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Variational problems involving non–local elliptic operators

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Contents

Introduction

This thesis deals with nonlinear elliptic problems like the following one

$$\begin{cases} -\mathcal{L}_K u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \,, \end{cases} \tag{0.0.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded set, the nonlinear term f satisfies suitable conditions which will be introduced case by case, while $\mathcal{L}_K$ is a general non–local operator defined as follows:

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \tag{0.0.2}$$

for all $x \in \mathbb{R}^n$. Let also $s \in (0,1)$, here the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfies the following conditions:

$$mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\left\{|x|^2\,, 1\right\}; \tag{0.0.3}$$

$$\text{there exists } \theta > 0 \text{ such that } K(x) \geqslant \theta\, |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,. \tag{0.0.4}$$

A typical model for K is given by $K(x) = |x|^{-(n+2s)}$. In this case, it follows that $\mathcal{L}_K = -(-\Delta)^s$ and problem (0.0.1) becomes

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{0.0.5}$$

where $-(-\Delta)^s$ is the fractional Laplace operator which may be defined as

$$-(-\Delta)^s u(x) = c(n,s) \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy \tag{0.0.6}$$

for $x \in \mathbb{R}^n$, where $c(n,s)$ is the normalizing constant given by

$$c(n,s) = \frac{1}{2}\left(\int_{\mathbb{R}^n} \frac{1-\cos(\xi_1)}{|\xi|^{n+2s}}\, d\xi\right)^{-1} \tag{0.0.7}$$

as defined in [40] (see this paper and the references therein for further details on fractional Laplacian).

Recently, in the literature a deep interest was shown for non–local operators, thanks to their intriguing analytical structure and in view of several applications in a wide range of contexts. From the physical point of view, these equations take into account long-range particle interactions with a power-law decay. When the decay at infinity is sufficiently weak, the long-range phenomena may prevail and the non–local effects persist even on large scales (see e.g. [29, 71]).

The probabilistic counterpart of these fractional equation is that the underlying diffusion is run by a stochastic process with power-law tail probability distribution (the so-called Pareto or Lévy distribution), see for instance [89, 91]. Since long relocations are allowed by the process, the diffusion obtained is sometimes referred to with the name of anomalous (in contrast with the classical one coming from Poisson distributions). Physical realizations of these models occur in different fields, such as fluid dynamics (and especially quasi-geostrophic and water wave equations), dynamical systems, elasticity and micelles (see among the others [38, 39, 82, 85]). In mathematical finance, these stochastic processes have been applied to American options for modelling the jump processes of the financial derivatives such as futures, options and swaps, as explained in [37] and references therein. Also, the scale invariance of the non–local probability distribution may combine with the intermittency and renormalization properties of other nonlinear dynamics and produce complex patterns with fractional features. For instance, there are indications that the distribution of food on the ocean surface has scale invariant properties (see e.g. [90] and references therein) and it is possible that optimal searches of predators reflect these patterns in the effort of locating abundant food in sparse environments, also considering that power-law distribution of movements allow the individuals to visit more sites than the classical Brownian situation (see e.g. [19, 53]).

Nonlinear elliptic problems modeled by

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.8}$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded set and the perturbation f is a function satisfying different growth conditions (asymptotically linear, superlinear, subcritical or critical, for instance), were widely studied in the literature (see, for instance, [5, 22, 67, 86, 92] and references therein). Mathematically speaking, a motivation for studying problem (0.0.5) (and more generally (0.0.1)) is trying to extend some important results which are well-known for the classical case of the Laplacian $-\Delta$ to a non–local setting.

After the seminal papers [25, 26, 83] of Caffarelli and Silvestre, many mathematicians studied non–local problems in different contexts. In particular, there is a wide literature regarding problem (0.0.5) with a superlinear term f. We refer to [11, 16, 17, 31, 60, 75, 80, 81, 88] for a critical case, that is when $f(x,u) = g(x,u) + |u|^{2^*-2}u$, where $2^* = 2n/(n-2s)$ is the fractional Sobolev exponent and g satisfies suitable subcritical growth conditions. In [23, 61, 74, 77] the authors take into account a subcritical growth for the superlinear term f. In all these works weak solutions of problem (0.0.5) can be seen as critical points of a Euler-Lagrange functional associated with the problem. Thus, existence results are obtained by using topological and variational methods, particularly by using the Mountain Pass Theorem and the Linking Theorem (see [66, 67]).

Inspired by the variational approach used in the papers cited above, in this thesis we mainly deal with non–local equations with asymptotically linear right-hand side. Very few attempts have been made to treat this kind of problems. For this, we develop a functional analytical setting that is inspired by (but not equivalent to) the fractional Sobolev spaces, in order to correctly encode the Dirichlet boundary datum in the variational formulation. In Chapter 1 we will introduce this functional setting by starting from the space X, introduced for the first time in [76]. In the recent papers cited above, the authors take into account the homogeneous space $X_0 = \{g \in X : \ g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$. In this thesis we will consider instead the linear space Z defined as the closure of $C_0^\infty(\Omega)$ in X. As we will show in Section 1.3, by considering more regularity on domain Ω the two functional spaces are equal. However, in general Z is a subset of X_0. For this reason, the choice of Z is an improvement and in all the thesis we will assume Ω is simply a bounded domain of $\mathbb{R}^n$ without further conditions.

Always in this first chapter, we will introduce some basic properties of Z which will be used in the sequel. In particular, we will study the following general eigenvalue problem for operator $-\mathcal{L}_K$

$$\begin{cases} -\mathcal{L}_K u + q(x)u = \lambda a(x)u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \,, \end{cases} \tag{0.0.9}$$

where q is a bounded, non-negative function on Ω while a is a positive and Lipschitz continuous function on $\overline{\Omega}$. We recall that there exists a non–decreasing sequence of positive eigenvalues λ_k for which (0.0.9) admits non–trivial weak solutions e_k. In this case, any weak solution e_k will be called an eigenfunction corresponding to the eigenvalue λ_k whose properties will be studied in Section 1.2. By considering these eigenfunctions it will be possible to split the functional space Z in two subspaces $\mathbb{P}_{k+1}$ and $\mathbb{H}_k$ (which is finite dimensional), as required in the variational approach used along this thesis.

In Chapter 2 we will study problem (0.0.1) with f satisfying a linear growth and an asymptotically linear condition. By setting

$$\liminf_{|t|\to\infty} \frac{f(x,t)}{t} := \underline{\alpha}(x) \quad \text{and} \quad \limsup_{|t|\to\infty} \frac{f(x,t)}{t} := \overline{\alpha}(x),$$

the variational technique used here changes depending on how the functions $\underline{\alpha}$, $\overline{\alpha}$ behaves with respect to the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ of $-\mathcal{L}_K$. When $\overline{\alpha} < \lambda_1$, by taking into account the properties of the first eigenvalue it is possible to obtain the existence of a weak solution of (0.0.1) by a minimization argument. If there exists $k \in \mathbb{N}$ such that $\lambda_k < \underline{\alpha} \leqslant \overline{\alpha} < \lambda_{k+1}$, we will prove that the functional associated to (0.0.1) satisfies the geometric features required by the Saddle Point Theorem by Rabinowitz (see [66, 67]) and the Palais-Smale compactness conditions.

Chapter 3 is devoted to the study of the following nonlinear problem

$$\begin{cases} -\mathcal{L}_K u = \lambda a(x)u + f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{0.0.10}$$

where a is a positive, Lipschitz continuous function on $\overline{\Omega}$ and f is a continuous, bounded function whose primitive goes to infinity. Here, problem (0.0.10) is treated in presence of resonance. That is, the parameter λ belongs to the spectrum of operator $-\mathcal{L}_K$. As in Chapter 2, problem (0.0.10) can be seen as the Euler–Lagrange equation of a suitable functional and it is possible to get a weak solution by using the Saddle Point Theorem.

In Chapter 4 we will study the following problem

$$\begin{cases} -\mathcal{L}_K u + q(x)u = \lambda u + f(u) + h(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{0.0.11}$$

where, as introduced in Chapter 1, q is a bounded, non-negative function, while f and h are sufficiently smooth functions. In this chapter we will consider both resonant and the non-resonant case, that is the case when λ belongs to the spectrum of the operator driving the equation and the one when λ does not, respectively. The approach used to solve these two cases is still variational, as in the previous chapters; however, the resonant case is more difficult than the non–resonant one. In order to solve problem (0.0.11) in a resonant setting we will need a more restrictive condition for f called the Landesman–Lazer condition, firstly introduced in [58]. We also require a basic condition regarding the nodal set of eigenfunctions of $-\mathcal{L}_K$. When $K(x) = |x|^{-(n+2s)}$, this condition is a direct consequence of the unique continuation principle proved by Fall and Felli in [42].

In Chapter 5 we will introduce a Kirchhoff type problem driven by a non–local integrodifferential operator, that is

$$\begin{cases} -M\left(\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy\right)\,\mathcal{L}_K u & \\ = \lambda f(x,u) + |u|^{2^*-2}\,u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{0.0.12}$$

where M and f are two continuous functions. The approach used here is still variational, based on the application of the Mountain Pass Theorem (see [66, 67]). Since the nonlinearity in (0.0.12) is of the critical form, the verification of the Palais-Smale compactness condition is more complicated, due to a lack of compactness at critical level L^{2^*}. To overcome this problem we will use a concentration–compactness principle, introduced in the fractional framework by Palatucci and Pisante in [65]. Furthermore, we will give later an alternative proof of the Palais–Smale condition mainly based on application of the celebrated Brezis & Lieb lemma (see [21]).

Finally, in the appendix we present some detailed motivation for Kirchhoff type problem in non–local setting, starting from some classical models for vibrating strings

The thesis is mainly based on the following works [46, 47, 48, 49, 50].

Chapter 1

Functional spaces

1.1 Basic properties

In this thesis we will mainly study problems like

$$\begin{cases} -\mathcal{L}_K u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{1.1.1}$$

by using variational methods. For this, the choice of the functional space where to work plays an important role. A natural space where finding solutions for them is the fractional Sobolev space $H^s(\mathbb{R}^n)$ (see [1, 40]). Note that in (1.1.1) the homogeneous Dirichlet datum is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, as it happens in the classical case of the Laplacian, consistently with the non–local character of the operator $\mathcal{L}_K$. In order to study (1.1.1) it is important to encode the 'boundary condition' $u = 0$ in $\mathbb{R}^n \setminus \Omega$ in the weak formulation. For this the usual fractional Sobolev space is not enough. The functional space that takes into account this boundary condition will be denoted by Z and it was introduced in [46] in the following way.

First, we denote by X the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y) \mapsto (g(x) - g(y))\sqrt{K(x-y)} \text{ is in } L^2(Q,\, dxdy)\,, \tag{1.1.2}$$

where $Q := \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. The space X is endowed with the norm defined as

$$\|u\|_X = \Big(\int_\Omega |u(x)|^2\, dx + \iint_Q |u(x) - u(y)|^2 K(x-y) dx\, dy \Big)^{1/2}. \tag{1.1.3}$$

It is immediate to observe that bounded and Lipschitz functions belong to X (see [74, 76] for further details on space X).

In many articles, like [60, 61, 62, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81], the authors worked in the following homogeneous space

$$X_0 = \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}\,. \tag{1.1.4}$$

Here, we will denote with Z the closure of $C_0^\infty(\Omega)$ in X; this space was introduced for the first time in [46]. As we will see in a forthcoming section, generally space X_0 contains Z. However, by assuming more regularity for the domain Ω it is possible to show that $X_0 = Z$.

In the sequel we will provide and prove some basic results of the space Z which will be useful along the thesis. In the next lemma we recall the connection between the space Z and the homogeneous fractional Sobolev spaces.

Lemma 1.1.1. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4).

Then, Z is continuously embedded in $H_0^s(\Omega)$ (for a detailed description see [40]) which is the closure of $C_0^\infty(\Omega)$ in the space $H^s(\Omega)$ of functions u defined on Ω for which is well defined the so-called Gagliardo norm

$$\|u\|_{H^s(\Omega)} = \Big(\int_\Omega |u(x)|^2\,dx + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \Big)^{1/2}.$$

Proof. We simply observe that by (0.0.4) we get

$$\theta \iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant \iint_Q |u(x)-u(y)|^2\,K(x-y)\,dx\,dy \tag{1.1.5}$$

and so

$$\|u\|_{H^s(\Omega)} \leqslant c(\theta)\|u\|_X\,,$$

with $c(\theta) = \max\{1, \theta^{-1/2}\}$. □

Now, we give a convergence property for bounded sequences in Z.

Lemma 1.1.2. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4).

Then, Z is compactly embedded in $L^p(\Omega)$ for any $p \in [1, 2^*)$, where 2^* is the fractional critical Sobolev exponent given by[1]

$$2^* := \begin{cases} \dfrac{2n}{n-2s} & \text{if } n > 2s,\\ +\infty & \text{if } n \leqslant 2s. \end{cases} \tag{1.1.6}$$

[1] Note that, when $s = 1$ the exponent 2^* reduces to the classical critical Sobolev exponent.

Proof. Let Ω' be a regular, open subset of $\mathbb{R}^n$ such that $\Omega \subseteq \Omega'$. For any $u \in H_0^s(\Omega)$ we can define

$$\widetilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega' \setminus \Omega. \end{cases}$$

It is clear that $\widetilde{u} \in H_0^s(\Omega')$. Indeed, if $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in $C_0^\infty(\Omega)$ which converges to u in $H_0^s(\Omega)$ then $\{\widetilde{u}_j\}_{j\in\mathbb{N}}$ is a sequence in $C_0^\infty(\Omega')$ which converges to $\widetilde{u}$ in $H_0^s(\Omega')$. Moreover, we also have

$$\|\widetilde{u}\|_{H^s(\Omega')} = \|u\|_{H^s(\Omega)} .$$

Thus, $H_0^s(\Omega')$ is isometric embedded in $H_0^s(\Omega)$. The conclusion follows by remembering that $H_0^s(\Omega')$ is compactly embedded in $L^p(\Omega')$ with $1 \leqslant p < 2^*$ (see [40, Theorem 6.7]). □

We conclude this section with the following result.

Lemma 1.1.3. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4).

Then, Z is a Hilbert space endowed with the following norm

$$\|v\|_Z = \Big(\iint_Q |v(x) - v(y)|^2 K(x-y) dx\, dy \Big)^{1/2}, \tag{1.1.7}$$

which is equivalent to the usual one defined in (1.1.3).

Proof. We start by claiming that there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leqslant C \left(\iint_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx\, dy \right)^{1/2} \tag{1.1.8}$$

for any $u \in H_0^s(\Omega)$. In fact, since Ω is bounded there is $R > 0$ such that $\Omega \subseteq B_R$ and $|B_R \setminus \Omega| > 0$. So, we get

$$\begin{aligned} &\iint_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx\, dy \geqslant \int_{\mathcal{C}\Omega} \left(\int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dy \right) dx \\ &= \int_{\mathcal{C}\Omega} \left(\int_\Omega \frac{|u(y)|^2}{|x-y|^{n+2s}} dy \right) dx \geqslant \int_{B_R \setminus \Omega} \left(\int_\Omega \frac{|u(y)|^2}{|2R|^{n+2s}} dy \right) dx = \frac{|B_R \setminus \Omega|}{(2R)^{n+2s}} \|u\|^2_{L^2(\Omega)} \end{aligned}$$

for any $u \in H_0^s(\Omega)$ (since $u = 0$ a.e. in $\mathcal{C}\Omega$), which proves our claim. Finally, by combining (1.1.5) and (1.1.8) we conclude the proof. □

Remark 1.1.4. *From now on, we will take* (1.1.7) *as norm on Z, apart from few cases. Note also that in* (1.1.3) *and* (1.1.7) *all the integrals can be extended to all $\mathbb{R}^n$ and $\mathbb{R}^{2n}$, since $u = 0$ a.e. in $\mathcal{C}\Omega$.*

1.2 An eigenvalue problem

This section is devoted to the study of the non-homogeneous eigenvalue problem

$$\begin{cases} -\mathcal{L}_K u + q(x)u = \lambda a(x)u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.2.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded set, $q : \Omega \to \mathbb{R}$ is such that $q \in L^\infty(\Omega)$ and $q(x) \geqslant 0$ for a.e. $x \in \Omega$, $a : \overline{\Omega} \to \mathbb{R}$ is a positive Lipschitz continuous function.

More precisely, we consider the weak formulation, which consists in the following eigenvalue problem

$$\begin{cases} \displaystyle\iint_{\mathbb{R}^{2n}} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dx\,dy \int_\Omega q(x)u(x)\varphi(x)dx \\ \qquad\qquad\qquad\qquad\qquad\qquad = \lambda \displaystyle\int_\Omega a(x)u(x)\varphi(x)dx \qquad \forall\, \varphi \in Z \\ u \in Z. \end{cases} \tag{1.2.2}$$

We recall that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.2.2) provided there exists a non–trivial solution $u \in Z$ of problem (1.2.2) and, in this case, any solution will be called an eigenfunction corresponding to the eigenvalue λ.

In order to generalize as much as possible, here we equip Z with the following norm

$$\|g\|_{Z,\,q} = \left(\iint_{\mathbb{R}^{2n}} |g(x) - g(y)|^2 K(x-y)\,dx\,dy + \int_\Omega q(x)\,|g(x)|^2\,dx\right)^{1/2}, \tag{1.2.3}$$

which is equivalent to the usual one defined in (1.1.3), as we prove in the following lemma:

Lemma 1.2.1. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying assumptions* (0.0.3) *and* (0.0.4) *and let $q : \Omega \to \mathbb{R}$ satisfy $q \in L^\infty(\Omega)$ and $q(x) \geqslant 0$ a.e. $x \in \Omega$.*

Then, the expression

$$\langle u, v\rangle_{Z,\,q} = \iint_{\mathbb{R}^{2n}} (u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy + \int_\Omega q(x)u(x)v(x)dx \tag{1.2.4}$$

defines on Z a scalar product that induces a norm, denoted with $\|\cdot\|_{Z,\,q}$, equivalent to the usual one defined in (1.1.3).

Proof. Since the expression (1.2.4) is a sum of two scalar products, it is immediate to observe that $\langle\cdot,\cdot\rangle_{Z,\,q}$ is a scalar product on Z which induces the norm defined in (1.2.3).

Now, we show that the norm defined in (1.2.3) is equivalent to the one given in

(1.1.3). For this, let $v \in Z$. It is easily seen that

$$\begin{aligned}\|v\|^2_{Z,\,q} &= \iint_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy + \int_\Omega q(x)\,|v(x)|^2\,dx \\ &\leqslant \iint_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy + \|q\|_{L^\infty(\Omega)}\,\|v\|^2_{L^2(\Omega)} \leqslant C_1\|v\|^2_X\,,\end{aligned} \tag{1.2.5}$$

where $C_1 = \max\Big\{1,\, \|q\|_{L^\infty(\Omega)}\Big\} > 0$.

Moreover, by [74, Lemma 6] we know that there is a constant $C_2 > 1$ such that

$$\|v\|^2_X \leqslant C_2 \iint_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy\,,$$

so that, by recalling that q is bounded and non-negative a.e. on Ω, we get

$$\begin{aligned}\frac{1}{C_2}\|v\|^2_X &\leqslant \iint_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy \\ &\leqslant \iint_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy + \int_\Omega q(x)\,|v(x)|^2\,dx = \|v\|^2_{Z,\,q}\,.\end{aligned} \tag{1.2.6}$$

By combining (1.2.5) and (1.2.6) we conclude the proof. □

Finally, we note that, since $a \in L^\infty(\Omega)$, all the embeddings properties of Z into the usual Lebesgue space $L^2(\Omega)$ still hold true in $L^2(\Omega,\,\mu)$, with $\mu(\cdot) = a(\cdot)dx$, defined as

$$\begin{aligned}L^2(\Omega,\,\mu) := \Big\{ & g : \Omega \to \mathbb{R} \text{ s.t. } g \text{ is measurable in } \Omega \text{ and} \\ & \textstyle\int_\Omega a(x)|g(x)|^2\,dx = \int_\Omega |g|^2\,d\mu < +\infty \Big\}\,.\end{aligned}$$

Now, we are ready to introduce the properties of eigenfunctions related to the operator $-\mathcal{L}_K + q$. These properties will play a crucial role in the study of asymptotically linear problems. In particular, we need them to check suitable geometrical features of the functional associated to the problem. We would also point out that these properties are the non–local transpositions of well–known results of eigenfunctions of the classical Laplace operator (see for instance [14, Section 1.7]).

For a detailed proof of the next result we refer to [77, Proposition 9 and Appendix A], where the problem (1.2.2) with $q \equiv 0$ and $a \equiv 1$ was considered. The proof of [77, Proposition 9] can be easily adapted in order to get the following proposition:

Proposition 1.2.2. *Let Ω be an open, bounded subset of $\mathbb{R}^n$ and let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying assumptions* (0.0.3) *and* (0.0.4)*. Moreover, let $q : \Omega \to \mathbb{R}$ be a function such that $q \in L^\infty(\Omega)$ and $q(x) \geqslant 0$ a.e. $x \in \Omega$, let $a : \overline{\Omega} \to \mathbb{R}$ be a positive Lipschitz continuous function.*

Then,

(i) *problem* (1.2.2) *admits an eigenvalue* λ_1 *which is positive and that can be characterized as follows*

$$\lambda_1 = \min_{\substack{u\in Z\\ \|u\|_{L^2(\Omega,\,\mu)}=1}} \left(\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy + \int_\Omega q(x)\,|u(x)|^2\,dx \right)$$

or, equivalently,

$$\lambda_1 = \min_{u\in Z\setminus\{0\}} \frac{\displaystyle\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy + \int_\Omega q(x)\,|u(x)|^2\,dx}{\displaystyle\int_\Omega a(x)\,|u(x)|^2\,dx}, \quad (1.2.7)$$

where $\|\cdot\|_{L^2(\Omega,\,\mu)}$ *denotes the* L^2*–norm with respect to the measure* $\mu(x) = a(x)dx$*;*

(ii) *there exists a non–negative function* $e_1 \in Z$*, which is an eigenfunction corresponding to* λ_1*, attaining the minimum in* (1.2.7)*, that is* $\|e_1\|_{L^2(\Omega,\,\mu)} = 1$ *and*

$$\lambda_1 = \iint_{\mathbb{R}^{2n}} |e_1(x)-e_1(y)|^2 K(x-y)dx\,dy + \int_\Omega q(x)\,|e_1(x)|^2\,dx;$$

(iii) λ_1 *is simple, that is if* $u\in Z$ *is a solution of the following equation*

$$\iint_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy + \int_\Omega q(x)\,|u(x)|^2\,dx$$
$$= \lambda_1 \int_\Omega a(x)u(x)\varphi(x)dx \quad \forall\varphi\in Z,$$

then $u = \zeta e_1$*, with* $\zeta\in\mathbb{R}$*;*

(iv) *the set of the eigenvalues of problem* (1.2.2) *consists of a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *with*[2]

$$0 < \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \ldots \quad (1.2.8)$$

and

$$\lambda_k \to +\infty \ \textit{as} \ \ k\to+\infty.$$

Moreover, for any $k\in\mathbb{N}$ *the eigenvalues can be characterized as follows:*

$$\lambda_{k+1} = \min_{\substack{u\in\mathbb{P}_{k+1}\\ \|u\|_{L^2(\Omega,\,\mu)}=1}} \left(\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)dx\,dy + \int_\Omega q(x)\,|u(x)|^2\,dx \right),$$

[2]As usual, here we call λ_1 the *first eigenvalue* of the operator $-\mathcal{L}_K + q$. This notation is justified by (1.2.8). Notice also that some of the eigenvalues in the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ may repeat, i.e. the inequalities in (1.2.8) may be not always strict.

or, equivalently,

$$\lambda_{k+1} = \min_{u \in \mathbb{P}_{k+1} \setminus \{0\}} \frac{\displaystyle\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y) dx\, dy + \int_\Omega q(x) |u(x)|^2 dx}{\displaystyle\int_\Omega a(x) |u(x)|^2 dx}, \tag{1.2.9}$$

where

$$\mathbb{P}_{k+1} := \left\{ u \in Z : \langle u, e_j \rangle_Z = 0 \quad \forall j = 1, \dots, k \right\} \quad (\textit{with } \mathbb{P}_1 := Z); \tag{1.2.10}$$

(v) for any $k \in \mathbb{N}$ there exists a function $e_{k+1} \in \mathbb{P}_{k+1}$, which is an eigenfunction corresponding to λ_{k+1}, attaining the minimum in (1.2.9)*, that is $\|e_{k+1}\|_{L^2(\Omega, \mu)} = 1$ and*

$$\lambda_{k+1} = \iint_{\mathbb{R}^{2n}} |e_{k+1}(x) - e_{k+1}(y)|^2 K(x-y) dx\, dy + \int_\Omega q(x) |e_{k+1}(x)|^2 dx; \tag{1.2.11}$$

(vi) the sequence $\{e_k\}_{k \in \mathbb{N}}$ of eigenfunctions corresponding to λ_k is an orthonormal basis of $L^2(\Omega, \mu)$ and an orthogonal basis of Z;

(vii) each eigenvalue λ_k has finite multiplicity; more precisely, if λ_k is such that

$$\lambda_{k-1} < \lambda_k = \ldots = \lambda_{k+h} < \lambda_{k+h+1}$$

for some $h \in \mathbb{N}_0$, then the set of all the eigenfunctions corresponding to λ_k agrees with

$$\textit{span}\, \{e_k, \dots, e_{k+h}\}.$$

Proof. The proof substantially follows by the general theory of functional analysis and by the compact embedding of Z in $L^2(\Omega)$, proved in Lemma 1.1.2.. □

Now, we point out that Proposition 1.2.2 gives a variational characterization of the eigenvalues λ_k of $-\mathcal{L}_K + q$ (see formulas (1.2.7) and (1.2.9)) . Another interesting characterization of the eigenvalues is given in the next result. For the proof we refer to [72, Proposition 5] , where the case $q \equiv 0$ and $a \equiv 1$ was treated (again, the general case can be proved likewise) .

Proposition 1.2.3. *Let Ω be an open, bounded subset of $\mathbb{R}^n$ and let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying assumptions* (0.0.3) *and* (0.0.4)*. Moreover, let $q : \Omega \to \mathbb{R}$ be a function such that $q \in L^\infty(\Omega)$ and $q(x) \geqslant 0$ a.e. $x \in \Omega$, let $a : \overline{\Omega} \to \mathbb{R}$ be a positive Lipschitz continuous function. Let $\left\{\lambda_k\right\}_{k \in \mathbb{N}}$ be the sequence of the* *eigenvalues given in Proposition 1.2.2 and let* $\{e_k\}_{k\in\mathbb{N}}$ *be the corresponding sequence of eigenfunctions.*

Then, for any $k \in \mathbb{N}$ *the eigenvalues can be characterized as follows:*

$$\lambda_k = \max_{u\in span\{e_1,\dots,e_k\}\setminus\{0\}} \frac{\displaystyle\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy + \int_\Omega q(x)\,|u(x)|^2\,dx}{\displaystyle\int_\Omega a(x)|u(x)|^2\,dx}\,.$$

1.3 A density result

Aim of this section is to show that, as we pointed out in the previous sections, Z is the better and natural space in order to study problem (1.1.1) from a variational point of view. Indeed, in the classical Laplace setting (i.e. by considering problem (1.1.1) with $-\Delta$ instead of $-\mathcal{L}_K$) the natural space where finding solutions is the Sobolev space $H^1_0(\Omega)$, which is defined as the closure of $C^\infty_0(\Omega)$ in the norm of $H^1(\Omega)$. Moreover, as we see in the next lemma, in general Z is a subset of the functional space X_0 introduced in (1.1.4).

Before proving our lemma we would note that, since X_0 is a space of functions defined in $\mathbb{R}^n$, in this section we denote by $C^\infty_0(\Omega)$ the space

$$C^\infty_0(\Omega) = \{g : \mathbb{R}^n \to \mathbb{R} : g \in C^\infty(\mathbb{R}^n) \text{ and } g = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}\,. \tag{1.3.1}$$

Lemma 1.3.1. *Let* Ω *be an open subset of* $\mathbb{R}^n$. *Let* $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ *be a function such that* (0.0.3) *holds true and satisfying*

$$K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.3.2}$$

Let X_0 *and* $C^\infty_0(\Omega)$ *be the spaces defined as in* (1.1.4) *and* (1.3.1), *respectively.*

Then, $C^\infty_0(\Omega) \subseteq X_0$ *.*

Proof. Let $\varphi \in C^\infty_0(\Omega)$. By using (1.3.2), it is easy to see that

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} |\varphi(x)-\varphi(y)|^2 K(x-y)\,dx\,dy = \\ &\iint_{\operatorname{Supp}\varphi\times\operatorname{Supp}\varphi} |\varphi(x)-\varphi(y)|^2 K(x-y)\,dx\,dy \\ &\quad + 2\iint_{\operatorname{Supp}\varphi\times\mathcal{C}(\operatorname{Supp}\varphi)} |\varphi(x)-\varphi(y)|^2 K(x-y)\,dx\,dy \\ &\leqslant 2\iint_{\operatorname{Supp}\varphi\times\mathbb{R}^n} |\varphi(x)-\varphi(y)|^2 K(x-y)\,dx\,dy.\end{aligned} \tag{1.3.3}$$

Now, we notice that for any $x, y \in \mathbb{R}^n$

$$|\varphi(x) - \varphi(y)| \leqslant \|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}|x - y|$$

and

$$|\varphi(x) - \varphi(y)| \leqslant 2\|\varphi\|_{L^\infty(\mathbb{R}^n)}\,,$$

thanks to the regularity of φ. Accordingly, for any $x, y \in \mathbb{R}^n$

$$|\varphi(x) - \varphi(y)| \leqslant 2\|\varphi\|_{C^1(\mathbb{R}^n)}\min\{|x - y|, 1\} = 2\|\varphi\|_{C^1(\mathbb{R}^n)}\sqrt{m(x - y)},$$

where m is defined in (0.0.3). Therefore, from (1.3.3) we deduce that

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} |\varphi(x) - \varphi(y)|^2 K(x - y)\,dx\,dy &\leqslant 2^3\|\varphi\|^2_{C^1(\mathbb{R}^n)} \iint_{\operatorname{Supp}\varphi\times\mathbb{R}^n} m(x - y)K(x - y)\,dx\,dy \\ &= 2^3|\operatorname{Supp}\varphi|\,\|\varphi\|^2_{C^1(\mathbb{R}^n)}\int_{\mathbb{R}^n} m(\xi)K(\xi)\,d\xi\,,\end{aligned}$$

where $|\operatorname{Supp}\varphi|$ denotes the Lebesgue measure of $\operatorname{Supp}\varphi$. Thus, Lemma 1.3.1 follows by (0.0.3) and by the fact that $\operatorname{Supp}\varphi$ is bounded. □

From Lemma 1.3.1 and by using the equivalence of the norms in Z given by (1.1.3) and (1.1.7), it easily follows the inclusion $Z \subset X_0$. However, under a further and more restrictive assumption on domain Ω, the two functional spaces Z and X_0 are equal. This equivalence follows by using the next density property of X_0, which we mention here without proof, since it is beyond our purposes (for a detailed proof see paper [49]).

Theorem 1.3.2. *[49] Let Ω be an open subset of $\mathbb{R}^n$, with continuous boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function such that* (0.0.3) *and* (0.0.4) *hold true and let X_0 be the space defined as in* (1.1.4)*.*

Then, for any $u \in X_0$ there exists a sequence $u_k \in C_0^\infty(\Omega)$ such that $u_k \to u$ in X_0 as $k \to +\infty$. In other words, $C_0^\infty(\Omega)$ is a dense subspace of X_0.

We think that it is an interesting problem to determine the “minimal” regularity assumptions on the domain Ω under which the density of the smooth functions, compactly supported in Ω, stated in Theorem 1.3.2, holds true. However, we remark that such property does not hold for *any* domain Ω, not even when $n = 1$, as the following counterexample shows.

Remark 1.3.3. *Let $\Omega := (-1, 0) \cup (0, 1)$, $s \in (1/2, 1)$, $\psi : \mathbb{R} \to \mathbb{R}$ be any fixed smooth function supported in $(-1, 1)$ with $\psi(0) = 1$, and define*

$$\varphi(x) := \begin{cases} \psi(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Then, since integrals disregard sets of measure zero, we have that for any $s \in (0,1)$

$$\|\varphi\|_{H^s(\mathbb{R})} = \|\psi\|_{H^s(\mathbb{R})} < +\infty\,,$$

hence $\varphi \in H^s(\mathbb{R})$*. Also,* φ *vanishes outside* Ω*, that is* $\varphi \in X_0$ *.*

Now, let η *be any smooth function supported in* Ω*. We have that* $\eta(0) = 0$ *and so, denoting by* $f := \varphi - \eta$*, by the fractional Sobolev embedding (see e.g. [40, Theorem 8.2] and [74, Lemmas 6 and 7]), we obtain that*

$$1 = \lim_{\Omega \ni x \to 0} f(x) \leqslant \|f\|_{L^\infty(\Omega)} \leqslant C\|f\|_{H^s(\Omega)} \leqslant C\|\varphi - \eta\|_X \leqslant \tilde{C}\|\varphi - \eta\|_{X_0},$$

where C *and* $\tilde{C}$ *are positive constants. Therefore, smooth functions compactly supported in* Ω *cannot approximate* φ *in* X_0*.*

Chapter 2

An asymptotically linear problem

2.1 Introduction

In general, nonlinear elliptic problems like the following one

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

has a variational nature and its solutions can be constructed as critical points of the associated Euler–Lagrange. For this, the assumptions on the perturbation f have a direct influence on the topological structure of the problem and the variational approach changes by depending on these assumptions. When the nonlinear term has a superlinear growth the Mountain Pass Theorem and the Linking Theorem are natural ways to face problem (0.0.8). While for asymptotically linear problems, namely those where the nonlinearity grows linearly at infinity, an application of the Saddle Point Theorem is most suitable. A natural question is whether or not these topological and variational methods may be adapted to a non–local framework in order to extend the classical results known for (0.0.8).

Aim of this chapter is to consider the non–local counterpart of problem (0.0.8) with an asymptotically linear perturbation. Here, we deal with the following problem

$$\begin{cases} -\mathcal{L}_K u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{2.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set and $\mathcal{L}_K$ is the non–local operator formally defined as in (0.0.2). For a fixed $s \in (0,1)$, the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfy conditions (0.0.3) and (0.0.4), introduced in the Introduction. Moreover, in view of our problem we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that:

$$\begin{aligned} &\text{there exist } a \in L^2(\Omega) \text{ and } b \geqslant 0 \text{ such that } |f(x,t)| \leqslant a(x) + b\,|t| \\ &\text{for any } t \in \mathbb{R} \text{ and a.e. } x \in \Omega. \end{aligned} \tag{2.1.2}$$

Now, we can state in a precise way problem (2.1.1) by writing it in the variational form:

$$\begin{cases} \displaystyle\iint_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy \\ \qquad\qquad\qquad\qquad = \displaystyle\int_\Omega f(x,u(x))\varphi(x)dx \quad \text{for any } \varphi \in Z \\ u \in Z, \end{cases} \tag{2.1.3}$$

where Z is the functional space introduced in Chapter 1. Thanks to our assumptions on Ω, f and K, all the integrals in (2.1.3) are well defined if u, $\varphi \in Z$. We also point out that the odd part of function K gives no contribution to the integral of the left-hand side of (2.1.3). Indeed, write $K = K_e + K_o$, where for all $x \in \mathbb{R}^n \setminus \{0\}$

$$K_e(x) = \frac{K(x)+K(-x)}{2} \quad \text{and} \quad K_o(x) = \frac{K(x)-K(-x)}{2}.$$

Then, it is apparent that for all u and $\varphi \in Z$

$$\langle u, \varphi \rangle_Z = \iint_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K_e(x-y)dxdy.$$

Therefore, it would be not restrictive to assume that K is even [1].

Now, we are ready to introduce the main result of the chapter. Here, we denote with $\lambda_1, \lambda_2, \ldots$ the eigenvalues of $-\mathcal{L}_K$ which we already introduced in Section 1.2.

Theorem 2.1.1. *Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4) *and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function verifying* (2.1.2)*. Moreover, by setting*

$$\liminf_{|t|\to\infty} \frac{f(x,t)}{t} := \underline{\alpha}(x) \quad \textit{and} \quad \limsup_{|t|\to\infty} \frac{f(x,t)}{t} := \overline{\alpha}(x) \quad \textit{for a.e. } x \in \Omega, \tag{2.1.4}$$

we assume that one of the two following conditions is satisfied: either $\overline{\alpha}(x) < \lambda_1$ for a.e. $x \in \Omega$, or there exists $k \in \mathbb{N}$ such that $\lambda_k < \underline{\alpha}(x) \leqslant \overline{\alpha}(x) < \lambda_{k+1}$ for a.e. $x \in \Omega$.

Then, problem (2.1.1) *admits a weak solution $u \in Z$.*

[1] As we shall see, this fact could hold true also for the next main problems (3.1.2), (4.1.1) and (5.1.1).

We notice that, in our framework, no solution of problem (2.1.3) is known from the beginning, unlike the cases treated in [72, 74, 75, 77, 80, 81], for example, where the variational problems considered admit the trivial solution $u = 0$ (indeed, in our case, $f(x, 0)$ may not vanish and $u = 0$ may not be a solution).

The proof of Theorem 2.1.1 relies on the Saddle Point Theorem (see, for instance, [66, 67]). In order to check the geometric assumptions needed for applying this result, we perform some energy estimates in fractional Sobolev spaces. Indeed, Theorem 2.1.1 is the fractional analog of a result valid for the classical Laplacian (see, e.g., [63, Theorem 4.1.1]).

It is an interesting question if weak solutions of problem (2.1.3) solve also problem (2.1.1) in an appropriate strong sense. Some interesting results about this problem can be found in [79]. Note also that, when f is a "good function", any weak solution is a classical solution. This can be seen in the fractional setting (with $\mathcal{L}_K = -(-\Delta)^s$) as follows. Let u be a weak solution of (2.1.1). Then, from [79, Proposition 7] we have the boundedness of u and by [69, Proposition 1.1] it follows that u is continuous up to the boundary. Finally, by considering $u * \eta_\varepsilon$ and $f * \eta_\varepsilon$, where η_ε is a standard mollifier, it is not difficult to see that u is regular in the interior of Ω by applying a standard bootstrap argument (see [18, Theorem 5]).

Also, it is worth pointing out that the solution found in Theorem 2.1.1 is unique, under a suitable condition on the nonlinearity.

Corollary 2.1.2. *Under the same assumptions of Theorem 2.1.1 and if in addition there exists a $k \in \mathbb{N}$ such that*

$$\lambda_k < \frac{f(x,\tau) - f(x,t)}{\tau - t} < \lambda_{k+1} \quad \text{for any } \tau,\, t \in \mathbb{R} \text{ with } \tau \neq t \text{ and a.e. } x \in \Omega\,. \tag{2.1.5}$$

Then the weak solution of problem (2.1.1) *is unique.*

The chapter is organized as follows. In Section 2.2 we collect some preliminary estimates on the primitive of f and an useful technical lemma. In Section 2.3 we prove Theorem 2.1.1 performing the classical Saddle Point Theorem.

2.2 Some preliminary estimates and a technical result

Here we use condition (2.1.2) on f to deduce some preliminary estimates involving its primitive F with respect to the second variable, that is

$$F(x,t) = \int_0^t f(x,\tau)d\tau. \tag{2.2.1}$$

At first we immediately notice that, by integrating (2.1.2), it follows that

$$|F(x,t)| \leqslant a(x)\,|t| + b\frac{|t|^2}{2} \text{ for any } t \in \mathbb{R} \text{ and a.e. } x \in \Omega. \tag{2.2.2}$$

Moreover, by also exploiting the notations introduced in (2.1.4) we get the following result.

Lemma 2.2.1. *Assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the condition* (2.1.2)*. Then, the primitive function F verifies the following inequalities*

$$\limsup_{|t|\to\infty} \frac{F(x,t)}{t^2} \leqslant \frac{\overline{\alpha}(x)}{2}, \tag{2.2.3}$$

and

$$\liminf_{|t|\to\infty} \frac{F(x,t)}{t^2} \geqslant \frac{\underline{\alpha}(x)}{2}, \tag{2.2.4}$$

for a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$, where $\overline{\alpha}$ and $\underline{\alpha}$ are defined as in (2.1.4)*.*

Proof. By (2.1.4), for all $\varepsilon > 0$ there exists $R > 0$ such that

$$\frac{f(x,t)}{t} - \overline{\alpha}(x) < \varepsilon \quad \forall\,|t| > R. \tag{2.2.5}$$

Integrating and recalling (2.2.2) we get

$$\begin{aligned} F(x,t) &\leqslant |F(x,R)| + \int_R^{|t|} f(x,\tau)d\tau \\ &\leqslant a(x)R + b\frac{R^2}{2} + \frac{\overline{\alpha}(x)+\varepsilon}{2}(t^2 - R^2) \quad \forall\,|t| > R. \end{aligned} \tag{2.2.6}$$

Therefore

$$\limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} \leqslant \frac{\overline{\alpha}(x)+\varepsilon}{2}$$

for every $\varepsilon > 0$, and so (2.2.3) follows. In analogous way one can prove (2.2.4). $\square$

In order to prove the compactness condition necessary to apply the variational theorem, we will need the following technical lemma.

Lemma 2.2.2. *Let Ω be a measurable subset of $\mathbb{R}^n$ and let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a sequence of functions of $L^2(\Omega)$. If*

$$\varphi_j \rightharpoonup \varphi \quad \text{in } L^2(\Omega), \tag{2.2.7}$$

and there exists two functions ψ_1, $\psi_2 \in L^2(\Omega)$ such that

$$\psi_1(x) \leqslant \liminf_{j\to+\infty} \varphi_j(x) \leqslant \limsup_{j\to+\infty} \varphi_j(x) \leqslant \psi_2(x)$$

a.e. in Ω, then

$$\psi_1(x) \leqslant \varphi(x) \leqslant \psi_2(x)$$

a.e. in Ω.

Proof. We prove that $\psi_1(x) \leqslant \varphi(x)$ (the proof that $\varphi(x) \leqslant \psi_2(x)$ is similar).

Case 1. *We first consider the case when*

$$\liminf_{j\to+\infty} \varphi_j(x) > \psi_1(x) \quad \textit{a.e. in } \Omega. \tag{2.2.8}$$

Let $\beta_j := (\varphi_j - \psi_1)^+ = \max\{\varphi_j - \psi_1, 0\}$. By the Fatou Lemma we have

$$\liminf_{j\to+\infty} \int_\Omega \beta_j(x)\eta(x)dx \geqslant \int_\Omega \liminf_{j\to+\infty} \beta_j(x)\eta(x)dx \geqslant 0 \tag{2.2.9}$$

for all $\eta \in L^2(\Omega)$ with $\eta \geqslant 0$ a.e. in Ω. Now we have that

$$\begin{aligned}\int_\Omega (\varphi_j - \psi_1)^+(x)\eta(x)dx &= \int_{\{x\in\Omega:\, \varphi_j(x)\geqslant\psi_1(x)\}} (\varphi_j - \psi_1)(x)\eta(x)dx \\ &= \int_\Omega (\varphi_j - \psi_1)(x)\eta(x)dx - \int_{\{x\in\Omega:\, \varphi_j(x)<\psi_1(x)\}} (\varphi_j - \psi_1)(x)\eta(x)dx.\end{aligned} \tag{2.2.10}$$

Moreover, by using Hölder inequality we get

$$\begin{aligned}&\left| \int_{\{x\in\Omega:\, \varphi_j(x)<\psi_1(x)\}} (\varphi_j - \psi_1)(x)\eta(x)dx \right| \\ \leqslant& 4\left(\int_\Omega (|\varphi_j(x)|^2 + |\psi_1(x)|^2)dx\right)^{1/2} \left(\int_{\{x\in\Omega:\, \varphi_j(x)<\psi_1(x)\}} |\eta(x)|^2\, dx\right)^{1/2}.\end{aligned} \tag{2.2.11}$$

Since by (2.2.7) the sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ (see [20, Proposition III.5]), the first term of the right-hand side of (2.2.11) is finite, therefore

$$\begin{aligned}&4\left(\int_\Omega (|\varphi_j(x)|^2 + |\psi_1(x)|^2)dx\right)^{1/2} \left(\int_{\{x\in\Omega:\, \varphi_j(x)<\psi_1(x)\}} |\eta(x)|^2\, dx\right)^{1/2} \\ \leqslant& C\left(\int_{\{x\in\Omega:\, \varphi_j(x)<\psi_1(x)\}} |\eta(x)|^2\, dx\right)^{1/2}\end{aligned} \tag{2.2.12}$$

with C a positive constant independent of j. Let $g_j := \eta^2\chi_{\{x\in\Omega:\, \varphi_j(x)<\psi_1(x)\}}$, denoting with χ the characteristic function on $\{x \in \Omega :\ \varphi_j(x) < \psi_1(x)\}$, and set

$$A := \{x \in \Omega :\ g_j(x) \not\to 0 \text{ as } j \to +\infty\}.$$

If $x \in A$ then we can construct a subsequence such that $\varphi_{j_k}(x) < \psi_1(x)$, so passing to the limit we get

$$\liminf_{j\to+\infty} \varphi_j(x) \leqslant \liminf_{k\to+\infty} \varphi_{j_k}(x) \leqslant \psi_1(x).$$

By comparing the last inequality and (2.2.8) we see that the measure of A is equal to 0 and so it follows that $g_j(x) \to 0$ a.e. in Ω. Since $|g_j| \leqslant \eta^2 \in L^1(\Omega)$, by the Lebesgue Dominated Convergence Theorem we have

$$\lim_{j\to+\infty} \int_\Omega g_j(x)dx = \lim_{j\to+\infty} \int_{\{x\in\Omega:\, \varphi_j(x)<\psi_1(x)\}} |\eta(x)|^2\, dx = 0. \tag{2.2.13}$$

By (2.2.9)–(2.2.13) we get

$$\liminf_{j\to+\infty}\int_\Omega(\varphi_j-\psi_1)(x)\eta(x)dx=\liminf_{j\to+\infty}\int_\Omega(\varphi_j-\psi_1)^+(x)\eta(x)dx\geqslant 0$$

and so by (2.2.7) it follows that

$$\int_\Omega(\varphi-\psi_1)(x)\eta(x)dx\geqslant 0\quad\forall\eta\in L^2(\Omega)\tag{2.2.14}$$

and from this we get

$$\varphi(x)\geqslant\psi_1(x)\quad\text{a.e. in}\,\Omega$$

concluding the proof.

Case 2. *Now we assume that*

$$\liminf_{j\to+\infty}\varphi_j(x)\geqslant\psi_1(x)\quad a.e.\ in\ \Omega.$$

For an arbitrary $\varepsilon>0$ we set $\gamma_j:=\varphi_j+\varepsilon$, therefore

$$\liminf_{j\to+\infty}\gamma_j(x)\geqslant\psi_1(x)+\varepsilon>\psi_1(x)\quad\text{a.e. in}\,\Omega$$

and

$$\gamma_j\rightharpoonup\varphi+\varepsilon\quad\text{in}\,L^2(\Omega).$$

So, by Case 1 we have

$$\varphi(x)+\varepsilon\geqslant\psi_1(x)\quad\text{a.e. in }\ \Omega$$

and for the arbitrariness of ε we can conclude the proof. □

2.3 Main results

For the proof of Theorem 2.1.1, we observe that problem (2.1.3) has a variational structure, indeed it is the Euler-Lagrange equation of the functional $\mathcal{J}:Z\to\mathbb{R}$ defined as follows

$$\mathcal{J}(u)=\frac{1}{2}\iint_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)\,dx\,dy-\int_\Omega F(x,u(x))dx\,.$$

Note that the functional $\mathcal{J}$ is Fréchet differentiable in $u\in Z$ and for any $\varphi\in Z$

$$\begin{aligned}\langle\mathcal{J}'(u),\varphi\rangle=&\iint_{\mathbb{R}^{2n}}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy\\&-\int_\Omega f(x,u(x))\varphi(x)\,dx\,.\end{aligned}$$

Thus, critical points of $\mathcal{J}$ are solutions of problem (2.1.3). In order to find these critical points we will divide the proof in two cases. At first, when $\overline{\alpha}(x) < \lambda_1$ the existence of the solution of problem (2.1.3) follows from the Weierstrass Theorem (i.e. by direct minimization). When $\lambda_k < \underline{\alpha}(x) \leqslant \overline{\alpha} < \lambda_{k+1}$ for some $k \in \mathbb{N}$, we will make use of the Saddle Point Theorem (see [66, 67]). For this, as usual for minimax theorems, we have to check that the functional $\mathcal{J}$ has a particular geometric structure (as stated, in our case, in conditions (I_3) and (I_4) of [67, Theorem 4.6]) and that it satisfies the Palais–Smale compactness condition (see, for instance, [67, page 3]).

2.3.1 The first case

In this subsection, in order to apply the Weierstrass Theorem we first verify that the functional $\mathcal{J}$ satisfies the following geometric feature.

Proposition 2.3.1. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4) *and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function verifying* (2.1.2)*. Moreover, let $\overline{\alpha}(x) < \lambda_1$ a.e. in Ω.*

Then, the functional $\mathcal{J}$ verifies

$$\liminf_{\|u\|_Z \to +\infty} \frac{\mathcal{J}(u)}{\|u\|_Z^2} > 0. \tag{2.3.1}$$

Proof. Let $\{u_j\}_{j \in \mathbb{N}}$ be a sequence in Z such that $\|u_j\|_Z \to +\infty$. Since Z is a reflexive space (being a Hilbert space, by Lemma 1.1.3), up to a subsequence, there exists $u \in Z$ such that $u_j / \|u_j\|_Z$ converges to u weakly in Z. Moreover, by applying Lemma 1.1.2 and [20, Theorem IV.9]

$$\begin{aligned} &\frac{u_j}{\|u_j\|_Z} \to u \quad \text{in } L^q(\mathbb{R}^n) \quad \forall q \in [1, 2^*) \\ &\frac{u_j}{\|u_j\|_Z} \to u \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{2.3.2}$$

as $j \to +\infty$ and $\|u\|_Z \leqslant 1$. Now, notice that $2^* > 2$, by (1.1.6). Therefore by (2.2.2) and the first observation in (2.3.2)

$$\frac{|F(x, u_j(x))|}{\|u_j\|_Z^2} \leqslant \frac{a(x)\,|u_j(x)| + b\dfrac{|u_j(x)|^2}{2}}{\|u_j\|_Z^2} \to \frac{b}{2}\,|u(x)|^2 \quad \text{in } L^1(\Omega). \tag{2.3.3}$$

So, by (2.3.3) and [20, Theorem IV.9], up to a subsequence, there exists a function $h \in L^1(\Omega)$ such that

$$\frac{|F(x, u_j(x))|}{\|u_j\|_Z^2} \leqslant h(x) \quad \text{a.e. in } \Omega. \tag{2.3.4}$$

By (2.3.4) and the generalized Fatou Lemma it follows that

$$\limsup_{j\to+\infty}\int_\Omega \frac{F(x,u_j(x))}{\|u_j\|_Z^2}dx \leqslant \int_\Omega \limsup_{j\to+\infty}\frac{F(x,u_j(x))}{\|u_j\|_Z^2}dx. \tag{2.3.5}$$

Now, we claim that

$$\limsup_{j\to+\infty}\frac{F(x,u_j(x))}{\|u_j\|_Z^2} \leqslant \frac{\overline{\alpha}(x)}{2}\,|u(x)|^2 \tag{2.3.6}$$

a.e. in Ω. By fixing $\varepsilon>0$ and $x\in\Omega$, by (2.3.2) there exists $j_{\varepsilon,\,x}>0$ such that

$$\left|\frac{|u_j(x)|^2}{\|u_j\|_Z^2}-|u(x)|^2\right|<\varepsilon \tag{2.3.7}$$

for $j\geqslant j_{\varepsilon,\,x}$. Moreover, by (2.2.3) there exists $t_{\varepsilon,\,x}>0$ such that

$$\frac{F(x,t)}{t^2}\leqslant\frac{\overline{\alpha}(x)}{2}+\varepsilon \tag{2.3.8}$$

for any $|t|\geqslant t_\varepsilon$. Now, if $|u_j(x)|>t_{\varepsilon,\,x}$ by (2.3.7) and (2.3.8) it follows that

$$\begin{aligned}\frac{F(x,u_j(x))}{\|u_j\|_Z^2} &\leqslant \left(\frac{\overline{\alpha}(x)}{2}+\varepsilon\right)\frac{|u_j(x)|^2}{\|u_j\|_Z^2}\\ &=\left(\frac{\overline{\alpha}(x)}{2}+\varepsilon\right)|u(x)|^2+\left(\frac{\overline{\alpha}(x)}{2}+\varepsilon\right)\left(\frac{|u_j(x)|^2}{\|u_j\|_Z^2}-|u(x)|^2\right)\\ &\leqslant\left(\frac{\overline{\alpha}(x)}{2}+\varepsilon\right)|u(x)|^2+\left(\frac{\overline{\alpha}^+(x)}{2}+\varepsilon\right)\varepsilon,\end{aligned} \tag{2.3.9}$$

for $j\geqslant j_{\varepsilon,\,x}$, with $\overline{\alpha}^+(x)=\max\{\overline{\alpha}(x),0\}$. Since $\|u_j\|_Z\to+\infty$, if $|u_j(x)|\leqslant t_{\varepsilon,\,x}$ for $j\geqslant j_{\varepsilon,\,x}$ sufficiently large, by (2.2.2) we get

$$\frac{F(x,u_j(x))}{\|u_j\|_Z^2}\leqslant\frac{\|a\|_{L^\infty(\mathbb{R}^n)}\,t_{\varepsilon,\,x}+\dfrac{b}{2}t^2_{\varepsilon,\,x}}{\|u_j\|_Z^2}\leqslant\left(\frac{\overline{\alpha}^+(x)}{2}+\varepsilon\right)\varepsilon. \tag{2.3.10}$$

By combining (2.3.9) and (2.3.10), for $j\geqslant j_{\varepsilon,\,x}$ we obtain

$$\frac{F(x,u_j(x))}{\|u_j\|_Z^2}\leqslant\left(\frac{\overline{\alpha}(x)}{2}+\varepsilon\right)|u(x)|^2+\left(\frac{\overline{\alpha}^+(x)}{2}+\varepsilon\right)\varepsilon,$$

and so by sending $j\to+\infty$ and then $\varepsilon\to0$ we get (2.3.6), proving our claim. Therefore, by (2.3.5), (2.3.6) and remembering that $\overline{\alpha}(x)<\lambda_1$, we have

$$\limsup_{j\to+\infty}\int_\Omega\frac{F(x,u_j(x))}{\|u_j\|_Z^2}dx\leqslant\int_\Omega\frac{\overline{\alpha}(x)}{2}\,|u(x)|^2\,dx\begin{cases}<\dfrac{\lambda_1}{2}\displaystyle\int_\Omega|u(x)|^2\,dx & \text{if } u\not\equiv0,\\ =0 & \text{if } u\equiv0,\end{cases}$$

so that

$$\liminf_{j\to+\infty} \frac{\mathcal{J}(u_j)}{\|u_j\|_Z^2} = \frac{1}{2} - \limsup_{j\to+\infty} \int_\Omega \frac{F(x,u_j(x))}{\|u_j\|_Z^2} dx$$
$$> \begin{cases} \dfrac{1}{2} - \dfrac{\lambda_1}{2}\displaystyle\int_\Omega |u(x)|^2\, dx & \text{if } u \not\equiv 0, \\ \dfrac{1}{2} & \text{if } u \equiv 0. \end{cases} \tag{2.3.11}$$

Now, by (1.1.7), (1.2.7) and remembering that $\|u\|_Z \leqslant 1$, we get

$$\frac{\lambda_1}{2}\int_\Omega |u(x)|^2\, dx \leqslant \frac{1}{2}\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\, dx\, dy = \frac{1}{2}\|u\|_Z \leqslant \frac{1}{2} \tag{2.3.12}$$

therefore

$$\liminf_{j\to+\infty} \frac{\mathcal{J}(u_j)}{\|u_j\|_Z^2} > \begin{cases} 0 & \text{if } u \not\equiv 0, \\ \dfrac{1}{2} & \text{if } u \equiv 0, \end{cases}$$

and so we have (2.3.1). $\square$

Proof of Theorem 2.1.1, when $\overline{\alpha}(x) < \lambda_1$

Let us note that the map $u \mapsto \|u\|_Z^2$ is lower semicontinuous in the weak topology of Z, while the map $u \mapsto \int_\Omega F(x,u)$ is continuous in the weak topology of Z. Indeed, if $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in Z such that $u_j \rightharpoonup u$ in Z, then by Lemma 1.1.2 and [20, Theorem IV.9], up to a subsequence, u_j converges to u strongly in $L^q(\Omega)$ for any $q \in [1, 2^*)$ and a.e. in Ω and it is dominated in $L^q(\Omega)$. Since F is a Carathéodory function and by (2.2.2) it follows that

$$|F(x,u_j(x))| \leqslant a(x)\,|u_j(x)| + \frac{b}{2}\,|u_j(x)|^2\,,$$

by applying the Lebesgue Dominated Convergence Theorem we have the continuity of $u \mapsto \int_\Omega F(x,u)$. So the functional $\mathcal{J}$ is lower semicontinuous and by using also (2.3.1) to obtain coerciveness we can apply the Weierstrass Theorem in order to find a minimum of $\mathcal{J}$ on Z, which is clearly a solution of problem (2.1.3).

2.3.2 The second case

Here, we assume that $\lambda_k < \underline{\alpha}(x) \leqslant \overline{\alpha}(x) < \lambda_{k+1}$ for some $k \in \mathbb{N}$. Before proving our results, we recall some notations introduced in Section 1.2.

In particular, in what follows, e_k will be the k-th eigenfunction corresponding to the eigenvalue λ_k of $-\mathcal{L}_K$ for any $k \in \mathbb{N}$. That is, e_k is a non–trivial weak solution of the following eigenvalue problem

$$\begin{cases} -\mathcal{L}_K u = \lambda_k u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

Also, we consider

$$\mathbb{P}_{k+1} := \left\{u \in Z : \ \langle u, e_j\rangle_Z = 0 \ \ \forall j = 1, \ldots, k\right\}$$

as defined in Proposition 1.2.2, while $\mathbb{H}_k := \operatorname{span}\{e_1, \ldots, e_k\}$ will denote the linear subspace generated by the first k eigenfunctions of $-\mathcal{L}_K$ for any $k \in \mathbb{N}$. It is immediate to observe that $\mathbb{P}_{k+1} = \mathbb{H}_k^{\perp}$ with respect to the scalar product in Z. Moreover, since Z is a Hilbert space (thanks to Lemma 1.1.3), we can divide it as $Z = \mathbb{H}_k \oplus \mathbb{P}_{k+1}$.

Now we prove that the functional $\mathcal{J}$ has the geometric features required by the Saddle Point Theorem.

Proposition 2.3.2. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4) *and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function verifying* (2.1.2)*. Moreover, assume there exists $k \in \mathbb{N}$ such that $\lambda_k < \underline{\alpha}(x) \leqslant \overline{\alpha}(x) < \lambda_{k+1}$ a.e. in Ω.*

Then, the functional $\mathcal{J}$ verifies

$$\limsup_{\substack{u \in \mathbb{H}_k \\ \|u\|_Z \to +\infty}} \frac{\mathcal{J}(u)}{\|u\|_Z^2} < 0. \tag{2.3.13}$$

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{H}_k$ such that $\|u_j\|_Z \to +\infty$. Since $\mathbb{H}_k$ is finite dimensional there exists $u \in \mathbb{H}_k$ such that $u_j / \|u_j\|_Z$ converges to u strongly in Z. Moreover, by applying Lemma 1.1.2 and [20, Theorem IV.9], up to a subsequence

$$\begin{aligned} &\frac{u_j}{\|u_j\|_Z} \to u \quad \text{in } L^q(\mathbb{R}^n) \quad \forall q \in [1, 2^*) \\ &\frac{u_j}{\|u_j\|_Z} \to u \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{2.3.14}$$

as $j \to +\infty$ and $\|u\|_Z = 1$. Now, by using (2.2.4) and proceeding as in the proof of claim (2.3.6), it follows that

$$\liminf_{j\to+\infty} \frac{F(x, u_j(x))}{\|u_j\|_Z^2} \geqslant \frac{\underline{\alpha}(x)}{2} |u(x)|^2, \tag{2.3.15}$$

a.e. in Ω. So by (2.3.15), the Fatou Lemma and the fact that $\underline{\alpha}(x) > \lambda_k$ we get

$$\limsup_{j\to+\infty} \frac{\mathcal{J}(u_j)}{\|u_j\|_Z^2} \leqslant \frac{1}{2} - \int_\Omega \frac{\underline{\alpha}(x)}{2} |u(x)|^2 \, dx < \frac{1}{2} - \frac{\lambda_k}{2} \int_\Omega |u(x)|^2 \, dx. \tag{2.3.16}$$

Now, since $u \in \mathbb{H}_k$, then we can write

$$u(x) = \sum_{i=1}^{k} u_i e_i(x)$$

with $u_i \in \mathbb{R}$, $i = 1, \ldots, k$. Moreover, since $\{e_1, \ldots, e_k, \ldots\}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal one of Z (see Proposition 1.2.2–(vi)) and by (1.2.8) and (1.2.11), we get

$$\|u\|_Z^2 = \sum_{i=1}^{k} u_i^2 \|e_i\|_Z^2 = \sum_{i=1}^{k} \lambda_i u_i^2 \leqslant \lambda_k \sum_{i=1}^{k} u_i^2 = \lambda_k \int_\Omega |u(x)|^2 \, dx. \tag{2.3.17}$$

So, by (2.3.16), (2.3.17) and the fact that $\|u\|_Z = 1$, we get (2.3.13). $\square$

Also, Proposition 2.3.2 has the following counterpart.

Proposition 2.3.3. *Let* $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ *satisfy assumptions* (0.0.3) *and* (0.0.4) *and let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function verifying* (2.1.2). *Moreover, assume there exists* $k \in \mathbb{N}$ *such that* $\lambda_k < \underline{\alpha}(x) \leqslant \overline{\alpha}(x) < \lambda_{k+1}$ *a.e. in* Ω.

Then, the functional $\mathcal{J}$ *verifies*

$$\liminf_{\substack{u \in \mathbb{P}_{k+1} \\ \|u\|_Z \to +\infty}} \frac{\mathcal{J}(u)}{\|u\|_Z^2} > 0. \tag{2.3.18}$$

Proof. The proof is similar to the proof of Proposition 2.3.1. In this case $\overline{\alpha}(x) < \lambda_{k+1}$ for some $k \in \mathbb{N}$, so (2.3.11) becomes

$$\liminf_{j \to +\infty} \frac{\mathcal{J}(u_j)}{\|u_j\|_Z^2} > \begin{cases} \dfrac{1}{2} - \dfrac{\lambda_{k+1}}{2} \displaystyle\int_\Omega |u(x)|^2 \, dx & \text{if } u \not\equiv 0, \\ \dfrac{1}{2} & \text{if } u \equiv 0. \end{cases}$$

In place of (2.3.12), by using (1.2.9), we get

$$\frac{\lambda_{k+1}}{2} \int_\Omega |u(x)|^2 \, dx \leqslant \frac{1}{2} \iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 \, K(x-y) \, dx \, dy \tag{2.3.19}$$

and from this point we can conclude exactly as in the proof of Proposition 2.3.1. $\square$

Now, as usual in variational methods, we prove the boundedness of a Palais-Smale sequence. We recall that $\{u_j\}_{j \in \mathbb{N}}$ is a Palais-Smale sequence for $\mathcal{J}$ at level $c \in \mathbb{R}$ if it verifies

$$\mathcal{J}(u_j) \to c, \tag{2.3.20}$$

and

$$\sup \left\{ \left| \left\langle \mathcal{J}'(u_j), \varphi \right\rangle \right| : \varphi \in Z, \, \|\varphi\|_Z = 1 \right\} \to 0, \tag{2.3.21}$$

as $j \to +\infty$.

Proposition 2.3.4. *Let* $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ *satisfy assumptions* (0.0.3) *and* (0.0.4) *and let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function verifying* (2.1.2). *Moreover,* *assume there exists $k \in \mathbb{N}$ such that $\lambda_k < \underline{\alpha}(x) \leqslant \overline{\alpha}(x) < \lambda_{k+1}$. Finally, let $c \in \mathbb{R}$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z verifying* (2.3.20) *and* (2.3.21).

Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z.

Proof. We argue by contradiction and suppose that $\{u_j\}_{j\in\mathbb{N}}$ is unbounded. As usual, up to a subsequence we can assume $\|u_j\|_Z \to +\infty$ as $j \to +\infty$ and there exists $u \in Z$ such that $u_j/\|u_j\|_Z$ converges to u weakly in Z, that is

$$\begin{aligned}&\iint_{\mathbb{R}^{2n}} \left(\frac{u_j(x)}{\|u_j\|_Z} - \frac{u_j(y)}{\|u_j\|_Z}\right)(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy \to \\ &\iint_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy \quad \text{for any } \varphi \in Z\end{aligned} \tag{2.3.22}$$

as $j \to +\infty$. Moreover, by applying Lemma 1.1.2 and [20, Theorem IV.9], up to a subsequence

$$\begin{aligned}&\frac{u_j}{\|u_j\|_Z} \to u \quad \text{in } L^q(\mathbb{R}^n) \quad \forall q \in [1, 2^*) \\ &\frac{u_j}{\|u_j\|_Z} \to u \quad \text{a.e. in } \mathbb{R}^n\end{aligned} \tag{2.3.23}$$

as $j \to +\infty$. Moreover, by (2.1.2) and the first observation in (2.3.23), since $2 < 2^*$, we get

$$\frac{|f(x,u_j)|}{\|u_j\|_Z} \leqslant \frac{a(x)}{\|u_j\|_Z} + b\frac{|u_j|}{\|u_j\|_Z} \to b\,|u| \quad \text{in } L^2(\Omega) \tag{2.3.24}$$

as $j \to +\infty$. So $\left\{f(x,u_j)/\|u_j\|_Z\right\}_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ and we can assume that there exists $w \in L^2(\Omega)$ such that

$$\frac{f(x,u_j)}{\|u_j\|_Z} \rightharpoonup w \quad \text{in } L^2(\Omega). \tag{2.3.25}$$

By (2.3.21) we have

$$\begin{aligned}\langle \mathcal{J}'(u_j), v\rangle = &\iint_{\mathbb{R}^{2n}} \big(u_j(x)-u_j(y)\big)\big(v(x)-v(y)\big)K(x-y)\,dx\,dy \\ &- \int_\Omega f(x,u_j(x))v(x)\,dx \to 0\end{aligned} \tag{2.3.26}$$

for all $v \in Z$. Moreover, by (2.3.22) with $\varphi = v$ and (2.3.25) we have

$$\begin{aligned}\frac{\langle \mathcal{J}'(u_j), v\rangle}{\|u_j\|_Z} \longrightarrow &\iint_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)K(x-y)\,dx\,dy \\ &- \int_\Omega w(x)v(x)dx \quad \text{in } L^2(\Omega).\end{aligned} \tag{2.3.27}$$

So, by combining (2.3.26) with (2.3.27) we get

$$\iint_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)K(x-y)\,dx\,dy - \int_\Omega w(x)v(x)dx = 0$$

for all $v \in Z$ and we deduce that u is a weak solution of problem

$$\begin{cases} -\mathcal{L}_K u(x) = w(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.3.28}$$

Now we claim that

$$\begin{aligned} &\text{there exists } m \in L^\infty(\Omega) \text{ such that } \underline{\alpha}(x) \leqslant m(x) \leqslant \overline{\alpha}(x) \text{ a.e. in } \Omega \\ &\text{and } w = mu. \end{aligned} \tag{2.3.29}$$

If $x \in \Omega$ so that $u(x) > 0$, then $u_j(x) \to +\infty$ and so, by using (2.1.4), it follows that

$$\liminf_{j\to+\infty} \frac{f(x,u_j(x))}{\|u_j\|_Z} = \liminf_{j\to+\infty} \frac{f(x,u_j(x))}{u_j(x)} \frac{u_j(x)}{\|u_j\|_Z} \geqslant \underline{\alpha}(x)u(x) \tag{2.3.30}$$

and in the same way

$$\limsup_{j\to+\infty} \frac{f(x,u_j(x))}{\|u_j\|_Z} \leqslant \overline{\alpha}(x)u(x). \tag{2.3.31}$$

On the other hand, if $x \in \Omega$ so that $u(x) < 0$, then $u_j(x) \to -\infty$ and we get the reversed sign inequality, with

$$\liminf_{j\to+\infty} \frac{f(x,u_j(x))}{\|u_j\|_Z} \leqslant \underline{\alpha}(x)u(x) \tag{2.3.32}$$

and with

$$\limsup_{j\to+\infty} \frac{f(x,u_j(x))}{\|u_j\|_Z} \geqslant \overline{\alpha}(x)u(x). \tag{2.3.33}$$

Finally, when $x \in \Omega$ so that $u(x) = 0$, by (2.1.2) we have

$$\frac{|f(x,u_j(x))|}{\|u_j\|_Z} \leqslant \frac{a(x)}{\|u_j\|_Z} + b\frac{|u_j(x)|}{\|u_j\|_Z} \to 0 \tag{2.3.34}$$

as $j \to +\infty$. So, by (2.3.25), (2.3.30)–(2.3.34) and Lemma 2.2.2, we get

$$\begin{aligned} \underline{\alpha}(x)u(x) \leqslant w(x) \leqslant \overline{\alpha}(x)u(x) \qquad & \text{if } u(x) > 0, \\ \overline{\alpha}(x)u(x) \leqslant w(x) \leqslant \underline{\alpha}(x)u(x) \qquad & \text{if } u(x) < 0 \text{ and} \\ w(x) = 0 \qquad & \text{if } u(x) = 0. \end{aligned}$$

Now, we set

$$m(x) := \begin{cases} \dfrac{w(x)}{u(x)} & \text{if } u \neq 0, \\ 0 & \text{if } u = 0, \end{cases} \tag{2.3.35}$$

and we observe that m is measurable and bounded, since $\lambda_k < \underline{\alpha}(x) \leqslant m(x) \leqslant \overline{\alpha}(x) < \lambda_{k+1}$ a.e. in Ω, and $w = mu$. This establishes (2.3.29).

So, by (2.3.28) and (2.3.29) we have proved that u is a weak solution of problem

$$\begin{cases} -\mathcal{L}_K u(x) = m(x)u(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.3.36}$$

Now, we can write $u = u_1 + u_2$, where $u_1 \in \mathbb{H}_k$ and $u_2 \in \mathbb{P}_{k+1}$. Multiplying the equation in (2.3.36) by u_1 and u_2, we obtain

$$\int_\Omega m(x)\left|u_1(x)\right|^2 dx + \int_\Omega m(x)u_1(x)u_2(x)dx = \iint_{\mathbb{R}^{2n}} \left|u_1(x) - u_1(y)\right|^2 K(x-y)dxdy,$$
$$\int_\Omega m(x)\left|u_2(x)\right|^2 dx + \int_\Omega m(x)u_1(x)u_2(x)dx = \iint_{\mathbb{R}^{2n}} \left|u_2(x) - u_2(y)\right|^2 K(x-y)dxdy$$

so that

$$\begin{aligned} &\iint_{\mathbb{R}^{2n}} \left|u_1(x) - u_1(y)\right|^2 K(x-y)\,dx\,dy - \int_\Omega m(x)\left|u_1(x)\right|^2\,dx \\ &\qquad = \iint_{\mathbb{R}^{2n}} \left|u_2(x) - u_2(y)\right|^2 K(x-y)\,dx\,dy - \int_\Omega m(x)\left|u_2(x)\right|^2\,dx. \end{aligned} \tag{2.3.37}$$

Now we apply (2.3.17) to the function $u_1 \in \mathbb{H}_k$ and we conclude that

$$\iint_{\mathbb{R}^{2n}} \left|u_1(x) - u_1(y)\right|^2 K(x-y)\,dx\,dy \leqslant \lambda_k \int_\Omega \left|u_1(x)\right|^2\,dx. \tag{2.3.38}$$

Also, by (1.2.9) and the fact that $u_2 \in \mathbb{P}_{k+1}$

$$\iint_{\mathbb{R}^{2n}} \left|u_2(x) - u_2(y)\right|^2 K(x-y)\,dx\,dy \geqslant \lambda_{k+1} \int_\Omega \left|u_2(x)\right|^2\,dx. \tag{2.3.39}$$

Therefore, by (2.3.37), (2.3.38) and (2.3.39) and by considering that

$$\lambda_k - m(x) \leqslant \lambda_k - \underline{\alpha}(x) < 0 \quad \text{and} \quad \lambda_{k+1} - m(x) \geqslant \lambda_{k+1} - \overline{\alpha}(x) > 0 \tag{2.3.40}$$

we get

$$0 \geqslant \int_\Omega (\lambda_k - m(x))\left|u_1(x)\right|^2 dx \geqslant \int_\Omega (\lambda_{k+1} - m(x))\left|u_2(x)\right|^2 dx \geqslant 0, \tag{2.3.41}$$

so that all integrals are zero. But by (2.3.40) we get $u_1 = u_2 = 0$ and so $u \equiv 0$.

Now, by (2.3.21)

$$0 \leftarrow \frac{\langle \mathcal{J}'(u_j), u_j\rangle}{\|u_j\|_Z^2} = 1 - \int_\Omega \frac{f(x, u_j(x))}{\|u_j\|_Z}\frac{u_j(x)}{\|u_j\|_Z}dx, \tag{2.3.42}$$

where, since $\left\{f(x,u_j)/\left\|u_j\right\|_Z\right\}_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ and by using the first observation in (2.3.23) and (2.3.25), the right-hand side verifies

$$\begin{aligned} \int_\Omega \frac{f(x, u_j(x))}{\|u_j\|_Z}\frac{u_j(x)}{\|u_j\|_Z}dx &= \int_\Omega \frac{f(x, u_j(x))}{\|u_j\|_Z}\left(\frac{u_j(x)}{\|u_j\|_Z} - u(x)\right)dx \\ &\quad + \int_\Omega \frac{f(x, u_j(x))}{\|u_j\|_Z}u(x)dx \to \int_\Omega w(x)u(x)dx. \end{aligned} \tag{2.3.43}$$

as $j \to +\infty$. So by (2.3.42) and (2.3.43), it follows that $\int_\Omega w(x)u(x)dx = 1$ and we get a contradiction since $u \equiv 0$. □

Proof of Theorem 2.1.1, when $\lambda_k < \underline{\alpha}(x) \leqslant \overline{\alpha}(x) < \lambda_{k+1}$

At first, we prove that $\mathcal{J}$ satisfies the geometric structure required by the Saddle Point Theorem. By Proposition 2.3.3 it follows that for any $M > 0$ there exists $R > 0$ such that if $u \in \mathbb{P}_{k+1}$ and $\|u\|_Z \geqslant R$ then $\mathcal{J}(u) \geqslant M$. If $u \in \mathbb{P}_{k+1}$ with $\|u\|_Z \leqslant R$, by applying (1.2.9), (2.2.2) and Hölder inequality we have

$$\begin{aligned}\mathcal{J}(u) &\geqslant -\int_\Omega F(x,u(x))dx \geqslant -\int_\Omega a(x)\,|u(x)|\,dx - \frac{b}{2}\int_\Omega |u(x)|^2\,dx\\ &\geqslant -\|a\|_{L^2(\Omega)}\,\|u\|_{L^2(\Omega)} - \frac{b}{2}\lambda_{k+1}^{-1}\,\|u\|_Z^2 \geqslant -C_R\end{aligned}$$

for some constant $C_R = C(R,\Omega) > 0$. So, we get

$$\mathcal{J}(u) \geqslant -C_R \qquad \forall u \in \mathbb{P}_{k+1}. \tag{2.3.44}$$

By Proposition 2.3.2 we can choose $T > 0$ in such way that for any $u \in \mathbb{H}_k$ with $\|u\|_Z = T$ we have

$$\sup_{\substack{u\in\mathbb{H}_k\\ \|u\|_Z=T}} \mathcal{J}(u) < -C_R \leqslant \inf_{u\in\mathbb{P}_{k+1}} \mathcal{J}(u), \tag{2.3.45}$$

We have thus proved that $\mathcal{J}$ has the geometric structure of the Saddle Point Theorem (see [67, Theorem 4.6]). Now it remains to check the validity of the Palais-Smale condition. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z that satisfies (2.3.20) and (2.3.21). Since, by Proposition 2.3.4, $\{u_j\}_{j\in\mathbb{N}}$ is bounded and Z is a reflexive space (being a Hilbert space, by Lemma 1.1.3), up to a subsequence, there exists $u \in Z$ such that u_j converges to u weakly in Z, that is

$$\begin{aligned}&\iint_{\mathbb{R}^{2n}} (u_j(x)-u_j(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy \to\\ &\iint_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy \quad \text{for any } \varphi\in Z\end{aligned} \tag{2.3.46}$$

as $j \to +\infty$. Moreover, by applying Lemma 1.1.2 and [20, Theorem IV.9]

$$\begin{aligned}&u_j \to u \quad \text{in } L^q(\mathbb{R}^n) \quad \forall q \in [1, 2^*)\\ &u_j \to u \quad \text{a.e. in } \mathbb{R}^n\end{aligned} \tag{2.3.47}$$

as $j \to +\infty$. By (2.3.21) we have

$$\begin{aligned}0 \leftarrow \langle \mathcal{J}'(u_j), (u_j - u)\rangle &= \iint_{\mathbb{R}^{2n}} |u_j(x)-u_j(y)|^2\,K(x-y)\,dx\,dy\\ &- \iint_{\mathbb{R}^{2n}} (u_j(x)-u_j(y))(u(x)-u(y))K(x-y)\,dx\,dy\\ &- \int_\Omega f(x,u_j(x))(u_j(x)-u(x))dx.\end{aligned} \tag{2.3.48}$$

Now, since $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, by (2.1.2) also $\{f(x,u_j)\}_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ and so, by (2.3.47) and Hölder inequality, we get

$$\left|\int_\Omega f(x,u_j(x))(u_j(x)-u(x))dx\right| \leqslant \|u_j-u\|_{L^2(\Omega)} \left(\int_\Omega |f(x,u_j(x))|^2\,dx\right)^{1/2} \to 0 \tag{2.3.49}$$

as $j\to+\infty$. By (2.3.46) with $\varphi=u$, (2.3.48) and (2.3.49) it follows that

$$\iint_{\mathbb{R}^{2n}} |u_j(x)-u_j(y)|^2\,K(x-y)\,dx\,dy \to \iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2\,K(x-y)\,dx\,dy$$

so that

$$\|u_j\|_Z \to \|u\|_Z \tag{2.3.50}$$

as $j\to+\infty$. Finally we have that

$$\begin{aligned}\|u_j-u\|_Z^2 &= \|u_j\|_Z^2 + \|u\|_Z^2 - 2\iint_{\mathbb{R}^{2n}} (u_j(x)-u_j(y))(u(x)-u(y))K(x-y)\,dx\,dy \\ &\to 2\,\|u\|_Z^2 - 2\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2\,K(x-y)\,dx\,dy = 0\end{aligned}$$

as $j\to+\infty$, thanks to (2.3.46) and (2.3.50). Thus, we have proved the Palais-Smale condition and we can make use of the Saddle Point Theorem in order to obtain a critical point $u\in Z$ of $\mathcal{J}$.

2.3.3 Proof of Corollary 2.1.2

We conclude this chapter by proving a uniqueness result for problem (2.1.3). For this we need the further condition (2.1.5) for the nonlinearity f.

Let u_1, $u_2\in Z$ be two solutions of problem (2.1.3). Then $w:=u_1-u_2$ is a solution of the following problem

$$\begin{cases} -\mathcal{L}_K w(x) = f(x,u_1(x)) - f(x,u_2(x)) & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^n\setminus\Omega. \end{cases} \tag{2.3.51}$$

Now, by setting

$$m(x) := \begin{cases} \dfrac{f(x,u_1(x))-f(x,u_2(x))}{u_1(x)-u_2(x)} & \text{if } u_1(x)\neq u_2(x), \\ 0 & \text{if } u_1(x)=u_2(x), \end{cases} \tag{2.3.52}$$

we get that w is a solution of problem (2.3.36). Moreover, by (2.1.5), m is a measurable function that verifies $\lambda_k < m(x) < \lambda_{k+1}$ a.e. in Ω. As seen in the proof of Proposition 2.3.4, problem (2.3.36) has a unique solution $u\equiv 0$ and so we get $u_1=u_2$ concluding the proof.

Chapter 3

A problem at resonance

3.1 Introduction

In this chapter we consider the non–local counterpart of semilinear elliptic partial differential equations of the type

$$\begin{cases} -\Delta u = \lambda u + f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.1.1}$$

namely

$$\begin{cases} -\mathcal{L}_K u = \lambda a(x) u + f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{3.1.2}$$

Here, $\Omega \subset \mathbb{R}^n$ is an open, bounded set, λ is a real parameter and $\mathcal{L}_K$ is the non–local operator defined as in (0.0.2) with the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfying (0.0.3) and (0.0.4) for a fixed $s \in (0,1)$. Moreover, the perturbation $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a function such that

$$f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}); \tag{3.1.3}$$

$$\text{there exists a constant } M > 0 \text{ such that } |f(x,t)| \leqslant M \text{ for any } (x,t) \in \Omega \times \mathbb{R}; \tag{3.1.4}$$

$$F(x,t) = \int_0^t f(x,\tau)d\tau \to +\infty \text{ as } |t| \to +\infty \text{ uniformly for } x \in \Omega. \tag{3.1.5}$$

While, $a : \overline{\Omega} \to \mathbb{R}$ is such that

$$a \text{ is a positive Lipschitz continuous function in } \overline{\Omega}\,. \tag{3.1.6}$$

One of the motivations for studying (3.1.2) is trying to extend some important results, which are well known for the classical case of the Laplacian $-\Delta$ (see, e.g., [67, Chapter 4]), to a non–local setting. The conditions we consider on a and f are classical in the nonlinear analysis (see, e.g., conditions $(p1)$, $(p2)$ and $(p7)$ in [67, Theorem 4.12]) and, roughly speaking, they state that problem (3.1.2) is a suitable perturbation from the following non-homogenous eigenvalue problem

$$\begin{cases} -\mathcal{L}_K u = \lambda a(x) u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{3.1.7}$$

We recall that there exists a non-decreasing sequence of positive eigenvalues λ_k for which (3.1.7) admits non–trivial solutions, as showed in Section 1.2 (with $q \equiv 0$).

Finally, note that, thanks to (3.1.5), the nonlinearity f cannot be the trivial function. As a model for f we can take the functions

$$f(x,t) = M > 0 \quad \text{or} \quad f(x,t) = b(x)\arctan(t)\,,$$

with $b \in Lip(\overline{\Omega})$ and $b > 0$ in Ω. In the first case $u \equiv 0$ does not solve (3.1.2), while in the second one the trivial function is a solution of (3.1.2). In general, the function $u \equiv 0$ in $\mathbb{R}^n$ is a solution of problem (3.1.2) if and only if $f(\cdot, 0) = 0$. This is an important difference with respect to the other works in the subject, such as [72, 74, 75, 77, 80, 81], where the trivial function is always a solution.

Such as in the previous chapter, the objective here is to find solutions for (3.1.2) via variational methods. For this, firstly we need the weak formulation of (3.1.2), which is given by the following problem

$$\begin{cases} \displaystyle\iint_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy = \lambda \int_\Omega a(x)u(x)\varphi(x)\,dx \\ \displaystyle\qquad\qquad\qquad\qquad\qquad\qquad + \int_\Omega f(x,u(x))\varphi(x)\,dx \quad \forall\,\varphi \in Z \\ u \in Z\,, \end{cases} \tag{3.1.8}$$

where Z is the functional space introduced in Chapter 1.

The main result of the present chapter can be stated as follows:

Theorem 3.1.1. *Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function satisfying* (0.0.3) *and* (0.0.4) *and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $a : \overline{\Omega} \to \mathbb{R}$ be two functions verifying* (3.1.3)–(3.1.5) *and* (3.1.6)*, respectively. Moreover, assume that λ is an eigenvalue of the non-homogeneous linear problem in* (3.1.7)*.*

Then, problem (3.1.2) *admits a weak solution $u \in Z$.*

In the classical case of the Laplacian $-\Delta$ the counterpart of Theorem 3.1.1 is given in [67, Theorem 4.12]: in this sense Theorem 3.1.1 may be seen as the natural extension of classical results to the non–local fractional setting.

This chapter is organized as follows. In Section 3.2 we will give some notations and we will state and prove some technical lemmas useful along the chapter. While in

Section 3.3 we will prove Theorem 3.1.1 by making use of the classical Saddle Point Theorem.

3.2 Some technical lemmas

In this section we prove some technical lemmas, which will be useful in order to apply the Saddle Point Theorem to problem (3.1.8). For this, we recall some notations introduced in Section 1.2.

Here, by denoting with $\mu(\cdot) = a(\cdot)dx$, we define

$$L^2(\Omega,\, \mu) := \Big\{ g : \Omega \to \mathbb{R} \text{ s.t. } g \text{ is measurable in } \Omega \text{ and} \\ \int_\Omega a(x)|g(x)|^2\, dx = \int_\Omega |g|^2\, d\mu < +\infty \Big\}.$$

By (3.1.6) it follows that $a \in L^\infty(\Omega)$ and so all the embeddings properties of Z into the usual Lebesgue space $L^2(\Omega)$ still hold true in $L^2(\Omega,\, \mu)$.

In what follows, without loss of generality, we will fix $\lambda = \lambda_k$ with $k \in \mathbb{N}$ such that $\lambda_k < \lambda_{k+1}$ and we will denote by $\mathbb{H}_k$ the linear subspace of Z generated by the first k eigenfunctions of $-\mathcal{L}_K$, i.e.

$$\mathbb{H}_k := \operatorname{span}\{e_1, \dots, e_k\}\,,$$

while $\mathbb{P}_{k+1}$ will be the space defined in (1.2.10). Here e_j and λ_j, $j \in \mathbb{N}$, are the eigenfunctions and the eigenvalues of $-\mathcal{L}_K$, as defined in Proposition 1.2.2.

It is immediate to observe that $\mathbb{P}_{k+1} = \mathbb{H}_k^\perp$ with respect to the scalar product in Z. Thus, since Z is a Hilbert space (thanks to Lemma 1.1.3), we can write it as a direct sum as follows

$$Z = \mathbb{H}_k \oplus \mathbb{P}_{k+1}\,.$$

Moreover, since $\{e_1\,, \dots\,, e_k\,, \dots\}$ is an orthogonal basis of Z, it follows that

$$\mathbb{P}_{k+1} = \overline{\operatorname{span}\{e_j :\ j \geqslant k+1\}}\,.$$

Also we will set

$$\mathbb{E}_k^0 := \operatorname{span}\{e_j :\ \lambda_j = \lambda_k\} \quad \text{and} \quad \mathbb{E}_k^- := \operatorname{span}\{e_j :\ \lambda_j < \lambda_k\}\,. \tag{3.2.1}$$

Note that with this notation, if $u \in \mathbb{H}_k$, then we can write it as

$$u = u^0 + u^-\,, \ \text{ with } u^0 \in \mathbb{E}_k^0 \text{ and } u^- \in \mathbb{E}_k^-\,.$$

Now, we are ready to introduce and prove some technical estimates from the properties of eigenvalues and eigenfunctions of $-\mathcal{L}_K$.

Lemma 3.2.1. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4) *and let $a : \overline{\Omega} \to \mathbb{R}$ verify* (3.1.6) *.*

Then, for any $u \in \mathbb{P}_{k+1}$

$$\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)\, dx\, dy - \lambda_k \int_\Omega a(x)|u(x)|^2\, dx \geqslant \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) \|u\|_Z^2 \,.$$

Proof. If $u \equiv 0$, then the assertion is trivial. Now, let $u \in \mathbb{P}_{k+1} \setminus \{0\}$. By the variational characterization of λ_{k+1} given in (1.2.9) we get that

$$\|u\|^2_{L^2(\Omega,\,\mu)} \leqslant \frac{1}{\lambda_{k+1}} \|u\|_Z^2 \,.$$

As a consequence of this and taking into account that λ_k is positive (since $\lambda_k \geqslant \lambda_1 > 0$), we obtain

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)\, dx\, dy - \lambda_k \int_\Omega a(x)|u(x)|^2\, dx &\geqslant \|u\|_Z^2 - \frac{\lambda_k}{\lambda_{k+1}} \|u\|_Z^2 \\ &= \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) \|u\|_Z^2,\end{aligned}$$

concluding the proof. □

Note that, if $\lambda_k = \lambda_{k+1}$, then Lemma 3.2.1 is trivial. The interesting case is when $\lambda_k < \lambda_{k+1}$.

Lemma 3.2.2. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4) *and let $a : \overline{\Omega} \to \mathbb{R}$ verify* (3.1.6) *.*

Then, there exists a positive constant M^*, depending on k, such that

$$\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)\, dx\, dy - \lambda_k \int_\Omega a(x)|u(x)|^2\, dx \leqslant -M^* \|u^-\|_Z^2$$

for all $u \in \mathbb{H}_k$, where $u = u^- + u^0$, $u^- \in E_k^-$ and $u^0 \in E_k^0$.

Proof. Of course, if $u \equiv 0$, then the assertion is trivial. Hence, assume that $u \in \mathbb{H}_k \setminus \{0\}$. Let $h \in \mathbb{N}$ be the multiplicity of λ_k (h is finite thanks to Proposition 1.2.2-(vii)), that is suppose that

$$\lambda_{k-h-1} < \lambda_{k-h} = \cdots = \lambda_k < \lambda_{k+1} \,. \tag{3.2.2}$$

With this notation, u can be written as follows

$$u = u^- + u^0 \,,$$

with

$$u^- \in \mathbb{E}_k^- = \operatorname{span}\{e_1, \ldots, e_{k-h-1}\} \quad \text{and} \quad u^0 \in \mathbb{E}_k^0 = \operatorname{span}\{e_{k-h}, \ldots, e_k\} \,.$$

Notice that u^0 is a linear combination of eigenfunctions corresponding to the same eigenvalue $\lambda_{k-h} = \cdots = \lambda_k$, hence it is also an eigenfunction corresponding to λ_k. Hence, by (1.2.2),

$$\|u^0\|_Z^2 = \lambda_k \|u^0\|^2_{L^2(\Omega,\mu)}.$$

Also, u^- and u^0 are orthogonal both in Z and in $L^2(\Omega,\, \mu)$, therefore

$$\begin{aligned} \|u\|_Z^2 - \lambda_k \|u\|^2_{L^2(\Omega,\,\mu)} &= \|u^-\|_Z^2 + \|u^0\|_Z^2 - \lambda_k \left(\|u^-\|^2_{L^2(\Omega,\,\mu)} + \|u^0\|^2_{L^2(\Omega,\,\mu)} \right) \\ &= \|u^-\|_Z^2 - \lambda_k \|u^-\|^2_{L^2(\Omega,\,\mu)}\,. \end{aligned} \tag{3.2.3}$$

Now, note that $u^- \in \mathbb{E}_k^- = \operatorname{span}\{e_1, \dots, e_{k-h-1}\}$. Hence, by this and Proposition 1.2.3 we get

$$\|u^-\|_Z^2 \leqslant \lambda_{k-h-1} \|u^-\|^2_{L^2(\Omega,\,\mu)}\,. \tag{3.2.4}$$

Finally, (3.2.3) and (3.2.4) yield

$$\begin{aligned} \|u\|_Z^2 - \lambda_k \|u\|^2_{L^2(\Omega,\,\mu)} &= \|u^-\|_Z^2 - \lambda_k \|u^-\|^2_{L^2(\Omega,\,\mu)} \\ &\leqslant \|u^-\|_Z^2 - \frac{\lambda_k}{\lambda_{k-h-1}} \|u^-\|_Z^2 \\ &= \left(1 - \frac{\lambda_k}{\lambda_{k-h-1}}\right) \|u^-\|_Z^2\,, \end{aligned}$$

which gives the desired assertion with

$$M^* := \frac{\lambda_k}{\lambda_{k-h-1}} - 1\,.$$

Note that $M^* > 0$, thanks to (3.2.2). ☐

Finally, in the next two results we discuss some properties of the function F defined as in (3.1.5).

Lemma 3.2.3. *Let* $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *satisfy* (3.1.3)–(3.1.5).

Then, there exists a positive constant $\widetilde{M}$*, depending on* Ω*, such that*

$$\left| \int_\Omega F(x, u(x))\, dx \right| \leqslant \widetilde{M} \|u\|_Z$$

for all $u \in Z$.

Proof. Using the definition of F and (3.1.4), it is easy to see that

$$\left| \int_\Omega F(x, u(x))\, dx \right| = \left| \int_\Omega \int_0^{u(x)} f(x,t)\, dt\, dx \right| \leqslant M \int_\Omega |u(x)|\, dx\,,$$

so that, by Hőlder inequality and Lemma 1.1.2 we get

$$\left| \int_\Omega F(x, u(x))\, dx \right| \leqslant M\, |\Omega|^{1/2}\, \|u\|_{L^2(\Omega)} \leqslant \widetilde{M} \|u\|_Z \tag{3.2.5}$$

for all $u \in Z$, where $\widetilde{M}$ is a positive constant depending on Ω. Hence, the assertion is proved. $\square$

Lemma 3.2.4. *Let* $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *satisfy* (3.1.3)–(3.1.5) .

Then,

$$\lim_{\substack{u \in \mathbb{E}_k^0 \\ \|u\|_Z \to +\infty}} \int_\Omega F(x, u(x))\, dx = +\infty\,.$$

Proof. We argue by contradiction and suppose that there exists a positive constant C and a sequence $\{u_j\}_{j\in\mathbb{N}} \subset E_k^0$ such that

$$t_j := \|u_j\|_Z \to +\infty \tag{3.2.6}$$

and

$$\int_\Omega F(x, u_j(x))dx \leqslant C\,. \tag{3.2.7}$$

Let $v_j := u_j/\|u_j\|_Z$. Of course, $\{v_j\}_{j\in\mathbb{N}}$ is bounded in Z. Hence, since $\mathbb{E}_k^0$ is finite dimensional, there exists $v \in \mathbb{E}_k^0$ such that v_j converges to v strongly in Z. Note also that $v \not\equiv 0$, since

$$\|v\| = \lim_{j\to+\infty} \|v_j\| = 1\,.$$

Furthermore, recalling Lemma 1.1.2,

$$v_j \to v \quad \text{in } L^q(\mathbb{R}^n) \quad \text{for any } q \in [1, 2^*) \tag{3.2.8}$$

and, by applying [20, Theorem IV.9], up to a subsequence (still denoted by v_j)

$$v_j \to v \quad \text{a.e. in } \mathbb{R}^n \tag{3.2.9}$$

as $j \to +\infty$.

Now, we define $i(r) := \inf\limits_{x\in\overline{\Omega},\, |t|\geqslant r} F(x, t)$ for $r > 0$. By (3.1.5) it follows that

$$\lim_{r\to+\infty} i(r) = +\infty. \tag{3.2.10}$$

Note that

$$\inf_{x\in\overline{\Omega},\, t\in\mathbb{R}} F(x, t) \;\text{ is finite.} \tag{3.2.11}$$

Indeed, by (3.1.5) it follows that for any $H > 0$ there exists $R > 0$ such that

$$F(x, t) > H \;\text{ for any } |t| > R \text{ and any } x \in \Omega. \tag{3.2.12}$$

Moreover, if $|t| \leqslant R$, by (3.1.4) we have

$$|F(x, t)| \leqslant M\, |t| \leqslant MR =: C_R, \tag{3.2.13}$$

for any $x \in \Omega$. Hence, by (3.2.12) and (3.2.13) we can conclude that

$$F(x,t) \geqslant -C_R \quad \text{for any} \quad (x,t) \in \Omega \times \mathbb{R}\,,$$

which implies (3.2.11).

As a consequence of (3.2.11), we may define

$$\omega^* := -\min\left\{-1, \inf_{x\in\overline{\Omega},\, t\in\mathbb{R}} F(x,t)\right\}.$$

Notice that $\omega^* \geqslant 0$ and $F(x,t) \geqslant -\omega^*$ for any $x \in \overline{\Omega}$ and any $t \in \mathbb{R}$. Now, we fix $h > 0$ and set $\Omega_{j,\,h} = \{x \in \Omega : \ |t_j v_j(x)| \geqslant h\}$. Thus, we get

$$\begin{aligned} \int_\Omega F(x, t_j v_j(x))dx &= \int_{\Omega_{j,\,h}} F(x, t_j v_j(x))dx + \int_{\Omega\setminus\Omega_{j,\,h}} F(x, t_j v_j(x))dx \\ &\geqslant |\Omega_{j,\,h}|\, i(h) - \omega^*\ |\Omega|\ . \end{aligned} \tag{3.2.14}$$

Since $v \not\equiv 0$, there exists a set $\Omega^\sharp$ with $\left|\Omega^\sharp\right| > 0$ and a constant $\delta > 0$ such that $|v(x)| \geqslant \delta$ a.e. $x \in \Omega^\sharp$. Then, by (3.2.9) and Egorov Theorem, there exists a measurable set $\Omega^* \subseteq \Omega^\sharp$ such that $|\Omega^*| \geqslant |\Omega^\sharp|/2 > 0$ and the limit in (3.2.9) is uniform in Ω^*. In particular, if j is large enough,

$$\sup_{x\in\Omega^*} |v_j(x) - v(x)| \leqslant \frac{\delta}{4}$$

and therefore $|v_j(x)| \geqslant 3\delta/4$ a.e. $x \in \Omega^*$. So, by (3.2.6), for h fixed above there exists j_h such that $|t_j v_j(x)| \geqslant h$ for any $j \geqslant j_h$ and a.e. $x \in \Omega^*$. As a consequence of this, we have that $\Omega^* \subseteq \Omega_{j,\,h}$ for $j \geqslant j_h$. Finally, by (3.2.7) and (3.2.14), we have

$$C \geqslant \int_\Omega F(x, t_j v_j(x))dx \geqslant |\Omega^*|\, i(h) - \omega^*\ |\Omega|$$

for $j \geqslant j_h$. Passing to the limit as $h \to +\infty$ and taking into account (3.2.10), we get a contradiction. This proves the assertion. $\square$

3.3 An existence result

This section is devoted to the proof of Theorem 3.1.1, which is the main result of the present chapter. At this purpose, first of all we observe that problem (3.1.8) has a variational structure, indeed it is the Euler–Lagrange equation of the functional $\mathcal{J} : Z \to \mathbb{R}$ defined as follows

$$\mathcal{J}(u) = \frac{1}{2}\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega a(x)|u(x)|^2\,dx - \int_\Omega F(x, u(x)dx\,, \tag{3.3.1}$$

where F was introduced in (3.1.5).

Note that the functional $\mathcal{J}$ is Fréchet differentiable in $u \in Z$ and for any $\varphi \in Z$

$$\langle \mathcal{J}'(u), \varphi\rangle = \iint_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \\ - \lambda \int_\Omega a(x)u(x)\varphi(x)\,dx - \int_\Omega f(x,u(x))\varphi(x)\,dx\,.$$

Thus, critical points of $\mathcal{J}$ are weak solutions of problem (3.1.2). In order to find these critical points, in the sequel we will apply the Saddle Point Theorem by Rabinowitz (see [66, 67]). For this, as in the previous chapter, we have to verify that the functional $\mathcal{J}$ satisfies both the appropriate geometric conditions (see (I_3) and (I_4) of [67, Theorem 4.6]) and the Palais–Smale compactness condition (see [67, p. 3]).

3.3.1 Geometry of the functional $\mathcal{J}$

In this subsection we will prove that the functional $\mathcal{J}$ has the geometric features required by the Saddle Point Theorem.

Proposition 3.3.1. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4)*. Moreover, let $\lambda = \lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$ and let f and a be two functions satisfying* (3.1.3)–(3.1.5) *and* (3.1.6)*, respectively.*

Then

$$\liminf_{\substack{u\in \mathbb{P}_{k+1} \\ \|u\|_Z\to+\infty}} \frac{\mathcal{J}(u)}{\|u\|_Z^2} > 0. \tag{3.3.2}$$

Proof. Since $u \in \mathbb{P}_{k+1}$, by Lemmas 3.2.1 and 3.2.3 we have

$$\mathcal{J}(u) \geqslant \frac{1}{2}\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)\|u\|_Z^2 - \widetilde{M}\|u\|_Z\,.$$

Hence, dividing both the sides of this expression by $\|u\|_Z^2$ and passing to the limit as $\|u\|_Z \to +\infty$, we get (3.3.2), since $\lambda_k < \lambda_{k+1}$ by assumption. □

Proposition 3.3.2. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4)*. Moreover, let $\lambda = \lambda_k < \lambda_{k+1}$ for some $k \in \mathbb{N}$ and let f and a be two functions satisfying* (3.1.3)–(3.1.5) *and* (3.1.6)*, respectively. Then*

$$\lim_{\substack{u\in \mathbb{H}_k \\ \|u\|_Z\to+\infty}} \mathcal{J}(u) = -\infty\,.$$

Proof. Since $u \in \mathbb{H}_k$, we can write $u = u^- + u^0$, with $u^- \in \mathbb{E}_k^-$ and $u^0 \in \mathbb{E}_k^0$. Also, $\mathcal{J}(u)$ can be written as follows

$$\begin{aligned}\mathcal{J}(u) = &\frac{1}{2}\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \frac{\lambda_k}{2}\int_\Omega a(x)|u(x)|^2 dx \\ &- \int_\Omega \Big(F(x,u^0(x)+u^-(x)) - F(x,u^0(x))\Big)\,dx - \int_\Omega F(x,u^0(x))\,dx.\end{aligned} \tag{3.3.3}$$

First of all, note that, by (3.1.4), Hőlder inequality and Lemma 1.1.2, it follows that

$$\begin{aligned}\Big|\int_\Omega \Big(F(x,u^0(x)+u^-(x)) - F(x,u^0(x))\Big)\,dx\Big| &= \left|\int_\Omega \int_{u^0(x)}^{u^0(x)+u^-(x)} f(x,t)dt\,dx\right| \\ &\leqslant M\int_\Omega \big|u^-(x)\big|\,dx \leqslant M\,|\Omega|^{1/2}\,\|u^-\|_{L^2(\Omega)} \\ &\leqslant \overline{M}\|u^-\|_Z\,,\end{aligned} \tag{3.3.4}$$

where $\overline{M}$ denotes a positive constant depending on Ω. Thus, by (3.3.3), (3.3.4) and Lemma 3.2.2, we get

$$\mathcal{J}(u) \leqslant -M^*\|u^-\|_Z^2 + \overline{M}\|u^-\|_Z - \int_\Omega F(x,u^0(x))\,dx. \tag{3.3.5}$$

Beware that the first norm in the right hand side of (3.3.5) is squared, while the second one is not. Moreover, by orthogonality we have

$$\|u\|_Z^2 = \|u^0\|_Z^2 + \|u^-\|_Z^2\,.$$

Then, as $\|u\|_Z \to +\infty$, we have that at least one of the two norms, either $\|u^0\|_Z$ or $\|u^-\|_Z$, goes to infinity.

Suppose that $\|u^0\|_Z \to +\infty$ (in this case $\|u^-\|_Z$ can be finite or not). Then, (3.3.5), the fact that $u^0 \in \mathbb{E}_k^0$ and Lemma 3.2.4 show that $\mathcal{J}(u) \to -\infty$ as $\|u\|_Z \to +\infty$ and so, Proposition 3.3.2 follows.

Otherwise, assume that $\|u^0\|_Z$ is finite. In this setting, of course,

$$\|u^-\|_Z \to +\infty\,. \tag{3.3.6}$$

and, by Lemma 3.2.3, $\displaystyle\int_\Omega F(x,u^0(x))\,dx$ is also finite.

Moreover, by (3.3.5) and (3.3.6), we have that $\mathcal{J}(u) \to -\infty$ as $\|u\|_Z \to +\infty$. This completes the proof of Proposition 3.3.2. □

3.3.2 The Palais–Smale condition

In this subsection we discuss a compactness property for the functional $\mathcal{J}$, given by the Palais–Smale condition.

First of all, as usual when using variational methods, we prove the boundedness of a Palais–Smale sequence for $\mathcal{J}$. We say that $\{u_j\}_{j\in\mathbb{N}}$ is a Palais–Smale sequence for $\mathcal{J}$ at level $c\in\mathbb{R}$ if

$$|\mathcal{J}(u_j)|\leqslant c, \tag{3.3.7}$$

and

$$\sup\Big\{|\langle \mathcal{J}'(u_j),\varphi\rangle| : \varphi\in Z,\ \|\varphi\|_Z=1\Big\}\to 0 \ \text{ as } j\to+\infty \tag{3.3.8}$$

hold true.

Proposition 3.3.3. *Let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4)*. Moreover, assume that $\lambda=\lambda_k<\lambda_{k+1}$ for some $k\in\mathbb{N}$ and let f and a be two functions satisfying* (3.1.3)–(3.1.5) *and* (3.1.6)*, respectively. Finally, let $c\in\mathbb{R}$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z verifying* (3.3.7) *and* (3.3.8)*.*

Then, the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z.

Proof. Let $u_j=u_j^0+u_j^-+u_j^+$, where $u_j^0\in\mathbb{E}_k^0$, $u_j^-\in\mathbb{E}_k^-$ and $u_j^+\in\mathbb{P}_{k+1}$. In order to prove Proposition 3.3.3, we will show that the sequences $\{u_j^0\}_{j\in\mathbb{N}}$, $\{u_j^-\}_{j\in\mathbb{N}}$ and $\{u_j^+\}_{j\in\mathbb{N}}$ are bounded in Z.

First of all, by (3.3.8), for large j, we get

$$\begin{aligned}\|u_j^\pm\|_Z &\geqslant \Big|\langle\mathcal{J}'(u_j),u_j^\pm\rangle\Big|\\ &=\Big|\iint_{\mathbb{R}^{2n}}\Big(u_j(x)-u_j(y)\Big)\Big(u_j^\pm(x)-u_j^\pm(y)\Big)K(x-y)\,dx\,dy\\ &\quad -\lambda_k\int_\Omega a(x)|u_j^\pm(x)|^2\,dx-\int_\Omega f(x,u_j(x))u_j^\pm(x)\,dx\Big|\,.\end{aligned} \tag{3.3.9}$$

While, by (3.1.4), the Hőlder inequality and Lemma 1.1.2

$$\Big|\int_\Omega f(x,u_j(x))u_j^\pm(x)\,dx\Big|\leqslant\tilde{M}\|u_j^\pm\|_Z, \tag{3.3.10}$$

with $\tilde{M}$ positive constant.

Finally, taking into account that $\big\{e_1,\dots,e_k\dots\big\}$ is a orthogonal basis of Z and of $L^2(\Omega,d\mu)$, $d\mu=a(\cdot)dx$, we get that

$$\begin{aligned}\langle\mathcal{J}'(u_j),u_j^\pm\rangle &=\iint_{\mathbb{R}^{2n}}|u_j^\pm(x)-u_j^\pm(y)|^2K(x-y)\,dx\,dy\\ &\quad-\lambda_k\int_\Omega a(x)|u_j^\pm(x)|^2\,dx-\int_\Omega f(x,u_j(x))u_j^\pm(x)\,dx\,.\end{aligned} \tag{3.3.11}$$

Now, by Lemma 3.2.1 (applied with $u=u_j^+\in\mathbb{P}_{k+1}$) and (3.3.9)–(3.3.11) we get

$$\left(1-\frac{\lambda_k}{\lambda_{k+1}}\right)\|u_j^+\|_Z^2-\tilde{M}\|u_j^+\|_Z\leqslant\|u_j^+\|_Z,$$

which shows that the sequence $\{u_j^+\}_{j\in\mathbb{N}}$ is bounded in Z.

Moreover, again by (3.3.9)–(3.3.11) and Lemma 3.2.2 (applied to $u_j^- \in \mathbb{E}_k^- \subset \mathbb{H}_k$), it follows that

$$\|u_j^-\|_Z \geqslant -\langle \mathcal{J}'(u_j), u_j^-\rangle \geqslant M^*\|u_j^-\|_Z^2 - \tilde{M}\|u_j^-\|_Z,$$

and so also $\{u_j^-\}_{j\in\mathbb{N}}$ is bounded in Z.

It remains to show that the sequence $\{u_j^0\}_{j\in\mathbb{N}}$ is bounded in Z. At this purpose, we point out that $u_j^0 \in \mathbb{E}_k^0$ and so, by (3.2.1), u_j^0 is an eigenfunctions corresponding to λ_k. Accordingly, by (1.2.2),

$$\frac{1}{2}\iint_{\mathbb{R}^{2n}} |u_j^0(x) - u_j^0(y)|^2 K(x-y)\,dx\,dy = \frac{\lambda_k}{2}\int_\Omega a(x)|u_j^0(x)|^2\,dx\,.$$

Therefore, by (3.3.7) and orthogonality, we see that

$$\begin{aligned}
c &\geqslant |\mathcal{J}(u_j)| \\
&= \Bigg|\frac{1}{2}\iint_{\mathbb{R}^{2n}} \Big(|u_j^+(x) - u_j^+(y)|^2 + |u_j^-(x) - u_j^-(y)|^2\Big) K(x-y)\,dx\,dy \\
&\quad -\frac{\lambda_k}{2}\int_\Omega a(x)\Big(|u_j^+(x)|^2 + |u_j^-(x)|^2\Big)\,dx - \int_\Omega \Big(F(x,u_j(x)) - F(x,u_j^0(x))\Big)dx \\
&\quad - \int_\Omega F(x,u_j^0(x))\,dx\Bigg|.
\end{aligned} \tag{3.3.12}$$

By Lemma 1.1.2 and the Hőlder inequality we get that there exists a positive constant C, possibly depending on Ω, such that

$$\left|\lambda_k \int_\Omega a(x)\left(|u_j^+(x)|^2 + |u_j^-(x)|^2\right)dx\right| \leqslant \lambda_k \|a\|_{L^\infty(\Omega)}\left(\|u_j^+\|_Z^2 + \|u_j^-\|_Z^2\right) \leqslant 2C\,, \tag{3.3.13}$$

and

$$\begin{aligned}
\left|\int_\Omega \Big(F(x,u_j(x)) - F(x,u_j^0(x))\Big)\,dx\right| &\leqslant \int_\Omega \left|\int_{u_j^0(x)}^{u_j^0(x)+u_j^-(x)+u_j^+(x)} f(x,t)dt\right| dx \\
&\leqslant M\int_\Omega \Big(|u_j^-(x)| + |u_j^+(x)|\Big)dx \\
&\leqslant M_*\left(\|u_j^-\|_Z + \|u_j^+\|_Z\right) \leqslant C\,,
\end{aligned} \tag{3.3.14}$$

since the sequences $\{u_j^-\}_{j\in\mathbb{N}}$ and $\{u_j^+\}_{j\in\mathbb{N}}$ are bounded in Z and (3.1.4) holds true.

Here M_* is a positive constant. Hence, by (3.3.12)–(3.3.14) it is easy to see that

$$\begin{aligned}\left|\int_\Omega F(x,u_j^0(x))\,dx\right| &\leqslant |\mathcal{J}(u_j)| \\ &\quad + \left|\frac{1}{2}\iint_{\mathbb{R}^{2n}}\Big(|u_j^+(x)-u_j^+(y)|^2+|u_j^-(x)-u_j^-(y)|^2\Big)K(x-y)\,dx\,dy\right. \\ &\quad -\frac{\lambda_k}{2}\int_\Omega a(x)\Big(|u_j^+(x)|^2+|u_j^-(x)|^2\Big)\,dx \\ &\quad \left.-\int_\Omega\Big(F(x,u_j(x))-F(x,u_j^0(x))\Big)dx\right| \\ &\leqslant c+\frac{1}{2}\Big(\|u^+\|_Z^2+\|u^-\|_Z^2\Big)+2C\leqslant\tilde{C}\end{aligned}$$

where $\tilde{C}$ is a positive constant independent of j. Here we have used again the fact that the sequences $\{u_j^-\}_{j\in\mathbb{N}}$ and $\{u_j^+\}_{j\in\mathbb{N}}$ are bounded in Z.

Hence, the integral $\displaystyle\int_\Omega F(x,u_j^0(x))\,dx$ is bounded. As a consequence, being $u^0\in\mathbb{E}_k^0$, by Lemma 3.2.4 it follows that also the sequence $\{u_j^0\}_{j\in\mathbb{N}}$ is bounded in Z, concluding the proof of Proposition 3.3.3. □

Now it remains to check the validity of the Palais–Smale condition, that is we have to show that every Palais–Smale sequence $\{u_j\}_{j\in\mathbb{N}}$ for $\mathcal{J}$ at level $c\in\mathbb{R}$ strongly converges in Z, up to a subsequence. This will be done in the next result.

Proposition 3.3.4. *Let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4)*. Moreover, assume that $\lambda=\lambda_k<\lambda_{k+1}$ for some $k\in\mathbb{N}$ and let f and a be two functions satisfying* (3.1.3)–(3.1.5) *and* (3.1.6)*, respectively. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z satisfying* (3.3.7) *and* (3.3.8)*.*

Then, there exists $u_\infty\in Z$ such that u_j strongly converges to some u_∞ in Z.

Proof. Since, by Proposition 3.3.3, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z and Z is a reflexive space (being a Hilbert space, by Lemma 1.1.3), up to a subsequence, there exists $u_\infty\in Z$ such that u_j converges to u_∞ weakly in Z, that is

$$\begin{aligned}&\iint_{\mathbb{R}^{2n}}(u_j(x)-u_j(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy\to \\ &\qquad\iint_{\mathbb{R}^{2n}}(u_\infty(x)-u_\infty(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy\end{aligned}\tag{3.3.15}$$

for any $\varphi\in Z$, as $j\to+\infty$. Moreover, by applying Lemma 1.1.2 and [20, Theorem IV.9], up to a subsequence

$$\begin{aligned}&u_j\to u_\infty \quad \text{in } L^q(\mathbb{R}^n) \text{ for any } q\in[1,2^*) \\ &u_j\to u_\infty \quad \text{a.e. in } \mathbb{R}^n\end{aligned}\tag{3.3.16}$$

as $j \to +\infty$.

By (3.3.8) we have

$$
\begin{aligned}
0 \leftarrow \langle \mathcal{J}'(u_j), u_j - u_\infty \rangle = & \iint_{\mathbb{R}^{2n}} |u_j(x) - u_j(y)|^2 \, K(x-y)\, dx\, dy \\
& - \iint_{\mathbb{R}^{2n}} \big(u_j(x) - u_j(y)\big)\big(u_\infty(x) - u_\infty(y)\big) K(x-y)\, dx\, dy \\
& - \lambda_k \int_\Omega a(x) u_j(x)(u_j(x) - u_\infty(x)) dx \\
& - \int_\Omega f(x, u_j(x))(u_j(x) - u_\infty(x)) dx
\end{aligned} \tag{3.3.17}
$$

as $j \to +\infty$. Now, by using the Hőlder inequality, (3.1.4) and (3.3.16), we get

$$
\begin{aligned}
& \left| \lambda_k \int_\Omega a(x) u_j(x)(u_j(x) - u_\infty(x)) dx + \int_\Omega f(x, u_j(x))(u_j(x) - u_\infty(x)) dx \right| \\
& \qquad \leqslant \Big(\lambda_k \|a\|_{L^\infty(\Omega)} \|u_j\|_{L^2(\Omega)} + M\, |\Omega|^{1/2} \Big) \|u_j - u_\infty\|_{L^2(\Omega)} \to 0
\end{aligned} \tag{3.3.18}
$$

as $j \to +\infty$. Hence, passing to the limit in (3.3.17) and taking into account (3.3.15) and (3.3.18), it follows that

$$
\iint_{\mathbb{R}^{2n}} |u_j(x) - u_j(y)|^2 \, K(x-y)\, dx\, dy \to \iint_{\mathbb{R}^{2n}} |u_\infty(x) - u_\infty(y)|^2 \, K(x-y)\, dx\, dy \,,
$$

that is

$$
\|u_j\|_Z \to \|u_\infty\|_Z \tag{3.3.19}
$$

as $j \to +\infty$.

Finally, we have that

$$
\begin{aligned}
\|u_j - u_\infty\|_Z^2 = & \, \|u_j\|_Z^2 + \|u_\infty\|_Z^2 \\
& - 2 \iint_{\mathbb{R}^{2n}} \big(u_j(x) - u_j(y)\big)\big(u_\infty(x) - u_\infty(y)\big) K(x-y)\, dx\, dy \\
& \to 2\|u_\infty\|_Z^2 - 2\|u_\infty\|_Z^2 = 0
\end{aligned}
$$

as $j \to +\infty$, thanks to (3.3.15) and (3.3.19). Hence, $u_j \to u_\infty$ strongly in Z as $j \to +\infty$ and this completes the proof of Proposition 3.3.4 . □

3.3.3 Proof of Theorem 3.1.1

In this subsection we will prove Theorem 3.1.1, as an application of the Saddle Point Theorem [67, Theorem 4.6] .

At first, we prove that $\mathcal{J}$ satisfies the geometric structure required by the Saddle Point Theorem. For this note that by Proposition 3.3.1 for any $H > 0$ there exists

$R > 0$ such that, if $u \in \mathbb{P}_{k+1}$ and $\|u\|_Z \geqslant R$, then

$$\mathcal{J}(u) \geqslant H\,. \tag{3.3.20}$$

While, if $u \in \mathbb{P}_{k+1}$ with $\|u\|_Z \leqslant R$, by applying (3.1.4), the Hőlder inequality and Lemma 1.1.2 we have

$$\begin{aligned}
\mathcal{J}(u) &\geqslant -\frac{\lambda_k}{2}\int_\Omega a(x)|u(x)|^2dx - \int_\Omega F(x,u(x))dx \\
&\geqslant -\frac{\lambda_k}{2}\|a\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)} - M\int_\Omega |u(x)|\,dx \\
&\geqslant -\frac{\lambda_k}{2}\|a\|_{L^\infty(\Omega)}\|u\|^2_Z - M_*\|u\|_Z \\
&\geqslant -\frac{\lambda_k}{2}\|a\|_{L^\infty(\Omega)}R^2 - M_*R =: -C_R\,.
\end{aligned} \tag{3.3.21}$$

Here M_* is a positive constant. Hence, by (3.3.20) and (3.3.21) we get

$$\mathcal{J}(u) \geqslant -C_R \text{ for any } u \in \mathbb{P}_{k+1}\,. \tag{3.3.22}$$

Moreover, by Proposition 3.3.2, there exists $T > 0$ such that, for any $u \in \mathbb{H}_k$ with $\|u\|_Z = T$, we have

$$\mathcal{J}(u) < -C_R\,. \tag{3.3.23}$$

Thus, by (3.3.22) and (3.3.23) it easily follows that

$$\sup_{\substack{u\in\mathbb{H}_k,\\ \|u\|_Z=T}} \mathcal{J}(u) < -C_R \leqslant \inf_{u\in\mathbb{P}_{k+1}} \mathcal{J}(u)\,,$$

so that the functional $\mathcal{J}$ has the geometric structure of the Saddle Point Theorem (see assumptions (I_3) and (I_4) of [67, Theorem 4.6]).

Since $\mathcal{J}$ satisfies also the Palais–Smale condition by Proposition 3.3.4, the Saddle Point Theorem provides the existence of a critical point $u \in Z$ for the functional $\mathcal{J}$. This concludes the proof of Theorem 3.1.1.

Chapter 4

Problems with a parameter

4.1 Introduction

Aim of the present chapter is to provide some existence results for the following non–local problems

$$\begin{cases} -\mathcal{L}_K u + q(x)u = \lambda u + f(u) + h(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{4.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded set, λ is a real parameter, $s \in (0,1)$, $\mathcal{L}_K$ is the non–local operator formally defined as in (0.0.2), whose kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfies conditions (0.0.3) and (0.0.4). Moreover, we suppose that in equation (4.1.1) the function $f : \mathbb{R} \to \mathbb{R}$ verifies the following assumptions:

$$f \in C^1(\mathbb{R}) \tag{4.1.2}$$

$$\text{there exists a constant } M > 0 \text{ such that } |f(t)| \leqslant M \text{ for any } t \in \mathbb{R}\,, \tag{4.1.3}$$

while q, $h : \Omega \to \mathbb{R}$ are such that

$$q \in L^\infty(\Omega), \;\; q(x) \geqslant 0 \text{ a.e. } x \in \Omega \tag{4.1.4}$$

and

$$h \in L^2(\Omega), \tag{4.1.5}$$

respectively.

When $f \equiv 0$ and $h \equiv 0$ problem (4.1.1) becomes the following eigenvalue problem

$$\begin{cases} -\mathcal{L}_K u + q(x)u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{4.1.6}$$

We recall that there exists a non-decreasing sequence of positive eigenvalues λ_k for which (4.1.6) admits a solution which we already introduced in Section 1.2 (with $a \equiv 1$).

Along this chapter we consider both the resonant and the non-resonant case, that is the case when λ belongs to the spectrum of the operator driving the equation and the one when λ does not, respectively. As for the resonant setting we would like to note that we are able to treat this case only requiring some extra conditions on the terms f and h . Precisely, denoting by

$$f_l = \lim_{t\to-\infty} f(t) \qquad \text{and} \qquad f_r = \lim_{t\to+\infty} f(t)\,,$$

we assume that

$$f_l \text{ and } f_r \text{ exist, are finite and such that } f_l > f_r \tag{4.1.7}$$

and

$$f_r \int_\Omega \varphi^-(x)dx - f_l \int_\Omega \varphi^+(x)dx < \int_\Omega h(x)\varphi(x)dx < f_l \int_\Omega \varphi^-(x)dx - f_r \int_\Omega \varphi^+(x)dx \quad \text{for any } \varphi \in E_\lambda \setminus \{0\}\,, \tag{4.1.8}$$

where $\varphi^+ = \max\{\varphi, 0\}$ and $\varphi^- = \max\{-\varphi, 0\}$ denote the positive and the negative part of the function φ, respectively, while E_λ is the linear space generated by the eigenfunctions related to λ (for a precise definition of E_λ we refer to Section 4.4).

We would remark that these extra conditions on f and h are exactly the same required in the resonant setting, when dealing with the classical Laplace operator (see [14, Section 4.4.3]). Moreover, we would point out that in (4.1.7) the limits f_l and f_r have to be different, but the case $f_l < f_r$ would work as well, with some modifications in the main arguments. Assumption (4.1.8) is the classical Landesman–Lazer condition, firstly introduced in [58], which represents one of the natural sufficient condition given in order to obtain an existence result in a resonant setting.

As a model for f we can take the function

$$f(t) = \begin{cases} \dfrac{1}{1+t^2} & \text{if } t \geqslant 0 \\[2ex] 1 & \text{if } t < 0\,. \end{cases}$$

We would also like to note that, in this case, f does not satisfy the assumptions required in Chapter 3, where always an asymptotically linear problem at resonance driven by a general non–local operator was considered. Indeed, in Chapter 3 the asymptotically linear case when the primitive of f goes to infinity was considered.

The main result of the present chapter concerns the existence of weak solutions for problem (4.1.1). For this, first of all, we have to write the weak formulation of (4.1.1), given by the following problem

$$\begin{cases} \displaystyle\iint_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy + \int_\Omega q(x)u(x)\varphi(x)dx \\ \displaystyle\qquad\qquad = \lambda\int_\Omega u(x)\varphi(x)dx + \int_\Omega f(u(x))\varphi(x)dx + \int_\Omega h(x)\varphi(x)dx \quad \forall\,\varphi\in Z \\ u\in Z, \end{cases} \tag{4.1.9}$$

where Z is the functional space introduced in Chapter 1. Before stating our existence result, we would like to note that, in general, the trivial function $u\equiv 0$ is not a solution of problem (4.1.1). On the other hand, if $h\equiv 0$ and $f(0)=0$, then $u\equiv 0$ solves the problem.

Now, we can state our main result as follows:

Theorem 4.1.1. *Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ be a function satisfying* (0.0.3) *and* (0.0.4) *and let $f:\mathbb{R}\to\mathbb{R}$, q, $h:\Omega\to\mathbb{R}$ be three functions verifying* (4.1.2)–(4.1.5).

Then, problem (4.1.1) *admits a weak solution $u\in Z$ provided either*

- *λ is not an eigenvalue of problem* (4.1.6)*, or*
- *λ is an eigenvalue of problem* (4.1.6) *and conditions* (4.1.7) *and* (4.1.8) *hold true.*

The proof of Theorem 4.1.1 is based on variational techniques. Precisely, we will find solutions of problem (4.1.1) as critical points of the Euler–Lagrange functional naturally associated with the problem. To this purpose we will perform the Saddle Point Theorem by Rabinowitz, see [67, Theorem 4.6]. Hence, as in the previous chapters, we have to study both the compactness properties of the functional associated with the problem and also its geometrical structure. In doing this we need to consider separately the case when the parameter λ is an eigenvalue of $-\mathcal{L}_K+q$ and the case when it does not, namely the resonant and the non-resonant situation.

The resonant setting is more difficult to be treated than the non-resonant one. Indeed, the resonant assumption affects both the compactness property and the geometry of the functional. For this reason, the extra assumptions (4.1.7)–(4.1.8) will be crucial both in proving the compactness and in showing the geometric properties possessed by the Euler–Lagrange functional associated with problem (4.1.1).

Theorem 4.1.1 extends the result obtained in [14, Theorem 4.4.11 and Theorem 4.4.17] (see also [14, Chapter 4] and references therein) in the case of the classical Laplacian operator to a general non–local framework.

The chapter is organized as follows. In Section 4.2 we will discuss the variational formulation of the problem, while Sections 4.3 and 4.4 will be devoted to the proof of Theorem 4.1.1, respectively in the non-resonant case and in the resonant one.

4.2 Variational formulation of the problem

For the proof of our main result, stated in Theorem 4.1.1, we first observe that problem (4.1.1) has a variational structure. Indeed, the weak formulation of problem (4.1.1), given in (4.1.9), represents the Euler-Lagrange equation of the functional $\mathcal{J} : Z \to \mathbb{R}$ defined as follows

$$\begin{aligned}\mathcal{J}(u) = &\frac{1}{2}\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy + \frac{1}{2}\int_\Omega q(x)\,|u(x)|^2\,dx \\ &-\frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \int_\Omega F(u(x))dx - \int_\Omega h(x)u(x)dx\,,\end{aligned} \tag{4.2.1}$$

where $F(t) = \displaystyle\int_0^t f(\tau)\,d\tau$.
Note that the functional $\mathcal{J}$ is well defined thanks to Lemma 1.2.1, the definition of F, assumptions (4.1.3)–(4.1.5) and since $Z \subseteq L^2(\Omega) \subseteq L^1(\Omega)$ (being Ω bounded and by Lemma (1.1.2)). Moreover, $\mathcal{J}$ is Fréchet differentiable at $u \in Z$ and for any $\varphi \in Z$

$$\begin{aligned}\langle \mathcal{J}'(u), \varphi\rangle = &\iint_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy + \int_\Omega q(x)u(x)\varphi(x)dx \\ &-\lambda\int_\Omega u(x)\varphi(x)dx - \int_\Omega f(u(x))\varphi(x)dx - \int_\Omega h(x)\varphi(x)dx\,.\end{aligned}$$

Thus, critical points of $\mathcal{J}$ are weak solutions of problem (4.1.1), that is solutions of (4.1.9).

At first, we need some notation. In what follows we will denote by

$$\lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots$$

the sequence of the eigenvalues of $-\mathcal{L}_K + q$ (see problem (4.1.6)) , while e_k will be the k-th eigenfunction corresponding to the eigenvalue λ_k . Moreover, we will set

$$\mathbb{P}_{k+1} := \Big\{u \in Z : \ \langle u, e_j\rangle_{Z,\,q} = 0 \ \ \forall j = 1, \ldots, k\Big\}$$

as defined in Proposition 1.2.2, while

$$\mathbb{H}_k := \operatorname{span}\{e_1, \ldots, e_k\}$$

will denote the linear subspace generated by the first k eigenfunctions of $-\mathcal{L}_K + q$ for any $k \in \mathbb{N}$.

In order to prove Theorem 4.1.1 we need some preliminary lemmas.

Lemma 4.2.1. *Let $q : \Omega \to \mathbb{R}$ satisfy* (4.1.4).

Then, the following inequality holds true

$$\|u\|^2_{Z,q} \leqslant \lambda_k \|u\|^2_{L^2(\Omega)}$$

for all $u \in \mathbb{H}_k$ and any $k \in \mathbb{N}$.

Proof. Let $u \in \mathbb{H}_k$. Then, we can write

$$u(x) = \sum_{i=1}^{k} u_i e_i(x)$$

with $u_i \in \mathbb{R}$, $i = 1, \dots, k$.

Since $\{e_1, \dots, e_k, \dots\}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal one of Z (see Proposition 1.2.2–(vi)), by (1.2.8) and (1.2.11), we get

$$\|u\|^2_{Z,\,q} = \sum_{i=1}^{k} u_i^2 \, \|e_i\|^2_{Z,\,q} = \sum_{i=1}^{k} \lambda_i u_i^2 \leqslant \lambda_k \sum_{i=1}^{k} u_i^2 = \lambda_k \|u\|^2_{L^2(\Omega)} \,,$$

which gives the desired assertion. □

Lemma 4.2.2. *Let $q : \Omega \to \mathbb{R}$ satisfy* (4.1.4).

Then, the following inequality holds true

$$\|u\|^2_{Z,q} \geqslant \lambda_{k+1} \|u\|^2_{L^2(\Omega)}$$

for all $u \in \mathbb{P}_{k+1}$ and any $k \in \mathbb{N}$.

Proof. If $u \equiv 0$, then the assertion is trivial, while if $u \in \mathbb{P}_{k+1} \setminus \{0\}$ it follows from the variational characterization of λ_{k+1} given in (1.2.9). □

To conclude this section we prove the following result:

Lemma 4.2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ and $h : \Omega \to \mathbb{R}$ be functions verifying* (4.1.2)–(4.1.3) *and* (4.1.5)*, respectively.*

Then, there exists a positive constant $\tilde{C}$ such that

$$\left| \int_\Omega F(u(x))dx + \int_\Omega h(x)u(x)dx \right| \leqslant \tilde{C} \, \|u\|_{Z,\,q}$$

for all $u \in Z$.

Proof. By (4.1.3), (4.1.5), the definition of F, the Hölder inequality, Lemma 1.2.1 and [74, Lemma 6], we get

$$\begin{aligned} \left| \int_\Omega F(u(x))dx + \int_\Omega h(x)u(x)dx \right| &\leqslant M \int_\Omega |u(x)| \, dx + \|h\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\leqslant M \, |\Omega|^{1/2} \, \|u\|_{L^2(\Omega)} + \tilde{\kappa} \, \|h\|_{L^2(\Omega)} \|u\|_{Z,\,q} \\ &\leqslant \tilde{C} \, \|u\|_{Z,\,q} \,, \end{aligned} \tag{4.2.2}$$

for a suitable $\tilde{C} > 0$ (here $|\Omega|$ denotes the measure of Ω and $\tilde{\kappa}$ is a positive constant). This gives the desired assertion. $\square$

Due to the variational nature of the problem, in order to find weak solutions for problem (4.1.1), in the following we will look for critical points of the functional $\mathcal{J}$ defined in (4.2.1). In doing this we need to study separately the resonant case and the non-resonant one, that is the case when the parameter λ is an eigenvalue of the operator $-\mathcal{L}_K + q$ and the one where λ is different from these eigenvalues, respectively. We will treat the non-resonant case in the forthcoming Section 4.3 and the resonant one in the next Section 4.4.

4.3 The non-resonant case

In this section we will prove Theorem 4.1.1 in the case when the parameter λ appearing in problem (4.1.1) is not an eigenvalue of the operator $-\mathcal{L}_K + q$. As we said before, the idea is to find critical points of the functional $\mathcal{J}$, given in formula (4.2.1). To this purpose, we will consider two different cases:

- $\lambda < \lambda_1$: in this setting the existence of a solution for problem (4.1.1) follows from the Weierstrass Theorem (i.e. by direct minimization);
- $\lambda > \lambda_1$: in this framework we will apply the Saddle Point Theorem (see [66, 67]) to the functional $\mathcal{J}$. As usual, for this we have to check that the functional $\mathcal{J}$ has a particular geometric structure (as stated, e.g., in conditions (I_3) and (I_4) of [67, Theorem 4.6]) and that it satisfies the Palais–Smale compactness condition (see, for instance, [67, page 3]).

4.3.1 The case $\lambda < \lambda_1$

In this subsection, in order to apply the Weierstrass Theorem, we first verify that the functional $\mathcal{J}$ satisfies some geometric features. For this we need a preliminary lemma.

Lemma 4.3.1. *Let $\lambda < \lambda_1$ and let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4). *Moreover, let $f : \mathbb{R} \to \mathbb{R}$, $q, h : \Omega \to \mathbb{R}$ be functions satisfying conditions* (4.1.2)–(4.1.5).

Then, the functional $\mathcal{J}$ verifies

$$\liminf_{\|u\|_{Z,\,q} \to +\infty} \frac{\mathcal{J}(u)}{\|u\|^2_{Z,\,q}} > 0.$$

Proof. By the variational characterization of λ_1 given in (1.2.7), we get

$$\lambda_1 \|u\|^2_{L^2(\Omega)} \leqslant \|u\|^2_{Z,\,q}$$

for any $u \in Z$ (of course, if $u \equiv 0$, this inequality is trivial).

Hence, as a consequence of this and Lemma 4.2.3, we get

$$\begin{aligned}
\mathcal{J}(u) &= \frac{1}{2}\,\|u\|^2_{Z,\,q} - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \int_\Omega F(u(x))dx - \int_\Omega h(x)u(x)dx \\
&\geqslant \begin{cases} \dfrac{1}{2}\big(1-\dfrac{\lambda}{\lambda_1}\big)\,\|u\|^2_{Z,\,q} - \tilde{C}\,\|u\|_{Z,\,q} & \text{if } \lambda > 0 \\[2ex] \dfrac{1}{2}\,\|u\|^2_{Z,\,q} - \tilde{C}\,\|u\|_{Z,\,q} & \text{if } \lambda \leqslant 0\,, \end{cases}
\end{aligned}$$

so that, dividing by $\|u\|^2_{Z,\,q}$ and passing to the limit as $\|u\|_{Z,\,q} \to +\infty$, we get the assertion, since $\lambda < \lambda_1$ by assumption. □

Proof of Theorem 4.1.1 in the non-resonant case, when $\lambda < \lambda_1$

Let us note that the map

$$u \mapsto \|u\|^2_{Z,\,q}$$

is lower semicontinuous in the weak topology of Z, while the map

$$u \mapsto \int_\Omega F(x, u(x))dx$$

is continuous in the weak topology of Z. Indeed, if $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in Z such that $u_j \rightharpoonup u$ in Z, then, by Lemma 1.1.2 and [20, Theorem IV.9], up to a subsequence, u_j converges to u strongly in $L^\nu(\Omega)$ and a.e. in Ω and it is dominated by some function $\kappa_\nu \in L^\nu(\Omega)$ for any $\nu \in [1, 2^*)$. Here and in the following 2^* is the fractional critical Sobolev exponent introduced in Section 1.1 and defined as in (1.1.6). Then, by (4.1.2) and (4.1.3) it follows

$$F(u_j(x)) \to F(u(x)) \ \text{ a.e. } x \in \Omega$$

as $j \to +\infty$ and

$$|F(u_j(x))| \leqslant M\,|u_j(x)| \leqslant M\kappa_1(x) \in L^1(\Omega)$$

a.e. $x \in \Omega$ and for any $j \in \mathbb{N}$. Hence, by applying the Lebesgue Dominated Convergence Theorem applied in $L^1(\Omega)$, we have that

$$\int_\Omega F(u_j(x))\,dx \to \int_\Omega F(u(x))\,dx$$

as $j \to +\infty$, that is the map

$$u \mapsto \int_\Omega F(x, u(x))dx$$

is continuous from Z with the weak topology to $\mathbb{R}$.

Moreover, again by Lemma 1.1.2, also the map

$$u \mapsto \frac{\lambda}{2} \int_\Omega |u(x)|^2 \, dx + \int_\Omega h(x)u(x) \, dx$$

is continuous in the weak topology of Z. Hence, the functional $\mathcal{J}$ is lower semicontinuous in the weak topology of Z.

Furthermore, Lemma 4.3.1 gives the coerciveness of $\mathcal{J}$. Thus, we can apply the Weierstrass Theorem in order to find a minimum u of $\mathcal{J}$ on Z. Clearly, u is a weak solution of problem (4.1.1).

4.3.2 The case $\lambda > \lambda_1$

In this subsection we can suppose that $\lambda_k < \lambda < \lambda_{k+1}$ for some $k \in \mathbb{N}$. This is due to the fact that the sequence of eigenvalues λ_k of the operator $-\mathcal{L}_K + q$ diverges to $+\infty$ as $k \to +\infty$ (see Proposition 1.2.2–(iv)).

In this framework we will look for critical points of the functional $\mathcal{J}$ using the Saddle Point Theorem. First of all, we need some preliminary lemmas.

Lemma 4.3.2. *Let $\lambda \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4) *and let $f : \mathbb{R} \to \mathbb{R}$, q, $h : \Omega \to \mathbb{R}$ be functions satisfying* (4.1.2)–(4.1.5).

Then, the functional $\mathcal{J}$ verifies

$$\limsup_{\substack{u \in \mathbb{H}_k \\ \|u\|_{Z,q} \to +\infty}} \frac{\mathcal{J}(u)}{\|u\|^2_{Z,q}} < 0\,.$$

Proof. Let $u \in \mathbb{H}_k$. By Lemma 4.2.1, Lemma 4.2.3 and the fact that $\lambda > 0$ (being $\lambda > \lambda_k \geqslant \lambda_1 > 0$) we get

$$\begin{aligned}\mathcal{J}(u) &= \frac{1}{2}\|u\|^2_{Z,q} - \frac{\lambda}{2}\int_\Omega |u(x)|^2 \, dx - \int_\Omega F(u(x))dx - \int_\Omega h(x)u(x)dx \\ &\leqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|u\|^2_{Z,q} + \tilde{C}\,\|u\|_{Z,q}\,.\end{aligned}$$

So, dividing by $\|u\|^2_{Z,q}$ and passing to the limit as $\|u\|_{Z,q} \to +\infty$, we get the assertion, since $\lambda > \lambda_k$. ∎

Note that Lemma 4.3.2 holds true for any $\lambda \in (\lambda_k, \lambda_{k+1}]$ for some $k \in \mathbb{N}$ and this will be used in the resonant case of problem (4.1.1), that is in the case when $\lambda = \lambda_{k+1}$.

Lemma 4.3.3. *Let $\lambda \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4) *and let $f : \mathbb{R} \to \mathbb{R}$, q, $h : \Omega \to \mathbb{R}$ be functions satisfying* (4.1.2)–(4.1.5).

Then, the functional $\mathcal{J}$ verifies

$$\liminf_{\substack{u\in\mathbb{P}_{k+1}\\ \|u\|_{Z,\,q}\to+\infty}} \frac{\mathcal{J}(u)}{\|u\|^2_{Z,\,q}} > 0\,.$$

Proof. Let $u \in \mathbb{P}_{k+1}$. In this case, by Lemma 4.2.2, Lemma 4.2.3 and the positivity of λ, we have

$$\mathcal{J}(u) \geqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\|u\|^2_{Z,\,q} - \tilde{C}\,\|u\|_{Z,\,q}\,,$$

so that, dividing by $\|u\|^2_{Z,\,q}$ and passing to the limit as $\|u\|_{Z,\,q} \to +\infty$, we get the assertion, being $\lambda < \lambda_{k+1}$. □

With these preliminary results we can prove that the functional $\mathcal{J}$ has the geometric structure required by the Saddle Point Theorem, according to the following result:

Proposition 4.3.4. *Let $\lambda \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4) *and let $f : \mathbb{R} \to \mathbb{R}$, q, $h : \Omega \to \mathbb{R}$ be functions satisfying* (4.1.2)–(4.1.5).

Then, there exist two positive constants C and T such that

$$\sup_{\substack{u\in\mathbb{H}_k\\ \|u\|_{Z,\,q}=T}} \mathcal{J}(u) < -C \leqslant \inf_{u\in\mathbb{P}_{k+1}} \mathcal{J}(u)\,.$$

Proof. By Lemma 4.3.3 it follows that for any $H > 0$ there exists $R > 0$ such that if $u \in \mathbb{P}_{k+1}$ and $\|u\|_Z \geqslant R$ then $\mathcal{J}(u) \geqslant H$.

On the other hand, if $u \in \mathbb{P}_{k+1}$ with $\|u\|_{Z,\,q} < R$, by applying Lemma 4.2.3, the Hölder inequality, Lemma 1.2.1 and [74, Lemma 6] we have

$$\begin{aligned}\mathcal{J}(u) &\geqslant -\frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \int_\Omega F(u(x))dx - \int_\Omega h(x)u(x)dx \geqslant \\ &\geqslant -\bar{\kappa}\,\|u\|^2_{Z,\,q} - \tilde{C}\,\|u\|_{Z,\,q} \\ &> -\bar{\kappa}R^2 - \tilde{C}R =: -C\,,\end{aligned}$$

thanks to the fact that $\lambda > 0$ (being $\lambda > \lambda_k \geqslant \lambda_1 > 0$ by (1.2.8)). Also, here $\bar{\kappa}$ is a positive constant.

So, we get

$$\mathcal{J}(u) \geqslant -C \qquad \text{for any } u \in \mathbb{P}_{k+1}\,. \tag{4.3.1}$$

Moreover, by Lemma 4.3.2 there exists $T > 0$ such that for any $u \in \mathbb{H}_k$ with $\|u\|_{Z,\,q} \geqslant T$ we have

$$\sup_{\substack{u\in\mathbb{H}_k\\ \|u\|_{Z,\,q}=T}} \mathcal{J}(u) \leqslant \sup_{\substack{u\in\mathbb{H}_k\\ \|u\|_{Z,\,q}\geqslant T}} \mathcal{J}(u) < -C\,. \tag{4.3.2}$$

Thus, Proposition 4.3.4 follows from (4.3.1) and (4.3.2). $\square$

Roughly speaking, Proposition 4.3.4 says that $\mathcal{J}$ has the geometric structure required by the Saddle Point Theorem.

Finally, we have to show that $\mathcal{J}$ satisfies the Palais–Smale condition. To this purpose, first of all we prove that every Palais–Smale sequence for $\mathcal{J}$ is bounded in Z.

Proposition 4.3.5. *Let $\lambda \in (\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4) *and let $f : \mathbb{R} \to \mathbb{R}$, q, $h : \Omega \to \mathbb{R}$ be functions satisfying* (4.1.2)–(4.1.5). *Let $c \in \mathbb{R}$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z such that*

$$\mathcal{J}(u_j) \leqslant c, \tag{4.3.3}$$

and

$$\sup\Big\{|\langle \mathcal{J}'(u_j), \varphi\rangle| : \varphi \in Z,\, \|\varphi\|_{Z,\,q} = 1\Big\} \to 0 \tag{4.3.4}$$

as $j \to +\infty$.

Then, the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z.

Proof. We argue by contradiction and we suppose that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is unbounded in Z. As a consequence, up to a subsequence, we can assume that

$$\|u_j\|_{Z,\,q} \to +\infty \quad \text{as } j \to +\infty\,. \tag{4.3.5}$$

Thus, there exists $u \in Z$ such that $u_j/\,\|u_j\|_{Z,\,q}$ converges to u weakly in Z, that is

$$\begin{aligned}&\iint_{\mathbb{R}^{2n}}\left(\frac{u_j(x)}{\|u_j\|_{Z,\,q}} - \frac{u_j(y)}{\|u_j\|_{Z,\,q}}\right)(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy + \int_\Omega q(x)\frac{u_j(x)}{\|u_j\|_{Z,\,q}}\varphi(x)dx\\ &\to \iint_{\mathbb{R}^{2n}}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy + \int_\Omega q(x)u(x)\varphi(x)dx\end{aligned} \tag{4.3.6}$$

as $j \to +\infty$, for any $\varphi \in Z$.

Hence, by applying Lemma 1.1.2 and [20, Theorem IV.9], up to a subsequence

$$\begin{aligned} &\frac{u_j}{\|u_j\|_{Z,\,q}} \to u \quad \text{in } L^\nu(\mathbb{R}^n) \ \text{ for any } \nu \in [1, 2^*) \\ &\frac{u_j}{\|u_j\|_{Z,\,q}} \to u \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{4.3.7}$$

as $j \to +\infty$. Here 2^* is the exponent defined as in (1.1.6) .

Furthermore, by (4.1.3), (4.1.5) and the Hölder inequality it follows that

$$\begin{aligned} &\frac{1}{\|u_j\|_{Z,\,q}} \left| \int_\Omega f(u_j(x))\varphi(x)\,dx + \int_\Omega h(x)\varphi(x)\,dx \right| \\ \leqslant &\frac{1}{\|u_j\|_{Z,\,q}} \Big(M\ \|\varphi\|_{L^1(\Omega)} + \|h\|_{L^2(\Omega)}\, \|\varphi\|_{L^2(\Omega)} \Big) \to 0 \end{aligned} \tag{4.3.8}$$

as $j \to +\infty$, for any $\varphi \in Z$, thanks to (4.3.5) .

So, by (4.3.6)–(4.3.8) we have

$$\begin{aligned} \frac{\langle \mathcal{J}'(u_j), \varphi \rangle}{\|u_j\|_{Z,\,q}} \to &\iint_{\mathbb{R}^{2n}} \big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x-y)\,dx\,dy \\ &+ \int_\Omega q(x)u(x)\varphi(x)\,dx - \lambda \int_\Omega u(x)\varphi(x)dx \end{aligned} \tag{4.3.9}$$

as $j \to +\infty$, for any $\varphi \in Z$.

Hence, by combining (4.3.4), (4.3.5) and (4.3.9) we get

$$\iint_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big) K(x-y)\,dx\,dy + \int_\Omega q(x)u(x)\varphi(x)\,dx = \lambda \int_\Omega u(x)\varphi(x)\,dx$$

for all $\varphi \in Z$ and we deduce that u is a weak solution of problem (4.1.6).

Let us now prove that $u \not\equiv 0$ in Z. Assume, by contradiction, that $u \equiv 0$ in Z. By (4.3.4) with $\varphi = u_j / \|u_j\|_{Z,\,q}$ we get

$$\begin{aligned} &\iint_{\mathbb{R}^{2n}} \frac{|u_j(x) - u_j(y)|^2}{\|u_j\|_{Z,\,q}} K(x-y)\,dx\,dy + \int_\Omega q(x)\frac{|u_j(x)|^2}{\|u_j\|_{Z,\,q}}dx - \lambda \int_\Omega \frac{|u_j(x)|^2}{\|u_j\|_{Z,\,q}}\,dx \\ &\quad - \int_\Omega f(u_j(x))\frac{u_j(x)}{\|u_j\|_{Z,\,q}}\,dx - \int_\Omega h(x)\frac{u_j(x)}{\|u_j\|_{Z,\,q}}\,dx \to 0 \end{aligned} \tag{4.3.10}$$

as $j \to +\infty$. Moreover, by (4.1.3), (4.1.5) and (4.3.7), since $u \equiv 0$, we get

$$\begin{aligned} &\left| \int_\Omega f(u_j(x))\frac{u_j(x)}{\|u_j\|_{Z,\,q}}\,dx + \int_\Omega h(x)\frac{u_j(x)}{\|u_j\|_{Z,\,q}}\,dx \right| \\ \leqslant &M\,\frac{\|u_j\|_{L^1(\Omega)}}{\|u_j\|_{Z,\,q}} + \frac{\|h\|_{L^2(\Omega)}\ \|u_j\|_{L^2(\Omega)}}{\|u_j\|_{Z,\,q}} \to 0 \end{aligned} \tag{4.3.11}$$

as $j \to +\infty$.

Hence, by combining (4.3.10) and (4.3.11) it follows that

$$\iint_{\mathbb{R}^{2n}} \frac{|u_j(x)-u_j(y)|^2}{\|u_j\|_{Z,\,q}} K(x-y)\,dx\,dy + \int_\Omega q(x)\frac{|u_j(x)|^2}{\|u_j\|_{Z,\,q}}\,dx - \lambda\int_\Omega \frac{|u_j(x)|^2}{\|u_j\|_{Z,\,q}}\,dx \to 0$$

so that, dividing by $\|u_j\|_{Z,\,q}$, we get

$$1-\lambda\frac{\|u_j\|^2_{L^2(\Omega)}}{\|u_j\|^2_{Z,q}} \to 0 \quad \text{as} \quad j\to+\infty\,.$$

This gives $1=0$, again by (4.3.7) and the fact that $u\equiv 0$ in Z. Of course, this is a contradiction and so $u\not\equiv 0$ in Z.

In this way we have constructed a non–trivial function u solving (4.1.6), but this contradicts the non-resonance assumption $\lambda_k<\lambda<\lambda_{k+1}$. Thus, the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z and this ends the proof of Proposition 4.3.5. □

Now, we can prove the following result, whose proof is quite standard and, differently from Proposition 4.3.5, it is not affected by the resonant/non-resonant assumptions:

Proposition 4.3.6. *Let $\lambda\in\mathbb{R}$. Let $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4) *and let $f:\mathbb{R}\to\mathbb{R}$, q, $h:\Omega\to\mathbb{R}$ be functions satisfying* (4.1.2)–(4.1.5)*. Let $\{u_j\}_{j\in\mathbb{N}}$ be a bounded sequence in Z such that* (4.3.4) *holds true.*

Then, there exists $u_\infty\in Z$ such that, up to a subsequence,

$$\|u_j-u_\infty\|_{Z,\,q}\to 0 \qquad \text{as } j\to+\infty.$$

Proof. Since $\{u_j\}_{j\in\mathbb{N}}$ is bounded by assumption and Z is a reflexive space (being a Hilbert space, by [74, Lemma 7]), up to a subsequence, there exists $u_\infty\in Z$ such that u_j converges to u_∞ weakly in Z, that is

$$\begin{aligned}&\iint_{\mathbb{R}^{2n}}(u_j(x)-u_j(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy+\int_\Omega q(x)u_j(x)\varphi(x)dx\to\\&\iint_{\mathbb{R}^{2n}}(u_\infty(x)-u_\infty(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy+\int_\Omega q(x)u_\infty(x)\varphi(x)dx\end{aligned}\tag{4.3.12}$$

as $j\to+\infty$, for any $\varphi\in Z$. Moreover, by applying Lemma 1.1.2 and [20, Theorem IV.9], up to a subsequence

$$\begin{aligned}&u_j\to u_\infty \quad \text{in } L^\nu(\mathbb{R}^n) \text{ for any } \nu\in[1,2^*)\\&u_j\to u_\infty \quad \text{a.e. in } \mathbb{R}^n\end{aligned}\tag{4.3.13}$$

as $j\to+\infty$. Again 2^* is defined as in (1.1.6).

By (4.3.4) we have

$$
\begin{aligned}
&0 \leftarrow \langle \mathcal{J}'(u_j), u_j - u_\infty \rangle = \iint_{\mathbb{R}^{2n}} |u_j(x) - u_j(y)|^2 K(x-y)\, dx\, dy \\
&+ \int_\Omega q(x)\, |u_j(x)|^2\, dx - \iint_{\mathbb{R}^{2n}} (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y)) K(x-y)\, dx\, dy \\
&- \int_\Omega q(x) u_j(x) u_\infty(x) dx - \lambda \int_\Omega u_j(x)\big(u_j(x) - u_\infty(x)\big) dx \\
&- \int_\Omega f(x, u_j(x))\big(u_j(x) - u_\infty(x)\big) dx - \int_\Omega h(x)\big(u_j(x) - u_\infty(x)\big) dx
\end{aligned}
\tag{4.3.14}
$$

as $j \to +\infty$.

Also note that, by the definition of norm in Z (see formula (1.1.7)), since $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z , then $\{u_j\}_{j\in\mathbb{N}}$ does in $L^2(\Omega)$. Hence, by using the Hölder inequality, (4.1.3), (4.1.5) and (4.3.13), we get

$$
\begin{aligned}
&\Bigg| \lambda \int_\Omega u_j(x)(u_j(x) - u_\infty(x)) dx + \int_\Omega f(x, u_j(x))(u_j(x) - u_\infty(x)) dx \\
&+ \int_\Omega h(x)(u_j(x) - u_\infty(x)) dx \Bigg| \\
&\leqslant \Big(\lambda \, \|u_j\|_{L^2(\Omega)} + M \; |\Omega|^{1/2} + \|h\|_{L^2(\Omega)} \Big) \, \|u_j - u_\infty\|_{L^2(\Omega)} \to 0
\end{aligned}
\tag{4.3.15}
$$

as $j \to +\infty$.

Then, by (4.3.12), (4.3.14) and (4.3.15) we obtain

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2n}} |u_j(x) - u_j(y)|^2 K(x-y)\, dx\, dy + \int_\Omega q(x)\, |u_j(x)|^2\, dx \\
&\qquad \to \iint_{\mathbb{R}^{2n}} |u_\infty(x) - u_\infty(y)|^2 K(x-y)\, dx\, dy + \int_\Omega q(x)\, |u_\infty(x)|^2\, dx \,,
\end{aligned}
$$

that is

$$
\|u_j\|_{Z,\,q} \to \|u_\infty\|_{Z,\,q} \tag{4.3.16}
$$

as $j \to +\infty$.

Finally, we have that

$$
\begin{aligned}
&\|u_j - u_\infty\|^2_{Z,\,q} = \|u_j\|^2_{Z,\,q} + \|u_\infty\|^2_{Z,\,q} \\
&- 2 \iint_{\mathbb{R}^{2n}} (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y)) K(x-y)\, dx\, dy - 2 \int_\Omega q(x) u_j(x) u_\infty(x) dx \\
&\to 2\, \|u_\infty\|^2_{Z,\,q} - 2 \iint_{\mathbb{R}^{2n}} |u_\infty(x) - u_\infty(y)|^2 K(x-y)\, dx\, dy - 2 \int_\Omega q(x)\, |u_\infty(x)|^2\, dx = 0
\end{aligned}
$$

as $j \to +\infty$, again thanks to (4.3.12) and (4.3.16). This concludes the proof. $\square$

Proof of Theorem 4.1.1 in the non-resonant case, when $\lambda > \lambda_1$

For the proof it is enough to observe that, by Proposition 4.3.4 the functional $\mathcal{J}$ satisfies the geometric assumptions required by the Saddle Point Theorem, while by Propositions 4.3.5 and 4.3.6 it verifies the Palais–Smale compactness condition. Hence, as a consequence of the Saddle Point Theorem, $\mathcal{J}$ possesses a critical point $u \in Z$, which, of course, is a weak solution of problem (4.1.1).

4.4 The resonant case

In this section we study problem (4.1.1) in presence of a resonance, namely when λ is an eigenvalue of the operator $-\mathcal{L}_K + q$. This kind of problem is harder to solve than the non-resonant one and we have to impose further conditions on the nonlinearities in the equation. Namely, we have to assume the extra conditions (4.1.7) and (4.1.8) on f and h. Also, we use the fractional counterpart of a well-known property of the eigenvalues in the standard case of the Laplacian (see [41, 54]), that is all the eigenfunctions are almost everywhere different from zero. In the non–local framework this result, recalled in the following theorem, is a direct consequence of the unique continuation principle proved by Fall and Felli in [42, Theorem 1.4].

Theorem 4.4.1. *[42] Let Ω be a bounded domain of $\mathbb{R}^n$. Let e be an eigenfunction corresponding to the eigenvalue λ of problem* (1.2.1)*.*

Then, by setting the nodal set $\mathcal{N}$ as

$$\mathcal{N} = \{x \in \Omega : \ e(x) = 0\},$$

it follows that $\mathcal{N}$ has a zero Lebesgue measure.

Without loss of generality, in the sequel we assume that for some k, $m \in \mathbb{N}$

$$\lambda_k < \lambda = \lambda_{k+1} = \ldots = \lambda_{k+m} < \lambda_{k+m+1}, \tag{4.4.1}$$

that is we suppose that λ is an eigenvalue of $-\mathcal{L}_K + q$ with multiplicity m.

As in the non-resonant framework, here the idea is to apply the Saddle Point Theorem. Hence, also in this case, we have to check that the functional $\mathcal{J}$ satisfies the Palais–Smale condition and possesses a suitable geometric structure. The resonant assumption (4.4.1) affects both these problems (i.e. the compactness and the geometric structure of the functional), making the proof more difficult than in the non-resonant setting.

Let us start by proving the compactness condition. If compared with the non-resonant case, in the resonant one the difference lies in the proof of the boundedness of the Palais–Smale sequence. Indeed, in order to show that the Palais–Smale sequence is bounded in Z , here we have to use different arguments, since the ones used in the non-resonant case are based mainly on the fact that the parameter λ is not an eigenvalue of the operator $-\mathcal{L}_K + q$. Precisely, we will argue by contradiction and we will use the Landesman–Lazer condition (4.1.8), which will be fundamental for our arguments. Also, it will be crucial for our proof the property stated in Theorem 4.4.1.

Proposition 4.4.2. *Let λ be as in* (4.4.1) *for some $k, m \in \mathbb{N}$. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (0.0.3) *and* (0.0.4). *Moreover, let $f : \mathbb{R} \to \mathbb{R}$, q, $h : \Omega \to \mathbb{R}$ be functions satisfying* (4.1.2)–(4.1.5), (4.1.7) *and* (4.1.8). *Let $c \in \mathbb{R}$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z such that* (4.3.3) *and* (4.3.4) *hold true.*

Then, the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z.

Proof. First of all, let us write $u_j = w_j + v_j$, with $w_j \in E_\lambda$ and $v_j \in E_\lambda^\perp$, where

$$E_\lambda := span\,\{e_{k+1}, \dots, e_{k+m}\}$$

is the linear space generated by the eigenfunctions related to $\lambda = \lambda_{k+1}$ (see assumption (4.4.1)) .

In order to prove Proposition 4.4.2, it is enough to show that both the sequences $\{w_j\}_{j\in\mathbb{N}}$ and $\{v_j\}_{j\in\mathbb{N}}$ are bounded in Z .

Let us prove first that the sequence $\{v_j\}_{j\in\mathbb{N}}$ is bounded in Z . For this, note that, since $w_j \in E_\lambda$, then

$$-\mathcal{L}_K w_j + q(x) w_j = \lambda w_j$$

in the weak sense, that is for any $\varphi \in Z$

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} \big(w_j(x) - w_j(y)\big)\big(\varphi(x) - \varphi(y)\big)\, dx\, dy + \int_\Omega q(x) w_j(x) \varphi(x)\, dx \\ - \lambda \int_\Omega w_j(x)\varphi(x)\, dx = 0\,.\end{aligned} \tag{4.4.2}$$

Moreover, by linearity, for any $\varphi \in Z$

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} \big(u_j(x) - u_j(y)\big)\big(\varphi(x) - \varphi(y)\big)\, dx\, dy = \iint_{\mathbb{R}^{2n}} \big(w_j(x) - w_j(y)\big)\big(\varphi(x) - \varphi(y)\big)\, dx\, dy \\ + \iint_{\mathbb{R}^{2n}} \big(v_j(x) - v_j(y)\big)\big(\varphi(x) - \varphi(y)\big)\, dx\, dy\end{aligned} \tag{4.4.3}$$

and

$$\int_\Omega q(x) u_j(x)\varphi(x)\, dx = \int_\Omega q(x) w_j(x)\varphi(x)\, dx + \int_\Omega q(x) v_j(x)\varphi(x)\, dx\,. \tag{4.4.4}$$

Hence, as a consequence of (4.4.2)–(4.4.4) and (4.3.4) we get that for any $\varphi \in Z$

$$
\begin{aligned}
0 \leftarrow \langle \mathcal{J}'(u_j), \varphi \rangle = \\
\iint_{\mathbb{R}^{2n}} &\big(w_j(x) - w_j(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x-y)\,dx\,dy + \int_\Omega q(x) w_j(x)\varphi(x)dx \\
&+ \iint_{\mathbb{R}^{2n}} \big(v_j(x) - v_j(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x-y)\,dx\,dy + \int_\Omega q(x) v_j(x)\varphi(x)dx \\
&- \lambda \int_\Omega u_j(x)\varphi(x)\,dx - \int_\Omega f(u_j(x))\varphi(x)\,dx - \int_\Omega h(x)\varphi(x)\,dx \\
= \iint_{\mathbb{R}^{2n}} &\big(v_j(x) - v_j(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x-y)\,dx\,dy + \int_\Omega q(x) v_j(x)\varphi(x)dx \\
&- \lambda \int_\Omega v_j(x)\varphi(x)\,dx - \int_\Omega f(u_j(x))\varphi(x)\,dx - \int_\Omega h(x)\varphi(x)\,dx
\end{aligned}
\tag{4.4.5}
$$

as $j \to +\infty$.

Now, assume by contradiction that $\|v_j\|_{Z,\,q} \to +\infty$ as $j \to +\infty$. Arguing exactly as in the proof of Proposition 4.3.5 one shows that $v_j / \|v_j\|_{Z,\,q}$ converges weakly in Z to an eigenfunction v relative to λ.

Of course $v \in E_\lambda \setminus \{0\}$, being an eigenfunction. On the other hand, since

$$v_j \in E_\lambda^\perp = \overline{span\{e_1, \dots, e_k, e_{k+m+1}, \dots\}},$$

then $v \in E_\lambda^\perp$. This leads to a contradiction since $v \not\equiv 0$ and $v \in E_\lambda \cap E_\lambda^\perp = \{0\}$. Then, $\{v_j\}_{j\in\mathbb{N}}$ is bounded in Z.

Now, it remains to prove that $\{w_j\}_{j\in\mathbb{N}}$ is bounded in Z. Also in this case we argue by contradiction and assume that

$$\|w_j\|_{Z,\,q} \to +\infty \tag{4.4.6}$$

as $j \to +\infty$.

Since E_λ is finite dimensional, there exists $w \in E_\lambda$ such that, up to a subsequence, $w_j / \|w_j\|_{Z,\,q}$ converges to w strongly in Z as $j \to +\infty$. Moreover, by applying Lemma 1.1.2 and [20, Theorem IV.9], up to a subsequence

$$
\begin{aligned}
&\frac{w_j}{\|w_j\|_{Z,\,q}} \to w \quad \text{in } L^\nu(\mathbb{R}^n) \text{ for any } \nu \in [1, 2^*) \\
&\frac{w_j}{\|w_j\|_{Z,\,q}} \to w \quad \text{a.e. in } \mathbb{R}^n
\end{aligned}
\tag{4.4.7}
$$

as $j \to +\infty$. The exponent 2^* is given in (1.1.6).

Note also that, since $w \in E_\lambda$, for any $\varphi \in Z$ we get

$$
\begin{aligned}
\iint_{\mathbb{R}^{2n}} \big(w(x) - w(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x-y)\,dx\,dy &+ \int_\Omega q(x) w(x)\varphi(x)dx \\
&= \lambda \int_\Omega w(x)\varphi(x)\,dx\,,
\end{aligned}
\tag{4.4.8}
$$

that is w is an eigenfunction of problem (4.1.6). Hence, by Theorem 4.4.1, the function w is almost everywhere different from zero, say

$$w(x) \neq 0 \text{ for any } x \in \Omega \setminus \mathcal{N}, \tag{4.4.9}$$

where $\mathcal{N} \subset \Omega$ has zero Lebesgue measure.

So, by using (4.4.6), (4.4.7), the fact that $\{v_j\}_{j\in\mathbb{N}}$ is bounded[1] in Z and (4.4.9), for a.e. $x \in \Omega$ we get

$$u_j(x) = w_j(x) + v_j(x) = \|w_j\|_{Z,\,q} \frac{w_j(x)}{\|w_j\|_{Z,\,q}} + v_j(x) \to \begin{cases} +\infty & \text{for a.e. } x \in \{w > 0\} \\ -\infty & \text{for a.e. } x \in \{w < 0\} \end{cases} \tag{4.4.10}$$

as $j \to +\infty$.

Let us define the function $f_\infty : \Omega \to \mathbb{R}$ as

$$f_\infty(x) := \begin{cases} f_r & \text{if } x \in \{w > 0\} \\ f_l & \text{if } x \in \{w < 0\}, \end{cases}$$

where f_l and f_r were introduced in (4.1.7). Note that f_∞ is well defined, thanks to (4.4.9).

By (4.1.2), (4.4.10) and the definition of f_∞ it follows that

$$f(u_j(x)) \to f_\infty(x) \quad \text{a.e. } x \in \Omega,$$

while, by (4.1.3), the fact that Ω is bounded and the Lebesgue Dominated Convergence Theorem we have

$$f(u_j) \to f_\infty \quad \text{in } L^\nu(\Omega) \text{ for any } \nu \in [1, +\infty) \tag{4.4.11}$$

as $j \to +\infty$.

Hence, by combining (4.4.5) with $\varphi = w$, (4.4.8) with $\varphi = v_j$ and (4.4.11), we obtain

$$\int_\Omega f_\infty(x) w(x) dx + \int_\Omega h(x) w(x) dx = 0,$$

namely, writing $w(x) = w^+(x) - w^-(x)$ and taking into account the definition of f_∞,

$$\int_\Omega h(x) w(x) dx = f_l \int_\Omega w^-(x) dx - f_r \int_\Omega w^+(x) dx.$$

This contradicts assumption (4.1.8). Thus, the sequence $\{w_j\}_{j\in\mathbb{N}}$ has to be bounded in Z and this concludes the proof of Proposition 4.4.2. □

[1] We stress that the boundedness in Z imply the convergence of v_j to some v in $L^1(\mathbb{R}^n)$ and a.e. – in particular, $|v(x)| \neq +\infty$ for a.e. $x \in \Omega$.

As a consequence of Proposition 4.3.6 and Proposition 4.4.2, the functional $\mathcal{J}$ has the Palais–Smale compactness property.

Finally, we prove that the functional $\mathcal{J}$ has the geometric feature required by the Saddle Point Theorem. As we said above, the resonance assumption affects also the proof of the particular geometric structure of the functional $\mathcal{J}$, making it more difficult than in the non-resonant setting. Indeed, here we can not use the arguments performed in the non-resonant framework, but we have to argue in a different way. For this, we will make use of Theorem 4.4.1 and of the Landesman–Lazer condition (4.1.8), which will be both crucial in the proof of the following proposition:

Proposition 4.4.3. *Let* λ *be as in* (4.4.1) *for some* $k, m \in \mathbb{N}$. *Let* $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ *satisfy assumptions* (0.0.3) *and* (0.0.4). *Moreover, let* $f : \mathbb{R} \to \mathbb{R}$, q, $h : \Omega \to \mathbb{R}$ *be functions satisfying* (4.1.2)*–*(4.1.5)*,* (4.1.7) *and* (4.1.8)*.*

Then, the functional $\mathcal{J}$ *verifies*

$$\inf_{u \in \mathbb{P}_{k+1}} \mathcal{J}(u) > -\infty. \tag{4.4.12}$$

Proof. In order to prove Proposition 4.4.3, we argue by contradiction and assume that there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $\mathbb{P}_{k+1}$ such that

$$\mathcal{J}(u_j) \to -\infty, \tag{4.4.13}$$

as $j \to +\infty$.

First of all, note that, by (4.4.1) and the orthogonality properties of $\{e_1, \ldots, e_k, \ldots\}$ (see Proposition 1.2.2-(vi)), we can write $\mathbb{P}_{k+1}$ as follows

$$\mathbb{P}_{k+1} = E_\lambda \oplus \mathbb{P}_{k+m+1}$$

(recall that $E_\lambda := span\{e_{k+1}, \ldots e_{k+m}\}$).

Then, for any $j \in \mathbb{N}$ the function u_j can be written as

$$u_j = w_j + v_j\,, \tag{4.4.14}$$

with $w_j \in E_\lambda$ and $v_j \in \mathbb{P}_{k+m+1}$, so that w_j and v_j are orthogonal both in Z and in $L^2(\Omega)$, again thanks to Proposition 1.2.2-(vi).

From now on we proceed by steps.

Claim 1. *The following assertion holds true:*

$$\|w_j\|_{Z,\,q} \to +\infty$$

as $j \to +\infty$.

Proof. First of all, since $w_j \in E_\lambda$, note that

$$\iint_{\mathbb{R}^{2n}} |w_j(x) - w_j(y)|^2 K(x-y)dx\,dy + \int_\Omega q(x)\,|w_j(x)|^2\,dx = \lambda \int_\Omega |w_j(x)|^2\,dx.$$

So, as a consequence of this, of (4.4.14), of the orthogonality of the w_j and v_j, of Lemma 4.2.2 (here applied in $\mathbb{P}_{k+m+1}$) and of the positivity of λ, we get

$$\begin{aligned}
\mathcal{J}(u_j) &= \frac{1}{2}\|u_j\|^2_{Z,\,q} - \frac{\lambda}{2}\int_\Omega |u_j(x)|^2\,dx - \int_\Omega F(u_j(x))dx - \int_\Omega h(x)u_j(x)dx \\
&= \frac{1}{2}\|w_j\|^2_{Z,\,q} + \frac{1}{2}\|v_j\|^2_{Z,\,q} - \frac{\lambda}{2}\|w_j\|^2_{L^2(\Omega)} - \frac{\lambda}{2}\|v_j\|^2_{L^2(\Omega)} - \int_\Omega F(u_j(x))dx \\
&\quad - \int_\Omega h(x)u_j(x)dx \\
&\geqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+m+1}}\right)\|v_j\|^2_{Z,\,q} - \int_\Omega F(u_j(x))dx - \int_\Omega h(x)u_j(x)dx \\
&\geqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+m+1}}\right)\|v_j\|^2_{Z,\,q} - \tilde{C}\,\|u_j\|_{Z,\,q} \\
&\geqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+m+1}}\right)\|v_j\|^2_{Z,\,q} - \tilde{C}\,\|v_j\|_{Z,\,q} - \tilde{C}\,\|w_j\|_{Z,\,q}\,,
\end{aligned} \tag{4.4.15}$$

also thanks to Lemma 4.2.3 . So, by combining (4.4.13) and (4.4.15) we get

$$\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+m+1}}\right)\|v_j\|^2_{Z,\,q} - \tilde{C}\,\|v_j\|_{Z,\,q} - \tilde{C}\,\|w_j\|_{Z,\,q} \to -\infty \tag{4.4.16}$$

which implies necessarily that

$$\|w_j\|_{Z,\,q} \to +\infty \;\text{ as }\; j \to +\infty\,,$$

since $\lambda = \lambda_{k+1} < \lambda_{k+m+1}$ by (4.4.1) . Hence, Claim 1 is proved. □

Now, since E_λ is finite dimensional, there exists $w \in E_\lambda$ such that, up to a subsequence,

$$w_j/\,\|w_j\|_{Z,\,q} \to w \;\text{ strongly in }\; Z \tag{4.4.17}$$

as $j \to +\infty$. Note that $w \not\equiv 0$, since $\|w\| = 1$. Also, w is an eigenfunction of problem (4.1.6) and so, by Theorem 4.4.1 w is almost everywhere different from zero, say

$$w(x) \neq 0 \;\text{ for any }\; x \in \Omega \setminus \mathcal{N}\,, \tag{4.4.18}$$

where $\mathcal{N} \subset \Omega$ has zero Lebesgue measure.

Moreover, by applying Lemma 1.1.2 and [20, Theorem IV.9], up to a subsequence, we also have

$$\begin{aligned}
&\frac{w_j}{\|w_j\|_{Z,\,q}} \to w \quad \text{in } L^\nu(\mathbb{R}^n) \text{ for any } \nu \in [1, 2^*) \\
&\frac{w_j}{\|w_j\|_{Z,\,q}} \to w \quad \text{a.e. in } \mathbb{R}^n
\end{aligned} \tag{4.4.19}$$

as $j \to +\infty$. Again here and in the sequel 2^* is the exponent given in (1.1.6).

Now, assume that $\|v_j\|_{Z,\,q} \neq 0$ for j sufficiently large. We will discuss the case when $\|v_j\|_{Z,\,q} = 0$ later on.

Again by applying Lemma 1.1.2 and [20, Theorem IV.9] we can say that there exists $v \in Z$ such that, up to a subsequence

$$\begin{aligned} &\frac{v_j}{\|v_j\|_{Z,\,q}} \to v \quad \text{in } L^\nu(\mathbb{R}^n) \ \text{ for any } \nu \in [1, 2^*) \\ &\frac{v_j}{\|v_j\|_{Z,\,q}} \to v \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{4.4.20}$$

as $j \to +\infty$.

Now, let us continue with some claims.

Claim 2. *The following assertion holds true:*

$$\frac{\|w_j\|_{Z,\,q}}{\|v_j\|_{Z,\,q}} \to +\infty$$

as $j \to +\infty$.

Proof. If $\{\|v_j\|_{Z,\,q}\}_{j\in\mathbb{N}}$ was bounded, then Claim 2 would follow by Claim 1. Assume that $\|v_j\|_{Z,\,q} \to +\infty$ as $j \to +\infty$. Writing (4.4.16) as follows

$$\|v_j\|_{Z,\,q}\left(\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+m+1}}\right)\|v_j\|_{Z,\,q} - \tilde{C} - \tilde{C}\frac{\|w_j\|_{Z,\,q}}{\|v_j\|_{Z,\,q}}\right) \to -\infty\,,$$

we would get necessarily that Claim 2 holds true, by assumption (4.4.1). This concludes the proof of Claim 2. □

Claim 3. *The following assertion holds true:*

$$\frac{F(u_j(x))}{\|w_j\|_{Z,\,q}} \to w(x) f_\infty(x) \quad \textit{a.e. } x \in \Omega$$

as $j \to +\infty$, *where* $f_\infty : \Omega \to \mathbb{R}$ *is the function defined as*

$$f_\infty(x) := \begin{cases} f_r & \textit{if } x \in \{w > 0\} \\ f_l & \textit{if } x \in \{w < 0\}\,, \end{cases} \tag{4.4.21}$$

with f_l *and* f_r *given in* (4.1.7) *and* w *as in* (4.4.17).

Proof. To prove this we first observe that

$$\lim_{t\to-\infty}\frac{F(t)}{t} = f_l \quad \text{and} \quad \lim_{t\to+\infty}\frac{F(t)}{t} = f_r. \tag{4.4.22}$$

We prove the identity for f_r, since the one for f_l is alike. If $f_r \neq 0$, we can use de l'Hôpital Theorem and get (4.4.22). On the other hand, when $f_r = 0$, for any $\varepsilon > 0$ there exists $T > 0$ such that $|f(t)| < \varepsilon$ for $t > T$. So, by (4.1.3) for $t > T$ it follows that

$$\left|\frac{F(t)}{t}\right| = \left|\frac{1}{t}\Big(\int_0^T f(\tau)\,d\tau + \int_T^t f(\tau)\,d\tau\Big)\right| \leqslant M\frac{T}{t} + \varepsilon\frac{(t-T)}{t}.$$

Passing to the limit as $t \to +\infty$ and as $\varepsilon \to 0$ we obtain (4.4.22) in this case too.

By (4.4.14), Claims 1 and 2, (4.4.18), (4.4.19) and (4.4.20) for a.e. $x \in \Omega$ we get

$$\begin{aligned} u_j(x) &= w_j(x) + v_j(x) \\ &= \|w_j\|_{Z,\,q}\left(\frac{w_j(x)}{\|w_j\|_{Z,\,q}} + \frac{\|v_j\|_{Z,\,q}}{\|w_j\|_{Z,\,q}}\frac{v_j(x)}{\|v_j\|_{Z,\,q}}\right) \\ &\to \begin{cases} +\infty & \text{for a.e. } x \in \{w > 0\} \\ -\infty & \text{for a.e. } x \in \{w < 0\}, \end{cases} \end{aligned} \tag{4.4.23}$$

as $j \to +\infty$. In particular, fixed any $x \in \Omega$, we have that $u_j(x) \neq 0$ for large j.

Now, again by (4.4.14) and Claim 1, we can write

$$\frac{F(u_j(x))}{\|w_j\|_{Z,\,q}} = \left(\frac{v_j(x)}{\|w_j\|_{Z,\,q}} + \frac{w_j(x)}{\|w_j\|_{Z,\,q}}\right)\frac{F(u_j(x))}{u_j(x)}. \tag{4.4.24}$$

By (4.4.22) and (4.4.23)

$$\frac{F(u_j(x))}{u_j(x)} \to \begin{cases} f_r & \text{for a.e. } x \in \{w > 0\} \\ f_l & \text{for a.e. } x \in \{w < 0\}, \end{cases}$$

that is

$$\frac{F(u_j(x))}{u_j(x)} \to f_\infty(x) \quad \text{a.e } x \in \Omega \tag{4.4.25}$$

as $j \to +\infty$, where f_∞ is given in (4.4.21) (this function is well defined, thanks to (4.4.18)).

Moreover, by Claim 2 and (4.4.20) it follows that

$$\frac{v_j(x)}{\|w_j\|_{Z,\,q}} = \frac{\|v_j\|_{Z,\,q}}{\|w_j\|_{Z,\,q}}\frac{v_j(x)}{\|v_j\|_{Z,\,q}} \to 0 \quad \text{a.e. } x \in \mathbb{R}^n \tag{4.4.26}$$

as $j \to +\infty$. So, by combining (4.4.24)–(4.4.26) and by using also (4.4.19), we get the assertion of Claim 3. □

Claim 4. *The following assertion holds true:*

$$\frac{F(u_j)}{\|w_j\|_{Z,\,q}} \to w f_\infty \quad \textit{in } L^1(\Omega)$$

as $j \to +\infty$, *where* w *is as in* (4.4.17) *and* f_∞ *is defined as in* (4.4.21).

Proof. Since $\left\{u_j/\left\|u_j\right\|_{Z,\,q}\right\}_{j\in\mathbb{N}}$ is bounded in Z, as usual by applying Lemma 1.1.2 and [20, Theorem IV.9], up to a subsequence, it converges strongly in $L^1(\Omega)$ and there exists $\kappa \in\ L^1(\Omega)$ such that for any $j \in \mathbb{N}$

$$\frac{|u_j(x)|}{\left\|u_j\right\|_{Z,\,q}} \leqslant \kappa(x) \quad \text{a.e. } x \in \Omega. \tag{4.4.27}$$

Moreover, by the orthogonality properties of v_j and w_j we get

$$\frac{\|u_j\|_{Z,\,q}}{\|w_j\|_{Z,\,q}} = 1 + \frac{\|v_j\|_{Z,\,q}}{\|w_j\|_{Z,\,q}}\,,$$

so that, by Claim 2 it follows that for any $j \in \mathbb{N}$

$$\frac{\left\|u_j\right\|_{Z,\,q}}{\left\|w_j\right\|_{Z,\,q}} \leqslant C$$

for some positive constant C.

As a consequence of this, (4.4.27) and (4.1.3) we get a.e. $x \in \Omega$

$$\frac{|F(u_j(x))|}{\left\|w_j\right\|_{Z,\,q}} \leqslant M\,\frac{|u_j(x)|}{\left\|w_j\right\|_{Z,\,q}} = M\,\frac{\|u_j\|_{Z,\,q}}{\left\|w_j\right\|_{Z,\,q}}\frac{|u_j(x)|}{\left\|u_j\right\|_{Z,\,q}} \leqslant \bar{C}\kappa(x) \in L^1(\Omega)$$

for a suitable positive constant $\bar{C}$. Then, the Lebesgue Dominated Convergence Theorem and Claim 3 yield the assertion of Claim 4. □

Claim 5. *The following assertion holds true:*

$$\lim_{j\to+\infty}\left(\int_\Omega \frac{F(u_j(x))}{\left\|w_j\right\|_{Z,\,q}}\,dx + \int_\Omega h(x)\frac{u_j(x)}{\left\|w_j\right\|_{Z,\,q}}\,dx\right) < 0\,.$$

Proof. First of all, note that

$$\frac{u_j}{\left\|w_j\right\|_{Z,\,q}} = \frac{w_j+v_j}{\left\|w_j\right\|_{Z,\,q}} = \frac{w_j}{\left\|w_j\right\|_{Z,\,q}} + \frac{v_j}{\left\|v_j\right\|_{Z,\,q}}\frac{\|v_j\|_{Z,\,q}}{\left\|w_j\right\|_{Z,\,q}} \to w \quad \text{in } L^2(\Omega)\,,$$

as $j \to +\infty$, thanks to (4.4.14), (4.4.19), (4.4.20) and Claim 2.

As a consequence of this and by Claim 4 and (4.4.21) we have

$$\begin{aligned}\lim_{j\to+\infty}&\left(\int_\Omega \frac{F(u_j(x))}{\left\|w_j\right\|_{Z,\,q}}\,dx + \int_\Omega h(x)\frac{u_j(x)}{\left\|w_j\right\|_{Z,\,q}}\,dx\right)\\ &= \int_\Omega f_\infty(x)w(x)dx + \int_\Omega h(x)w(x)dx\\ &= f_r\int_\Omega w^+(x)dx - f_l\int_\Omega w^-(x)dx + \int_\Omega h(x)w(x)dx < 0\,,\end{aligned} \tag{4.4.28}$$

since (4.1.8) holds true. This ends the proof of Claim 5. □

Now, we can conclude the proof of Proposition 4.4.3 . Indeed, arguing as (4.4.15) and using (4.4.1), we get

$$
\begin{aligned}
\mathcal{J}(u_j) &\geqslant \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+m+1}}\right)\|v_j\|^2_{Z,\,q} - \int_\Omega F(u_j(x))dx - \int_\Omega h(x)u_j(x)dx \\
&\geqslant -\|w_j\|_{Z,\,q}\left(\int_\Omega \frac{F(u_j(x))}{\|w_j\|_{Z,\,q}}\,dx + \int_\Omega h(x)\frac{u_j(x)}{\|w_j\|_{Z,\,q}}\,dx\right)\,,
\end{aligned}
$$

so that, by Claim 1 and Claim 5, we deduce

$$
\mathcal{J}(u_j) \to +\infty \quad \text{as } j \to +\infty\,,
$$

which contradicts (4.4.13). Hence, Proposition 4.4.3 holds true in the case when $\|v_j\|_{Z,\,q} \neq 0$ for j large enough.

Finally, it remains to consider the case when $\|v_j\|_{Z,\,q} = 0$ for j sufficiently large (up to a subsequence). In this setting, using the same arguments as above, the proof can be repeated in a simpler way. For the sake of clarity and for reader's convenience we prefer to give full details.

Since $\|v_j\|_{Z,\,q} = 0$ for j sufficiently large, it easily follows that

$$
v_j \to 0 \;\text{ in }\; Z \tag{4.4.29}
$$

as $j \to +\infty$. Hence, by Lemma 1.1.2 and [20, Theorem IV.9] up to a subsequence

$$
\begin{aligned}
&v_j \to 0 \quad \text{in } L^\nu(\mathbb{R}^n) \;\text{ for any } \nu \in [1, 2^*) \\
&v_j \to 0 \quad \text{a.e. in } \mathbb{R}^n
\end{aligned} \tag{4.4.30}
$$

as $j \to +\infty$.

As a consequence of this and by (4.4.14), (4.4.19) and Claim 1, we get that

$$
\frac{u_j}{\|w_j\|_{Z,q}} = \frac{w_j}{\|w_j\|_{Z,q}} + \frac{v_j}{\|w_j\|_{Z,q}} \to w \;\text{ in }\; L^\nu(\mathbb{R}^n) \;\text{ for any } \nu \in [1, 2^*)\,, \tag{4.4.31}
$$

so that

$$
\frac{u_j(x)}{\|w_j\|_{Z,q}} \to w(x) \quad \text{a.e } x \in \Omega
$$

as $j \to +\infty$, and, for any $j \in \mathbb{N}$ and a.e. $x \in \Omega$

$$
\frac{|u_j(x)|}{\|w_j\|_{Z,q}} \leqslant \kappa_\nu(x) \tag{4.4.32}
$$

for some $\kappa_\nu \in L^\nu(\Omega)$.

Also, again by (4.4.14), Claim 1, (4.4.19) and (4.4.30), we deduce that a.e. $x \in \Omega$

$$
u_j(x) = w_j(x) + v_j(x) = \|w_j\|_{Z,q}\frac{w_j(x)}{\|w_j\|_{Z,q}} + v_j(x) \to \begin{cases} +\infty & \text{if } x \in \{w > 0\} \\ -\infty & \text{if } x \in \{w < 0\}, \end{cases} \tag{4.4.33}
$$

as $j \to +\infty$, thanks to (4.4.18) .

Hence, by (4.4.22) and (4.4.33), also in this case we get

$$\frac{F(u_j(x))}{u_j(x)} \to f_\infty(x) \quad \text{a.e. } x \in \Omega \tag{4.4.34}$$

as $j \to +\infty$, where f_∞ is the function defined in (4.4.21) .

Now, we have that

$$\frac{F(u_j(x))}{\|w_j\|_{Z,\,q}} = \left(\frac{v_j(x)}{\|w_j\|_{Z,\,q}} + \frac{w_j(x)}{\|w_j\|_{Z,\,q}} \right) \frac{F(u_j(x))}{u_j(x)} \to w(x) f_\infty(x) \ \text{ a.e. } x \in \Omega\,, \tag{4.4.35}$$

as $j \to +\infty$, thanks to (4.4.14), (4.4.19), (4.4.30) and (4.4.34) .

Furthermore, by (4.1.3) and (4.4.32) we get that a.e. $x \in \Omega$ and for any $j \in \mathbb{N}$

$$\frac{|F(u_j(x))|}{\|w_j\|_{Z,\,q}} \leqslant M \, \frac{|u_j(x)|}{\|w_j\|_{Z,\,q}} \leqslant M \, \kappa_1(x) \in L^1(\Omega)\,,$$

so that, using also (4.4.35), we obtain

$$\frac{F(u_j)}{\|w_j\|_{Z,\,q}} \to w f_\infty \ \text{ in } L^1(\Omega) \tag{4.4.36}$$

as $j \to +\infty$.

Now, with (4.1.8), (4.4.31) (here used with $\nu = 2 < 2^*$) and (4.4.36), arguing as in Claim 5, we can show that

$$\lim_{j \to +\infty} \left(\int_\Omega \frac{F(u_j(x))}{\|w_j\|_{Z,\,q}} \, dx + \int_\Omega h(x) \frac{u_j(x)}{\|w_j\|_{Z,\,q}} \, dx \right) < 0\,.$$

Thus, the conclusion of Proposition 4.4.3 follows as in the previous case. This ends the proof of Proposition 4.4.3 . □

Finally, we are ready to prove Theorem 4.1.1, in the resonant case.

Proof of Theorem 4.1.1 in the resonant setting

First of all, let us check the geometric structure of the functional $\mathcal{J}$. For this, let

$$I = \inf_{u \in \mathbb{P}_{k+1}} \mathcal{J}(u)\,.$$

By Proposition 4.4.3 and the fact that $\mathcal{J} \not\equiv +\infty$, we have that $I \in \mathbb{R}$. Moreover, by Lemma 4.3.2, there exists $R > 0$ such that for any $u \in \mathbb{H}_k$ with $\|u\|_{Z,\,q} \geqslant R$ it holds true that

$$\mathcal{J}(u) < -|I| \leqslant I\,.$$

Then, as a consequence of this, we get

$$\sup_{\substack{u\in\mathbb{H}_k\\ \|u\|_{Z,\,q}=R}} \mathcal{J}(u) \leqslant \sup_{\substack{u\in\mathbb{H}_k\\ \|u\|_{Z,\,q}\geqslant R}} \mathcal{J}(u) < I = \inf_{u\in\mathbb{P}_{k+1}} \mathcal{J}(u)\,,$$

that is $\mathcal{J}$ has the geometry required by the Saddle Point Theorem (see [67, Theorem 4.6]).

Finally, by Proposition 4.3.6 (which holds true for any $\lambda \in \mathbb{R}$) and Proposition 4.4.2, the functional $\mathcal{J}$ satisfies the Palais–Smale condition.

Hence, we can make use of the Saddle Point Theorem in order to obtain a critical point $u \in Z$ of $\mathcal{J}$. This concludes the proof of Theorem 4.1.1 in the resonant case.

Chapter 5

A Kirchhoff type problem

5.1 Introduction

The purpose of this chapter is to investigate the existence of non-negative solutions for a Kirchhoff type problem driven by a non–local integrodifferential operator, that is

$$\begin{cases} -M\left(\|u\|_Z^2\right)\,\mathcal{L}_K u = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{5.1.1}$$

where $n > 2s$ with $s \in (0,1)$, $2^* = 2n/(n-2s)$, λ is a positive parameter, $\Omega \subset \mathbb{R}^n$ is an open, bounded set, M and f are two continuous functions whose properties will be introduced later and $\mathcal{L}_K$ is the non–local operator defined as in (0.0.2) whose kernel $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ still satisfies conditions (0.0.3) and (0.0.4).

The particularity of this kind of problem is due to the non–local behaviour of the term $M\left(\|u\|_Z^2\right)$. For this, the equation in (5.1.1) is no longer a pointwise identity, so the treatment of this problem presents such mathematical difficulties which make the study particularly interesting.

This kind of problem has been widely studied in recent years: we refer to [2, 34, 44, 45, 59] for Kirchhoff problems involving the classical Laplace operator, to [8, 10, 13, 35] for the p-Laplacian case and to [93] for a Kirchhoff operator with critical exponent. In [2, 44] the approach used is mainly based on the variational method joined with a concentration compactness argument, as in the present chapter. In particular in [44], the authors use a truncation argument to control the non–local term M. Also in [34, 59] a variational method is still used, but the main difference compared with the previous papers is that the critical Kirchhoff problem is set in all $\mathbb{R}^N$, resulting in lack of compactness of the embedding of $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$. In [8, 10, 13, 35, 93] the so-called *degenerate case* was taken into account in the elliptic case: in such a case, the function M verifies $M(0) = 0$, while in this chapter M is bounded below by a positive constant, that is substantially the *non-degenerate case*. While non-degenerate problems have been widely studied, only few attempts have been made to cover also the degenerate case [1].

Inspired by the above articles, in this chapter we would like to investigate the existence of a non–trivial solution for problem (5.1.8), by extending the results dealt with in [44] for the classical Laplacian case

In view of our problem, we suppose that $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ verifies the following conditions:

$$M \text{ is an increasing and continuous function;} \tag{5.1.2}$$

$$\text{there exists } m_0 > 0 \text{ such that } M(t) \geqslant m_0 = M(0) \text{ for any } t \in \mathbb{R}^+ . \tag{5.1.3}$$

A typical example for M is given by $M(t) = m_0 + tb$ with $b \geqslant 0$.

Also, we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies:

$$\lim_{|t|\to 0} \frac{f(x,t)}{t} = 0 \text{ uniformly in } x \in \Omega; \tag{5.1.4}$$

$$\text{there exists } q \in (2, 2^*) \text{ such that } \lim_{|t|\to\infty} \frac{f(x,t)}{t^{q-1}} = 0 \text{ uniformly in } x \in \Omega; \tag{5.1.5}$$

$$\begin{aligned} &\text{there exists } \sigma \in (2, 2^*) \text{ such that for any } x \in \Omega \text{ and } t > 0 \\ &0 < \sigma F(x,t) = \sigma \int_0^t f(x,\tau) d\tau \leqslant t f(x,t) . \end{aligned} \tag{5.1.6}$$

We would observe that conditions (5.1.4)–(5.1.5) give a subcritical growth for f since $q < 2^*$. While assumption (5.1.6) represents the well–known Ambrosetti–Rabinowitz superlinear condition (see [6]), with also the restriction $\sigma < 2^*$.

Moreover, since we intend to find non-negative solutions, we assume this further condition for f

$$f(x,t) = 0 \quad \text{for any } x \in \Omega \text{ and } t \leqslant 0. \tag{5.1.7}$$

An example of a function satisfying the conditions (5.1.4)–(5.1.7) is given by

$$f(x,t) = \begin{cases} 0 & \text{if } t \leqslant 0, \\ a(x)t^{q-1} & \text{if } 0 < t < 1, \\ a(x)t^{q_1-1} & \text{if } t \geqslant 1, \end{cases}$$

with $2 < q_1 < q$, $a \in L^\infty(\Omega)$ and $a(x) > 0$ for any $x \in \Omega$.

[1] After this thesis was concluded, we complete the study of problem (5.1.1) in [9], by covering also the degenerate case.

Now, before stating our main result we recall the weak formulation of (5.1.1), given by the following problem

$$
\begin{cases} M(\|u\|_Z^2) \displaystyle\iint_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy \\ \qquad = \lambda \displaystyle\int_\Omega f(x,u(x))\varphi(x)\,dx + \int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)dx \quad \forall\varphi\in Z \\ u\in Z. \end{cases} \tag{5.1.8}
$$

Thanks to our assumptions on Ω, M, f and K, all the integrals in (5.1.8) are well defined if u, $\varphi \in Z$.

Theorem 5.1.1. *Let $s\in(0,1)$, $n>2s$ and Ω be a bounded open subset of $\mathbb{R}^n$. Assume that the functions $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$, $M:\mathbb{R}_0^+\to\mathbb{R}^+$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy conditions* (0.0.3) *and* (0.0.4) *and* (5.1.2)–(5.1.7).

Then there exists $\lambda^*>0$ such that problem (5.1.1) *has a non–trivial weak solution u_λ for all $\lambda\geqslant\lambda^*$. Such solution also verifies*

$$
\lim_{\lambda\to\infty}\|u_\lambda\|_Z = 0. \tag{5.1.9}
$$

The chapter is organized as follows. In Section 5.2 we introduce a truncated problem whose weak solution will be a weak solution of the original problem (5.1.1). Section 5.3 is devoted to the study of our main results, by proving first some technical lemmas and the existence of a solution for the truncated problem.

5.2 A truncated problem

In order to prove Theorem 5.1.1 we need to control the non–local term $M\left(\|u\|_Z^2\right)$. For this, inspired by the truncation argument used in [3, 44], we first study an auxiliary truncated problem. Clearly, here we are assuming that M is unbounded, otherwise the truncation on M is not necessary. Given σ as in (5.1.6) and $a\in\mathbb{R}$ such that $m_0<a<\dfrac{\sigma}{2}m_0$, by (5.1.2) there exists $t_0>0$ such that $M(t_0)=a$. Now, by setting

$$
M_a(t) := \begin{cases} M(t) & \text{if } 0\leqslant t\leqslant t_0,\\ a & \text{if } t\geqslant t_0, \end{cases}
$$

we can introduce the following auxiliary problem

$$
\begin{cases} -M_a(\|u\|_Z^2)\mathcal{L}_K u = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega \end{cases} \tag{5.2.1}
$$

with f and λ defined as in Problem (5.1.1). By (5.1.2) we note also that

$$
M_a(t)\leqslant a \quad \text{for any } t\geqslant 0. \tag{5.2.2}
$$

As we show in the sequel, the proof of Theorem 5.1.1 is based on a careful study of the weak solution of problem (5.2.1). For this, we first prove the following result.

Theorem 5.2.1. *Let $s \in (0,1)$, $n > 2s$ and Ω be a bounded open subset of $\mathbb{R}^n$. Assume that the functions $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$, $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy conditions* (0.0.3) *and* (0.0.4) *and* (5.1.2)–(5.1.7).

Then there exists $\lambda_0 > 0$ such that problem (5.2.1) *has a non–trivial weak solution, for all $\lambda \geqslant \lambda_0$ and for all $a \in (m_0, \frac{\sigma}{2} m_0)$.*

5.3 Variational formulation and main results

For the proof of Theorem 5.2.1, we observe that problem (5.2.1) has a variational structure, indeed it is the Euler-Lagrange equation of the functional $\mathcal{J}_{a,\lambda} : Z \to \mathbb{R}$ defined as follows

$$\mathcal{J}_{a,\lambda}(u) = \frac{1}{2}\widehat{M_a}(\|u\|_Z^2) - \lambda \int_\Omega F(x,u(x))dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*} dx .$$

where

$$\widehat{M_a}(t) = \int_0^t M_a(\tau) d\tau.$$

Note that the functional $\mathcal{J}_{a,\lambda}$ is Fréchet differentiable in $u \in Z$ and for any $\varphi \in Z$

$$\begin{aligned}\left\langle \mathcal{J}'_{a,\lambda}(u), \varphi \right\rangle &= M_a(\|u\|_Z^2) \iint_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big) K(x-y)\, dx\, dy \\ &\quad - \lambda \int_\Omega f(x,u(x))\varphi(x)\, dx - \int_\Omega |u(x)|^{2^*-2} u(x)\varphi(x) dx .\end{aligned} \tag{5.3.1}$$

Thus critical points of $\mathcal{J}_{a,\lambda}$ are weak solutions of problem (5.2.1). Unlike previous chapters, the nonlinearity appearing on the right-hand side of main problems (5.1.1) and (5.2.1) is not asymptotically. For this, here we change variational theorem by applying the Mountain Pass Theorem (see [67, 86]) to prove Theorem 5.2.1. Thus, as usual, we have to check that $\mathcal{J}_{a,\lambda}$ posses a suitable geometric structure (as stated e.g. in [86, Theorem 6.1]) and it satisfies the Palais–Smale condition (see for instance [86, page 70]).

To prove all these properties, we need appropriate lower and upper bounds for f and its primitive. Now, assumptions (5.1.4) and (5.1.5) give subcritical growths. That is, for any $\varepsilon > 0$ there exists $c_\varepsilon = c(\varepsilon) > 0$ such that

$$|f(x,t)| \leqslant 2\varepsilon\, |t| + q c_\varepsilon\, |t|^{q-1} \quad \text{for any } (x,t) \in \Omega \times \mathbb{R} \tag{5.3.2}$$

by considering also (5.1.7) and so for the primitive

$$|F(x,t)| \leqslant \varepsilon\, |t|^2 + c_\varepsilon\, |t|^q \quad \text{for any } (x,t) \in \Omega \times \mathbb{R}. \tag{5.3.3}$$

Finally, (5.1.6) implies that $F(x,t) \geqslant c(x)t^\sigma$ for all $(x,t) \in \Omega \times [1,\infty)$, where $c(x) = F(x,1)$ is in $L^\infty(\Omega)$ by (5.3.3), with $\varepsilon = t = 1$. In conclusion, for any $(x,t) \in \Omega \times \mathbb{R}^+$

$$F(x,t) \geqslant c(x)t^\sigma - C(x), \quad C(x) = \max_{t\in[0,1]} \left|F(x,t) - c(x)t^\sigma\right|. \tag{5.3.4}$$

Again $C \in L^\infty(\Omega)$ by (5.3.3).

5.3.1 Geometry for auxiliary functional

Here we first prove that the functional $\mathcal{J}_{a,\,\lambda}$ has the geometric features required by the Mountain Pass Theorem.

Lemma 5.3.1. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$, $M : \mathbb{R}^+_0 \to \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be three functions satisfying* (0.0.3) *and* (0.0.4) *and* (5.1.2)*–*(5.1.7)*.*

Then, for any $\lambda > 0$ there exist two positive constants ρ and α such that

$$\mathcal{J}_{a,\,\lambda}(u) \geqslant \alpha > 0, \tag{5.3.5}$$

for any $u \in Z$ with $\|u\|_Z = \rho$.

Proof. Fix $\lambda > 0$. By (5.1.3) and (5.3.3) we get

$$\mathcal{J}_{a,\,\lambda}(u) \geqslant \frac{m_0}{2}\|u\|_Z^2 - \varepsilon\lambda\int_\Omega |u(x)|^2\,dx - c_\varepsilon\lambda\int_\Omega |u(x)|^q\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx.$$

So, by using a fractional Sobolev inequality (see [40, Theorem 6.5]), there is a positive constant $C = C(\Omega)$ such that

$$\mathcal{J}_{a,\,\lambda}(u) \geqslant \left(\frac{m_0}{2} - \varepsilon\lambda C\right)\|u\|_Z^2 - c_\varepsilon\lambda C\,\|u\|_Z^q - C\,\|u\|_Z^{2^*}.$$

Therefore, by fixing ε such that $m_0 > 2\varepsilon\lambda C$, since $2 < q < 2^*$, the result follows by choosing ρ sufficiently small. $\square$

Lemma 5.3.2. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$, $M : \mathbb{R}^+_0 \to \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be three functions satisfying* (0.0.3) *and* (0.0.4) *and* (5.1.2)*–*(5.1.7)*.*

Then, for any $\lambda > 0$ there exists an $e \in Z$ with $e \geqslant 0$ a.e. in $\mathbb{R}^n$, $\mathcal{J}_{a,\,\lambda}(e) < 0$ and $\|e\|_Z > \rho$, where ρ is given in Lemma 5.3.1.

Proof. Fix $\lambda > 0$ and take $u_0 \in Z$ such that $\|u_0\|_Z = 1$ and $u_0 \geqslant 0$ a.e. in $\mathbb{R}^n$. Now, let $t > 0$. By using (5.2.2) and (5.3.4), we get

$$\mathcal{J}_{a,\,\lambda}(tu_0) \leqslant a\frac{t^2}{2} - t^\sigma\lambda\int_\Omega c(x)\,|u_0(x)|^\sigma\,dx + \lambda\,\|C\|_{L^1(\Omega)} - \frac{t^{2^*}}{2^*}\int_\Omega |u_0(x)|^{2^*}\,dx.$$

Since $2 < \sigma < 2^*$, passing to the limit as $t \to +\infty$, we get that $\mathcal{J}_{a,\,\lambda}(tu_0) \to -\infty$, so that the assertion follows by taking $e = t_*u_0$, with $t_* > 0$ large enough. $\square$

5.3.2 The Palais–Smale condition

In this subsection we discuss a compactness property for the functional $\mathcal{J}_{a,\lambda}$, given by the Palais–Smale condition at a suitable level. For this, we fix $\lambda > 0$ and we set

$$c_{a,\lambda} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{J}_{a,\lambda}(\gamma(t))$$

where

$$\Gamma := \{\gamma \in C([0,1], Z) : \gamma(0) = 0, \mathcal{J}_{a,\lambda}(\gamma(1)) < 0\}.$$

Clearly, $c_{a,\lambda} > 0$ by Lemma 5.3.1. We recall that $\{u_j\}_{j\in\mathbb{N}}$ is a Palais–Smale sequence for $\mathcal{J}_{a,\lambda}$ at level $c_{a,\lambda}$ if

$$\mathcal{J}_{a,\lambda}(u_j) \to c_{a,\lambda}, \tag{5.3.6}$$

and

$$\sup\left\{\left|\left\langle \mathcal{J}'_{a,\lambda}(u_j), \varphi\right\rangle\right| : \varphi \in Z, \|\varphi\|_Z = 1\right\} \to 0, \tag{5.3.7}$$

as $j \to +\infty$. Also, we say that $\mathcal{J}_{a,\lambda}$ satisfies the Palais–Smale condition at level $c_{a,\lambda}$ if any Palais–Smale sequence $\{u_j\}_{j\in\mathbb{N}}$ at level $c_{a,\lambda}$ admits a convergent subsequence in Z.

Before proving the relatively compactness of the Palais–Smale sequences, we introduce an asymptotic condition for the level $c_{a,\lambda}$. This result will be crucial not only to get (5.1.9), but above all to overcome the lack of compactness due to the presence of a critical nonlinearity.

Lemma 5.3.3. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$, $M : \mathbb{R}^+_0 \to \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be three functions satisfying* (0.0.3) *and* (0.0.4) *and* (5.1.2)–(5.1.7).

Then

$$\lim_{\lambda\to+\infty} c_{a,\lambda} = 0.$$

Proof. Fix $\lambda > 0$ and let $e \in Z$ be the function given by Lemma 5.3.2. Since $\mathcal{J}_{a,\lambda}$ satisfies the Mountain Pass geometry, it follows that there exists $t_\lambda > 0$ verifying $\mathcal{J}_{a,\lambda}(t_\lambda e) = \max_{t\geqslant 0} \mathcal{J}_{a,\lambda}(te)$. Hence, $\left\langle \mathcal{J}'_{a,\lambda}(t_\lambda e), e\right\rangle = 0$ and by (5.3.1) we get

$$t_\lambda \|e\|_Z^2 M_a(t_\lambda^2 \|e\|_Z^2) = \lambda \int_\Omega f(x, t_\lambda e(x))e(x)\,dx + t_\lambda^{2^*-1} \int_\Omega |e(x)|^{2^*}\,dx. \tag{5.3.8}$$

Now, by construction $e \geqslant 0$ a.e. in $\mathbb{R}^n$. So, by (5.1.6), (5.2.2) and (5.3.8) it follows

$$a\|e\|_Z^2 \geqslant t_\lambda^{2^*-2} \int_\Omega |e(x)|^{2^*}\,dx,$$

which implies that $\{t_\lambda\}_{\lambda\in\mathbb{R}^+}$ is bounded.

Fix now a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ such that $\lambda_j \to +\infty$ as $j \to +\infty$. Clearly $\{t_{\lambda_j}\}_{j\in\mathbb{N}}$ is bounded. Hence, there exist a subsequence of $\{\lambda_j\}_{j\in\mathbb{N}}$ and a constant $\beta \geqslant 0$ such that $t_{\lambda_j} \to \beta$ as $j \to +\infty$. So, by using also (5.2.2) and (5.3.8) there exists $D > 0$ such that

$$\lambda_j \int_\Omega f(x, t_{\lambda_j} e(x)) e(x)\, dx + t_{\lambda_j}^{2^*-1} \int_\Omega |e(x)|^{2^*} dx = t_{\lambda_j} M_a(t_{\lambda_j}^2 \|e\|_Z^2) \leqslant D \tag{5.3.9}$$

for any $j \in \mathbb{N}$.

We claim that $\beta = 0$. Indeed, if $\beta > 0$ then by (5.3.2) and the Dominated Convergence Theorem,

$$\int_\Omega f(x, t_{\lambda_j} e(x)) e(x)\, dx \to \int_\Omega f(x, \beta e(x)) e(x)\, dx \quad \text{as } j \to +\infty.$$

By remembering that $\lambda_j \to +\infty$, we get

$$\lim_{j\to+\infty} \lambda_j \int_\Omega f(x, t_{\lambda_j} e(x)) e(x)\, dx + t_{\lambda_j}^{2^*-1} \int_\Omega |e(x)|^{2^*} dx = +\infty$$

which contradicts (5.3.9). Thus, we have that $\beta = 0$. Now, we consider the following path $\gamma_*(t) = te$ for $t \in [0,1]$ which belongs to Γ. By using (5.1.6) and Lemma 5.3.1 we get

$$0 < c_{a,\lambda} \leqslant \max_{t\in[0,1]} \mathcal{J}_{a,\lambda}(\gamma_*(t)) \leqslant \mathcal{J}_{a,\lambda}(t_\lambda e) \leqslant \frac{1}{2} \widehat{M_a}(t_\lambda^2 \|e\|_Z^2). \tag{5.3.10}$$

By (5.1.2) and by remembering that $\beta = 0$ we have

$$\lim_{\lambda\to+\infty} \widehat{M_a}(t_\lambda^2 \|e\|_Z^2) = 0,$$

and so by using also (5.3.10) we can conclude the proof. □

Now, as usual we first prove the boundedness of a Palais–Smale sequence for $\mathcal{J}_{a,\lambda}$ at level $c_{a,\lambda}$.

Lemma 5.3.4. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$, $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be three functions satisfying* (0.0.3) *and* (0.0.4) *and* (5.1.2)–(5.1.7)*. For any $\lambda > 0$, let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z verifying* (5.3.6) *and* (5.3.7)*.*

Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z.

Proof. Fix $\lambda > 0$. By (5.3.6) and (5.3.7) there exists $C > 0$ such that

$$|\mathcal{J}_{a,\lambda}(u_j)| \leqslant C \text{ and } \left| \left\langle \mathcal{J}'_{a,\lambda}(u_j), \frac{u_j}{\|u_j\|_Z} \right\rangle \right| \leqslant C, \tag{5.3.11}$$

for any $j \in \mathbb{N}$. Moreover, by (5.1.3), (5.1.6), and (5.2.2) it follows that

$$\begin{aligned} &\mathcal{J}_{a,\lambda}(u_j) - \frac{1}{\sigma} \left\langle \mathcal{J}'_{a,\lambda}(u_j), u_j \right\rangle \\ &\qquad \geqslant \frac{1}{2} \widehat{M_a}(\|u_j\|_Z^2) - \frac{1}{\sigma} M_a(\|u_j\|_Z^2) \|u_j\|_Z^2 \geqslant \left(\frac{1}{2} m_0 - \frac{1}{\sigma} a \right) \|u_j\|_Z^2 \end{aligned} \tag{5.3.12}$$

So, by combining (5.3.11) with (5.3.12) and by remembering that $m_0 < a < \frac{\sigma}{2}m_0$, we can conclude the proof. $\square$

We are now ready to prove the Palais-Smale condition. As usual in elliptic equations with critical nonlinearities, the main difficulty relies in the verification of the compactness property of the associated functional. This is due to the lack of compactness at critical level. To overcome this problem, we need the following concentration-compactness principle proved by Palatucci and Pisante in [65, Theorem 5] in a non–local setting.

Theorem 5.3.5. *[65] Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded subset and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $H^s_0(\Omega)$ weakly converging to u as $j \to +\infty$ and let μ, ν be two positive measures in $\mathbb{R}^n$ such that*

$$\left|(-\Delta)^{s/2}u_j\right|^2 dx \overset{*}{\rightharpoonup} \mu \quad \textit{and} \quad |u_j|^{2^*} dx \overset{*}{\rightharpoonup} \nu \quad \textit{in } \mathcal{M}(\mathbb{R}^n).$$

Then, there exist a (at most countable) set of distinct point $\{x_i\}_{i\in J}$, positive numbers $\{\nu_i\}_{i\in J}$, $\{\mu_i\}_{i\in J}$ and a positive measure $\widetilde{\mu}$ with $Supp\,\widetilde{\mu} \subset \overline{\Omega}$ such that

$$\nu = |u|^{2^*} dx + \sum_{i\in J} \nu_i \delta_{x_i},$$

and

$$\mu = \left|(-\Delta)^{s/2}u\right|^2 dx + \widetilde{\mu} + \sum_{i\in J} \mu_i \delta_{x_i}, \qquad \nu_i \leqslant S^{-2^*/2}\mu_i^{2^*/2},$$

with S the best constant of the fractional Sobolev embedding.

Before applying Theorem 5.3.5 we first need a sort of integration by parts. For the estimate of each terms deriving by the integration, we will exploit Theorem 5.3.5 and also the following result proved in [17, Lemmas 2.8 and 2.9].

Lemma 5.3.6. *[17] Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded subset and let $\{u_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $H^s_0(\Omega)$. Let $\psi \in C^\infty_0(\mathbb{R}^n)$ be a radial cut–off function and define $\psi_\delta(x) := \psi(x/\delta)$.*

Then

$$\lim_{\delta\to 0}\lim_{j\to\infty}\left|\int_{\mathbb{R}^n} u_j(x)(-\Delta)^{s/2}u_j(x)(-\Delta)^{s/2}\psi_\delta(x)dx\right| = 0$$

and

$$\lim_{\delta\to 0}\lim_{j\to\infty}\left|\int_{\mathbb{R}^n} (-\Delta)^{s/2}u_j(x)\int_{\mathbb{R}^n}\frac{(u_j(x)-u_j(y))(\psi_\delta(x)-\psi_\delta(y))}{|x-y|^{n+s}}dxdy\right| = 0.$$

Lemma 5.3.7. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$, $M : \mathbb{R}^+_0 \to \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be three functions satisfying* (0.0.3) *and* (0.0.4) *and* (5.1.2)–(5.1.7). *Let $\{u_j\}_{j\in\mathbb{N}}$ be a bounded sequence in Z verifying* (5.3.6) *and* (5.3.7).

Then there exists $\lambda_0 > 0$ such that for any $\lambda \geqslant \lambda_0$ the functional $\mathcal{J}_{a,\lambda}$ satisfies the Palais–Smale condition at level $c_{a,\lambda}$.

Proof. By Lemma 5.3.3 there exists $\lambda_0 > 0$ such that

$$c_{a,\lambda} < \left(\frac{1}{\theta} - \frac{1}{2^*}\right)\left[\frac{\theta\, m_0\, S}{c(n,s)}\right]^{n/2s} \tag{5.3.13}$$

for any $\lambda \geqslant \lambda_0$, where $c(n,s)$ is given in (0.0.7) and S is the best constant of the fractional Sobolev embedding (see [1, Theorem 7.58]) defined as

$$S = \inf_{\substack{v \in H^s(\mathbb{R}^n)\\ v \neq 0}} \frac{\|v\|^2_{H^s(\mathbb{R}^n)}}{\|v\|^2_{L^{2^*}(\mathbb{R}^n)}}. \tag{5.3.14}$$

Fix now $\lambda \geqslant \lambda_0$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in Z verifying (5.3.6) and (5.3.7). Since by Lemma 5.3.4 the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z, by applying also Lemma 1.1.2 and [20, Theorem 4.9], up to a subsequence, there exists $u \in Z$ such that u_j converges to u weakly in Z, strongly in $L^r(\Omega)$ with $r \in [1, 2^*)$ and a.e. in Ω. Also, in particular there exists $h \in L^r(\Omega)$ such that

$$|u_j(x)| \leqslant h(x) \quad \text{for any } j \in \mathbb{N} \text{ and a.e. } x \in \Omega.$$

We point out the above inequality and convergences are also verified in all $\mathbb{R}^n$, since $u_j = 0 = u$ a.e. in $\mathbb{R}^n \setminus \Omega$; in particular we shall assume that $h(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus \Omega$. Moreover, up to a subsequence, there is $\alpha \geqslant 0$ such that $\|u_j\|_Z \to \alpha$, so by using (5.1.2) it follows that $M_a(\|u_j\|_Z^2) \to M_a(\alpha^2)$ as $j \to +\infty$.

Now, we claim that

$$\|u_j\|_Z^2 \to \|u\|_Z^2 \qquad \text{as } j \to +\infty, \tag{5.3.15}$$

which clearly implies that $u_j \to u$ in Z as $j \to +\infty$. By Lemma 1.1.1 we know that $\{u_j\}_{j\in\mathbb{N}}$ is also bounded in $H_0^s(\Omega)$. So, by Phrokorov's Theorem we may suppose that there exist two positive measures μ and ν on $\mathbb{R}^n$ such that

$$\left|(-\Delta)^{s/2} u_j\right|^2 dx \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |u_j|^{2^*} dx \overset{*}{\rightharpoonup} \nu \tag{5.3.16}$$

in the sense of measures. Moreover, by Theorem 5.3.5 we obtain an at most countable set of distinct points $\{x_i\}_{i\in J}$, positive numbers $\{\nu_i\}_{i\in J}$, $\{\mu_i\}_{i\in J}$ and a positive measure $\widetilde{\mu}$ with $Supp\, \widetilde{\mu} \subset \overline{\Omega}$ such that

$$\nu = |u|^{2^*} dx + \sum_{i\in J} \nu_i \delta_{x_i}, \tag{5.3.17}$$

and

$$\mu = \left|(-\Delta)^{s/2}u\right|^2 dx + \widetilde{\mu} + \sum_{i\in J} \mu_i \delta_{x_i}, \qquad \nu_i \leqslant S^{-2^*/2}\mu_i^{2^*/2}, \tag{5.3.18}$$

with S the constant given in (5.3.14).

Our goal is to show that the set J is empty. We argue by contradiction and suppose $J \neq \emptyset$. Then we fix $i \in J$ and for any $\delta > 0$ we set $\psi_\delta(x) := \psi((x - x_i)/\delta)$ where $\psi \in C_0^\infty(\mathbb{R}^n, [0,1])$ is such that $\psi \equiv 1$ in $B(0,1)$ and $\psi \equiv 0$ in $\mathbb{R}^n \setminus B(0,2)$. Since for a fixed $\delta > 0$ the sequence $\{\psi_\delta u_j\}_{j\in\mathbb{N}}$ is bounded in Z uniformly in j, by (5.3.7) it follows that $\left\langle \mathcal{J}'_{a,\lambda}(u_j), \psi_\delta u_j\right\rangle \to 0$ as $j \to +\infty$. From this, by applying also (0.0.4) we get

$$\begin{aligned} &o_j(1) + \lambda \int_\Omega f(x, u_j(x))\psi_\delta(x)u_j(x)dx + \int_\Omega |u_j(x)|^{2^*} \psi_\delta(x)dx \\ &\qquad \geqslant \theta M_a(\|u_j\|_Z^2) \iint_{\mathbb{R}^{2n}} \frac{\big(u_j(x) - u_j(y)\big)\big(\psi_\delta(x)u_j(x) - \psi_\delta(y)u_j(y)\big)}{|x-y|^{n+2s}}\, dxdy, \end{aligned} \tag{5.3.19}$$

as $j \to +\infty$.

By [40, Proposition 3.6] we know that for any $v \in Z$

$$\iint_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}\, dxdy = c(n,s)^{-1} \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}v(x)\right|^2 dx,$$

with $c(n,s)$ the dimensional constant defined in (0.0.7) and, by taking derivative of the above equality, for any $v, w \in Z$ we obtain

$$\iint_{\mathbb{R}^{2n}} \frac{(v(x) - v(y))(w(x) - w(y))}{|x-y|^{n+2s}} dxdy = c(n,s)^{-1} \int_{\mathbb{R}^n} (-\Delta)^{s/2}v(x)(-\Delta)^{s/2}w(x)dx. \tag{5.3.20}$$

Furthermore, for any $v, w \in Z$ we have

$$(-\Delta)^{s/2}(vw)(x) = v(x)(-\Delta)^{s/2}w(x) + w(x)(-\Delta)^{s/2}v(x) - 2I_{s/2}(v,w)(x), \tag{5.3.21}$$

where the last term is defined, in the principal value sense, as follows

$$I_{s/2}(v,w)(x) = P.V. \int_{\mathbb{R}^n} \frac{(v(x) - v(y))(w(x) - w(y))}{|x-y|^{n+s}} dy.$$

Thus, by (5.3.20) and (5.3.21) the integral in the right–hand side of (5.3.19) becomes

$$\begin{aligned} &\iint_{\mathbb{R}^{2n}} \frac{\big(u_j(x) - u_j(y)\big)\big(\psi_\delta(x)u_j(x) - \psi_\delta(y)u_j(y)\big)}{|x-y|^{n+2s}}\, dxdy \\ &\quad = c(n,s)^{-1} \int_{\mathbb{R}^n} u_j(x)(-\Delta)^{s/2}u_j(x)(-\Delta)^{s/2}\psi_\delta(x)dx \\ &\quad\quad + c(n,s)^{-1} \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}u_j(x)\right|^2 \psi_\delta(x)dx \\ &\quad\quad - 2c(n,s)^{-1} \int_{\mathbb{R}^n} (-\Delta)^{s/2}u_j(x) \int_{\mathbb{R}^n} \frac{(u_j(x) - u_j(y))(\psi_\delta(x) - \psi_\delta(y))}{|x-y|^{n+s}} dxdy. \end{aligned}$$

From this, by using (5.3.16) and Lemma 5.3.6 we get

$$\lim_{\delta\to 0}\lim_{j\to\infty}\left[\iint_{\mathbb{R}^{2n}}\frac{\big(u_j(x)-u_j(y)\big)\big(\psi_\delta(x)u_j(x)-\psi_\delta(y)u_j(y)\big)}{|x-y|^{n+2s}}dxdy\right]=c(n,s)^{-1}\mu_i. \tag{5.3.22}$$

Moreover, by (5.3.2) and the Dominated Convergence Theorem we get

$$\int_{B(x_i,\delta)} f(x,u_j(x))u_j(x)\psi_\delta(x)dx \to \int_{B(x_i,\delta)} f(x,u(x))u(x)\psi_\delta(x)dx$$

as $j\to+\infty$. So by sending $\delta\to 0$ we observe that

$$\lim_{\delta\to 0}\lim_{j\to\infty}\int_{B(x_i,\delta)} f(x,u_j(x))u_j(x)\psi_\delta(x)dx=0. \tag{5.3.23}$$

Furthermore, by (5.3.16) it follows that

$$\int_\Omega |u_j(x)|^{2^*}\psi_\delta(x)dx \to \int_\Omega \psi_\delta(x)d\nu \quad \text{as } j\to+\infty$$

and by combining this last formula with (5.3.19), (5.3.22) and (5.3.23), using also(5.3.16), we obtain

$$\nu_i \geqslant \theta M_a(\alpha^2)c(n,s)^{-1}\mu_i,$$

recalling that $M_a(\|u_j\|_Z^2)\to M_a(\alpha^2)$ as $j\to+\infty$. By using (5.1.3) we conclude that $\nu_i\geqslant \theta\, m_0\, c(n,s)^{-1}\mu_i$ and by using also the inequality in (5.3.18), since $\nu_i>0$ we get

$$\nu_i \geqslant \left[\frac{\theta\, m_0\, S}{c(n,s)}\right]^{n/2s}. \tag{5.3.24}$$

Now we shall prove that the above expression cannot occur and so, since $i\in J$ was arbitrary, the set J is empty. By (5.3.6) and (5.3.7) we have

$$\lim_{j\to+\infty}\left(\mathcal{J}_{a,\lambda}(u_j)-\frac{1}{\sigma}\left\langle \mathcal{J}'_{a,\lambda}(u_j),u_j\right\rangle\right)=c_{a,\lambda}. \tag{5.3.25}$$

Moreover, by (5.1.3), (5.1.6), (5.2.2) and remembering that $m_0<a<\dfrac{\sigma}{2}m_0$ we have

$$\begin{aligned}
&\mathcal{J}_{a,\lambda}(u_j)-\frac{1}{\sigma}\left\langle \mathcal{J}'_{a,\lambda}(u_j),u_j\right\rangle\\
&\qquad\geqslant \frac{1}{2}\widehat{M_a}(\|u_j\|_Z^2)-\frac{1}{\sigma}M_a(\|u_j\|_Z^2)\,\|u_j\|_Z^2+\left(\frac{1}{\sigma}-\frac{1}{2^*}\right)\int_\Omega |u_j(x)|^{2^*}dx\\
&\qquad\geqslant \left(\frac{1}{2}m_0-\frac{1}{\sigma}a\right)\|u_j\|_Z^2+\left(\frac{1}{\sigma}-\frac{1}{2^*}\right)\int_\Omega |u_j(x)|^{2^*}dx\\
&\qquad\geqslant \left(\frac{1}{\sigma}-\frac{1}{2^*}\right)\int_\Omega \psi_\delta(x)\,|u_j(x)|^{2^*}dx,
\end{aligned} \tag{5.3.26}$$

since also $0 \leqslant \psi_\delta \leqslant 1$. By combining (5.3.25) and (5.3.26), using also (5.3.16), we get

$$c_{a,\,\lambda} \geqslant \left(\frac{1}{\sigma} - \frac{1}{2^*}\right) \int_\Omega \psi_\delta(x) d\nu,$$

from which, by sending $\delta \to 0$ and by using (5.3.24), it follows that

$$c_{a,\,\lambda} \geqslant \left(\frac{1}{\sigma} - \frac{1}{2^*}\right) \left[\frac{\theta\, m_0\, S}{c(n,s)}\right]^{n/2s},$$

which clearly contradicts (5.3.13). Thus, J is empty and by (5.3.16) and (5.3.17) it follows that u_j converges to u in $L^{2^*}(\Omega)$. So, by (5.3.7) with $\varphi = u_j$, (5.3.2) and the Dominated Convergence Theorem we have

$$\lim_{j\to+\infty} M_a(\|u_j\|_Z^2)\, \|u_j\|_Z^2 = \lambda \int_\Omega f(x, u(x)) u(x) dx + \int_\Omega |u(x)|^{2^*} dx. \tag{5.3.27}$$

Moreover, by remembering that $u_j \rightharpoonup u$ in Z, $M_a(\|u_j\|_Z^2) \to M_a(\alpha^2)$ and by using (5.3.2) and the Dominated Convergence Theorem, (5.3.7) we have

$$M_a(\alpha^2)\, \langle u,\, \varphi\rangle_Z = \lambda \int_\Omega f(x, u(x))\varphi(x)\, dx - \int_\Omega |u(x)|^{2^*-2} u(x)\varphi(x) dx\,, \tag{5.3.28}$$

for any $\varphi \in Z$. So, by combining (5.3.27) and (5.3.28) it follows that

$$M_a(\|u_j\|_Z^2)\, \|u_j\|_Z^2 \to M_a(\alpha^2)\, \|u\|_Z^2 \qquad \text{as } j \to +\infty,$$

from which we conclude the proof of claim (5.3.15). $\square$

Before concluding the proof of our main results we will give an alternative proof of Lemma 5.3.7. This new approach does not need of Lemmas 5.3.5 and 5.3.6. Indeed it is mainly based on the celebrated Brezis & Lieb lemma (see [21]). In our factional framework the application of this lemma is different compared to the classical case, since we do not have derivatives of solutions in Z, but a sort of integro–differentiation (see (1.1.7)). The idea for this approach is given by paper [9] where we studied problem (5.1.1) in a degenerate setting. In the degenerate case the proof based on a concentration–compactness principle does not work.

An alternative proof of Lemma 5.3.7. Take $\lambda > 0$ and let $\{u_j\}_{j\in\mathbb{N}} \subset Z$ be a sequence in Z verifying (5.3.6) and (5.3.7).

Since by Lemma 5.3.4 the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in Z, by applying also Lemma 1.1.2 and [20, Theorem 4.9], there exists $u_\lambda \in Z$ such that, up to a subsequence, it follows that

$$\begin{aligned} &u_j \rightharpoonup u_\lambda \text{ in } Z \text{ and in } L^{2^*}(\Omega), && \|u_j\|_Z \to \alpha_\lambda, \\ &u_j \to u_\lambda \text{ in } L^q(\Omega) \text{ and in } L^2(\Omega), && \|u_j - u_\lambda\|_{L^{2^*}(\Omega)} \to \ell_\lambda, \\ &u_j \to u_\lambda \text{ a.e. in } \Omega, && |u_j| \leqslant h \text{ a.e. in } \Omega, \end{aligned} \tag{5.3.29}$$

with $h \in L^2(\Omega) \cap L^q(\Omega)$. Furthermore, by (5.1.2) and (5.1.3) we have $M_a(\|u_j\|_Z^2) \to M_a(\alpha_\lambda^2) > 0$ as $j \to +\infty$.

We first assert that

$$\lim_{\lambda \to +\infty} \alpha_\lambda = 0. \tag{5.3.30}$$

Otherwise $\limsup_{\lambda \to +\infty} \alpha_\lambda = \alpha > 0$. Hence there is a sequence, say $k \to \lambda_k \uparrow +\infty$ such that $\alpha_{\lambda_k} \to \alpha$ as $k \to +\infty$. Now, by (5.1.3), (5.1.6), and (5.2.2) it follows that

$$\begin{aligned}
&\mathcal{J}_{a,\,\lambda_k}(u_j) - \frac{1}{\sigma}\left\langle \mathcal{J}'_{a,\,\lambda_k}(u_j), u_j \right\rangle \\
&\qquad \geqslant \frac{1}{2}\widehat{M_a}(\|u_j\|_Z^2) - \frac{1}{\sigma} M_a(\|u_j\|_Z^2)\,\|u_j\|_Z^2 \geqslant \left(\frac{1}{2}m_0 - \frac{1}{\sigma}a\right)\|u_j\|_Z^2
\end{aligned}$$

and letting $j \to +\infty$ and $k \to +\infty$ we get from Lemma 5.3.3 that

$$0 \geqslant \left(\frac{1}{2}m_0 - \frac{1}{\sigma}a\right)\alpha^2 > 0$$

since $m_0 < a < \dfrac{\sigma}{2} m_0$, which is the desired contradiction and proves the assertion (5.3.30).

Moreover, $\|u_\lambda\|_Z \leqslant \lim_{j \to +\infty} \|u_j\|_Z = \alpha_\lambda$ since $u_j \rightharpoonup u_\lambda$ in Z, so that (5.3.30) implies at once by the fractional Sobolev inequality

$$\lim_{\lambda \to +\infty} \|u_\lambda\|_{L^{2^*}(\Omega)} = \lim_{\lambda \to +\infty} \|u_\lambda\|_Z = 0. \tag{5.3.31}$$

By (5.3.2), (5.3.29) and the fact that $|u_j|^{2^*-2} u_j \rightharpoonup |u_\lambda|^{2^*-2} u_\lambda$ in $L^{2^{*'}}(\Omega)$, where $2^{*'} = 2n/(n+2s)$ is the Hölder conjugate of 2^*, we have

$$M_a(\alpha_\lambda^2)\,\langle u_\lambda, \varphi\rangle_Z = \lambda \int_\Omega f(x, u_\lambda(x))\varphi(x)dx + \int_\Omega |u_\lambda(x)|^{2^*-2} u_\lambda(x)\varphi(x)dx$$

for any $\varphi \in Z$. Hence, u_λ is a critical point of the $C^1(Z)$ functional

$$\mathcal{J}_{\alpha_\lambda}(u) = \frac{1}{2} M_a(\alpha_\lambda^2)\|u\|_Z^2 - \lambda \int_\Omega F(x, u(x))dx - \frac{1}{2^*}\|u\|_{L^{2^*}(\Omega)}^{2^*}. \tag{5.3.32}$$

In particular, (5.3.7) and (5.3.29) imply that as $j \to +\infty$

$$\begin{aligned}
o_j(1) &= \langle \mathcal{J}'_{a,\,\lambda}(u_j) - \mathcal{J}'_{\alpha_\lambda}(u_\lambda), u_j - u_\lambda\rangle = M_a(\|u_j\|_Z^2)\|u_j\|_Z^2 + M_a(\alpha_\lambda^2)\|u_\lambda\|_Z^2 \\
&\quad - \langle u_j, u_\lambda\rangle_Z \left[M_a(\|u_j\|_Z^2) + M_a(\alpha_\lambda^2)\right] - \lambda \int_\Omega \big[f(x,u_j) - f(x,u_\lambda)\big](u_j - u_\lambda)\,dx \\
&\quad - \int_\Omega \big(|u_j|^{2^*-2}u_j - |u_\lambda|^{2^*-2}u_\lambda\big)(u_j - u_\lambda)dx \\
&= M_a(\alpha_\lambda^2)\big(\alpha_\lambda^2 - \|u_\lambda\|_Z^2\big) - \|u_j\|_{L^{2^*}(\Omega)}^{2^*} + \|u_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + o_j(1) \\
&= M_a(\alpha_\lambda^2)\|u_j - u_\lambda\|_Z^2 - \|u_j - u_\lambda\|_{L^{2^*}(\Omega)}^{2^*} + o_j(1).
\end{aligned} \tag{5.3.33}$$

Indeed, by (5.3.2) and (5.3.29),

$$\lim_{j\to+\infty}\int_\Omega \big[f(x,u_j(x))-f(x,u_\lambda(x))\big](u_j(x)-u_\lambda(x))\,dx=0.$$

Moreover, again by (5.3.29) and the Brezis & Lieb lemma (see [21]), as $j\to+\infty$

$$\|u_j\|_Z^2=\|u_j-u_\lambda\|_Z^2+\|u_\lambda\|_Z^2+o_j(1),\qquad \|u_j\|_{L^{2^*}(\Omega)}^{2^*}=\|u_j-u_\lambda\|_{L^{2^*}(\Omega)}^{2^*}+\|u_\lambda\|_{L^{2^*}(\Omega)}^{2^*}+o_j(1).$$

Finally, we have used the fact that $\|u_j\|_Z\to\alpha_\lambda$. Therefore, we have proved the main formula

$$M_a(\alpha_\lambda^2)\lim_{j\to+\infty}\|u_j-u_\lambda\|_Z^2=\lim_{j\to+\infty}\|u_j-u_\lambda\|_{L^{2^*}(\Omega)}^{2^*}. \tag{5.3.34}$$

By (5.1.3), (5.1.6), (5.2.2) and remembering that $m_0<a<\dfrac{\sigma}{2}m_0$ we have

$$\begin{aligned}
&\mathcal{J}_{a,\lambda}(u_j)-\frac{1}{\sigma}\left\langle \mathcal{J}'_{a,\lambda}(u_j),u_j\right\rangle\\
&\qquad\geqslant \frac12\widehat{M_a}(\|u_j\|_Z^2)-\frac{1}{\sigma}M_a(\|u_j\|_Z^2)\,\|u_j\|_Z^2+\left(\frac1\sigma-\frac{1}{2^*}\right)\int_\Omega |u_j(x)|^{2^*}\,dx\\
&\qquad\geqslant \left(\frac12 m_0-\frac1\sigma a\right)\|u_j\|_Z^2+\left(\frac1\sigma-\frac{1}{2^*}\right)\int_\Omega |u_j(x)|^{2^*}\,dx\\
&\qquad\geqslant \left(\frac1\sigma-\frac{1}{2^*}\right)\int_\Omega |u_j(x)|^{2^*}\,dx.
\end{aligned}$$

So, using (5.3.29) and the Brezis & Lieb lemma, we attain as $j\to+\infty$

$$\begin{aligned}
c_\lambda+o_j(1)=\mathcal{J}_{a,\lambda}(u_j)-\frac1\sigma\left\langle\mathcal{J}'_{a,\lambda}(u_j),u_j\right\rangle &\geqslant \left(\frac1\sigma-\frac{1}{2^*}\right)\|u_j\|_{L^{2^*}(\Omega)}^{2^*}\\
&=\left(\frac1\sigma-\frac{1}{2^*}\right)\left\{\ell_\lambda^{2^*}+\|u_\lambda\|_{L^{2^*}(\Omega)}^{2^*}\right\}+o_j(1).
\end{aligned}$$

Thus, by Lemma 5.3.3 and (5.3.31) we also obtain

$$\lim_{\lambda\to+\infty}\ell_\lambda=0. \tag{5.3.35}$$

Denote by $\widetilde{S}$ the main fractional Sobolev constant, that is

$$\widetilde{S}=\inf_{\substack{v\in Z\\ v\neq 0}}\frac{\|v\|_Z^2}{\|v\|_{L^{2^*}(\Omega)}^2}.$$

By (5.3.34) and the notation in (5.3.29), for all $\lambda\in\mathbb{R}^+$

$$\ell_\lambda^{2^*}\geqslant \widetilde{S}\,M_a(\alpha_\lambda^2)\,\ell_\lambda^2\geqslant \widetilde{S}\,m_0\,\ell_\lambda^2, \tag{5.3.36}$$

by (5.1.3). This last inequality, together with (5.3.35), yields at once that there exists $\lambda_0>0$ such that $\ell_\lambda=0$ for all $\lambda\geqslant\lambda_0$.

Hence, for any $\lambda \geqslant \lambda_0$

$$\lim_{j\to+\infty} \|u_j - u_\lambda\|_{L^{2^*}(\Omega)}^{2^*} = 0.$$

Thus, $u_j \to u_\lambda$ in Z as $j \to +\infty$ for all $\lambda \geqslant \lambda_0$ by (5.3.34), being $M_a(\alpha_\lambda^2) > 0$ by (5.1.3). $\square$

5.3.3 Proofs of Theorems 5.1.1 and 5.2.1

In this subsection we conclude the proofs of our main theorems. We first show that the auxiliary problem (5.2.1) admits a non–trivial weak solution. Then we will see the same weak solution solves also the main problem (5.1.1).

Proof of Theorem 5.2.1. Lemmas 5.3.1, 5.3.2 and 5.3.7 guarantee that for any $\lambda \geqslant \lambda_0$ the functional $\mathcal{J}_{a,\lambda}$ satisfies all the assumptions of the Mountain Pass theorem. Hence, for any $\lambda \geqslant \lambda_0$ there exists a critical point $u \in Z$ for the functional $\mathcal{J}_{a,\lambda}$ at level $c_{a,\lambda}$. Since $\mathcal{J}_{a,\lambda}(u) = c_{a,\lambda} > 0 = \mathcal{J}_{a,\lambda}(0)$ we conclude that $u \not\equiv 0$. $\square$

Proof of Theorem 5.1.1. By Theorem 5.2.1, for any $\lambda \geqslant \lambda_0$ let u_λ be a weak solution of problem (5.2.1). Now, we claim that

$$\text{there exists } \lambda^* \geqslant \lambda_0 \text{ such that } \|u_\lambda\|_Z \leqslant t_0 \text{ for any } \lambda \geqslant \lambda^*, \tag{5.3.37}$$

where t_0 is given as at the beginning of Section 5.2. We argue by contradiction and suppose that there is a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ such that $\|u_{\lambda_j}\|_Z \geqslant t_0$. Since u_{λ_j} is a critical point of the functional $\mathcal{J}_{a,\lambda_j}$, by using also (5.1.3), (5.1.6) and (5.2.2) it follows that

$$\begin{aligned} c_{a,\lambda_j} &\geqslant \frac{1}{2}\widehat{M_a}(\|u_{\lambda_j}\|_Z^2) - \frac{1}{\sigma}M_a(\|u_{\lambda_j}\|_Z^2)\,\|u_{\lambda_j}\|_Z^2 \\ &\geqslant \left(\frac{1}{2}m_0 - \frac{1}{\sigma}a\right)\|u_{\lambda_j}\|_Z^2 \geqslant \left(\frac{1}{2}m_0 - \frac{1}{\sigma}a\right)t_0^2, \end{aligned}$$

which contradicts Lemma 5.3.3 since $m_0 < a < \dfrac{\sigma}{2}m_0$. So, by (5.3.37) we get $M_a(\|u_\lambda\|_Z^2) = M(\|u_\lambda\|_Z^2)$ which implies that u_λ is a weak solution of problem (5.1.1) for any $\lambda \geqslant \lambda_0$.

Moreover, arguing as above we have

$$c_{a,\lambda} \geqslant \left(\frac{1}{2}m_0 - \frac{1}{\sigma}a\right)\|u_\lambda\|_Z^2,$$

and so, since $m_0 < a < \dfrac{\sigma}{2}m_0$ and by Lemma 5.3.3, it follows that $\displaystyle\lim_{\lambda\to+\infty}\|u_\lambda\|_Z = 0$. $\square$

5.3.4 Existence of non-negative solutions

In this subsection we study the sign of weak solutions of problem (5.1.1). For this, we first introduce the following technical lemma.

Lemma 5.3.8. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy* (0.0.3) *and* (0.0.4). *Let $u \in Z$.*

Then the absolute value of u, denoted by $|u|$, is in Z.

Proof. We fix $a > 0$. Since $u \in Z$, by costruction there exists $w \in C_0^\infty(\Omega)$ such that

$$\|u - w\|_X < \frac{a}{2}. \tag{5.3.38}$$

Now, for any $\varepsilon > 0$ and $x \in \mathbb{R}^n$, we set $v_\varepsilon(x) := \big(\varepsilon^2 + w^2(x)\big)^{1/2} - \varepsilon$. We observe that $v_\varepsilon = 0 = w$ in $\mathbb{R}^n \setminus \Omega$ and it is a smooth function by construction. Hence, $v_\varepsilon \in C_0^\infty(\Omega)$. Also, we have $v_\varepsilon(x) \to |w(x)|$ for a.e. $x \in \mathbb{R}^n$ as $\varepsilon \to 0$. Since $|v_\varepsilon| \leqslant |w|$ for any $\varepsilon > 0$, by the Dominated Convergence Theorem, $v_\varepsilon \to |w|$ in $L^2(\mathbb{R}^n)$ as $\varepsilon \to 0$.

On the other hand,

$$|\nabla v_\varepsilon| = \frac{|w|\,|\nabla w|}{(\varepsilon^2 + w^2)^{1/2}} \leqslant |\nabla w|\,,$$

uniformly in ε. Therefore, by the boundedness and Lipschitz regularity of w it follows that

$$\begin{aligned} &|v_\varepsilon(x) - |w(x)| - v_\varepsilon(y) + |w(y)|\,|^2 \, K(x-y) \\ &\qquad \leqslant 2\left(|v_\varepsilon(x) - v_\varepsilon(y)|^2 + |\,|w(x)| - |w(y)|\,|^2\right) K(x-y) \\ &\qquad \leqslant C \, \min\left\{1, \, |x-y|^2\right\} K(x-y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n). \end{aligned}$$

thanks to (0.0.3). Thus, by the Dominated Convergence Theorem we get $v_\varepsilon \to |w|$ in X as $\varepsilon \to 0$, in particular

$$\|v_\varepsilon - |w|\|_X < \frac{a}{2} \tag{5.3.39}$$

for ε sufficiently small, say $\varepsilon \leqslant \bar{\varepsilon}$, with $\bar{\varepsilon} = \bar{\varepsilon}(a) > 0$.

By (5.3.38) and (5.3.39) it is easy to see that

$$\||u| - v_{\bar{\varepsilon}}\|_X \leqslant \||u| - |w|\|_X + \||w| - v_{\bar{\varepsilon}}\|_X \leqslant \|u - w\|_X + \||w| - v_{\bar{\varepsilon}}\|_X < a.$$

This concludes the proof. □

Corollary 5.3.9. *Let all the assumptions of Theorem 5.1.1 be satisfied and assume* (5.1.7) *in addition.*

Then problem (5.1.8) *has a non-negative solution u_λ for all $\lambda \geqslant \lambda^*$, where λ^* is the parameter given in Theorem 5.1.1.*

Proof. We fix $\lambda \geqslant \lambda^*$. Let $u_\lambda \in Z$ be a solution of problem (5.1.8), given by Theorem 5.1.1. By Lemma 5.3.8 we have $u_\lambda^- \in Z$. So, by (5.1.8) with $\varphi = u_\lambda^-$ we get

$$\begin{aligned} M(\|u_\lambda\|_Z^2) & \iint_{\mathbb{R}^{2n}} (u_\lambda(x) - u_\lambda(y))(u_\lambda^-(x) - u_\lambda^-(y))K(x-y)dx\,dy \\ &= \lambda \int_\Omega f(x, u_\lambda(x))u_\lambda^-(x)\,dx + \int_\Omega \left|u_\lambda^-(x)\right|^{2^*} dx. \end{aligned} \tag{5.3.40}$$

Now, we observe that

$$\begin{aligned} &(u_\lambda(x) - u_\lambda(y))(u_\lambda^-(x) - u_\lambda^-(y)) \\ &\qquad = -u_\lambda^+(x)u_\lambda^-(y) - u_\lambda^-(x)u_\lambda^+(y) - (u_\lambda^-(x) - u_\lambda^-(y))^2 \leqslant -\left|u_\lambda^-(x) - u_\lambda^-(y)\right|^2, \end{aligned}$$

for a.e. x, $y \in \mathbb{R}^n$. Moreover, by (5.1.7) we get $f(x, u_\lambda(x))u_\lambda^-(x) = 0$ for a.e. $x \in \mathbb{R}^n$. Thus, by (5.3.40) and being $M \geqslant 0$ by (5.1.3), it follows that

$$0 \geqslant -M(\|u_\lambda\|_Z^2) \iint_{\mathbb{R}^{2n}} \left|u_\lambda^-(x) - u_\lambda^-(y)\right|^2 K(x-y)dx\,dy \geqslant \left\|u_\lambda^-\right\|_{L^{2^*}(\Omega)}^{2^*}$$

which implies $u_\lambda^- \equiv 0$. $\square$

Appendix A

Some motivation for a fractional Kirchhoff equation

The goal of these last pages is to give some motivation for the problem studied in Chapter 5. For this, we would like first to recall some basic facts on the classical Kirchhoff equation: our explanations will be oversimplified, and even crude in some parts, and we will not attempt a rigorous mathematical justification of all the asymptotics that we are going to discuss heuristically.

We will consider the one-dimensional case for simplicity. For this we take the physical model of an elastic string constrained at the extrema. For concreteness, the string will be represented by the graph of a function $u : [-1, 1] \times [0, +\infty) \to \mathbb{R}$, and the end-point constraint reads $u(-1, t) = u(1, t) = 0$ for any $t \geqslant 0$. As usual we will write $u = u(x, t)$, where x is the space variable and t is the time.

For further use, we can indeed identify this finite string with an infinite string, that is constrained outside $(-1, 1)$, i.e. consider the function $u : [-1, 1] \times [0, +\infty) \to \mathbb{R}$, with $u(x, t) = 0$ for any $x \in \mathbb{R} \setminus (-1, 1)$ and any $t \geqslant 0$.

Then, the acceleration u_{tt} of the vertical displacement u of the vibrating string (that from now on will be assumed suitably small with respect to the length of the string) must be compensated, by Newton's law, by the elastic force of the string and by the external force field f: so we obtain the classical equation for the vibrating string:

$$u_{tt} = M u_{xx} + f.$$

If we look for stationary solutions, i.e. solutions $u(x)$ that do not depend on time, the equation boils down to

$$M u_{xx} + f = 0. \tag{A.0.1}$$

To a first approximation, for homogeneous strings, the elastic tension term M is simply a positive constant m_0. Several corrections to the model were proposed in order to take into account some discrepancies between the theory and the experimental data, since *"it is well known that the classical linearized analysis of the vibrating string can lead to results which are reasonably accurate only when the minimum (rest position) tension and the displacements are of such magnitude that the relative change in tension during the motion is small"*, see [32].

A classical modification of the above model is then to suppose that the tension increases if so does the length of the string. This ansatz is coherent with the common experience that a taut string reacts more strongly than a slack one. It is conceivable then to make the above ansatz quantitative and suppose, for simplicity, that the tension, for small deformations of the string, takes (at least for small elongations of the string) the linear form

$$M(\ell) = m_0 + 2b\ell, \tag{A.0.2}$$

where $b > 0$ is constant and ℓ is the increment in the length of the string with respect to its rest position (in which the string has length 2), i.e.

$$\ell = \int_{-1}^{1} \sqrt{1 + u_x^2}\, dx - 2. \tag{A.0.3}$$

For small deformations, $\sqrt{1 + u_x^2} = 1 + \frac{u_x^2}{2}$ up to higher order terms, and so

$$\ell = \frac{1}{2} \int_{-1}^{1} u_x^2\, dx.$$

By plugging this into (A.0.2) we obtain

$$M = m_0 + b \int_{-1}^{1} u_x^2\, dx = m_0 + b \int_{\mathbb{R}} u_x^2\, dx,$$

where we used the notation for which u is defined to vanish outside $(-1, 1)$. By inserting this into (A.0.1), one obtains the classical version of the Kirchhoff equation

$$M\left(\int_{-\infty}^{+\infty} u_x^2\, dx\right) u_{xx} + f = 0, \tag{A.0.4}$$

with $M(t) = m_0 + bt$. As a historical remark, we mention that the equation was first introduced in [55, 56] and then, probably independently, proposed in [32, 33]; see also [64] for a comparison between the theory and the experimental data.

We observe that the first term in (A.0.4) can be interpreted in a variational way, as arising from an energy of the form

$$\frac{1}{2}\widehat{M}\left(\int_{-\infty}^{+\infty} u_x^2\, dx\right), \tag{A.0.5}$$

where $\widehat{M}$ is a primitive of M.

With this respect, the Kirchhoff equation of non–local type that we studied originates from the idea that the energy in (A.0.5) does not depend on the H^1 norm of the function that parameterizes the graph of the string, but rather on its H^s norm, namely we replaced (A.0.5) with

$$\frac{1}{2}\widehat{M}\left(\int_{\mathbb{R}^2}\frac{(u(x)-u(y))^2}{|x-y|^{1+2s}}\,dx\,dy\right),$$

or even with more general kinds of fractional norms. In this sense, while the "non–local" feature of the tension in the classical Kirchhoff equation surfaces from the average of a "local" object (namely u_x^2), in the equation we took into account the "non–local" aspect of the tension arises from an object which is "non–local" as well. In general, we think it could be interesting to study even more general models in which the tension of the string is related to "non–local" measurments of the modification of the string from its rest position. Some of these models may be variational in nature (as the one considered here), some others may be not.

Now, we present another way of obtaining the model we study from the classical Kirchhoff equation. Following [24], for $\sigma \in (0,1)$, we consider the σ-length of the string as follows. Let $E := \{(x_1,x_2)\in\mathbb{R}^2 \text{ s.t. } x_2 < u(x_1)\}$ be the subgraph of u. We assume that the oscillation of the string does not exceed a size of order ε, i.e. $|u| < \varepsilon$ and so $\partial E \subset \{(x_1,x_2)\in\mathbb{R}^2 \text{ s.t. } |x_2| < \varepsilon\}$. Then we define the length of the string in the set $Q := [-1,1]\times[-\varepsilon,\varepsilon]$ as

$$\ell_\sigma(u) := I(E\cap Q, \mathbb{R}^2\setminus E) + I(Q\setminus E, E\setminus Q),$$

where, for any couple of disjoint measurable sets X, $Y \subset \mathbb{R}^2$ we set

$$I(X,Y) := \int_{X\times Y}\frac{dx\,dy}{|x-y|^{2+\sigma}}.$$

It is known that (up to a suitable rescaling) ℓ_σ tends to the classical length of the string as $\sigma \to 1$ (see [7, 30]). Of course, the fractional length of the string at rest here is simply $\ell_\sigma(0)$, and so the difference between the fractional length of the string and its original value is

$$\ell_\sigma := \ell_\sigma(u) - \ell_\sigma(0).$$

So it is conceivable to replace in the model the dependence from the classical length with the dependence of this "non–local" version of length, i.e. to substitute (A.0.2) with

$$M(\ell_\sigma) = m_0 + 2b\ell_\sigma. \tag{A.0.6}$$

Moreover, ℓ_σ may be computed in terms of u thanks to the following geometric observation. Let

$$\begin{aligned} & E^+ := \{(x_1,x_2)\in\mathbb{R}^2 \text{ s.t. } 0<x_2<u(x_1)\}, \\ & E^- := \{(x_1,x_2)\in\mathbb{R}^2 \text{ s.t. } u(x_1)<x_2<0\}, \\ & W^+ := \mathbb{R}\times(0,+\infty), \\ & W^- := \mathbb{R}\times(-\infty,0) \\ \text{and} \qquad & Q^\pm := Q\cap W^\pm. \end{aligned}$$

Then

$$\begin{aligned} & \ell_\sigma(u) \\ = \; & I((Q^-\setminus E^-)\cup E^+,(W^+\setminus E^+)\cup E^-)+I((Q^+\setminus E^+)\cup E^-,W^-\setminus Q) \\ = \; & I(Q^-\setminus E^-,W^+\setminus E^+)+I(Q^-\setminus E^-,E^-) \\ & +I(E^+,W^+\setminus E^+)+I(E^+,E^-) \\ & +I(Q^+\setminus E^+,W^-\setminus Q)+I(E^-,W^-\setminus Q) \end{aligned}$$

and

$$\ell_\sigma(0)=I(Q^-,W^+)+I(Q^+,W^-\setminus Q).$$

Moreover

$$\begin{aligned} & I(Q^-,W^+)-I(Q^-\setminus E^-,W^+\setminus E^+) \\ = \; & I(Q^-\setminus E^-,E^+)+I(E^-,W^+\setminus E^+)+I(E^-,E^+) \end{aligned}$$

and

$$I(Q^+,W^-\setminus Q)-I(Q^+\setminus E^+,W^-\setminus Q)=I(E^+,W^-\setminus Q).$$

As a consequence

$$\begin{aligned} \ell_\sigma \;=\; & I(Q^-\setminus E^-,E^-)+I(E^+,W^+\setminus E^+) \\ & +I(E^-,W^-\setminus Q)-I(Q^-\setminus E^-,E^+) \\ & -I(E^-,W^+\setminus E^+)-I(E^+,W^-\setminus Q). \end{aligned}$$

By collecting all the terms involving E^+ and E^- and using that $I(X,Y)=I(Y,X)$ we obtain

$$\begin{aligned} \ell_\sigma \;=\; & I(E^+,W^+\setminus E^+)-I(E^+,W^-\setminus E^-) \\ & +I(E^-,W^-\setminus E^-)-I(E^-,W^+\setminus E^+). \end{aligned} \tag{A.0.7}$$

We now write separately the first two terms. For typographical convenience we use the notation of writing the integrating variables next to their integral sign. Also, we set $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$: notice that $u^\pm \geqslant 0$ and $u = u^+ - u^-$. In this way, $E^+ = \{(x_1, x_2) \in \mathbb{R}^2 \text{ s.t. } 0 < x_2 < u^+(x_1)\}$, $E^- := \{(x_1, x_2) \in \mathbb{R}^2 \text{ s.t. } -u^-(x_1) < x_2 < 0\}$,

$$\begin{aligned} & I(E^+, W^+ \setminus E^+) \\ = & \int_{\mathbb{R}} dx_1 \int_0^{u^+(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \int_{u^+(y_1)}^{+\infty} dy_2 \big(|x_1 - y_1|^2 + |x_2 - y_2|^2\big)^{-(2+\sigma)/2} \end{aligned}$$

and

$$\begin{aligned} & I(E^+, W^- \setminus E^-) \\ = & \int_{\mathbb{R}} dx_1 \int_0^{u^+(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \int_{-\infty}^{-u^-(y_1)} dy_2 \big(|x_1 - y_1|^2 + |x_2 - y_2|^2\big)^{-(2+\sigma)/2}. \end{aligned}$$

Thus, we set $\psi = \psi(x_1, y_1, z_2) := \big(|x_1 - y_1|^2 + |z_2|^2\big)^{-(2+\sigma)/2}$, we make the substitution $z_2 := y_2 - x_2$ and we get

$$\begin{aligned} & I(E^+, W^+ \setminus E^+) - I(E^+, W^- \setminus E^-) \\ & = \int_{\mathbb{R}} dx_1 \int_0^{u^+(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \left[\int_{u^+(y_1)-x_2}^{+\infty} dz_2 - \int_{-\infty}^{-u^-(y_1)-x_2} dz_2 \right] \psi. \end{aligned} \tag{A.0.8}$$

Now we observe that

$$\int_{-\infty}^0 dz_2\ \psi = \int_0^{+\infty} dz_2\ \psi,$$

since ψ is even in z_2. Therefore

$$\begin{aligned} & \left[\int_{u^+(y_1)-x_2}^{+\infty} dz_2 - \int_{-\infty}^{-u^-(y_1)-x_2} dz_2 \right] \psi \\ = & \left[\int_{u^+(y_1)-x_2}^{0} dz_2 + \int_0^{+\infty} dz_2 - \int_{-\infty}^0 dz_2 - \int_0^{-u^-(y_1)-x_2} dz_2 \right] \psi \\ = & \left[\int_{u^+(y_1)-x_2}^{0} dz_2 - \int_0^{-u^-(y_1)-x_2} dz_2 \right] \psi, \end{aligned}$$

hence (A.0.8) becomes

$$\begin{aligned} & I(E^+, W^+ \setminus E^+) - I(E^+, W^- \setminus E^-) \\ & = - \int_{\mathbb{R}} dx_1 \int_0^{u^+(x_1)} dx_2 \int_{\mathbb{R}} dy_1 \left[\int_0^{u^+(y_1)-x_2} dz_2 + \int_0^{-u^-(y_1)-x_2} dz_2 \right] \psi. \end{aligned} \tag{A.0.9}$$

At this point, we make the crude approximation

$$\int_0^{\varepsilon'} dz_2\ \psi \simeq \psi\Big|_{z_2=0} \varepsilon' = |x_1 - y_1|^{-(2+\sigma)} \varepsilon', \tag{A.0.10}$$

when ε' is of the order of ε. As a matter of fact, such approximation is not fully justified when x_1 and y_1 are in a neighborhood of size much smaller than ε, due to the singularity of the kernel: since this appendix is mainly motivational, and should not be interpreted in a strictly rigorous mathematical language, we neglect this subtle point and just take the ansatz that (A.0.10) is reasonable for most of the points of integration x_1 and y_1 and see what happens. Similarly, we observe that, for $s := \frac{\sigma+1}{2}$, at least formally and in the principal value sense

$$\begin{aligned} \|u\|^2_{H^s(\mathbb{R})} &= \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u(x_1) - u(y_1)|^2}{|x_1 - y_1|^{1+2s}} \\ &= \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u(x_1)|^2 - u(x_1)u(y_1) + |u(y_1)|^2 - u(y_1)u(x_1)}{|x_1 - y_1|^{2+\sigma}} \\ &= 2\int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u(x_1)|^2 - u(x_1)u(y_1)}{|x_1 - y_1|^{2+\sigma}}, \end{aligned} \tag{A.0.11}$$

thanks to the symmetric role played by x_1 and y_1.

From (A.0.10) we obtain the approximation

$$\begin{aligned} &\left[\int_0^{u^+(y_1)-x_2} dz_2 + \int_0^{-u^-(y_1)-x_2} dz_2\right]\psi \\ &\qquad \simeq |x_1 - y_1|^{-(2+\sigma)}\big(u^+(y_1) - u^-(y_1) - 2x_2\big) \\ &\qquad = |x_1 - y_1|^{-(2+\sigma)}\big(u(y_1) - 2x_2\big). \end{aligned}$$

Therefore, up to terms that we neglected,

$$\begin{aligned} &\int_0^{u^+(x_1)} dx_2 \left[\int_0^{u^+(y_1)-x_2} dz_2 + \int_0^{-u^-(y_1)-x_2} dz_2\right]\psi \\ = &\int_0^{u^+(x_1)} dx_2 |x_1 - y_1|^{-(2+\sigma)}\big(u(y_1) - 2x_2\big) \\ = &-|x_1 - y_1|^{-(2+\sigma)}\big(|u^+(x_1)|^2 - u^+(x_1)u(y_1)\big). \end{aligned}$$

Consequently, (A.0.9) becomes

$$\begin{aligned} &I(E^+, W^+ \setminus E^+) - I(E^+, W^- \setminus E^-) \\ &\quad = \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u^+(x_1)|^2 - u^+(x_1)u(y_1)}{|x_1 - y_1|^{2+\sigma}}. \end{aligned} \tag{A.0.12}$$

Notice also that a reflection of the vertical variable transforms the set E^+ of the function u into the set E^- for the function $-u$, and also $(-u)^+ = u^-$. Hence the symmetric version of (A.0.12) reads

$$\begin{aligned} &I(E^-, W^- \setminus E^-) - I(E^-, W^+ \setminus E^+) \\ &= \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u^-(x_1)|^2 + u^-(x_1)u(y_1)}{|x_1 - y_1|^{2+\sigma}}. \end{aligned} \tag{A.0.13}$$

Moreover, since, at any point x_1 either $u^+(x_1) = 0$ or $u^-(x_1) = 0$, we see that

$$|u(x_1)|^2 = |u^+(x_1)|^2 + |u^-(x_1)|^2.$$

Accordingly, by plugging (A.0.12) and (A.0.13) into (A.0.7) and we obtain the approximation

$$\ell_\sigma = \int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dy_1 \frac{|u(x_1)|^2 - u(x_1)u(y_1)}{|x_1 - y_1|^{2+\sigma}} = \frac{1}{2}\|u\|^2_{H^s(\mathbb{R})},$$

where in the last step we used (A.0.11). By inserting this expression into (A.0.6) we obtain the approximated tension

$$M = m_0 + 2b\|u\|^2_{H^s(\mathbb{R})}.$$

Hence, a non–local model for the vibrating string may be obtained from (A.0.1), by considering the above tension and by replacing the local spatial second derivative with the non–local operator $-(-\Delta)^s$: in this way we obtain the non–local equation

$$-M\left(\|u\|^2_{H^s(\mathbb{R})}\right)(-\Delta)^s u + f = 0.$$

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