LACK OF GROUND STATE FOR NLS ON BRIDGE-TYPE GRAPHS

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Abstract. We prove the nonexistence of ground states for NLS on bridge-like graphs, i.e. graphs with two halflines and four vertices, of which two at infinity, with Kirchhoff matching conditions. By ground state we mean any minimizer of the energy functional among all functions with the same mass.

1. Introduction

A *graph* $\mathcal{G}$ consists of a set V of points $v_1, \dots, v_{N_v}$ called *vertices* and a set A of *edges* $e_1, \dots e_{N_e}$ joining pairs of vertices. Multiple connections between the same couple of vertices (i.e. several edges between the same vertices) and also edges connecting a vertex with itself, called *self-loops*, are allowed. We assume that the number N_v of vertices as well as the number N_e of edges, are finite.
We require $\mathcal{G}$ to be a *metric* graph, that is we identify every edge with a real interval, namely

$$e_j \longmapsto I_j := (0, l_j)$$

with $l_j \in (0, +\infty]$.

Notice that a given vertex v can act both as the left endpoint for an edge and as the right endpoint for another one. It is then meaningful to define on the set A the functions R and L, such that $L(e_j) = v$ if v is the left endpoint of e_j, and $R(e_j) = v$ if v is the right endpoint of e_j.
Owing to the metric structure, it is natural to define functions $u : \mathcal{G} \longrightarrow \mathbb{C}$ as

$$u := (u_1, \dots u_{N_e}),$$

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where $u_j : I_j \longrightarrow \mathbb{C}$ is the restriction of u to the edge e_j. The definition of u is made complete by specifying the value of u at any vertex of $\mathcal{G}$. We define L^p spaces on $\mathcal{G}$ according to the norm

$$\|u\|_{L^p(\mathcal{G})}^p \;=\; \sum_{k=1}^{N_e} \|u_k\|_{L^p(I_k)}^p.$$

(In the following we will use the shorthand notation $\|u\|_p \;=\; \|u\|_{L^p(\mathcal{G})}$ and $\|u_j\|_p \;=\; \|u_j\|_{L^p(I_j)}$.)
Analogously, we define the space $H^1(\mathcal{G})$ as the subspace of $L^2(\mathcal{G})$ consisting of functions u such that $u' := (u'_1, \ldots u'_{N_e})$ is an element of $L^2(\mathcal{G})$ too, and satisfies the *continuity condition* at vertices, that states that the limit of $u(x)$ as x approaches a vertex v exists and is independent of the particular edge on which x runs.

Owing to the previous definitions, one can introduce the *energy functional*

$$E(u, \mathcal{G}) \;=\; \frac{1}{2}\|u'\|_2^2 - \frac{1}{p}\|u\|_p^p, \tag{1.1}$$

defined on any function $u \in H^1(\mathcal{G})$. It is well-known that this functional corresponds to the conserved energy of the equation

$$i\partial_t u(t) = -\Delta u(t) - |u(t)|^{p-2}u(t),$$

i.e., a nonlinear Schrödinger equation on $\mathcal{G}$ with nonlinearity power $p-1$ (for a general introduction to NLS see [13]). In such equation multiplication and powers are to be understood componentwise (i.e. edge by edge), while the definition of the Laplacian has to be completed by Kirchhoff boundary conditions (see the end of this Section, and [18] for the classification of all self-adjoint vertex conditions).
There is nowadays a huge literature on *quantum graphs*, i.e. metric graphs with differential or pseudo-differential operators acting on functions defined on it ([21, 7, 8, 15, 19, 20]). However, most of such a literature is concerned with linear systems, while for nonlinear systems the research started more than two decades ago ([3]), got relevant results, but has remained less extensive ([9, 12, 11, 10, 22]). Important

results have been obtained for dispersive estimates, that are often used to link linear and nonlinear evolutions ([4, 5]).

NLS-type equations are currently used to model several systems where the propagation of waves in branched structures is relevant: Bose-Einstein condensates in ramified traps, optical fibers, T-junctions and others. In all applications it proves important to get information on stationary solutions (i.e. the modes of the system) and on their stability. The stationary solutions for which one can typically state a stability result are the *ground states* of the systems, i.e., the minimizers, possibly under suitable constraints, of the functional (1.1) ([14, 23, 16]). Observe indeed that this functional is not bounded from below, since, for all non-trivial u, $E(\lambda u) \to -\infty$ as $\lambda \to +\infty$. However, as soon as the nonlinearity power p is *subcritical*, i.e.

$$2 \; < \; p \; < 6,$$

the restriction of E to the manifold of functions u sharing the same, fixed, value μ for the *mass*, namely the constraint

$$\|u\|_2^2 \; = \; \mu > 0,$$

is bounded from below. Indeed, by the Gagliardo-Nirenberg inequality

$$\|u\|_p \; \leq \; C \, \|u\|_2^{\frac{1}{2}+\frac{1}{p}} \|u\|_{H^1}^{\frac{1}{2}-\frac{1}{p}}$$

that can be easily extended to graphs with a finite number of edges, one has

$$E(u, \mathcal{G}) \; \geq \; \frac{1}{2}\|u'\|_2^2 - C\|u'\|_2^{\frac{p}{2}-1} - C$$

where the mass constraint was taken into account, and lower boundedness immediately follows. So the following questions arise.

(i) Is the infimum of E on the space $H^1_\mu = \{u \in H^1, \; \|u\|_2^2 = \mu\}$ larger or smaller than the infimum on the line?

(ii) Is the infimum attained?

Question (ii) can be rephrased in a more physical language, as follows: does there exist a ground state?

The answers to (i) and (ii) depend on the nature of $\mathcal{G}$, as the following examples illustrate.