v G_\lambda \right)_{L^2},$$

for any $v \in V$.

Denote

$$H'_R = P^c_T \left[ \dot{\omega} \frac{dP^c}{d\omega} \psi + \dot{\gamma} P^c J \psi + \rho J g + \dot{\gamma} (P^d_T - P^d) J f \right],$$

and

$$H''_R = 8\pi \sqrt{\lambda} P^c_T \left( \frac{\sqrt{\omega} - \sqrt{\omega_T}}{4\pi} \mathbb{T} q_f + P^c J N(q_\chi) - J N_2(q_\psi) \right) G_\lambda.$$

We recall that by $\Pi^\pm$ we denote (see Appendix C) the projections onto the branches $\mathcal{C}_\pm$ of the continuous spectrum separately. Analogously we denote by $\Pi^\pm_T$ the corresponding projections of the linear generator $L_T$ frozen at time $T$. The following lemma is useful in the following.

**Lemma II.8.** *There exists a constant $C > 0$ such that for each $h \in X^c_T$ holds*

$$\left\| [P^c_T J - i(\Pi^+_T - \Pi^-_T)] h \right\|_{L^1_w} \leq C \|h\|_{L^\infty_{w^{-1}}}.$$

The proof is in Appendix VI D for any $t > 0$. Finally, let us define

$$L_M(t) = L_T + i\rho(t)(\Pi_T^+ - \Pi_T^-), \tag{34}$$

then the previous equation becomes

$$\left(\frac{dh}{dt}, v\right)_{L^2} = Q_{L_M}(h, v) + (8\pi P_T^c J N_2(q_\psi) G_\lambda, q_v G_\lambda)_{L^2} + (\widetilde{H}_R, v)_{L^2} + (H_R'', q_v G_\lambda)_{L^2}, \tag{35}$$

for any $v \in V$, where we denoted

$$\widetilde{H}_R = H_R' + \rho[P_T^c J - i(\Pi_T^+ - \Pi_T^-)]h.$$

Finally, let us expand the second summand in the right hand side of (35), and get

$$\left(\frac{dh}{dt}, v\right)_{L^2} = Q_{L_M}(h, v) + (z^2 H_{20} + z\overline{z} H_{11} + \overline{z}^2 H_{02})\overline{q_v} + (\widetilde{H}_R, v)_{L^2} + (H_R'', q_v G_\lambda)_{L^2}, \tag{36}$$

for any $v \in V$, where

$$\begin{aligned} H_{20} &= (8\pi\sqrt{\lambda} P_T^c J N_2(q_\Psi) G_\lambda, G_\lambda)_{L^2}, \\ H_{11} &= 2(8\pi\sqrt{\lambda} P_T^c J N_2(q_\Psi, q_{\Psi^*}) G_\lambda, G_\lambda)_{L^2}, \\ H_{02} &= (8\pi\sqrt{\lambda} P_T^c J N_2(q_{\Psi^*}) G_\lambda, G_\lambda)_{L^2}. \end{aligned}$$

Thanks to the estimates done for the other equations and Lemma II.8, one can estimate the remainders in the following way:

$$\|H_R'\|_{L^1_w} \le C\left(|z|(|\dot{\omega}| + |\dot{\gamma}|) + \mathcal{R}_1(\omega)(|\omega - \omega_T| + |\dot{\gamma}|\|f\|_{L^\infty_{w^{-1}}})\right) \le$$

$$\le \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})\left(|z|^3 + |z|\|f\|_{L^\infty_{w^{-1}}} + \|f\|^2_{L^\infty_{w^{-1}}} + |\omega - \omega_T|\|f\|_{L^\infty_{w^{-1}}}\right),$$

hence

$$\|\widetilde{\mathcal{H}}_R\|_{L^1_w} \le \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})\left(|z|^3 + |z|\|f\|_{L^\infty_{w^{-1}}} + \|f\|^2_{L^\infty_{w^{-1}}} + |\omega - \omega_T|\|f\|_{L^\infty_{w^{-1}}}\right), \tag{37}$$

and

$$\|H_R''\|_{L^1_w} \le \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})\left(|z|^3 + |z|\|f\|_{L^\infty_{w^{-1}}} + \|f\|^2_{L^\infty_{w^{-1}}} + |\omega - \omega_T|(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})\right). \tag{38}$$

**Remark II.9.** *In the same way one directly expands the equation for the function $f$ getting*

$$\left(\frac{df}{dt}, v\right)_{L^2} = Q_L(f, v) + (z^2 F_{20} + z\overline{z} F_{11} + \overline{z}^2 F_{02})\overline{q_v} + (\widetilde{F}_R, v)_{L^2} + (F''_R, q_v G_\lambda)_{L^2}, \tag{39}$$

*for any $v \in V$, where*

$$F_{20} = (8\pi\sqrt{\lambda} J N_2(q_\Psi) G_\lambda, G_\lambda)_{L^2},$$
$$F_{11} = 2(8\pi\sqrt{\lambda} J N_2(q_\Psi, q_{\Psi^*}) G_\lambda, G_\lambda)_{L^2},$$
$$F_{02} = (8\pi\sqrt{\lambda} J N_2(q_{\Psi^*}) G_\lambda, G_\lambda)_{L^2}.$$

*and*

$$\widetilde{F}_R = \dot{\omega} \frac{dP^c}{d\omega} \psi + \dot{\gamma} P^c J \psi + \dot{\gamma}(P^d_T - P^d) J f,$$

$$F''_R = 8\pi\sqrt{\lambda} \left( \frac{\sqrt{\omega} - \sqrt{\omega_T}}{4\pi} \mathbb{T} q_f + P^c J N(q_\chi) - J N_2(q_\psi) \right) G_\lambda.$$

*Furthermore, the $L^1_w$ norms of the remainders $\widetilde{F}_R$ and $F''_R$ can be estimated by the corresponding norms of the remainders $\widetilde{H}_R$ and $H''_R$.*

## III. CANONICAL FORM OF THE EQUATIONS

In this section we would like to use the technique of normal coordinates in order to transform the modulation equations for $\omega$, $\gamma$, $z$, and $h$ to a simpler canonical form. We will also try to keep the estimates of the remainders as much close as possible to the original ones.

### A. Canonical form of the equation for $h$

Our goal is to exploit a change of variable in such a way that the function $h$ is mapped in a new function decaying in time at least as $t^{-3/2}$. For this purpose one could expand $h$ as

$$h = h_1 + k + k_1, \tag{40}$$

where

$$k = a_{20} z^2 + a_{11} z\overline{z} + a_{02} \overline{z}^2,$$

with some coefficients $a_{ij} = a_{ij}(x, \omega)$ such that $a_{ij} = \overline{a_{ji}}$, and

$$k_1 = -\exp\left(\int_0^t L_M(s) ds\right) k(0),$$

where the operator $L_M$ was defined in (34).

Note that $h_1(0) = h(0)$, since $k_1(0) = -k(0)$.

**Proposition III.1.** *There exist $a_{ij} \in L^\infty_{w^{-1}}(\mathbb{R}^3)$, for $i$, $j = 0, 1, 2$, such that the equation for $h_1$ has the form*

$$\left(\frac{dh_1}{dt}, v\right)_{L^2} = Q_{L_M}(h_1, v) + (\widehat{H}_R, v)_{L^2} + (H''_R, q_v G_\lambda)_{L^2}, \tag{41}$$

*for all $v \in V$, where $\widehat{H}_R = \widetilde{H}_R + \overline{H}_R$ with*

$$\overline{H}_R = -\left[\dot{\omega}\left(\frac{da_{20}}{d\omega}z^2 + \frac{da_{11}}{d\omega}z\overline{z} + \frac{da_{02}}{d\omega}\overline{z}^2\right) + (2a_{20}z + a_{11}\overline{z})(\dot{z} - i\xi_T z) + \right. \tag{42}$$

$$\left. +(a_{11}z + 2a_{20}\overline{z})(\dot{\overline{z}} + i\xi_T\overline{z}) - \rho(\Pi_T^+ - \Pi_T^-)k\right].$$

*Proof.* The thesis is proved substituting (40) into (35) and equating the coefficients of the quadratic powers of $z$ which leads to the system

$$\begin{cases} Q_{L_T}(a_{20}, v) + \Re(H_{20}\overline{q_v}) - (2i\xi_T a_{20}, v)_{L^2} = 0 \\ Q_{L_T}(a_{11}, v) + \Re(H_{11}\overline{q_v}) = 0 \\ Q_{L_T}(a_{02}, v) + \Re(H_{02}\overline{q_v}) + (2i\xi_T a_{02}, v)_{L^2} = 0 \end{cases}, \tag{43}$$

for all $v \in V$. The former system admits the solution

$$a_{11} = -L_T^{-1}H_{11}$$
$$a_{20} = -(L_T - 2i\xi_T - 0)^{-1}H_{20}$$
$$a_{02} = \overline{a_{02}} = -(L_T + 2i\xi_T - 0)^{-1}H_{02}$$

□

**Remark III.2.** *From the explicit structure of the remainder $\widehat{H}_R$ it follows that it still satisfies estimate* (37).

We will need to apply the next lemma which can be proved as Proposition 2.3 in [18].

**Lemma III.3.** *If $\sigma \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$ and $f \in V \cap L^1_w$, then there exists some constant $C > 0$ such that for any $t \geq 0$*

$$\|e^{-L_T t}(L_T + 2i\xi_T - 0)^{-1}P_T^c f\|_{L^\infty_{w^{-1}}} \leq C(1+t)^{-3/2}\|f\|_{L^1_w}.$$

**Remark III.4.** *Let us note that*

$$h = P_T^c h = P_T^c h_1 + P_T^c k + P_T^c k_1,$$

*hence, in order to estimate the decay of* $\|h\|_{L^\infty_{w^{-1}}}$*, it suffices to estimate the decay of*

$$\|P_T^c h_1\|_{L^\infty_{w^{-1}}}, \quad \|P_T^c k\|_{L^\infty_{w^{-1}}}, \quad \textit{and} \quad \|P_T^c k_1\|_{L^\infty_{w^{-1}}}.$$

### B. Canonical form of the equation for $\omega$

Since $\Omega_{11} = 0$, we can exploit the method by Buslaev and Sulem in [9], Proposition 4.1 and get the following proposition.

**Proposition III.5.** *There exist coefficients* $b_{ij} = b_{ij}(\omega)$*, with* $i$*,* $j = 0, 1, 2, 3$*, and vector functions* $b'_{ij} = b'_{ij}(x, \omega)$*, with* $i$*,* $j = 0, 1$*, such that function*

$$\omega_1 = \omega + b_{20} z^2 + b_{11} z\overline{z} + b_{02}\overline{z}^2 + b_{30} z^3 + b_{21} z^2\overline{z} + b_{12} z\overline{z}^2 + b_{03}\overline{z}^3 +$$

$$+ z(f, b'_{10})_{L^2} + \overline{z}(f, b'_{01})_{L^2},$$

*solves a differential equation of the form*

$$\dot{\omega}_1 = \widehat{\Omega}_R,$$

*for some remainder* $\widehat{\Omega}_R$*.*

*Proof.* Substituting the equations (29), (31), and (39) into the derivative with respect to time of the expression for $\omega_1$ and equating the coefficients of $z^2$, $z\overline{z}$, $\overline{z}^2$, $z$, and $\overline{z}$ one gets the following system

$$\begin{cases} \Omega_{20} + 2i\xi b_{20} = 0 \\ \Omega_{02} - 2i\xi b_{02} = 0 \\ \Omega_{30} + 3i\xi b_{30} + 2Z_{20} b_{20} + \Re(F_{20}\overline{q_{b'_{10}}}) = 0 \\ \Omega_{03} - 3i\xi b_{03} + 2Z_{02} b_{02} + \Re(F_{02}\overline{q_{b'_{01}}}) = 0 \\ \Omega_{21} + i\xi b_{21} + 2Z_{11} b_{20} + 2Z_{20} b_{02} + \Re(F_{11}\overline{q_{b'_{10}}} + F_{20}\overline{q_{b'_{01}}}) = 0 \\ \Omega_{12} - i\xi b_{12} + 2Z_{11} b_{02} + 2Z_{20} b_{20} + \Re(F_{11}\overline{q_{b'_{01}}} + F_{20}\overline{q_{b'_{10}}}) = 0 \\ (q_f, \Omega'_{10}) + i\xi (f, b'_{10})_{L^2} + Q_L(f, b'_{10}) = 0 \\ (q_f, \Omega'_{01}) + i\xi (f, b'_{01})_{L^2} + Q_L(f, b'_{01}) = 0 \end{cases}.$$

The last two equations of this system can be solved in a way similar to the ones system (43), and the proof follows. □

**Remark III.6.** *From the proof of the previous proposition it also follows that the remainder $\widehat{\Omega}_R$ can be estimated as $\Omega_R$, namely*

$$|\widehat{\Omega}_R| \leq \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2.$$

In the next lemma we prove a uniform bound for $|\omega_T - \omega|$ on the interval $[0, T]$. For later convenience let us denote

$$\mathcal{R}_2(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}}) = \mathcal{R}\left(\max_{0\leq t\leq T} |\omega_T - \omega|, \max_{0\leq t\leq T} (|z| + \|f\|_{L^\infty_{w^{-1}}})\right).$$

**Remark III.7.** *Let us note that $|\omega| \leq |\omega_0| + |\omega_0 - \omega_T| + |\omega - \omega_T|$, then*

$$\max_{0\leq t\leq T} \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}}) = \mathcal{R}\left(\max_{0\leq t\leq T} |\omega_T - \omega|, \max_{0\leq t\leq T} (|z| + \|f\|_{L^\infty_{w^{-1}}})\right).$$

The next lemma can be proved as in Section 3.5 of [18].

**Lemma III.8.** *For any $t \in [0, T]$ we have*

$$|\omega_T - \omega| \leq \mathcal{R}_2(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}}) \left[\int_t^T (|z(\tau)| + \|f(\tau)\|_{L^\infty_{w^{-1}}})^2 d\tau + \right.$$

$$\left. + (|z_T| + \|f_T\|_{L^\infty_{w^{-1}}})^2 + (|z| + \|f\|_{L^\infty_{w^{-1}}})^2\right].$$

### C. Canonical form of the equation for $\gamma$

Equations (30) for $\gamma$ and (29) for $\omega$ differ just because in general $\Gamma_{11} \neq 0$. But we can perform the same change of variable in the previous subsection, namely

$$\gamma_1 = \gamma + d_{20}z^2 + d_{02}\overline{z}^2 + d_{30}z^3 + d_{21}z^2\overline{z} + d_{12}z\overline{z}^2 + d_{03}\overline{z}^3 + z(f, d'_{10})_{L^2} + \overline{z}(f, d'_{01})_{L^2},$$

for some suitable coefficients $d_{ij} = d_{ij}(\omega)$, with $i$, $j = 0$, 1, 2, 3, and vector functions $d'_{ij} = d'_{ij}(x, \omega)$, with $i$, $j = 0$, 1. Then the function $\gamma_1$ solves the differential equation

$$\dot{\gamma}_1 = \Gamma_{11}(\omega) z\overline{z} + \widehat{\Gamma}_R,$$

for some remainder $\widehat{\Gamma}_R$, which can be estimated as $\Gamma_R$, i.e.

$$|\widehat{\Gamma}_R| \leq \mathcal{R}(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2.$$

### D. Canonical form of the equation for $z$

Exploiting the change of variable (40) used to obtain the canonical form of equation (35) for $h$, one can prove the following proposition.

**Proposition III.9.** *There exist coefficients $c_{ij} = c_{ij}(\omega)$, with $i,\ j = 0,\ 1,\ 2,\ 3,$ such that function*

$$z_1 = z + c_{20}z^2 + c_{11}z\overline{z} + c_{02}\overline{z}^2 + c_{30}z^3 + c_{21}z^2\overline{z} + c_{03}\overline{z}^3,$$

*solves a differential equation of the form*

$$\dot{z}_1 = i\xi z_1 + iK|z_1|^2 z_1 + \widehat{Z}_R, \tag{44}$$

*where*

$$iK = Z_{21} + Z'_{21} + \frac{i}{\xi}Z_{20}Z_{11} - \frac{i}{\xi}Z_{11}^2 - \frac{2i}{3\xi}Z_{02}^2,$$

*with the coefficient $Z_{ij}$, $i,\ j = 0, 1, 3,$ defined in* (32), *and*

$$\widetilde{Z}_R = (g + P_T^c h_1 + P_T^c k_1, Z'_{10})z + (g + P_T^c h_1 + P_T^c k_1, Z'_{01})\overline{z} + Z_R.$$

The proof is a matter of calculation, but we give it explicitly to stress the role of the functions $a_{ij}$, $i,\ j = 0,\ 1,\ 2$.

*Proof.* Substituting (40) in the equation (31) the differential equation for $z$ becomes

$$\dot{z} = i\xi z + Z_{20}z^2 + Z_{11}z\overline{z} + Z_{02}\overline{z}^2 + Z_{30}z^3 + Z_{21}z^2\overline{z} + Z_{12}z\overline{z}^2 + Z_{03}\overline{z}^3 + \tag{45}$$

$$+ Z'_{30}z^3 + Z'_{21}z^2\overline{z} + Z'_{12}z\overline{z}^2 + Z'_{03}\overline{z}^3 + \widetilde{Z}_R,$$

where

$$Z'_{30} = \Re(q_{a_{20}}\overline{Z'_{10}}),$$

$$Z'_{03} = \Re(q_{a_{02}}\overline{Z'_{01}}),$$

$$Z'_{21} = \Re(q_{a_{11}}\overline{Z'_{10}}) + \Re(q_{a_{20}}\overline{Z'_{01}}),$$

$$Z'_{12} = \Re(q_{a_{11}}\overline{Z'_{01}}) + \Re(q_{a_{02}}\overline{Z'_{10}}),$$

and the remainder $\widetilde{Z}_R$ is as in the statement of the proposition.

Inserting equation (45) into the time derivative of the expression for $z_1$ and equating the coefficients of $z^2$, $z\overline{z}$, $\overline{z}^2$, $z^3$, $z\overline{z}^2$, and $\overline{z}^3$ one obtains the system

$$\begin{cases} i\xi c_{20} + Z_{20} = 0 \\ -i\xi c_{11} + Z_{11} = 0 \\ -3i\xi c_{02} + Z_{02} = 0 \\ 2i\xi c_{30} + Z_{30} + Z'_{30} + 2c_{20}Z_{20} + c_{11}Z_{20} = 0 \\ Z_{12} + Z'_{12} + 2c_{20}Z_{20} + c_{11}(Z_{11} + Z_{02}) + 2c_{02}Z_{11} - 2i\xi c_{12} = 0 \\ -4i\xi c_{03} + Z_{03} + Z'_{03} + c_{11}Z_{02} = 0 \end{cases}$$

The theorem follows from the fact the the above system is solvable and in particular

$$c_{20} = \frac{i}{\xi} Z_{20}, \quad c_{11} = -\frac{i}{\xi} Z_{11}, \quad \text{and} \quad c_{02} = \frac{i}{3\xi} Z_{02}.$$

$\square$

**Remark III.10.** *For later convenience let us note that, since $Z_{21}$, $Z_{20}$, $Z_{11}$, and $Z_{02}$ are purely imaginary, one has*

$$\Re(iK) = \Re(Z'_{21}).$$

*Moreover, we need the following lemma.*

**Lemma III.11.** *There exists $\sigma^* \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$ such that if $\sigma \in \left(\frac{1}{\sqrt{2}}, \sigma^*\right)$, then*

$$\Re(Z'_{21}) < 0,$$

*$\forall \omega$ belonging to an open neighbourhood of $\omega_0$.*

*Proof.* First of all recall that $\xi_T = 2\sigma\sqrt{1-\sigma^2}\omega_T$, then one can compute

$$\kappa = -(\Psi, J\Psi)_{L^2} = \frac{i}{4\pi\sqrt{\omega_T}} \left( \frac{1}{\sqrt{1 - 2\sigma\sqrt{1-\sigma^2}}} - \frac{(\sqrt{1-\sigma^2}-1)^2}{\sigma^2} \frac{1}{\sqrt{1+2\sigma\sqrt{1-\sigma^2}}} \right) = \tag{46}$$

$$= \frac{i}{4\pi\sqrt{\omega_T}} \frac{\sigma^2\sqrt{1+2\sigma\sqrt{1-\sigma^2}} - (\sqrt{1-\sigma^2}-1)^2\sqrt{1-2\sigma\sqrt{1-\sigma^2}}}{\sigma^2(2\sigma^2-1)}.$$

Since $\kappa$ is purely imaginary with positive imaginary part and $L_T^{-1}2P_T^cJ$ is self-adjoint, for the first summand in the expression for $\Re(Z'_{21})$ one gets

$$\Re(q_{a_{11}}\overline{Z'_{10}}) = -2\Re\left(\frac{q_{L_T^{-1}2P_T^cJN_2(q_\Psi,q_{\Psi^*})}\overline{JN_2(q_\Psi,q_{\Psi^*})}}{\kappa}\right) = 0.$$

Hence,

$$\Re(Z'_{21}) = -2\Re\left(\frac{q_{a_{20}}\overline{JN_2(q_\Psi)}}{\kappa}\right).$$

By direct computations one has

$$a_{20}(x) = (L_T - (2i\xi_T+0))^{-1}H_{20} = A\frac{e^{-\sqrt{\omega_T+2\xi_T}|x|}}{4\pi|x|}\begin{pmatrix}1\\-i\end{pmatrix} + C\frac{e^{-i\sqrt{-\omega_T+2\xi_T}|x|}}{4\pi|x|}\begin{pmatrix}1\\i\end{pmatrix},$$

with

$$\begin{array}{l} A = -\frac{4\pi}{d}[((2\sigma+1)\sqrt{\omega_T} - i\sqrt{-\omega_T+2\xi_T})(H_{20})_1 + (i\sqrt{\omega_T} + \sqrt{-\omega_T+2\xi_T})(H_{20})_2] \\ C = \frac{4\pi}{d}[((2\sigma+1)\sqrt{\omega_T} - \sqrt{\omega_T+2\xi_T})(H_{20})_1 - (i\sqrt{\omega_T} - i\sqrt{\omega_T+2\xi_T})(H_{20})_2] \end{array},$$

where $d = 2i(2\sigma+1)\omega_T + 2(\sigma+1)\sqrt{\omega_T}\sqrt{-\omega_T+2\xi_T} - 2i(\sigma+1)\sqrt{\omega_T}\sqrt{\omega_T+2\xi_T} - 2\sqrt{\omega_T+2\xi_T}\sqrt{-\omega_T+2\xi_T}$. From which follows

$$q_{a_{20}} = \frac{4\pi}{d}\left[\begin{pmatrix} (i\sqrt{-\omega_T+2\xi_T} - \sqrt{\omega_T+2\xi_T})(H_{20})_1 \\ ((2\sigma+1)\sqrt{\omega_T} - \sqrt{\omega_T+2\xi_T} - \sqrt{-\omega_T+2\xi_T})(H_{20})_1 \end{pmatrix} + \right.$$

$$\left. + \begin{pmatrix} -i(2\sqrt{\omega_T} + \sqrt{\omega_T+2\xi_T} + i\sqrt{-\omega_T+2\xi_T})(H_{20})_2 \\ (-\sqrt{\omega_T+2\xi_T} + i\sqrt{-\omega_T+2\xi_T})(H_{20})_2 \end{pmatrix}\right].$$

Hence

$$\begin{aligned} \Re((q_{a_{20}})_1) = &\ \tfrac{16\pi}{|d|^2}[i(H_{20})_1(-(\sigma+1)\sqrt{\omega_T}\sqrt{\omega_T+2\xi_T}\sqrt{-\omega_T+2\xi_T} + ((\sigma+1)\omega_T+\xi_T)\sqrt{-\omega_T+2\xi_T})+ \\ &(H_{20})_2(-(2(\sigma+1)\xi_T + (2\sigma+1)\omega_T)\sqrt{\omega_T} + (\xi_T + (2\sigma+1)\omega_T)\sqrt{\omega_T+2\xi_T})] \\ \Im((q_{a_{20}})_2) = &\ \tfrac{16\pi}{|d|^2}[i(H_{20})_1((2(\sigma+1)^2\omega_T+\xi_T)\sqrt{-\omega_T+2\xi_T} - (3\sigma+2)\sqrt{\omega_T}\sqrt{\omega_T+2\xi_T}\sqrt{-\omega_T+2\xi_T})+ \\ &+(H_{20})_2((\sigma+1)\omega_T^{3/2} + ((\sigma+1)\omega_T - \xi_T)\sqrt{\omega_T+2\xi_T})]. \end{aligned} \tag{47}$$

Moreover, by (27) one gets

$$JN_2(q_\Psi) = \begin{pmatrix} -2\sigma|q_{\omega_T}|^{2\sigma-1}(q_\Psi)_1(q_\Psi)_2 \\ \sigma|q_{\omega_T}|^{2\sigma-1}(3(q_\Psi)_1^2 + (q_\Psi)_2^2) + 2\sigma(\sigma-1)|q_{\omega_T}|^{2\sigma-1}(q_\Psi)_1^2 \end{pmatrix} = \tag{48}$$

$$= \begin{pmatrix} -2i\sigma|q_{\omega_T}|^{2\sigma-1}\left(1-\frac{\sqrt{1-\sigma^2}-1}{\sigma}\right)\left(1+\frac{\sqrt{1-\sigma^2}-1}{\sigma}\right) \\ 2\sigma|q_{\omega_T}|^{2\sigma-1}\left(1-\frac{\sqrt{1-\sigma^2}-1}{\sigma}\right) \end{pmatrix},$$

which implies

$$H_{20} = (8\pi\sqrt{\omega_T}P_T^c JN_2(q_\Psi)G_{\omega_T}, G_{\omega_T})_{L^2} = \tag{49}$$

$$= JN_2(q_\Psi) - \frac{(JN_2(q_\Psi))_1|q_{\omega_T}|}{16\pi\Delta\omega^{3/2}}\left(\frac{1}{\sigma}-1\right)\begin{pmatrix}1\\0\end{pmatrix}+$$

$$+\frac{\sqrt{\omega_T}}{\kappa}\begin{pmatrix} -(JN_2(q_\Psi))_2\left(\frac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}-\frac{\sqrt{1-\sigma^2}-1}{\sigma}\frac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2 \\ (JN_2(q_\Psi))_1\left(\frac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}+\frac{\sqrt{1-\sigma^2}-1}{\sigma}\frac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2 \end{pmatrix}.$$

Let us notice that (48) and (49) imply

$$\begin{aligned}
i(H_{20})_1 i(JN_2(q_\Psi))_1 = {} & -\tfrac{1}{2\sigma-1}(JN_2(q_\Psi))_1^2+ \\
& +\tfrac{\sqrt{\omega_T}}{4\pi i\kappa}\left(\tfrac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}-\tfrac{\sqrt{1-\sigma^2}-1}{\sigma}\tfrac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2 i(JN_2(q_\Psi))_1(JN_2(q_\Psi))_2 \\
(H_{20})_2 i(JN_2(q_\Psi))_1 = {} & (JN_2(q_\Psi))_2 i(JN_2(q_\Psi))_1+ \\
& -\tfrac{\sqrt{\omega_T}}{4\pi i\kappa}\left(\tfrac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}+\tfrac{\sqrt{1-\sigma^2}-1}{\sigma}\tfrac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2 (JN_2(q_\Psi))_1^2 \\
i(H_{20})_1 (JN_2(q_\Psi))_2 = {} & \tfrac{1}{2\sigma-1}i(JN_2(q_\Psi))_1(JN_2(q_\Psi))_2+ \\
& +\tfrac{\sqrt{\omega_T}}{4\pi i\kappa}\left(\tfrac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}-\tfrac{\sqrt{1-\sigma^2}-1}{\sigma}\tfrac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2 (JN_2(q_\Psi))_2^2 \\
(H_{20})_2 (JN_2(q_\Psi))_2 = {} & (JN_2(q_\Psi))_2^2+ \\
& +\tfrac{\sqrt{\omega_T}}{4\pi i\kappa}\left(\tfrac{1}{\sqrt{\omega_T-\xi_T}+\sqrt{\omega_T}}+\tfrac{\sqrt{1-\sigma^2}-1}{\sigma}\tfrac{1}{\sqrt{\omega_T+\xi_T}+\sqrt{\omega_T}}\right)^2 i(JN_2(q_\Psi))_1(JN_2(q_\Psi))_2,
\end{aligned}$$

then by (46) and (47) it follows

$$\Re(Z'_{21}) = -2\Re\left(\frac{q_{a_{20}}\overline{JN_2(q_\Psi)}}{\kappa}\right) = \frac{2}{i\kappa}(\Re((q_{a_{20}})_1)i(JN_2(q_\Psi))_1 + \Im((q_{a_{20}})_2)(JN_2(q_\Psi))_2) =$$

$$= \frac{128\pi\omega_T^{3/2}|q_{\omega_T}|^{4\sigma-2}}{i\kappa|d|^2}\sigma^2\left(1-\frac{\sqrt{1-\sigma^2}-1}{\sigma}\right)^2 f(\sigma),$$

with

$$f(\sigma) = \left(\left(2(1+\sigma)^2+2\sigma\sqrt{1-\sigma^2}\right)\sqrt{-1+4\sigma\sqrt{1-\sigma^2}} - (2+3\sigma)\sqrt{-1+4\sigma\sqrt{1-\sigma^2}}\sqrt{1+4\sigma\sqrt{1-\sigma^2}}+\right.$$

$$+\left(1+\frac{-1+\sqrt{1-\sigma^2}}{\sigma}\right)\left((-1-\sigma)\sqrt{-1+16\sigma^2-16\sigma^4}+\left(1+\sigma+2\sigma\sqrt{1-\sigma^2}\right)\sqrt{-1+4\sigma\sqrt{1-\sigma^2}}\right)\cdot$$

$$\cdot\left(\frac{1}{-1+2\sigma}-\frac{\left(\frac{1}{1+\sqrt{1-2\sigma\sqrt{1-\sigma^2}}}-\frac{-1+\sqrt{1-\sigma^2}}{\sigma+\sigma\sqrt{1+2\sigma\sqrt{1-\sigma^2}}}\right)^2}{\frac{1}{\sqrt{1-2\sigma\sqrt{1-\sigma^2}}}-\frac{\left(-1+\sqrt{1-\sigma^2}\right)^2}{\sigma^2\sqrt{1+2\sigma\sqrt{1-\sigma^2}}}}\right)+$$

$$+\left(1+\sigma+\left(1+\sigma-2\sigma\sqrt{1-\sigma^2}\right)\sqrt{1+4\sigma\sqrt{1-\sigma^2}}+\right.$$

$$\left.+\left(1+\frac{-1+\sqrt{1-\sigma^2}}{\sigma}\right)\left(-1-2\sigma+\left(-4\sigma-4\sigma^2\right)\sqrt{1-\sigma^2}+\left(1+2\sigma+2\sigma\sqrt{1-\sigma^2}\right)\sqrt{1+4\sigma\sqrt{1-\sigma^2}}\right)\right)\cdot$$

$$\cdot\left(1-\frac{\left(\frac{1}{1+\sqrt{1-2\sigma\sqrt{1-\sigma^2}}}+\frac{-1+\sqrt{1-\sigma^2}}{\sigma+\sigma\sqrt{1+2\sigma\sqrt{1-\sigma^2}}}\right)^2}{\frac{1}{\sqrt{1-2\sigma\sqrt{1-\sigma^2}}}-\frac{\left(-1+\sqrt{1-\sigma^2}\right)^2}{\sigma^2\sqrt{1+2\sigma\sqrt{1-\sigma^2}}}}\right).$$

Notice that one has $f(\sigma)\to\widetilde{f}>0$, $d\to\widetilde{d}\neq 0$, and $i\kappa\to-\infty$ as $\sigma\to 1/\sqrt{2}$; this implies

$$\lim_{\sigma\to 1/\sqrt{2}}\Re(Z'_{21})=\frac{128\sqrt{2}\,\omega_T^{3/2}|q_{\omega_T}|^{2\sqrt{2}-2}}{\pi|\widetilde{d}|^2}\widetilde{f}\lim_{\sigma\to 1/\sqrt{2}}\frac{1}{i\kappa}=0^-.$$

Hence there is a neighborhood of $\frac{1}{\sqrt{2}}$ where $\Re(Z'_{21})$ is strictly negative. A *Mathematica* plot of the function $f(\sigma)$ in the range $\left(\frac{1}{\sqrt{2}},\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$ is given in figure III.10.

FIG. 1. $f(\sigma)$

Summing up, one can conclude that there exists $\sigma^*\in\left(\frac{1}{\sqrt{2}},\frac{\sqrt{3}+1}{2\sqrt{2}}\right]$ such that $\Re(Z'_{21})<0$ for $\sigma\in\left(\frac{1}{\sqrt{2}},\sigma^*\right)$. □

**Remark III.12.** *The following reformulation of the equation for $z_1$ will be useful. First of all, if we denote $K_T=K(\omega_T)$, then the ordinary differential equation for $z_1$ becomes*

$$\dot{z}_1=i\xi z_1+iK_T|z_1|^2z_1+\widehat{\widehat{Z}}_R,$$

*for some remainder $\widehat{\widehat{Z}}_R$.*

*Secondly, let us notice that $z_1$ is oscillating while $y = |z_1|^2$ decreases at infinity. Hence, it is easier to deal with the variable $y$, which satisfies the equation*

$$\dot{y} = 2\Re(iK_T)y^2 + Y_R, \tag{50}$$

*where $Y_R$ is some suitable remainder.*

**Remark III.13.** *From Lemma II.2 we have*

$$|(g + P_T^c h_1 + P_T^c k_1, Z'_{10})| \leq \mathcal{R}(\omega)(\|g\|_{L^\infty_{w^{-1}}} + \|P_T^c h_1\|_{L^\infty_{w^{-1}}} + \|P_T^c k_1\|_{L^\infty_{w^{-1}}}) \leq$$

$$\leq \mathcal{R}_1(\omega)(|\omega_T - \omega|\|h\|_{L^\infty_{w^{-1}}} + \|P_T^c h_1\|_{L^\infty_{w^{-1}}} + \|P_T^c k_1\|_{L^\infty_{w^{-1}}}),$$

*hence*

$$|Y_R| = |\widehat{\widehat{Z}}_R||z| = |\widetilde{Z}_R + i(K - K_T)|z_1|^2 z_1||z| \leq$$

$$\leq \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})|z|[(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2 + |z||\omega_T - \omega|(|z|^2 + \|h\|_{L^\infty_{w^{-1}}}) +$$

$$+ |z|\|P_T^c k_1\|_{L^\infty_{w^{-1}}} + |z|\|P_T^c h_1\|_{L^\infty_{w^{-1}}}].$$

## IV. MAJORANTS

In this section we exploit the so-called majorant method to prove large time asymptotic for the solutions of the modulation equations. Preliminarily, we need some assumptions on the initial conditions.

### A. Initial conditions

Let us fix some $\epsilon > 0$ to be chosen later in order to obtain a uniform control in the estimates. Then we assume that

$$\begin{aligned} &|z(0)| \leq \epsilon^{1/2} \\ &\|f(0)\|_{L^1_w} \leq c\epsilon^{1/2}, \end{aligned} \tag{51}$$

where $c > 0$ is some positive constant.

From the definition of $z_1$ (see Proposition III.5) one has

$$z_1 - z = \mathcal{R}(\omega)|z|^2.$$

Then the following estimate holds

$$y(0) = |z_1(0)|^2 \leq |z(0)|^2 + \mathcal{R}(\omega, |z(0)|)|z(0)|^3 \leq \epsilon + \mathcal{R}(\omega, |z(0)|)\epsilon^{3/2}.$$

We also need an estimate for the initial datum of the function $h(t)$. For this purpose recall that, from the definition of $f$ and $h$ one has the decomposition $h = f + (P^d - P_T^d)f$ . Hence,

$$\|h(0)\|_{L^1_w} \leq \|f(0)\|_{L^1_w} + \|(P^d - P_T^d)f(0)\|_{L^1_w} \leq c\epsilon^{3/2} + \mathcal{R}_1(\omega)|\omega_T - \omega|\|f(0)\|_{L^\infty_{w^{-1}}},$$

for some constant $c > 0$.

Thanks to the former estimates, one can prove the following lemma.

**Lemma IV.1.** *Let us assume conditions* (51) *on the initial data. Then*

$$\|P_T^c k_1\|_{L^\infty_{w^{-1}}} \leq c\frac{|z(0)|^2}{(1+t)^{3/2}} \leq \frac{c\epsilon}{(1+t)^{3/2}},$$

*for all* $t \geq 0$.

*Proof.* Let us denote $\zeta = \int_0^t \rho(\tau)d\tau$ (the quantity $\rho$ was defined in (33)).

From the definition of the exponential and the idempotency of the projections one gets

$$e^{i\zeta\Pi_T^\pm} = \Pi_T^\pm e^{i\zeta} + \Pi_T^\mp + P_T^d.$$

Then it follows

$$e^{i\zeta(\Pi_T^+ - \Pi_T^-)} = (\Pi_T^+ e^{i\zeta} + \Pi_T^- + P_T^d)(\Pi_T^- e^{-i\zeta} + \Pi_T^+ + P_T^d) = \Pi_T^+ e^{i\zeta} + \Pi_T^- e^{-i\zeta} + P_T^d.$$

The lemma follows from the fact that $L_T$ commutes with the projectors $\Pi_T^\pm$, the definition (34) of the operator $L_M$ and the decay of the evolution of the functions $P_T^c a_{ij}$, $i$, $j = 0$, 1, 2, stated in Lemma III.3, namely

$$\|P_T^c k_1\|_{L^\infty_{w^{-1}}} = \left\| e^{\int_0^t L_M(\tau)d\tau} P_T^c k(0) \right\|_{L^\infty_{w^{-1}}} =$$

$$= \|e^{L_T t} P_T^c (e^{i\zeta}\Pi_T^+ + e^{-i\zeta}\Pi_T^- + P_T^d)(a_{20}z^2(0) + a_{11}z(0)\overline{z(0)} + a_{02}\overline{z(0)}^2)\|_{L^\infty_{w^{-1}}} \leq$$

$$\leq c\frac{|z(0)|^2}{(1+t)^{3/2}} \leq \frac{c\epsilon}{(1+t)^{3/2}}.$$

$\square$

### B. Definition of the majorants

We are now in the position to define the majorants:

$$M_0(T) = \max_{0\le t\le T} |\omega_T - \omega| \left(\frac{\epsilon}{1+\epsilon t}\right)^{-1} \tag{52}$$

$$M_1(T) = \max_{0\le t\le T} |z(t)| \left(\frac{\epsilon}{1+\epsilon t}\right)^{-1/2} \tag{53}$$

$$M_2(T) = \max_{0\le t\le T} \|P_T^c h_1(t)\|_{L^\infty_{w^{-1}}} \left(\frac{\epsilon}{1+\epsilon t}\right)^{-3/2} \tag{54}$$

We shall use the following vector notation

$$M = (M_0, M_1, M_2). \tag{55}$$

**Remark IV.2.** *From the estimates on $g$, $k_1$ and the definitions of the majorants follows*

$$\|f\|_{L^\infty_{w^{-1}}} = \|g + P_T^c h_1 + P_T^c k + P_T^c k_1\|_{L^\infty_{w^{-1}}} \le$$

$$\le \mathcal{R}_1(\omega)\left(|\omega_T - \omega| + |z|^2 + \frac{\epsilon}{(1+t)^{3/2}}\|P_T^c h_1\|_{L^\infty_{w^{-1}}}\right) \le$$

$$\le \frac{\epsilon}{1+\epsilon t}\mathcal{R}_1(\omega)(M_1^2 + \epsilon^{1/2}M_2).$$

From the assumptions (51) on the initial data one obtains

$$y(0) \le \epsilon + \mathcal{R}(\epsilon^{1/2}M)\epsilon^{3/2} \le \epsilon(1 + \mathcal{R}(\epsilon^{1/2}M)\epsilon^{1/2}),$$

$$\|h(0)\|_{L^1_w} \le c\epsilon^{3/2}\mathcal{R}(\epsilon^{1/2}M)\epsilon^2 M_0(1 + M_1^2 + \epsilon^{1/2}M_2).$$

### C. The equation for $y$

We aim at studying the asymptotic behavior of the solution of equation (50) for the variable $y$ introduced in Remark 2.18. To do that we need the following lemma which is the analogous of Lemma 4.1 in [18].

**Lemma IV.3.** *The remainder $Y_R$ in equation* (50) *satisfies the estimate*

$$|Y_R| \le \mathcal{R}(\epsilon^{1/2}M)\frac{\epsilon^{5/2}}{(1+\epsilon t)^2\sqrt{\epsilon t}}(1+|M|)^5.$$

Hence, equation (50) is of the form

$$\dot{y} = 2\Re(iK_T)y^2 + Y_R, \tag{56}$$

with

$$\Re(iK_T) < 0,$$
$$y(0) \leq \epsilon y_0,$$
$$|Y_R| \leq \overline{Y}\frac{\epsilon^{5/2}}{(1+\epsilon t)^2\sqrt{\epsilon t}},$$

where $y_0$ and $\overline{Y} > 0$ are some constants. Then we can apply Proposition 5.6 in [9] and get the next lemma.

**Lemma IV.4.** *Assuming the initial condition and the source term of equation* (50) *as above, the solution $y(t)$ is bonded as follows for any $t > 0$*

$$\left| y(t) - \frac{y(0)}{1 + 2\Im(K_T)y_0 t} \right| \leq c\overline{Y}\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2},$$

*where $c = c(y_0, \Im(K_T))$.*

### D. The equation for $P_T^c h_1$

As a first step let us estimate the remianders in the equation (41) for $h_1$. This is done in the next two lemmas.

**Lemma IV.5.** *The remainders $\widetilde{H}_R$ and $H_R''$ can be estimated as*

$$\|P_T^c \widetilde{H}_R\|_{L^1_w} \leq \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}((1+M_1)^3 + \epsilon^{1/2}(1+|M|)^4),$$

*and*

$$\|P_T^c H_R''\|_{L^1_w} \leq \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}((1+M_1)^3 + \epsilon^{1/2}(1+|M|)^4).$$

*Proof.* From the estimate (37) on $\widetilde{H}_R$ one has

$$\|P_T^c \widetilde{H}_R\|_{L^1_w} \leq \mathcal{R}_2(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})[|z|^3 + (|z| + |\omega_T - \omega|)(|z|^2 + \|P_T^c k_1\|_{L^\infty_{w^{-1}}} +$$

$$+ \|P_T^c h_1\|_{L^\infty_{w^{-1}}}) + (|z|^2 + \|P_T^c k_1\|_{L^\infty_{w^{-1}}} + \|P_T^c h_1\|_{L^\infty_{w^{-1}}})^2] \leq$$

$$\leq \mathcal{R}(\epsilon^{1/2}M)\left[\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}M_1^3+\right.$$

$$+\left(\left(\frac{\epsilon}{1+\epsilon t}\right)^{1/2}M_1+\frac{\epsilon}{1+\epsilon t}M_0\right)\left(\frac{\epsilon}{1+\epsilon t}M_1^2+\frac{\epsilon}{(1+t)^{3/2}}+\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}M_2\right)+$$

$$\left.+\left(\frac{\epsilon}{1+\epsilon t}M_1^2+\frac{\epsilon}{(1+t)^{3/2}}+\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}M_2\right)\right]\leq$$

$$\leq \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}((1+M_1)^3+\epsilon^{1/2}(1+|M|)^4).$$

The bound for $H_R''$ follows in the same way from the estimate (38). □

In the next lemma we get a estimate the evolution under the linear operator $L_T$ of the remainder $P_T^c\overline{H}_R$.

**Lemma IV.6.** *For any $t$, $s \geq 0$ the following estimate holds*

$$\|e^{L_T t}P_T^c\overline{H}_R(s)\|_{L^\infty_{w^{-1}}}(1+t)^{3/2}\leq \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2}(M_1^3+\epsilon^{1/2}(1+|M|)^3).$$

*Proof.* From the analytic expression (42) of $\overline{H}_R$ and the estimates of the evolution of the functions $a_{20}$, $a_{11}$, and $a_{02}$ stated in Lemma III.3, one has

$$\|e^{L_T t}P_T^c\overline{H}_R(s)\|_{L^\infty_{w^{-1}}}(1+t)^{3/2}\leq$$

$$\leq \mathcal{R}_2(\omega,|z|+\|f\|_{L^\infty_{w^{-1}}})|z|[|z||\omega_T-\omega|+(|z|+\|k_1\|_{L^\infty_{w^{-1}}}+\|h_1\|_{L^\infty_{w^{-1}}})^2]\leq$$

$$\leq \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon s}\right)^{1/2}M_1\left[\left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2}M_0M_1+\right.$$

$$\left.+\left(\left(\frac{\epsilon}{1+\epsilon s}\right)^{1/2}M_1+\frac{\epsilon}{(1+s)^{3/2}}+\left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2}M_2\right)^2\right]\leq$$

$$\leq \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2}(M_1^3+\epsilon^{1/2}(1+|M|)^3).$$

□

From the two previous lemmas we can get the following result.

**Lemma IV.7.** *Let us consider the equation for $P_T^c h_1$*

$$\left(\frac{dP_T^c h_1}{dt}, v\right)_{L^2} = Q_{L_M}(P_T^c h_1, v) + (P_T^c \widehat{H}_R, v)_{L^2} + (P_T^c H_R'', q_v G_\lambda)_{L^2},$$

*with initial condition and source terms satisfying*

$$\|h_1(0)\|_{L^1_w} \leq \epsilon^{3/2} h_0,$$

$$\widehat{H}_R = \widetilde{H}_R + H_R'',$$

*such that*

$$\|P_T^c \widetilde{H}_R\|_{L^1_w} \leq \overline{H}_1 \left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2},$$

$$\|P_T^c H_R''\|_{L^1_w} \leq \overline{H}_2 \left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2},$$

$$\|e^{L_T t} P_T^c \overline{H}_R(s)\|_{L^\infty_{w^{-1}}}(1+t)^{3/2} \leq \overline{H}_3 \left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2} (M_1^3 + \epsilon^{1/2}(1+|M|)^3).$$

*for some positive constant $h_0$, $\overline{H}_1$, $\overline{H}_2$ and $\overline{H}_3$. Then its solution is bounded as follows*

$$\|P_T^c h_1\|_{L^\infty_{w^{-1}}} \leq c \left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2} (h_0 + \overline{H}_1 + \overline{H}_2 + \overline{H}_3),$$

*where $c = c(\omega_T) > 0$.*

*Proof.* By the Duhamel representation one has

$$(P_T^c h_1, v)_{L^2} = \left(e^{\int_0^t L_M(\tau)d\tau} h_1(0) + \int_0^t e^{\int_s^t L_M(\tau)d\tau} P_T^c \widehat{H}_R(s) ds, v\right)_{L^2} +$$

$$+ \left(\int_0^t e^{\int_s^t L_M(\tau)d\tau} P_T^c H_R''(s) ds, q_v G_\lambda\right)_{L^2},$$

for all $v \in V$.

Then from the dispersive estimate in Theorem VI.3 and the estimates on the remainders proved above in the duality paring defined by the inner product $L^2$, one has

$$\|P_T^c h_1\|_{L^\infty_{w^{-1}}} = \sup_{0 \neq v \in L^1_w} \frac{(P_T^c h_1, v)_{L^2}}{\|v\|_{V \cap L^1_w}} \leq$$

$$\leq c(\omega_T)\left(\frac{1}{(1+t)^{3/2}}\|h_1(0)\|_{L^1_w}+\int_0^t\frac{1}{(1+t-s)^{3/2}}(\|P^c_T\widetilde{H}_R(s)\|_{L^1_w}+\|P^c_TH''_R(s)\|_{L^1_w})ds+\right.$$

$$\left.+\int_0^t\|e^{L_T(t-s)}P^c_T\overline{H}_R(s)\|_{L^\infty_{w^{-1}}}ds\right)\leq$$

$$\leq c(\omega_T)\left(\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}h_0+\int_0^t\frac{1}{(1+t-s)^{3/2}}\left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2}ds(\overline{H}_1+\overline{H}_2+\overline{H}_3)\right).$$

The lemma follows from the fact that

$$\int_0^t\frac{1}{(1+t-s)^{3/2}}\left(\frac{\epsilon}{1+\epsilon s}\right)^{3/2}ds\leq c\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2},$$

for some constant $c>0$. ☐

### E. Uniform bounds for the majorants

To prove that the majorants are uniformly bounded, the following lemma will be useful.

**Lemma IV.8.** *For any $T>0$ the majorants $M_0$, $M_1$, and $M_2$ satisfy the following inequalities*

$$\begin{aligned}&M_0(T)\leq\mathcal{R}(\epsilon^{1/2}M)[(1+M_1)^4+\epsilon(1+|M|)^2],\\&(M_1(T))^2\leq\mathcal{R}(\epsilon^{1/2}M)[1+\epsilon^{1/2}(1+|M|)^5],\\&M_2(T)\leq\mathcal{R}(\epsilon^{1/2}M)[(1+M_1)^3+\epsilon^{1/2}(1+|M|)^4].\end{aligned}$$

*Proof.* It follows form Lemma IV.4 and IV.7 as Lemma 4.6 in [18], but we give the proof for sake of completeness.

`Step 1.` Let us begin noting that

$$|z|^2+\|f\|_{L^\infty_{w^{-1}}}\leq\mathcal{R}_2(\omega,|z|+\|f\|_{L^\infty_{w^{-1}}})(|z|^2+\|P^c_Tk_1\|_{L^\infty_{w^{-1}}}+\|P^c_Th_1\|_{L^\infty_{w^{-1}}})\leq$$

$$\leq\mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{(1+t)^{3/2}}+\frac{\epsilon}{1+\epsilon t}M_1^2+\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}M_2\right)\leq$$

$$\leq\mathcal{R}(\epsilon^{1/2}M)\frac{\epsilon}{1+\epsilon t}(1+M_1^2+\epsilon^{1/2}M_2).$$

Then by the definition of $M_0$ and the bound on $|\omega_T-\omega|$:

$$M_0(T)\leq\max_{0\leq t\leq T}\left[\left(\frac{\epsilon}{1+\epsilon t}\right)^{-1}\mathcal{R}(\epsilon^{1/2}M)\left(\int_t^T\left(\frac{\epsilon}{1+\epsilon\tau}\right)^2(1+M_1(\tau)^2+\right.\right.$$

$$+\epsilon^{1/2}M_2(\tau))^2 d\tau + \left(\frac{\epsilon}{1+\epsilon t}\right)^2 (1+M_1^2+\epsilon^{1/2}M_2)^2\Big)\Big] \leq$$

$$\leq \mathcal{R}(\epsilon^{1/2}M)[(1+M_1)^4 + \epsilon(1+|M|)^2].$$

`Step 2.` Since $y = |z_1|^2$, we can exploit the inequality proved in Lemma IV.4, the fact that $\overline{Y} = \mathcal{R}(\epsilon^{1/2}M)(1+|M|)^5$ and $y(0) \leq \epsilon y_0$, one gets

$$y \leq \mathcal{R}(\epsilon^{1/2}M)\left[\frac{\epsilon}{1+\epsilon t} + \left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}(1+|M|)^5\right].$$

From which follows

$$|z|^2 \leq y + \mathcal{R}(\omega)|z|^3 \leq$$

$$\leq \mathcal{R}(\epsilon^{1/2}M)\left[\frac{\epsilon}{1+\epsilon t} + \left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}(1+|M|)^5 + \left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2} M_1^3\right] \leq \mathcal{R}(\epsilon^{1/2}M)[1+\epsilon^{1/2}(1+|M|)^5].$$

`Step 3.` Recall that

$$\|h(0)\|_{L^1_w} \leq c\epsilon^{3/2}\mathcal{R}(\epsilon^{1/2}M)\epsilon^2 M_0(1+M_1^2+\epsilon^{1/2}M_2),$$

$$\overline{H}_1 = \mathcal{R}(\epsilon^{1/2}M)((1+M_1)^3 + \epsilon^{1/2}(1+|M|)^4),$$

$$\overline{H}_2 = \mathcal{R}(\epsilon^{1/2}M)((1+M_1)^3 + \epsilon^{1/2}(1+|M|)^4),$$

$$\overline{H}_3 = \mathcal{R}(\epsilon^{1/2}M)(M_1^3 + \epsilon^{1/2}(1+|M|)^3).$$

Hence from Lemma IV.7 follows

$$\|P_T^c h_1\|_{L^\infty_{w^{-1}}} \leq \mathcal{R}(\epsilon^{1/2}M)\left(\frac{\epsilon}{1+\epsilon t}\right)^{3/2}((1+M_1)^3 + \epsilon^{1/2}(1+|M|)^4),$$

which implies the inequality for $M_2$. □

We are now in the position to prove the uniform boundedness of the majorants.

**Proposition IV.9.** *If $\epsilon > 0$ is sufficiently small, there exist a positive constant $\overline{M}$ independent of $T$ and $\epsilon$ such that*

$$|M(T)| \leq \overline{M},$$

*for all $T > 0$.*

*Proof.* From the previous lemma follows

$$|M|^2 \le \mathcal{R}(\epsilon^{1/2}M)[(1+M_1)^8 + \epsilon^{1/2}(1+|M|)^8] \le \mathcal{R}(\epsilon^{1/2}M)(1+\epsilon^{1/2}F(M)),$$

where in the last inequality we have replaced the estimate for $M_1^2$, and $F(M)$ is a suitable polynomial function.

Furthermore, $M(0)$ is small and $M(T)$ is a continuous function. Hence it follows that $|M|$ is bounded independent of $\epsilon \ll 1$. ☐

The last proposition gives a summary of the behavior of the functions $\omega(t)$, $z(t)$, $P_T^c h_1(t)$, and $f(t)$.

**Corollary IV.10.** *There exists a finite limit $\omega_\infty$ for the function $\omega(t)$ as $t \to +\infty$. Moreover the following holds for all $t > 0$*

$$\begin{aligned}
&|\omega_\infty - \omega(t)| \le \overline{M}\tfrac{\epsilon}{1+\epsilon t},\\
&|z(t)| \le \overline{M}\left(\tfrac{\epsilon}{1+\epsilon t}\right)^{1/2},\\
&\|P_T^c h_1(t)\|_{L^\infty_{w^{-1}}} \le \overline{M}\left(\tfrac{\epsilon}{1+\epsilon t}\right)^{3/2},\\
&\|f(t)\|_{L^\infty_{w^{-1}}} \le \overline{M}\tfrac{\epsilon}{1+\epsilon t}.
\end{aligned}$$

## V. LARGE TIME BEHAVIOR OF THE SOLUTION AND SCATTERING ASYMPTOTICS

### A. Large time behavior of the solution of equation (1)

The results of the previous section lead to the following theorem.

**Theorem V.1.** *Let $u(t)$ be a solution of equation (1) with initial datum $u_0 \in V \cap L^1_w$ of the form*

$$u_0(x) = e^{i\theta_0}\Phi_{\omega_0}(x) + z_0\Psi(x) + \overline{z}_0\Psi^*(x) + f_0(x),$$

*where $\theta_0 \in \mathbb{R}$, $\omega_0 > 0$, $z_0 \in \mathbb{C}$ with*

$$|z(0)| \le \epsilon^{1/2}, \qquad \|f_0\|_{L^1_w} \le c\epsilon^{3/2},$$

*for some $\epsilon$, $c > 0$. Then, provided $\epsilon$ is small enough, there exist $\omega(t)$, $\gamma(t)$, $z(t) \in C^1([0,+\infty))$ solutions of the modulation equations (21)-(23), and two constants $\omega_\infty$, $\overline{M} > 0$ such that $\omega_\infty = \lim_{t\to+\infty}\omega(t)$ and for all $t \ge 0$*

$$u(t,x) = e^{i(\int_0^t \omega(s)ds + \gamma(t))}\left(\Phi_{\omega(t)}(x) + z(t)\Psi(t,x) + \overline{z(t)}\Psi^*(t,x) + f(t,x)\right),$$

*where*

$$|\omega_\infty - \omega(t)| \le \overline{M}\frac{\epsilon}{1+\epsilon t}, \quad |z(t)| \le \overline{M}\left(\frac{\epsilon}{1+\epsilon t}\right)^{1/2}, \quad \|f(t)\|_{L^\infty_{w^{-1}}} \le \overline{M}\frac{\epsilon}{1+\epsilon t}.$$

*Proof.* Let us recall that the decomposition of the function $f$ as

$$f = g + h_1 + k + k_1$$

depends on the quantity $\omega(T)$. On the other hand Corollary IV.10 claims that the function $\omega(t)$ converges to some $\omega_\infty > 0$ as $t \to +\infty$.
As a consequence, one can reformulate the decomposition by choosing $T = +\infty$. Moreover, all the estimates obtained before for finite $T$ can be extended to $T = +\infty$ without modification. Hence the theorem. □

The next goal is to construct precise asymptotic expressions for $\omega(t)$, $\gamma(t)$, and $z(t)$. For later convenience let us define (recall that $\xi$ depends explicitly on $\omega$, see (12); and similarly for $K$, see (44) and subsequent, and $\gamma$)

$$\xi_\infty = \xi(\omega_\infty),$$

$$\gamma_\infty = \gamma(\omega_\infty),$$

$$K_\infty = K(\omega_\infty).$$

**Lemma V.2.** *Under the assumption of Theorem V.1 the functions $\omega(t)$, $\gamma(t)$, and $z(t)$ have the following asymptotic behavior as $t \to +\infty$:*

$$\omega(t) = \omega_\infty + \frac{q_1}{1+\epsilon k_\infty t} + \frac{q_2}{1+\epsilon k_\infty t}\cos(2\xi_\infty t + a_1 \log(1+\epsilon k_\infty t) + a_2) + O(t^{-3/2}),$$

$$\gamma(t) = \gamma_\infty + b_1 \log(1+\epsilon k_\infty t) + O(t^{-1}),$$

$$z(t) = z_\infty \frac{e^{i\int_0^t \xi(\tau)d\tau}}{(1+\epsilon k_\infty t)^{\frac{1-i\delta}{2}}} + O(t^{-1}),$$

*where*

$$z_\infty = z_1(0) + \int_0^{+\infty} e^{-i\int_0^s \xi(\tau)d\tau}(1+\epsilon k_\infty s)^{\frac{1-i\delta}{2}} Z_1(s)ds,$$

$\epsilon k_\infty = 2\Im(K_\infty)y_0$, $\delta = \frac{\Re(K_\infty)}{\Im(K_\infty)}$, *and $q_1$, $q_2$, $a_1$, $a_2$, $b_1$ are constants.*

*Proof.* We will prove the asymptotics for $z(t)$ only; the formulas for $\omega(t)$ and $\gamma(t)$ can be deduced as in Sections 6.1 and 6.2 of [9].

In order to do that let us recall that the equation for $z_1(t)$ can be written as

$$\dot{z_1} = i\xi z_1 + iK_\infty |z_1|^2 z_1 + \widehat{\widehat{Z}}_R,$$

moreover Remark III.13 and the inequalities satisfied by the majorants in Lemma IV.8 justify the following estimates on $\widehat{\widehat{Z}}_R$

$$|\widehat{\widehat{Z}}_R| \leq \mathcal{R}_1(\omega, |z| + \|f\|_{L^\infty_{w^{-1}}})[(|z|^2 + \|f\|_{L^\infty_{w^{-1}}})^2 + |z||\omega_T - \omega|(|z|^2 + \|h\|_{L^\infty_{w^{-1}}})+$$

$$+|z|\|P^c_T k_1\|_{L^\infty_{w^{-1}}} + |z|\|P^c_T h_1\|_{L^\infty_{w^{-1}}}] \leq$$

$$\leq \mathcal{R}(\epsilon^{1/2} M)\frac{\epsilon^2}{(1+\epsilon t)^{3/2}\sqrt{\epsilon t}}(1+\overline{M}^4) = O(t^{-2}),$$

as $t \to +\infty$. On the other hand, Lemma IV.4 implies

$$y(t) = \frac{y(0)}{1 + 2\Im(K_\infty)y(0)t} + O(t^{-3/2}), \qquad \text{as } t \to +\infty.$$

Let us note that $|z_1|$ satisfies the same bound of $|z|$, namely

$$|z_1| \leq \overline{M}\left(\frac{\epsilon}{1+\epsilon t}\right)^{1/2},$$

then the equation for $z_1(t)$ can be rewritten in the formulas

$$\dot{z_1} = i\xi z_1 + iK_\infty \frac{y(0)}{1 + 2\Im(K_\infty)y(0)t} z_1 + Z_1,$$

where $Z_1 = O(t^{-2})$ as $t \to +\infty$.

Since $y(0) = \epsilon y_0$, one has $\epsilon K_\infty y_0 = \frac{i}{2}\epsilon k_\infty(1 - i\delta)$ and the equation for $z_1(t)$ becomes

$$\dot{z_1} = \left(i\xi - \frac{i}{2}\epsilon k_\infty(1-i\delta)\frac{1}{1+\epsilon k_\infty t}\right) z_1 + Z_1.$$

Hence, one gets

$$z_1(t) = \frac{e^{i\int_0^t \xi(\tau)d\tau}}{(1+\epsilon k_\infty t)^{\frac{1-i\delta}{2}}}\left(z_1(0) + \int_0^s e^{-i\int_0^t \xi(\tau)d\tau}(1+\epsilon k_\infty s)^{\frac{1-i\delta}{2}} ds\right) = z_\infty \frac{e^{i\int_0^t \xi(\tau)d\tau}}{(1+\epsilon k_\infty t)^{\frac{1-i\delta}{2}}} + z_R,$$

where $z_\infty$ is as in the statement of the lemma and

$$z_R = -\int_t^{+\infty} e^{i\int_s^t \xi(\tau)d\tau}\left(\frac{1+\epsilon k_\infty s}{1+\epsilon k_\infty t}\right)^{\frac{1-i\delta}{2}} Z_1(s)ds.$$

The bound on $Z_1$ implies $z_R = O(t^{-1})$. Therefore $z(t)$ has the asymptotic behavior as $t \to +\infty$ stated in the lemma because

$$z(t) = z_1(t) + O(t^{-1}) = z_\infty \frac{e^{i\int_0^t \xi(\tau)d\tau}}{(1+\epsilon k_\infty t)^{\frac{1-i\delta}{2}}} + O(t^{-1}).$$

□

### B. Scattering asymptotics

Let us make the following ansatz

$$u(t,x) = s(t,x) + \zeta(t,x) + f(t,x),$$

where

$$s(t,x) = e^{i\Theta(t)}\Phi_{\omega(t)}(x),$$

is the modulated soliton and

$$\zeta(t,x) = e^{i\Theta(t)}[(z(t)+\overline{z}(t))\Psi_1(x) + i(z(t)-\overline{z}(t))\Psi_2(x)]$$

is the fluctuating component. Recall that the functions $\Phi_\omega$, $\Psi_1$ and $\Psi_2$ satisfy

$$\omega\Phi_\omega = -H_\alpha\Phi_\omega,$$

$$\omega\Psi_1 = -i\xi\Psi_2 - H_{\alpha_1}\Psi_1,$$

$$\omega\Psi_2 = i\xi\Psi_1 - H_{\alpha_2}\Psi_2.$$

Therefore from equation (1) one gets

$$\left(i\frac{df}{dt}, v\right)_{L^2} = Q_0(f,v) - \nu(|q_u|^{2\sigma}q_u - |q_s|^{2\sigma}q_s - \alpha_1 q_{(z+\overline{z})\Psi_1} - \alpha_2 q_{(z-\overline{z})\Psi_2})\overline{q_v} +$$

$$+(\dot\gamma(s+\zeta) - i\dot\omega\frac{d}{d\omega}(s+\zeta) - ie^{i\Theta}[(\dot z - i\xi z)(\Psi_1 + i\Psi_2) + (\dot{\overline{z}} - i\xi\overline{z})(\Psi_1 - i\Psi_2)], v)_{L^2},$$

for all $v \in V$, where $Q_0$ is the quadratic form of the free Laplacian. Hence, as in [1], the solution $f(t)$ can be formally expressed as

$$f(t,x) = U_t * f_0(x) + i\int_0^t U_{t-\tau}(x)q_f(\tau)d\tau - i\int_0^t U_{t-\tau} * G(\tau)d\tau,$$

where we have denoted

$$G(t) = \dot{\gamma}(t)(s(t)+\zeta(t)) - i\dot{\omega}(t)\frac{d}{d\omega}(s(t)+\zeta(t)) +$$

$$-ie^{i\Theta(t)}[(\dot{z}(t)-i\xi z(t))(\Psi_1(t)+i\Psi_2(t)) + (\dot{\overline{z}}(t)-i\xi\overline{z}(t))(\Psi_1(t)-i\Psi_2(t))]$$

and $U_t(x) = \frac{e^{i\frac{|x|^2}{4t}}}{(4\pi i t)^{3/2}}$ is the propagator of the free Laplacian in $\mathbb{R}^3$.
In order to prove the asymptotic stability result we need the two following lemmas.

**Lemma V.3.** *If the assumptions of Theorem V.1 hold true, then*

$$\int_0^t U_{t-\tau}(x)q_f(\tau)d\tau = U_t * \int_0^{+\infty} U_{-\tau}(x)q_f(\tau)d\tau - \int_t^{+\infty} U_{t-\tau}(x)q_f(\tau)d\tau = U_t * \phi_0 + r_0,$$

*where $\phi_0 \in L^2(\mathbb{R}^3)$ and $r_0 = O(t^{-1/4})$ as $t \to +\infty$ in $L^2(\mathbb{R}^3)$.*

*Proof.* The strategy is similar to the one exploited in the case without eigenvalues (see the proof of Theorem 7.1 in [3]): since $\phi_0(x) = \frac{1}{(4\pi i)^{3/2}}\widetilde{\phi}_0\left(\frac{|x|^2}{4}\right)$, for some function $\widetilde{\phi}_0 : \mathbb{R}^+ \to \mathbb{C}$, one gets

$$\|\phi_0\|_{L^2}^2 = \frac{1}{(4\pi)^2}\int_0^{+\infty}\left|\widetilde{\phi}_0\left(\frac{r^2}{4}\right)\right|^2 r^2 dr = \frac{1}{(2\pi)^2}\int_0^{+\infty}|\widetilde{\phi}_0(y)|^2\sqrt{y}dy.$$

Hence $\phi_0 \in L^2(\mathbb{R}^3)$ if and only if $\widetilde{\phi}_0 \in L^2(\mathbb{R}^+, \sqrt{y}dy)$. On the other hand, one can make the change of variables $u = \frac{1}{\tau}$ in the integral that defines the function $\widetilde{\phi}_0$ and get

$$\widetilde{\phi}_0(y) = \int_0^{+\infty} e^{-iyu}\frac{1}{u}q_f\left(\frac{1}{u}\right)\sqrt{u}du,$$

then $\widehat{\widetilde{\phi}_0} = \frac{1}{u}q_f\left(\frac{1}{u}\right)$. Moreover, by Corollary IV.10 one has

$$\|\widehat{\widetilde{\phi}_0}\|_{L^2}^2 = \int_0^{+\infty}\frac{1}{u^2}\left|q_f\left(\frac{1}{u}\right)\right|^2\sqrt{u}du \le C\int_0^{+\infty}\frac{\sqrt{u}}{(u+\epsilon)^2}du \le C,$$

for some constant $C > 0$, hence the Plancherel identity implies

$$\widetilde{\phi}_0 \in L^2(\mathbb{R}^+, \sqrt{y}dy).$$

In the same way, for any $t > 0$ the following holds

$$\|r_0\|_{L^2}^2 = \frac{1}{(2\pi)^2}\left\|\frac{1}{u}q_f\left(t+\frac{1}{u}\right)\right\|_{L^2(\mathbb{R}^+,\sqrt{u}du)}^2 \le C\frac{1}{\sqrt{1+\epsilon t}},$$

for some constant $C > 0$ independent of $t$. Which concludes the proof. ◻

The analogous result for the integral function $\int_0^t U_{t-\tau} * G(\tau)d\tau$ requires different tools.

**Lemma V.4.** *Assume that the assumptions of Theorem V.1 hold true, then*

$$\int_0^t U_{t-\tau} * G(\tau)d\tau = U_t * \int_0^{+\infty} U_{-\tau} * G(\tau)d\tau - U_t * \int_t^{+\infty} U_{-\tau} * G(\tau)d\tau = U_t * \phi_1 + r_1,$$

*where* $\phi_1 \in L^2(\mathbb{R}^3)$ *and* $r_1 = O(t^{-1/2})$ *as* $t \to +\infty$ *in* $L^2(\mathbb{R}^3)$.

*Proof.* We exploit the idea used in [18] to prove Lemma 5.5.

`Step 1: restriction to the leading terms.`

From the expansions (29), (30) and (31) for $\dot{\omega}(t)$, $\dot{\gamma}(t)$ and $\dot{z}(t) - i\xi z(t)$ follow that the function $G(t)$ is made by a quadratic part consisting in the terms multiplied $e^{i\Theta(t)} z_\infty^2$, $e^{i\Theta(t)}\overline{z_\infty}^2$ or $e^{i\Theta(t)}|z_\infty|^2$, with

$$z_\infty = \frac{e^{i\xi_\infty t}}{\sqrt{1 + \epsilon k_\infty t}},$$

which are of order $t^{-1}$ and a remainder of order $t^{-3/2}$. The convergence and the decay of the remainder is trivial from the unitarity of $U_t$. Furthermore, from the analytic definition of $G$ it follows that it is a complex linear combination of functions of the form

$$Q(x) = e^{-\sqrt{\alpha}|x|^2}, \qquad \alpha = \omega_\infty, \omega_\infty + \nu_\infty, \omega_\infty - \nu_\infty.$$

Hence it suffices to prove the lemma for the functions $\Pi(t)Q(x)$, where $\Pi(t)$ is one between $e^{i\Theta(t)} z_\infty^2$, $e^{i\Theta(t)}\overline{z_\infty}^2$ and $e^{i\Theta(t)}|z_\infty|^2$.

`Step 2: decomposition of` $U_t * Q$.

Let us note that we can rewrite the convolution product as follows

$$U_t * Q = \frac{e^{i\frac{|x|^2}{4t}}}{(4\pi i t)^{3/2}} \int_{\mathbb{R}^3} e^{-i\frac{(x,y)}{2t}} Q(y)dy + \frac{e^{i\frac{|x|^2}{4t}}}{(4\pi i t)^{3/2}} \int_{\mathbb{R}^3} e^{-i\frac{(x,y)}{2t}} (e^{i\frac{|y|^2}{4t}} - 1)Q(y)dy =$$

$$= \frac{e^{i\frac{|x|^2}{4t}}}{(2it)^{3/2}} \widehat{Q}\left(\frac{x}{2t}\right) + \frac{e^{i\frac{|x|^2}{4t}}}{(2it)^{3/2}} \widehat{Q_t}\left(\frac{x}{2t}\right), \tag{57}$$

where $Q_t(y) = (e^{i\frac{|y|^2}{4t}} - 1)Q(y)$.

Since $|e^{i\theta} - 1| \leq \theta$ and the function $G(y)$ is exponentially decaying as $|y| \to +\infty$, the $L^2$ norm of the second term of (57) can be estimated in the following way for any $t > 1$

$$\frac{1}{(2t)^{3/2}} \left\|\widehat{Q_t}\left(\frac{\cdot}{2t}\right)\right\|_{L^2} = \|\widehat{Q_t}(\cdot)\|_{L^2} \leq \frac{1}{4t} \left(\int_{\mathbb{R}^3} |y|^4 |Q(y)|^2 dy\right)^{1/2} \leq \frac{C}{t},$$

for some constant $C > 0$. Hence, recalling that $\Pi(\tau) \le (1+\epsilon k_\infty \tau)^{-1}$, we obtain

$$\int_0^{+\infty} \Pi(\tau) U_\tau * Q_t d\tau \in L^2(\mathbb{R}^3),$$

and

$$\int_t^{+\infty} \Pi(\tau) U_\tau * Q_t d\tau = O(t^{-1}),$$

as $t \to +\infty$ in $L^2(\mathbb{R}^3)$.

`Step 3: Analysis of the first term in (57) in a particular case.`

Let us first show how to treat the terms with the phase $\Theta(t)$ replaced by $\omega_\infty t$.

Note that

$$\widehat{Q}(x) = \frac{1}{\alpha + |x|^2},$$

Hence, in the case of the summands with $|z_\infty|^2$ it suffices to prove the integrability of the function

$$I(x) = \int_0^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\sqrt{\tau}}{(1+\epsilon k_\infty \tau)(|x|^2 + 4\alpha\tau^2)} d\tau =$$

$$= A(x) \int_0^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \left( \frac{\sqrt{\tau}}{(1+\epsilon k_\infty \tau)} - \frac{4\alpha}{\epsilon k_\infty} \frac{\tau\sqrt{\tau}}{(|x|^2 + 4\alpha\tau^2)} \right) d\tau +$$

$$+ \frac{4\alpha}{\epsilon^2 k_\infty^2} A(x) \int_0^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\sqrt{\tau}}{(|x|^2 + 4\alpha\tau^2)} d\tau = I_1(x) + I_2(x),$$

and the decay of

$$I_t(x) = \int_t^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\sqrt{\tau}}{(1+\epsilon k_\infty \tau)(|x|^2 + 4\alpha\tau^2)} d\tau =$$

$$= A(x) \int_t^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \left( \frac{\sqrt{\tau}}{(1+\epsilon k_\infty \tau)} - \frac{4\alpha}{\epsilon k_\infty} \frac{\tau\sqrt{\tau}}{(|x|^2 + 4\alpha\tau^2)} \right) d\tau +$$

$$+ \frac{4\alpha}{\epsilon^2 k_\infty^2} A(x) \int_t^\infty e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\sqrt{\tau}}{(|x|^2 + 4\alpha\tau^2)} d\tau = I_{1,t}(x) + I_{2,t}(x),$$

where $A(x) = \frac{\epsilon^2 k_\infty^2}{4\alpha + \epsilon^2 k_\infty^2 |x|^2}$.

For the function $I_2(x)$ one has

$$|I_2(x)| \le \frac{4\alpha}{\epsilon^2 k_\infty^2} A(x) \int_0^\infty \frac{\sqrt{\tau}}{(|x|^2 + 4\alpha\tau^2)} d\tau = C \frac{A(x)}{\sqrt{|x|}} \in L^2(\mathbb{R}^3).$$

With the same estimate it is trivial to prove

$$I_{2,t}(x) = O(t^{-1/2})$$

as $t \to +\infty$, in $L^2(\mathbb{R}^3)$.

In order to treat $I_1$ note that

$$\frac{\sqrt{\tau}}{(1+\epsilon k_\infty t)} - \frac{4\alpha}{\epsilon k_\infty}\frac{\tau\sqrt{\tau}}{(|x|^2+4\alpha\tau^2)} = -\frac{1}{\epsilon k_\infty\sqrt{\tau}(1+\epsilon k_\infty \tau)} + \frac{|x|^2}{\epsilon k_\infty\sqrt{\tau}(|x|^2+4\alpha\tau^2)}.$$

Since $\frac{1}{\epsilon k_\infty\sqrt{\tau}(1+\epsilon k_\infty\tau)} = O(t^{-3/2})$ as $t \to +\infty$, is integrable on $(0,+\infty)$ and $A(x) \in L^2(\mathbb{R}^3)$ one has to prove

$$\frac{|x|^2 A(x)}{\epsilon k_\infty}\int_0^{+\infty} e^{i(\omega_\infty\tau - \frac{|x|^2}{4\tau})}\frac{1}{\sqrt{\tau}(|x|^2+4\alpha\tau^2)}d\tau =$$

$$= A(x)\int_0^{+\infty} e^{i\omega_\infty(\tau-\frac{|x|^2}{4\omega_\infty\tau})}\frac{1}{\sqrt{\tau}}d\tau - 4\alpha A(x)\int_0^{+\infty} e^{i(\omega_\infty\tau-\frac{|x|^2}{4\tau})}\frac{\tau^{3/2}}{(|x|^2+4\alpha\tau^2)}d\tau \in L^2(\mathbb{R}^3).$$

From formulas 3.871.3 and 3.871.4 in [16] one has

$$A(x)\int_0^{+\infty} e^{i\omega_\infty(\tau-\frac{|x|^2}{4\omega_\infty\tau})}\frac{1}{\sqrt{\tau}}d\tau = \frac{e^{i\pi/4}}{\sqrt{\pi\omega_\infty}}A(x)|x|^{3/2}e^{-\sqrt{\omega_\infty}|x|} \in L^2(\mathbb{R}^3).$$

It remains to handle with the second integral in the former sum which can be done integrating by parts in the following way

$$\left|A(x)\int_0^{+\infty} e^{i(\omega_\infty\tau-\frac{|x|^2}{4\tau})}\frac{\tau^{3/2}}{(|x|^2+4\alpha\tau^2)}d\tau\right| =$$

$$= 4A(x)\left|\int_0^{+\infty} e^{i(\omega_\infty\tau-\frac{|x|^2}{4\tau})}\frac{d}{d\tau}\left[\frac{\tau^{7/2}}{(|x|^2+4\alpha\tau^2)(|x|^2+4\omega_\infty\tau^2)}\right]d\tau\right| \leq$$

$$\leq CA(x)\int_0^{+\infty}\frac{\tau^{5/2}}{(|x|^2+4\min\{\alpha,\omega_\infty\}\tau^2)^2}d\tau \leq C\frac{A(x)}{\sqrt{|x|}} \in L^2(\mathbb{R}^3).$$

Then we are done.

In order to estimate the decay of $I_{1,t}$ it suffices to study the decay of

$$\frac{|x|^2 A(x)}{\epsilon k_\infty}\int_t^{+\infty} e^{i(\omega_\infty\tau-\frac{|x|^2}{4\tau})}\frac{1}{\sqrt{\tau}(|x|^2+4\alpha\tau^2)}d\tau =$$

$$= A(x)\int_t^{+\infty} e^{i\omega_\infty(\tau-\frac{|x|^2}{4\omega_\infty\tau})}\frac{1}{\sqrt{\tau}}d\tau - 4\alpha A(x)\int_t^{+\infty} e^{i(\omega_\infty\tau-\frac{|x|^2}{4\tau})}\frac{\tau^{3/2}}{(|x|^2+4\alpha\tau^2)}d\tau,$$

which can be done integrating by parts as before. Let us do that for the second term (the computation for the first one are analogous and simpler):

$$\left| A(x) \int_t^{+\infty} e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{\tau^{3/2}}{(|x|^2 + 4\alpha\tau^2)} d\tau \right| =$$

$$= 4A(x) \left| \int_t^{+\infty} e^{i(\omega_\infty \tau - \frac{|x|^2}{4\tau})} \frac{d}{d\tau} \left[ \frac{\tau^{7/2}}{(|x|^2 + 4\alpha\tau^2)(|x|^2 + 4\omega_\infty \tau^2)} \right] d\tau \right| \leq$$

$$\leq CA(x) \left[ t^{-1/2} + \int_t^{+\infty} \frac{\tau^{5/2}}{(|x|^2 + 4\min\{\alpha, \omega_\infty\}\tau^2)^2} d\tau \right] \leq$$

$$\leq CA(x) \left( 1 + \frac{1}{\sqrt{|x|}} \right) t^{-1/2}.$$

The case of the summands with $z_\infty^2$ is analogous, while the case of $\overline{z_\infty}^2$ is more difficult because $|x|^2 + 4(\omega_\infty - 2\xi_\infty)\tau^2 = 0$ for

$$\tau = t^* = \frac{|x|}{2\sqrt{2\xi_\infty - \omega_\infty}}.$$

Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function with the properties:

$$0 < g(t^*) < t^* \quad \forall t^* > 0, \qquad \text{and} \qquad A(x)g(t^*) \in L^2(\mathbb{R}^3).$$

It follows that $g(t^*) = O(t^*) = O(|x|)$ as $|x| \to +\infty$. Hence, one can represent $(0, +\infty) = (0, t^* - g(t^*)] \cup (t^* - g(t^*), t^* + g(t^*)] \cup (t^* + g(t^*), +\infty)$. Integrating by parts once more one has

$$\left| A(x) \int_0^{t^* - g(t^*)} e^{i((\omega_\infty - 2\xi_\infty)\tau - \frac{|x|^2}{4\tau})} \frac{t^{3/2}}{|x|^2 + 4\alpha\tau^2} d\tau \right| \leq$$

$$\leq CA(x) \Bigg( (t^* - g(t^*))^{-1/2} + \int_0^{t^* - g(t^*)} \frac{t^{5/2}}{(|x|^2 + 4\alpha\tau^2)||x|^2 + 4(\omega_\infty - 2\xi_\infty)\tau^2|} d\tau +$$

$$+ \int_0^{t^* - g(t^*)} \frac{t^{9/2}}{(|x|^2 + 4\alpha\tau^2)^2||x|^2 + 4(\omega_\infty - 2\xi_\infty)\tau^2|} d\tau +$$

$$+ \int_0^{t^* - g(t^*)} \frac{t^{9/2}}{(|x|^2 + 4\alpha\tau^2)||x|^2 + 4(\omega_\infty - 2\xi_\infty)\tau^2|^2} d\tau \Bigg) \leq$$

$$\leq CA(x)((t^* - g(t^*))^{-1/2} + (t^* - g(t^*))^{3/8}) \in L^2(\mathbb{R}^3),$$

where the last inequality follows from formula 3.194.1 in [16]. In the same way (exploiting formula 3.194.2 instead of 3.194.1 in [16]), one has

$$\left| A(x) \int_{t^*+g(t^*)}^{\infty} e^{i((\omega_\infty - 2\xi_\infty)\tau - \frac{|x|^2}{4\tau})} \frac{t^{3/2}}{|x|^2 + 4\alpha\tau^2} d\tau \right| \leq$$

$$\leq CA(x)((t^* + g(t^*))^{-1/8} + (t^* + g(t^*))^{-9/8} + (t^* - g(t^*))^{-1/2}) \in L^2(\mathbb{R}^3).$$

Finally,

$$\left| A(x) \int_{t^*-g(t^*)}^{t^*+g(t^*)} e^{i((\omega_\infty - 2\xi_\infty)\tau - \frac{|x|^2}{4\tau})} \frac{t^{3/2}}{|x|^2 + 4\alpha\tau^2} d\tau \right| \leq$$

$$\leq CA(x) \int_{t^*-g(t^*)}^{t^*+g(t^*)} \frac{1}{\sqrt{\tau}} d\tau \leq C \frac{A(x) g(t^*)}{\sqrt{t^* - g(t^*)}} \in L^2(\mathbb{R}^3).$$

Summing up, the integrability of the integral function

$$\int_0^{+\infty} \Pi(\tau) U_\tau * Q d\tau$$

is achieved. It is left to study the decay of

$$A(x) \int_t^{+\infty} e^{i((\omega_\infty - 2\xi_\infty)\tau - \frac{|x|^2}{4\tau})} \frac{t^{3/2}}{|x|^2 + 4\alpha\tau^2} d\tau.$$

First of all, let us note that integrating by parts one obtains

$$\left| A(x) \int_{(0, t^*-g(t^*)] \cap [t, +\infty)} e^{i((\omega_\infty - 2\xi_\infty)\tau - \frac{|x|^2}{4\tau})} \frac{t^{3/2}}{|x|^2 + 4\alpha\tau^2} d\tau \right| \leq$$

$$\leq CA(x) \int_t^{t^*-g(t^*)} \left| \frac{d}{d\tau} \frac{t^{7/2}}{(|x|^2 + 4\alpha\tau^2)(|x|^2 + 4(\omega_\infty - 2\xi_\infty)\tau^2)} \right| d\tau \leq$$

$$\leq CA(x) \left( t^{-1/2} + \int_t^{t^*-g(t^*)} \frac{\sqrt{\tau}}{|x|^2 + 4\alpha\tau^2} d\tau + \int_t^{t^*-g(t^*)} \frac{\sqrt{\tau}}{||x|^2 + 4(\omega_\infty - 2\xi_\infty)\tau^2|} d\tau + \right.$$

$$\left. + \int_t^{t^*-g(t^*)} \frac{\tau^{9/2}}{(|x|^2 + 4\alpha\tau^2)||x|^2 + 4(\omega_\infty - 2\xi_\infty)\tau^2|^2} d\tau \right).$$

The three integrals in the last inequality can be estimated in the following way:

(i) $\int_t^{t^*-g(t^*)} \frac{\sqrt{\tau}}{|x|^2+4\alpha\tau^2} d\tau \leq C \int_t^{+\infty} \tau^{-3/2} d\tau \leq C t^{-1/2}$;

(ii) $\int_t^{t^*-g(t^*)} \frac{\sqrt{\tau}}{|x|^2+4(2\xi_\infty-\omega_\infty)\tau^2} d\tau = \int_t^{t^*-g(t^*)} \frac{\sqrt{\tau}}{(|x|+2\sqrt{2\xi_\infty-\omega_\infty}\tau)||x|-2\sqrt{2\xi_\infty-\omega_\infty}\tau|} d\tau$
$\leq C t^{-1/2} \int_0^{t^*-g(t^*)} \frac{1}{||x|-2\sqrt{2\xi_\infty-\omega_\infty}\tau|} d\tau \leq C t^{-1/2}$;

(iii) $\int_t^{t^*-g(t^*)} \frac{\tau^{9/2}}{(|x|^2+4\alpha\tau^2)||x|^2+4(\omega_\infty-2\xi_\infty)\tau^2|^2} d\tau \leq C t^{-1/2} \int_0^{t^*-g(t^*)} \frac{\tau}{||x|-2\sqrt{\omega_\infty-2\xi_\infty}\tau|^2} d\tau$
$\leq C t^{-1/2}(1+\ln||x|-2\sqrt{\omega_\infty-2\xi_\infty}(t^*-g(t^*))|)$.

Hence, since $A(x)\ln||x|-2\sqrt{\omega_\infty-2\xi_\infty}(t^*-g(t^*))| \in L^2(\mathbb{R}^3)$, one can conclude

$$A(x)\int_{(0,t^*-g(t^*)]\cap[t,+\infty)} e^{i((\omega_\infty-2\xi_\infty)\tau-\frac{|x|^2}{4\tau})} \frac{t^{3/2}}{|x|^2+4\alpha\tau^2} d\tau = O(t^{-1/2})$$

as $t \to +\infty$, in $L^2(\mathbb{R}^3)$.

Let us now observe that

$$\left| A(x)\int_{(t^*-g(t^*),+\infty)\cap[t,+\infty)} e^{i((\omega_\infty-2\xi_\infty)\tau-\frac{|x|^2}{4\tau})} \frac{t^{3/2}}{|x|^2+4\alpha\tau^2} d\tau \right| \leq$$

$$\leq CA(x)\int_t^{+\infty} \left| \frac{d}{d\tau} \frac{t^{7/2}}{(|x|^2+4\alpha\tau^2)(|x|^2+4(\omega_\infty-2\xi_\infty)\tau^2)} \right| d\tau \leq$$

$$\leq B(x)A(x)\left( t^{-1/2} + \int_t^{+\infty} \frac{\sqrt{\tau}}{|x|^2+4\alpha\tau^2} d\tau \right) \leq CB(x)A(x)t^{-1/2} \in L^2(\mathbb{R}^3),$$

where $B : \mathbb{R}^3 \to \mathbb{R}^+$ is a continuous bounded function.

Finally,

$$\left| A(x)\int_{(t^*-g(t^*),(t^*+g(t^*)]\cap[t,+\infty)} e^{i((\omega_\infty-2\xi_\infty)\tau-\frac{|x|^2}{4\tau})} \frac{t^{3/2}}{|x|^2+4\alpha\tau^2} d\tau \right| \leq$$

$$\leq CA(x)\int_t^{t^*+g(t^*)} \frac{1}{\sqrt{\tau}} d\tau \leq CA(x)g(t^*)t^{-1/2} \in L^2(\mathbb{R}^3).$$

Summing up, thanks to the unitarity of $U_t$, we proved

$$U_t * \int_t^{+\infty} \Pi(\tau)U_\tau * Q d\tau = O(t^{-1/2}),$$

as $t \to +\infty$, in $L^2(\mathbb{R}^3)$.

`Step 4: conclusion of the proof.`

The conclusions of the previous step hold true if the phase $\omega_\infty t$ is replaced by $\Theta(t)$. In fact, the estimates which involve the integral of the absolute value are totally unaffected by change of phase, then it is only left to adjust the argument involving integration by parts. This can be done integrating by parts exactly as before, which leaves a factor $e^{i(\Theta(t)-\omega_\infty t)}$ in the integrand. Then, the boundary terms can be treated in the same way because $|e^{i(\Theta(t)-\omega_\infty t)}| = 1$. Finally, the extra contribution to the integrand can be estimated as it is done for the summand arising from differentiation of $t^{7/2}$ since $|\dot{\Theta}(t) - \omega_\infty| \leq \frac{C}{1+\epsilon k_\infty t}$ for all $t > 0$, where $C$ is a positive constant. □

Summing up, we have proved the following asymptotic stability result.

**Theorem V.5.** *Let* $\sigma \in \left(\frac{1}{\sqrt{2}}, \sigma^*\right)$, *for a certain* $\sigma^* \in \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$ *and* $u(t) \in C(\mathbb{R}^+, V)$ *be a solution of equation* (1) *with*

$$u(0) = u_0 = e^{i\omega_0 t+\gamma_0}\Phi_{\omega_0} + e^{i\omega_0 t+\gamma_0}[(z_0 + \overline{z_0})\Psi_1 + i(z_0 - \overline{z_0})\Psi_2] + f_0 \in V \cap L^1_w(\mathbb{R}^3),$$

*for some* $\omega_0 > 0$, $\gamma_0$, $z_0 \in \mathbb{R}$ *and* $f_0 \in L^2(\mathbb{R}^3) \cap L^1_w(\mathbb{R}^3)$. *Furthermore, assume that the initial datum* $u_0$ *is close to a solitary wave, i.e.*

$$|z_0| \leq \epsilon^{1/2} \qquad \text{and} \qquad \|f_0\|_{L^1_w} \leq c\epsilon^{3/2},$$

*where* $c$, $\epsilon > 0$.

*Then, if* $\epsilon$ *is sufficiently small, the solution* $u(t)$ *can be asymptotically decomposed as follows*

$$u(t) = e^{i\omega_\infty t + ib_1 \log(1+\epsilon k_\infty t)}\Phi_{\omega_\infty} + U_t * \phi_\infty + r_\infty(t), \quad \text{as } t \to +\infty,$$

*where* $\omega_\infty$, $\epsilon k_\infty > 0$, $b_1 \in \mathbb{R}$ *and* $\phi_\infty$, $r_\infty(t) \in L^2(\mathbb{R}^3)$ *with*

$$\|r_\infty(t)\|_{L^2} = O(t^{-1/4}) \quad \text{as } t \to +\infty,$$

*in* $L^2(\mathbb{R}^3)$.

**Remark V.6.** *Numerical evidences (see Lemma III.11) suggest* $\sigma^* = \frac{\sqrt{3}+1}{2\sqrt{2}} \simeq 0,96$.

## VI. APPENDICES

In the following appendices we collect auxiliary material on the model studied. In Appendices A and B we recall results from [3] regarding resolvent, spectrum and dispersive behavior of linearization operator $L$ (see (11)) to help the independent reading of the present paper. In the subsequent appendices C,D,E we state and prove further details regarding spectral properties of the linearized operators, in particular the structure of eigenfunctions associated to the discrete spectrum, the structure of the generalized eignevectors, and the proof of Lemma II.8.

### A. Resolvent and spectrum of linearization

We denote

$$G_{\omega\pm i\lambda}(x) = \frac{e^{i\sqrt{-\omega\mp i\lambda}|x|}}{4\pi|x|} \qquad \omega > 0, \lambda \in \mathbb{C}, \tag{58}$$

with the prescription $\Im\sqrt{-\omega \pm i\lambda} > 0$.

Furthermore, we make use of the notation $\langle g, h\rangle := \int_{\mathbb{R}^3} g(x)h(x)\, dx$.

The resolvent of linearized operator $L$ is described in the following

**Theorem VI.1.** *The resolvent $R(\lambda) = (L - \lambda I)^{-1}$ of the operator $L$ is given by*

$$R(\lambda) = \begin{bmatrix} -\lambda\mathcal{G}_{\lambda^2}* & -\Gamma_{\lambda^2}* \\ \Gamma_{\lambda^2}* & -\lambda\mathcal{G}_{\lambda^2}* \end{bmatrix} + \frac{4\pi}{W(\lambda^2)} i \begin{bmatrix} \Lambda_1 & i\Sigma_2 \\ -i\Sigma_1 & \Lambda_2 \end{bmatrix}, \tag{59}$$

*where*

$$W(\lambda^2) = 32\pi 2\alpha_1\alpha_2 - 4i\pi(\alpha_1+\alpha_2)\left(\sqrt{-\omega+i\lambda} + \sqrt{-\omega-i\lambda}\right) - 2\sqrt{-\omega+i\lambda}\sqrt{-\omega-i\lambda},$$

*and formula* (59) *holds for all* $\lambda \in \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \ W(\lambda^2) = 0, \quad or \quad \Re(\lambda) = 0 \ and \ |\Im(\lambda)| \geq \omega\}$. *Furthermore, the symbol $*$ in* (59) *denotes the convolution and*

$$\mathcal{G}_{\lambda^2}(x) = \frac{1}{2i\lambda}\left(G_{\omega-i\lambda}(x) - G_{\omega+i\lambda}(x)\right), \quad \Gamma_{\lambda^2}(x) = \frac{1}{2}\left(G_{\omega-i\lambda}(x) + G_{\omega+i\lambda}(x)\right).$$

*Finally, the entries of the second matrix are finite rank operators whose action on $f \in L^2(\mathbb{R}^3)$ reads*

$$\Lambda_1 f = [i\lambda(4\pi\alpha_2 - i\sqrt{-\omega+i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle - (4\pi\alpha_1 - i\sqrt{-\omega+i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega+i\lambda} + \tag{60}$$

$$+[i\lambda(4\pi\alpha_2 - i\sqrt{-\omega-i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle + (4\pi\alpha_1 - i\sqrt{-\omega-i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega-i\lambda},$$

$$\Lambda_2 f = [i\lambda(4\pi\alpha_1 - i\sqrt{-\omega+i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle - (4\pi\alpha_2 - i\sqrt{-\omega+i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega+i\lambda} +$$

$$+[i\lambda(4\pi\alpha_1 - i\sqrt{-\omega-i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle + (4\pi\alpha_2 - i\sqrt{-\omega-i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega-i\lambda},$$

$$\Sigma_1 f = -[i\lambda(4\pi\alpha_2 - i\sqrt{-\omega+i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle - (4\pi\alpha_1 - i\sqrt{-\omega+i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega+i\lambda}+$$

$$+[i\lambda(4\pi\alpha_2 - i\sqrt{-\omega-i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle + (4\pi\alpha_1 - i\sqrt{-\omega-i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega-i\lambda},$$

$$\Sigma_2 f = -[i\lambda(4\pi\alpha_1 - i\sqrt{-\omega+i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle - (4\pi\alpha_2 - i\sqrt{-\omega+i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega+i\lambda}+$$

$$+[i\lambda(4\pi\alpha_1 - i\sqrt{-\omega-i\lambda})\langle\mathcal{G}_{\lambda^2}, f\rangle + (4\pi\alpha_2 - i\sqrt{-\omega-i\lambda})\langle\Gamma_{\lambda^2}, f\rangle]G_{\omega-i\lambda}.$$

The resolvent determines also the spectrum of the operator $L$, as given in the following

**Proposition VI.2.** *The spectrum of the linearized operator $L$ has the following structure*

*(a)* $\sigma_{ess}(L) = \{\lambda \in \mathbb{C} :\ \Re(\lambda) = 0 \text{ and } |\Im(\lambda)| \geq \omega\}$

*(b) If $\sigma \in (0, 1/\sqrt{2})$, the only eigenvalue of $L$ is $0$ with algebraic multiplicity $2$.*

*(c) If $\sigma = 1/\sqrt{2}$, $L$ has resonances $\pm i\omega$ at the border of the essential spectrum and the eigenvalue $0$ with algebraic multiplicity $2$.*

*(d) If $\sigma \in (1/\sqrt{2}, 1)$, $L$ has two simple eigenvalues $\pm i\xi == \pm i2\sigma\sqrt{1-\sigma^2}\omega$ and the eigenvalue $0$ with algebraic multiplicity $2$.*

*(e) If $\sigma = 1$, the only eigenvalue of $L$ is $0$ with algebraic multiplicity $4$.*

*(f) If $\sigma \in (1, +\infty)$, $L$ has two simple eigenvalues $\pm 2\sigma\sqrt{\sigma^2-1}\omega$ and the eigenvalue $0$ with algebraic multiplicity $2$.*

## B. Dispersive estimates

We recall that the propagator $e^{-tL}$ is the inverse Laplace transform of the resolvent. So the dispersive behaviour associated to the linearized dynamics projected on the continuous spectrum is controlled by the following result, proved in [3], Theorem 4.8.

**Theorem VI.3.** *Let $\sigma \neq \frac{1}{\sqrt{2}}$. There exists a constant $C > 0$ such that*

$$\left|\frac{1}{2\pi i}\int_{\mathbb{R}^3}\int_{\mathcal{C}_+\cup\mathcal{C}_-}(R(\lambda+0) - R(\lambda-0))(x)e^{-\lambda t}f(y)\,d\lambda dy\right| \leq C\left(1+\frac{1}{|x|}\right)t^{-\frac{3}{2}}\int_{\mathbb{R}^3}\left(1+\frac{1}{|y|}\right)|f(y)|dy$$

*for any $f \in L^1_w(\mathbb{R}^3)$, where*

$$\mathcal{C}_+ = \{\lambda \in \mathbb{C} :\ \Re(\lambda) = 0 \text{ and } \Im(\lambda) \geq \omega\}, \quad \mathcal{C}_- = \{\lambda \in \mathbb{C} :\ \Re(\lambda) = 0 \text{ and } \Im(\lambda) \leq -\omega\}\ .$$

### C. Eigenfunctions associated to $\pm i\xi$ and generalized eigenfunctions

#### 1. *The eigenfunctions associated to $\pm i\xi$*

Here we describe the eigenspaces associated to the simple purely imaginary eigenvalues $\pm i\xi = \pm i2\sigma\sqrt{1-\sigma^2}\omega$.

Let us start with the eigenvalue $i\xi$. The following proposition holds true.

**Proposition VI.4.** *The eigenspace associated to $i\xi$ is spanned by*

$$\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix} = \frac{e^{-\sqrt{\omega-\xi}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix} - \frac{\sqrt{1-\sigma^2}-1}{\sigma}\frac{e^{-\sqrt{\omega+\xi}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

*Proof.* In order to prove the proposition we need to solve the equation

$$L\Psi = i\xi\Psi$$

in $D(L)$. For $x \neq 0$, the previous equation is equivalent to the system

$$\begin{cases} (-\triangle+\omega)^2\Psi_1 - \xi^2\Psi_1 = 0 \\ \Psi_2 = \frac{i}{\xi}(-\triangle+\omega)\Psi_1 \end{cases},$$

from which follows that $\Psi_1$ must belong to $L^2(\mathbb{R}^3)$ and solve the equation

$$(-\triangle+\omega-\xi)(-\triangle+\omega+\xi)\Psi_1 = 0.$$

Hence, the solutions in $L^2(\mathbb{R}^3)$ are of the form

$$\begin{cases} \Psi_1(x) = A\frac{e^{-\sqrt{\omega-\xi}|x|}}{4\pi|x|} + B\frac{e^{-\sqrt{\omega+\xi}|x|}}{4\pi|x|} \\ \Psi_2(x) = iA\frac{e^{-\sqrt{\omega-\xi}|x|}}{4\pi|x|} - iB\frac{e^{-\sqrt{\omega+\xi}|x|}}{4\pi|x|} \end{cases},$$

for any $A$, $B \in \mathbb{C}$.

It is left to look for $A$, $B \in \mathbb{C}$ such that $\Psi_i \in D(L_i)$ for $i = 1, 2$, i.e.

$$\begin{cases} -\frac{\sqrt{\omega-\xi}}{4\pi}A - \frac{\sqrt{\omega+\xi}}{4\pi}B = -(2\sigma+1)\frac{\sqrt{\omega}}{4\pi}(A+B) \\ -i\frac{\sqrt{\omega-\xi}}{4\pi}A + i\frac{\sqrt{\omega+\xi}}{4\pi}B = -\frac{\sqrt{\omega}}{4\pi}(iA-iB) \end{cases}.$$

Exploiting the fact that $\xi = 2\sigma\sqrt{1-\sigma^2}\omega$ one can show that the two equations of the previous system are linearly dependent and

$$B = -\frac{\sqrt{1-\sigma^2}+1}{\sigma}A.$$

The thesis follows by setting $A = 1$. □

Let us note that in the previous proof we have chosen the constant in such a way that $\Psi_1(x) \in \mathbb{R}$ and $\Psi_2(x) \in i\mathbb{R}$ for any $x \in \mathbb{R}^3 \setminus \{0\}$. This fact will be used to prove the next proposition.

**Proposition VI.5.** *The eigenspace associated to $-i\xi$ is spanned by*

$$\Psi^* = \begin{pmatrix} \Psi_1 \\ -\Psi_2 \end{pmatrix}.$$

*Proof.* In the previous proposition we proved that

$$\begin{cases} L_2\Psi_2 = i\xi\Psi_1 \\ -L_1\Psi_1 = i\xi\Psi_2 \end{cases},$$

with $\Psi_1$ real and $\Psi_2$ purely imaginary.

Taking the conjugate of both equations and recalling that the operators $L_i$, $i = 1, 2$ act on the real and imaginary parts separately, one has

$$\begin{cases} L_2(-\Psi_2) = -i\xi\Psi_1 \\ -L_1\Psi_1 = -i\xi(-\Psi_2) \end{cases},$$

which is equivalent to

$$L\Psi^* = -i\xi\Psi^*,$$

because the operators $L_i$, $i = 1, 2$ are linear. The proof is complete. □

### 2. *The generalized eigenfunctions*

Our goal is to compute the generalized eigenfunctions associated to the continuous spectrum. In order to do that, we treat the two branches $\mathcal{C}_+$ and $\mathcal{C}_-$ of the continuous spectrum separately.

**Proposition VI.6.** *The generalized eigenfunctions associated to $\mathcal{C}_+$ are*

$$\Psi_+(x) = A\frac{e^{-\sqrt{\omega+\eta}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ -i \end{pmatrix} + C\frac{e^{-i\sqrt{\eta-\omega}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix} + D\frac{e^{i\sqrt{\eta-\omega}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix},$$

*for any $\eta \in [\omega, +\infty)$ and $D \in \mathbb{C}$, with*

$$\begin{aligned} A &= \tfrac{\sigma\sqrt{\omega}}{\sqrt{\omega+\eta}-(\sigma+1)\sqrt{\omega}}(C+D), \\ C &= \tfrac{(2\sigma+1)\omega+(\sigma+1)\sqrt{\omega}(i\sqrt{\eta-\omega}-\sqrt{\eta+\omega})-i\sqrt{\eta^2-\omega^2}}{-(2\sigma+1)\omega+(\sigma+1)\sqrt{\omega}(i\sqrt{\eta-\omega}+\sqrt{\eta+\omega})-i\sqrt{\eta^2-\omega^2}}D. \end{aligned}$$

*Proof.* For any $\eta \in [\omega, +\infty)$, we need to solve the system

$$L\Psi_+ = i\eta\Psi_+,$$

where $\Psi_+ \in L^\infty(\mathbb{R}^3)$ does not necessary belongs to $L^2(\mathbb{R}^3)$. As in the computation for the eigenfunction at $\pm i\xi$, if $x \neq 0$ the former equation is equivalent to the system

$$\begin{cases} (-\triangle + \omega - \xi)(-\triangle + \omega + \xi)(\Psi_+)_1 = 0 \\ (\Psi_+)_2 = \frac{i}{\xi}(-\triangle + \omega)(\Psi_+)_1 \end{cases},$$

which leads to

$$\Psi_+(x) = A\frac{e^{-\sqrt{\omega+\eta}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ -i \end{pmatrix} + B\frac{e^{\sqrt{\omega+\eta}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix} + C\frac{e^{-i\sqrt{\eta-\omega}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix} + D\frac{e^{i\sqrt{\eta-\omega}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix},$$

for some $A$, $B$, $C$, $D \in \mathbb{C}$. Since we require $\Psi_+ \in L^\infty(\mathbb{R}^3)$, we get $B = 0$. Moreover, the boundary conditions in the domain of the operators $L_1$ and $L_2$ must be satisfied by $(\Psi_+)_1$ and $(\Psi_+)_2$ respectively. Then $A$, $C$, and $D$ solve the system

$$\begin{cases} -\frac{\sqrt{\omega+\eta}}{4\pi}A - i\frac{\sqrt{\eta+\omega}}{4\pi}C + i\frac{\sqrt{\eta+\omega}}{4\pi}D = -\frac{(2\sigma+1)\sqrt{\omega}}{4\pi}(A + C + D) \\ i\frac{\sqrt{\omega+\eta}}{4\pi}A + \frac{\sqrt{\eta+\omega}}{4\pi}C - \frac{\sqrt{\eta+\omega}}{4\pi}D = -\frac{\sqrt{\omega}}{4\pi}(-iA + iC + iD) \end{cases},$$

which concludes the proof. $\square$

In the same way, one can prove the analogous result about $\mathcal{C}_-$.

**Proposition VI.7.** *The generalized eigenfunctions associated to $\mathcal{C}_-$ are*

$$\Psi_-(x) = A\frac{e^{-\sqrt{\omega-\eta}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ i \end{pmatrix} + C\frac{e^{-i\sqrt{-(\eta+\omega)}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ -i \end{pmatrix} + D\frac{e^{i\sqrt{-(\eta+\omega)}|x|}}{4\pi|x|}\begin{pmatrix} 1 \\ -i \end{pmatrix},$$

*for any $\eta \in (-\infty, -\omega]$, where $D \in \mathbb{C}$ and*

$$\begin{aligned} A &= \frac{\sigma\sqrt{\omega}}{\sqrt{\omega-\eta}-(\sigma+1)\sqrt{\omega}}(C + D), \\ C &= \frac{(2\sigma+1)\omega+(\sigma+1)\sqrt{\omega}(i\sqrt{-(\eta+\omega)}-\sqrt{\omega-\eta})-i\sqrt{\eta^2-\omega^2}}{-(2\sigma+1)\omega+(\sigma+1)\sqrt{\omega}(i\sqrt{-(\eta+\omega)}+\sqrt{\eta+\omega})-i\sqrt{\eta^2-\omega^2}}D. \end{aligned}$$

It is easy to see that the projection operators from $L^2(\mathbb{R}^3)$ onto $X^0$, $X^1$ and $X^c$ are given by

$$\begin{aligned} P^0 f &= -\frac{2}{\Delta}\Omega\left(f, \frac{d\Phi_\omega}{d\omega}\right)J\Phi_\omega + \frac{2}{\Delta}\Omega\left(f, J\Phi_\omega\right)\frac{d\Phi_\omega}{d\omega}, \quad \Delta = \tfrac{d}{d\omega}\|\Phi_\omega\|_{L^2}, \\ P^1 f &= \frac{\Omega(f,\Psi)}{\kappa}\Psi + \frac{\Omega(f,\Psi^*)}{\kappa}\Psi^*, \qquad \kappa = \Omega(\Psi, \Psi^*), \\ P^c f &= f - P^0 f - P^1 f, \end{aligned}$$

Moreover, we denote with $\Pi^\pm$ the projections onto the branches $\mathcal{C}_\pm$ of the continuous spectrum separately.

## D. Proof of Lemma II.8

In this appendix we prove Lemma II.8 whose statement is recalled for the reader's convenience.

**Lemma VI.8.** *There exists a constant $C > 0$ such that for each $h \in X^c$ holds*

$$\left\|[P^c J - i(\Pi^+ - \Pi^-)]h\right\|_{L^1_w} \leq C\|h\|_{L^\infty_{w^{-1}}}.$$

*Proof.* From the definitions of the operators $P^c$ and $\Pi^\pm$ one gets

$$P^c J - i(\Pi^+ - \Pi^-) = \Pi^+(J - iI) + \Pi^-(J + iI) =$$

$$= \frac{1}{2\pi i}\left[\int_{\mathcal{C}^+}(R(\lambda+0) - R(\lambda-0))(J - iI)d\lambda + \int_{\mathcal{C}^-}(R(\lambda+0) - R(\lambda-0))(J + iI)d\lambda\right].$$

We will estimate just the first integral because the second one can be handled in the same way. Exploiting the explicit form of the resolvent (VI.1) it follows that

$$R(\lambda)(J - iI) = (i\lambda\mathcal{G}_{\lambda^2} * + \Gamma_{\lambda^2} *)\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} + \frac{4\pi}{D(\lambda^2)}\begin{bmatrix} \Lambda_1 + \Sigma_2 & i(\Lambda_1 + \Sigma_2) \\ -i(\Lambda_2 + \Sigma_1) & \Lambda_2 + \Sigma_1 \end{bmatrix} =$$

$$= R_*(\lambda)(J - iI) + R_m(\lambda)(J - iI),$$

where $R_*$ and $R_m$ correspond to the convolution term of the resolvent and the multiplicative term. Note that

$$i\lambda\mathcal{G}_{\lambda^2}(x-y) + \Gamma_{\lambda^2}(x-y) = 2G_{\omega - i\lambda}(x-y) = \frac{e^{i\sqrt{-\omega+i\lambda}|x-y|}}{2\pi|x-y|}$$

is continuous on $\mathcal{C}_+$. Hence, the integral on $\mathcal{C}_+$ of the convolution addends vanishes.

Let us now consider the multiplicative addends in the integral on $\mathcal{C}_+$. From the explicit formulas for $\Lambda_1$ and $\Sigma_2$ given in Proposition VI.1 one can compute

$$(\Lambda_1 + \Sigma_2)(x,y) = 8\pi(\alpha_2 - \alpha_1)G_{\omega - i\lambda}(y)G_{\omega + i\lambda}(x) + [8\pi(\alpha_2 + \alpha_1) - 4i\sqrt{-\omega - i\lambda}]G_{\omega - i\lambda}(y)G_{\omega - i\lambda}(x) =$$

$$= 4\sigma\sqrt{\omega}\frac{e^{i\sqrt{-\omega+i\lambda}|y|}}{4\pi|y|}\frac{e^{i\sqrt{-\omega-i\lambda}|x|}}{4\pi|x|} - [4(\sigma+1)\sqrt{\omega} - 4i\sqrt{-\omega - i\lambda}]\frac{e^{i\sqrt{-\omega+i\lambda}(|x|+|y|)}}{(4\pi)^2|x||y|}.$$

Denote

$$D_\pm(\lambda^2) = D((\lambda \pm 0)^2).$$

Then it follows

$$\int_{\mathcal{C}^+}[(R_m(\lambda+0)-R_m(\lambda-0))(J-iI)]_{1,1}d\lambda=$$

$$=\int_{\mathcal{C}^+}\frac{\sigma\sqrt{\omega}e^{i\sqrt{-\omega+i\lambda}|y|}e^{-i\sqrt{-\omega-i\lambda}|x|}+((\sigma+1)\sqrt{\omega}-i\sqrt{-\omega-i\lambda})e^{i\sqrt{-\omega+i\lambda}(|x|+|y|)}}{\pi|x||y|D_+(\lambda^2)}d\lambda+$$

$$-\int_{\mathcal{C}^+}\frac{\sigma\sqrt{\omega}e^{i\sqrt{-\omega+i\lambda}|y|}e^{i\sqrt{-\omega-i\lambda}|x|}+((\sigma+1)\sqrt{\omega}+i\sqrt{-\omega-i\lambda})e^{i\sqrt{-\omega+i\lambda}(|x|+|y|)}}{\pi|x||y|D_-(\lambda^2)}d\lambda.$$

If we compute the change of variable $k=\sqrt{-\omega-i\lambda}$ in the first integral of the last equality, and $k=-\sqrt{-\omega-i\lambda}$ in the second one, then one has

$$\left|\int_{\mathcal{C}^+}[(R_m(\lambda+0)-R_m(\lambda-0))(J-iI)]_{1,1}d\lambda\right|=$$

$$=\left|\frac{4i}{\pi|x||y|}\left(\int_{-\infty}^{+\infty}\sigma\sqrt{\omega}k\frac{e^{-\sqrt{k^2+2\omega}|y|}}{D(k)}e^{-ik|x|}dk+\int_{-\infty}^{+\infty}2ik^2\frac{e^{-\sqrt{k^2+2\omega}(|x|+|y|)}}{D(k)}dk\right)\right|\leq$$

$$\leq C\frac{e^{-\sqrt{2\omega}|y|}}{|y||x|}\min\left\{\frac{1}{|x|},e^{-\sqrt{2\omega}|x|}\right\}\leq C\frac{e^{-\sqrt{2\omega}|y|}}{|y|}\frac{e^{-\sqrt{2\omega}|x|}}{|x|},$$

where the first inequality is obtained integrating by parts both integrals.
The integral of the other three elements of the matrix operator $(R_m(\lambda+0)-R_m(\lambda-0))(J-iI)$ can be estimated in the same way and this implies the statement of the lemma. $\square$

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