Aspects of Lagrange's Mechanics and their Legacy

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Abstract

From the "vibrating string" and "Kepler's equation" theories to relativistic quantum fields, perturbation theory, (divergent) series resummations, KAM theory.

Key words: *Vibrating string, Quantum fields, Perturbation theory, Divergent series resummation, KAM* [1]

1 Continuity and the vibrating string

In 1759 an important open problem was how general was the solution by Euler of the wave equation for a string starting from an initial configuration in which it is given a shape $u_0(x) = \varphi(x)$ and no initial velocity:

$$\begin{aligned}\partial_t^2 u =& c^2\partial_x^2 u, \qquad x\in[0,a],\ u(a)=u(b)=0\\ u(x,t) =& \varphi(x-ct)+\varphi(x+ct)\end{aligned} \tag{1.1}$$

According to D'Alembert's arguments it should have been necessary that $\varphi(x)$ be at least a *smooth* (twice differentiable in present notations or, in D'Alembert's language, subject to the "loi de continuté) function of $x \in R$ which is *periodic with period* $2a$ and odd around 0 and a. In his notations φ had to satisfy conditions at $0, a$ allowing its continuation to a $2a$–periodic function of class $\mathcal{C}_2$, odd around 0 and a.

Euler claimed that Eq.(2.1) would be a solution $u(x,t) = \varphi(x-ct)+\varphi(x+ct)$ if $\varphi(x)$ was simply defined outside the interval $[0, a]$ just continuing it as a periodic function of period $2a$ odd around 0 and a. He insisted that the

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($\mathcal{C}_1$) smoothness of $\varphi(x)$ is sufficient: this means that as time t becomes > 0 the form of the string appears (in general) to keep points with discontinuous curvature: *i.e.* points in which the second derivative is not defined. This led D'Alembert to think that when the continuation of φ was not smooth the solutions did not make sense.

Neither was able to produce a rigorous argument. Other theories were due to Taylor who had proposed that a general solution of the string motion (starting from an initial configuration in which the string is given a shape $u_0(x) = \varphi(x)$) is a sum $u(x) = \sum_n \alpha_n \sin(\frac{2\pi}{2a}nx)\cos(\frac{2\pi}{2a}nct)$: a view supported by D. Bernoulli (according to Lagrange). This is criticized by Euler who objects that such expressions only represent D'Alembert's solutions (which is true *if the series converges* in class $\mathcal{C}_2$).

All solutions had been obtained by arguments relying on unproved properties based on intuition or experience and this is the reason identified by Lagrange as the source of the controversies.

Lagrange's idea is that the problem must first be (in modern locution) *regularized*: this means imagining the string to consist of an indefinite number m of small particles aligned (in the rest state) and elastically interacting with the nearest neighbors possibly moving orthogonally to the rest line and with extremes $y_0 = y_m = 0$ fixed.

In this way the problem becomes a clearly posed mechanical problem and the equations of motion are readily derived: the only question is therefore finding the properties of their solutions and of the limit in which the mass of the particles tends to zero while their number and the strength of the elastic force tend to infinity so that a continuum string motion emerges, [1, T.I, p.71]:

Il resulte de tout cet exposé que l'Analyse que nous avons proposée dans le Chapitre précédent est peut-être, la seule qui puisse jeter sur ces matières obscures une lumière suffisante à éclaircir les doutes qu'on forme de part et d'autre.

The analysis is today well known: he notices that the problem is (in modern language) the diagonalization of a $(m-1) \times (m-1)$ tridiagonal symmetric matrix. The solution is perhaps the first example of the diagonalization procedure of a large matrix. The result is the representation of the general motion of the chain with general initial data for positions *and velocities*, [1, T.I, p.97]:

.. je ne crois pas qu'on ait jamais donné pour cela une formule générale, telle que nous venons de la trouver.

Lagrange's very remarkable three memories, the first two constitute a veritable monograph, consist in showing that the general motion has, for suitable ω_h, the form (translated in modern language)

$$y^{(\delta)}(\xi,t)=\sum_{h=1}^{m-1}\Big\{\widetilde{A}_h\sqrt{\frac{2}{m}}\sin\frac{\pi\,h}{a}\xi\cdot\cos\omega_h t+\widetilde{B}_h\sqrt{\frac{2}{m}}\sin\frac{\pi\,h}{a}\xi\cdot\sin\omega_h t\Big\},\quad (1.2)$$

where, denoting by δ (called dx by Lagrange) the mesh of the discretized positions so that the number of small masses, located at points $\xi=i\delta$, is $\frac{a}{\delta}$ and $\widetilde{A}_h,\widetilde{B}_h,\omega_h$ are derived from the initial profiles of positions $Z(\xi)$ and velocities $U(\xi)$, [1, T.I., p.163]: via the expressions

$$\begin{aligned}
&\sqrt{\frac{2}{m}}\widetilde{A}_h=\frac{2}{m}\sum_{i=1}^{m-1}(\,\sin\frac{\pi h}{a}\xi)Z(\xi)\xrightarrow[\delta\to 0]{}\frac{2}{a}\int_0^a Z(x)(\,\sin\frac{\pi h}{a}x)\,dx\\
&\sqrt{\frac{2}{m}}\widetilde{B}_h=\frac{2}{\omega_h\,m}\sum_{i=1}^{m-1}(\,\sin\frac{\pi h}{a}\xi)U(\xi)\xrightarrow[\delta\to 0]{}\frac{2}{\overline{\omega}_h\,a}\int_0^a U(x)(\,\sin\frac{\pi h}{a}x)\,dx,\\
&\omega_h=c\sqrt{2\frac{1-\cos(\frac{\pi h\delta}{a})}{\delta^2}}\xrightarrow[\delta\to 0]{}\overline{\omega}_h=c\frac{\pi h}{a}
\end{aligned}\quad (1.3)$$

where c is $\sqrt{\frac{\tau}{\mu}}$ where τ is the tension and μ is the density of the string. The formulae do not require smoothness assumptions on the data Z,U other than integrability as stated in answer to D'Alembert's critique, [1, T.I, p.324]:

Mais je le prie de faire attention que, dans ma solution, la détermination de la figure de la corde à chaque instant dépend uniquement des quantités Z et U, lesquelles n'entrent point dans l'opération dont il s'agit. Je conviens que la formule à laquelle j'applique la méthode de M. Bernoulli est assujettie à la loi de continuité; mais il ne me parait pas s'ensuivre que les quantités Z et U, qui constituent le coefficient de cette formule, le soient aussi, comme M. d'Alembert le prétend.

Hence, setting $\xi=x$, the general solution:

$$\begin{aligned}u(x,t)=\sum_{h=0}^{\infty}\sin\frac{\pi h}{a}x\Big\{&(\frac{2}{a}\int_0^a Z(x')\sin\frac{\pi h}{a}x'\,dx')\cos\overline{\omega}(h)t\\
&+(\frac{2}{a}\int_0^a U(x')\sin\frac{\pi h}{a}x'\,dx')\frac{\sin\overline{\omega}(h)t}{\overline{\omega}(h)}\Big\}\end{aligned}\quad (1.4)$$

is found: and it is far more general than the previous solutions because it permits initial data with $U\neq 0$, *i.e.* with initial data in which the string is already in motion. In the case $U = 0$ it coincides with Euler's as it can be written $\varphi(x+ct)+\varphi(x-ct)$ with φ odd around 0 and a and $2a$-periodic aside from the obvious convergence and exchange of sum and limits problems.

Check of Eq.(1.3) is based on some trigonometric properties: basically on "Cote's formula", [1, T.I, p.75], $|a^m - b^m| = \prod_{h=0}^{m-1}(a^2 - 2ab\cos\frac{2\pi}{m}h + b^2)^{\frac{1}{2}}$ (derivable from $a^m - b^m = \prod_{h=0}^{m-1}(ae^{i\frac{2\pi}{m}h} - b)$) which is used instead of the modern $\sum_{h=0}^{m-1} e^{i\frac{2\pi}{m}h(p-q)} = m\delta_{p,q}$).

The continuum limit Eq.(1.4) is identified with Euler's result: it is remarkably found together with interesting considerations and attempted justifications of resummations of divergent series like, [1, T.I, p.111],

$$\cos x + \cos 2x + \cos 3x + \ldots = -\frac{1}{2} \tag{1.5}$$

an incorrect statement immediately criticized by D'Alembert but whose use Lagrange defended correctly, [1, T.I, p.322].[1]

The analysis might appear not rigorous in today sense: not really because of the statement Eq.(1.5) nor because the statement that $\sin\frac{2\pi}{2}m(\frac{x}{a} \pm \frac{Ht}{T}) = 0$ as consequence of $m = \infty$, [1, T.I, p.102] which may make the eyebrows frown, as of course D'Alembert's did, and for which Lagrange almost "apologized":

"*Je conviens que je ne me suis pas exprimé assex exactment ..*",

[1, T.I, p.322], while making it more plausible (it seems to be an early version of what is today the "Lebesgue theorem" for Fourier series).

Also several other objections by D'Alembert and D. Bernoulli do not appear to have been answered very convincingly by the present standards in the memory in defense of the theory in [1, T.I, p.319-332]. The short note is still of great interest as it shows the Lagrange struggles and gets very close to the modern notion of "weak solution" of a PDE, see also [1, T.I, p.177], and to a formalization of a theory of resummation of divergent series.

[1]Lagrange uses this formula to infer the value of the difference $\Delta(x,\theta) = \frac{1}{2}\sum_{N=1}^{\infty}\cos n(x-\theta) - \frac{1}{2}\sum_{N=1}^{\infty}\cos n(x+\theta) = 0$ which "today" would instead be $(-\frac{1}{2} + \pi\delta(x-\theta) + \frac{1}{2} - \pi\delta(x+\theta)) = \pi(\delta(x-\theta) - \delta(x+\theta)) \neq 0$ unless $\theta = 0$. His argument is to consider first Eq.(1.5), which equals $-\frac{1}{2} + \pi\delta(x)$, and erroneously sets it $-\frac{1}{2}$, and the claim that $\Delta(x,0) = 0$ would be correct. Looking at the consequence of this error on Lagrange's theory it appears, as he correctly states in [1, T.I, p.111], that what is really used is the truncated version of the above relations and the Eq.(1.5) was only an alternative proposition: *il ne sera pas hors de propos de démontrer encore la même proposition d'une autre manière*, [1, T.I, p.109].

Today a mathematician will probably consider it not completely rigorous only because it applies to piecewise differentiable initial data Z, U for the positions and speeds of the string elements: but a proof of the exchanges of limits (and their existence) needed to "pass to the continuum limit" is not even mentioned.

Its need did not occur to Lagrange, in his 23-d year of age, as well as to Euler himself: the easy proof is in all textbooks, *e.g.* [2, Ch.4.5], and it is rightly considered that, *de facto*, Lagrange solved at the same time the controversies on the proof of completeness of the basis $\sin\frac{\pi}{a}nx$ for the functions on $[0, a]$ vanishing at $0, a$, *i.e* showed the convergence of Fourier's series for functions piecewise continuously differentiable.

The theory of the vibrating strings, in Lagrange, was motivated and became part of a long and detailed study of the propagation of sound in two exhaustive Memoires (followed by a short one to answer criticism): in which sound propagation, *i.e.* the wave equation, is analyzed in the one dimensional case and for three dimensional spherical waves (whose equation is taken from Euler and which he reduces to the one dimensional theory via the remarkable change of variables $u(x) = x^{-2}\int^x z(x)xdx$).

In the second Memory the ($\mathcal{C}_2$) continuity problem is also examined from a new viewpoint: namely to study the motion of the *internal points* of a string with extremes fixed. It is proved that the points move following the solution found with the discretization method. This is interesting also because the method presented will be recognized to be a precursor of the modern notion of *weak solution* of a PDE, [1, T.I, p.177]:

Les transformations dont je fais usage dans cette occasion sont celles qu'on appelle intégrations par parties, et qui se démontrent ordinairement par les principes du calcul différentiel; mais il n'est pas difficile de voir qu'elles ont leur fondement dans le calcul général des sommes et des différences; d'où il suit qu'on n'a point à craindre d'introduire par là dans notre calcul aucune loi de continuité entre les différentes valeurs de z.

The vibrating string theory and the continuum as a limit of a microscopically discrete reality was developed by Lagrange at a time when the atomistic conceptions were being established. The approach he adopted is a key legacy at the basis of methods currently employed in the most diverse fields, see for instance [3]: a further example will be discussed in the next section.

2 QFT: quantum elastic string

The key idea in the vibrating string has been that it is a continuous system which should be regarded as, and behaves as, a limit case of a system of infinitely many adjacent particles whose motion should be described via the ordinary equations *without* requiring new principles.

Interestingly this problem has reappeared essentially for the same reasons in recent times. The theory of elementary particles requires at the same time quantum mechanics and (at least) special relativity: it became soon clear that it could be appropriately formulated as a theory of quantized fields which met immediately impressive successes in the description of electromagnetic interactions of photons and electrons and of weak interactions. Particles were naturally represented by particular states of a field which could describe waves as well, [4, Sec.I].

The simplest example is a "scalar field", $\varphi(x)$, in space time dimension 2 corresponding classically to the Lagrangian

$$\mathcal{L} =\frac{\mu}{2}\int_\alpha^\beta \Big(\dot\varphi(x)^2-c^2(\frac{d\varphi}{dx}(x))^2-(\frac{m_0c^2}{\hbar})^2\varphi(x)^2\,-\,I(\varphi(x))\Big)\,dx \qquad (2.1)$$

where $I(\varphi)$ is some function of φ. If $I=0$ this is a vibrating string with density μ, tension $\tau=\mu c^2$ and an elastic pinning force $\mu(\frac{m_0c^2}{\hbar})^2$.

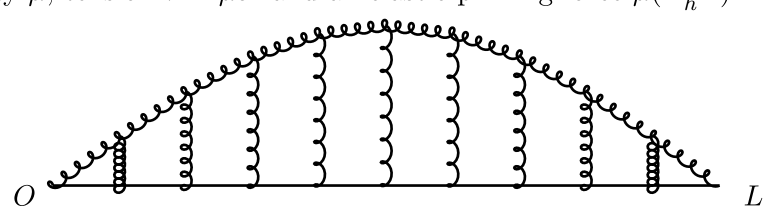

Fig.1

Figure 4.1: chain of oscillators elastically bound by nearest neighbors and to centers aligned on an axis orthogonal to their vibrations and Dirichlet's boundary condition.

The nonlinearity of the resulting wave equation produces the result that when two or more wave packets collide they emerge out of the collision quite modified and do not just go through each other as in the case of the linear string, so that their interaction is nontrivial.

Naively the quantum states will be, by the "natural extension of the usual quantization rules", functions $F(\varphi)$ of the function φ describing the configurational shape of the elastic deformations. The Hamiltonian operator acts on the wave function F as

$$\begin{aligned}(\mathcal{H}F)(\varphi) = \int_0^L \Big(&-\frac{\hbar^2}{2\mu}\frac{\delta^2 F}{\delta\,\varphi(x)^2}(\varphi) + \frac{\mu}{2}\Big(c^2(\frac{\partial\varphi}{\partial\mathbf{x}}(\mathbf{x}))^2 + \\ &+(\frac{m_0c^2}{\hbar})^2\varphi(x)^2 + I(\varphi(\mathbf{x}))\Big)\,F(\varphi)\Big)\;d^D\mathbf{x}\end{aligned} \tag{2.2}$$

where $\frac{\delta}{\delta\varphi(x)}$ is the functional derivative operator (a notion also due to Lagrange and to his calculus of variations) and it should be defined in the space $L_2(\text{“}d\varphi''\,)$, where the scalar product ought to be $(F,G)=\int\overline{F(\varphi)}\,G(\varphi)\,''d\varphi''$ and $''d\varphi''=\prod_{\mathbf{x}\in[0,L]}d\varphi(\mathbf{x})$.

Even though by now the mathematical meaning that one should try to attach to expressions like the above as “infinite dimensional elliptic operators” and “functional integrals” is quite well understood, particularly when $I\equiv 0$, formulae like the above are still quite shocking for conservative mathematicians, even more so because they turn out to be very useful and deep.

One possible way to give meaning to (2.2) is to go back to first principles and recall the classical interpretation of the vibrating string as a system of finitely many oscillators, following Lagrange's brilliant theory of the discretized wave equation and of the related Fourier series summarized in Sec.1.

Suppose, for simplicity, that the string has periodic boundary conditions instead of the Dirichlet's conditions studied by Lagrange; replace it with a lattice Z_δ with mesh $\delta>0$ and such that L/δ is an integer. In every point $n\delta$ of Z_δ put an oscillator with mass $\mu\,\delta$, described by a coordinate $\varphi_{n\delta}$ giving the elongation of the oscillator over its equilibrium position, and subject to elastic pinning force with potential energy $\frac12\mu\delta\,(\frac{m_0c^2}{\hbar})^2\varphi_{n\delta}^2$, to a nonlinear pinning force with potential energy $\frac12\mu\delta I(\varphi_{n\varepsilon})$. and finally to a linear elastic tension, coupling nearest neighbors at positions $n\delta,(n+1)\delta$, with potential energy $\frac12\mu\delta^{-1}c^2(\varphi_{n\delta}-\varphi_{(n+1)\delta})^2$.

Therefore the Lagrangian of the system, in the more general case of space dimension $D\ge 1$ ($D=1$ is the vibrating string, $D=2$ is the vibrating film, *etc*, *i.e.* space time dimension $d=D+1$), is

$$\mathcal{L}=\frac{\mu}{2}\delta^D\sum_{n\delta\in\Lambda_0}\Big(\dot\varphi_{n\delta}^2-c^2\sum_{j=1}^D\frac{(\varphi_{n\delta+e_j\delta}-\varphi_{n\delta})^2}{\delta^2}-(\frac{m_0c^2}{\hbar})^2\varphi_{n\delta}^2-I(\varphi_{n\delta})\Big) \tag{2.3}$$

where e_j is a unit vector oriented as the directions of the lattice; if $n\delta+e_j\delta$ is not in Λ_0 but $n\delta$ is in Λ_0 then the j-th coordinate equals L and $n\delta+e_j\delta$ has to be interpreted as the point whose j-th coordinate is replaced by 1; *i.e.*

is interpreted with periodic boundary conditions with coordinates identified modulo L.

It should be remarked that for $I = 0$ and $m_0 = 0$ Eq.(2.3) is, for $D' = 1$, the Lagrangian of the vibrating string introduced by Lagrange in his theory of sound.

Of course there is no conceptual problem in quantizing the system: it will correspond to the familiarly elliptic operator on $L_2(\prod_{n\varepsilon} d\varphi_{n\varepsilon}) = L_2(R^{\frac{L}{\delta}})$:

$$\begin{aligned}\mathcal{H}_\delta = &- \frac{\hbar^2}{2\mu\delta} \sum_{n\delta\in\Lambda_o} \frac{\partial^2}{\partial\varphi_{n\delta}^2} + \frac{\mu\delta}{2} \sum_{n\delta\in\Lambda_o} \\ &\Big(c^2 \frac{(\varphi_{n\delta+e\delta} - \varphi_{n\delta})^2}{\delta^2} + (\frac{m_0 c^2}{\hbar})^2 \varphi_{n\delta}^2 + I(\varphi_{n\delta})\Big)\end{aligned} \tag{2.4}$$

with $\mathcal{C}_0^\infty(R^{\frac{L}{\delta}})$ as domain (of essential self-adjointness) provided $I(\varphi)$ is assumed bounded below, as it should always be: *e.g.* in the case of the so called $\lambda\varphi^4$ *theory* with $I(\varphi) = \int_0^L (\lambda\varphi(x)^4 + \mu\varphi(x)^2 + \nu)dx$ with $\lambda > 0$.

At this point it could be claimed, with Lagrange [1, T.I, p.55],

Ces équations, comme il est aisé de le voir, sont en même nombre que les particules dont on cherche les mouvements; c'est pourqoi, le problème étant déjà absolument determiné par leur moyen, on est obligé de s'en tenir là, de sorte que toute condition étrangère ne peut pas manquer de rendre la solution insuffisante et même fautive.

This can be appreciated by recalling that in developing the theory, *even in the simplest case of the $\lambda\varphi^4$ field, just mentioned*, difficulties arise which have given rise to many discussions.

In particular the attempts to study the properties of the operator $\mathcal{H}$ via expansions in λ, μ, ν immediately lead to nonsensical results (infinities or indeterminate expressions). At the beginning of QFT the results were corrected by adding "*counterterms*" amounting, in the case of the string *i.e.* of space-time of dimension 2, to make μ, ν infinite.

The subtraction prescriptions, known as "*renormalization*") were not really arbitrary: the remark was that *all results* were given by many integrals which were divergent but in which a divergent part could be naturally isolated by bounding first the integration domains and by determining μ, ν as functions of the parameters defining the domains boundaries so that the infinities disappear when the boundaries of the domains of integration are removed.

Reassuringly the choice of the divergent "counterterms" μ, ν turned out to be essentially independent of the particular result that was being computed.

Even though the *renormalization* procedure was unambiguous (at least it was claimed to be such) it was not clear (at least not to many) that some new rule, *i.e.* a new physical assumption, was not implicitly introduced in the process. This is analogous to the controversy on the vibrating string solved by Lagrange with his theory of sound via discretization of the string, solution of its motion and remotion of the regularization, *i.e.* taking the particles mass to 0, the tension to ∞ at proper rates as $\delta \to 0$ so that all results remained well defined and converging to limits.

In the 1960's reducibility of renormalization theory to the rigorous study of the properties of the operator in Eq.(2.4) was started and at least in the cases of the quantum string and of the quantum film (*i.e.* the cases of space time dimension 2 and 3, respectively) it was fully understood in a few (very few) cases via the work of Nelson, Glimm-Jaffe, Wilson who showed clearly that no infinities really arise if the problem is correctly studied, [4]: *i.e.* taking seriously the discretized Hamiltonian computing physically relevant quantities and passing to the continuum limit, just as Lagrange did in his theory of sound.

This has been a major success of Physics and Analysis. The problem remains open in the case of $d = D + 1 = 4$ space-time dimensions and the reason is precisely that the nature of the regularization becomes essential in higher dimension: it seems that the naive discretization on a square lattice described here implies assumptions on the Physics at very small scale so that it might be impossible to perform the continuum limit $\delta \to 0$ unless the regularization of the formal functional derivatives in the Hamiltonian is chosen conveniently. It is conjectured that a naive discretization like the above Eq.(2.4) cannot lead to a nontrivial result if $d = 4$ and $I(\varphi)$ is a fourth order polynomial.

Therefore a challenge remains to explain why the renormalized quantum electrodynamics in 4-space time dimensions, only defined as a formal power expansion in the elementary electric charge, gives amazingly precise results perfectly agreeing with experiments although the known regularizations are all conjectured to yield trivial result if studied under full mathematical rigor. In recent times the regularization choice appears to have been related to very new conceptions of the nature at very small space-time scale (QCD and string theory): even in this respect Lagrange's method of attacking problems avoiding the introduction of new principles or of understanding their necessity is still fertile.

3 Perturbations in field theory

The problems of QFT are often studied by perturbation theory and a key role is played by *i.e.* scale invariant theories. For instance consider the elliptic operator $\mathcal{H}_\delta$ in Eq.(2.4) for a quantum string, written as $H^0_\delta + I_\delta$ separating the interaction $I_\delta(\varphi) = \lambda\varphi^4 + \mu\varphi^2 + \nu$.

Perturbation theory as developed by Lagrange might seem at first sight quite far from its use in QFT: yet it is quite close as it will be discussed in this section after giving some details on the form in which it arises in renormalization theory and how it appears as an implicit functions problem.

Restricting the discussion to the scalar φ^4-systems, Eq.(2.4), the breakthrough has been Wilson's theory which shows that the values of physical observables can be can be constructed, for the φ^4 system in dimensions $d = 2, 3$, as power series in parameters $\Lambda_k \stackrel{def}{=} (\lambda_k, \mu_k, \nu_k)$, $k = 0,\ 1, 2, \ldots$, called *running coupling constants*, which are related by

$$\Lambda_k = M\Lambda_{k+1} + B(\{\Lambda_r\}_{r=k+1}^{\infty}), \qquad k = 0, 1, \ldots \tag{3.1}$$

where M is a diagonal matrix with elements m_1, m_2, m_3. The physical meaning of the running constants is that they control the physical phenomena that occur on length scale $2^{-k} ell_0$ here ℓ_0 is a natural length scale associated with the system, *e.g.* $\ell_0 = \frac{\hbar}{m_0 c^2}$.

The series for the observable quantities values are *well defined and no infinities appear provided* the sequence Λ_k consists of uniformly bounded elements.

Hence the physical observables values can be expressed in terms of a well defined power series in few parameters, the Λ_k's, which are in turn functions of three independent parameters (the Λ_0, for instance) whose values define the theory. However the values of physical observables are very singular functions and their expansion in powers of the physical parameters (*i.e.* Λ_0, for instance) would be meaningless. In other words the infinities appearing in the heuristic theory are due to singularities in the Λ_k as functions of a single one among them: they are due to overexpanding the solution.

This leaves however open the problems:

(1) existence of a bounded sequence of "running couplings" Λ_k

(2) convergence of the series for the observables values

The second question has a negative answer because it is immediate that $\lambda_k < 0$ cannot be expected to be allowed: in the φ^4-systems they can be shown to be asymptotic series in space-time dimensions $2, 3$.

The first question has an answer in space-time dimensions $2, 3$ (ultimately due to $m_1 < 1$ and $m_2, m_3 > 1$) while in 4 dimensions space-time one more expansion parameter α_k is needed and $m_1, m_4 = 1$ but it seems impossible to have a bounded sequence with $\lambda_k > 0$ (the *triviality conjecture* proposes actual impossibility but it is a delicate and open problem).

The relation with Lagrange's work arises from Eq.(3.1) considered as an equation for $\Lambda \stackrel{def}{=} \{\Lambda_k\}_{k=0}^{\infty}$ of the form $A = \Lambda - \mathcal{M}(\Lambda) - \mathcal{B}(\Lambda)$ with $A = 0$. In the development of perturbation theory the request of a bounded solution of the latter implicit function problem arises in the form that is naturally identified with the boundedness requirement on the solution that Lagrange's theory of "litteral equations" would yield (see below) for Eq.(3.1).

In [1, T.III, p.25] the following formula is derived

$$\alpha = x - \varphi(x), \quad \longleftrightarrow \quad \psi(x) = \psi(\alpha) + \sum_{k=1}^{\infty} \frac{1}{k!} \partial_\alpha^{k-1} (\varphi(\alpha)^k \partial_\alpha \psi(\alpha)) \tag{3.2}$$

for all ψ. In particular for $\psi(\alpha) \equiv \alpha$

$$x(\alpha) = \alpha + \sum_{k=1}^{\infty} \frac{1}{k!} \partial_\alpha^{k-1} (\varphi(\alpha)^k), \qquad x(0) = \sum_{k=1}^{\infty} \frac{1}{k!} \partial_\alpha^{k-1} (\varphi(\alpha)^k) \Big|_{\alpha=0} \tag{3.3}$$

are expressions for the inverse function and for a solution of $x = \varphi(x)$. For the remarkable formula to works it is necessary that the series converges: it does not always select among the roots the closest to α and this is pointed out in [5], criticizing a related statement by Lagrange; it provides an interesting discussion of the properties of the root obtained by replacing φ by $t\varphi$ and studying which among the roots is associated with the sum of the series if the latter converges up to $t = 1$, theorem 3, p.18.

The formulae can be generalized to α, x in R^n, $n \geq 1$ and are used in the derivation and on the detailed analysis of Eq.(3.1) via the remark that the fixed point equation of Lagrange is equivalent to the following *tree expansion* of the $k - th$ term in Eq.(3.2) (when $\psi(x) = x$):

$$\frac{1}{k!} \partial_\alpha^{k-1} (\varphi(\alpha)^k) = \sum_\theta \mathrm{Val}(\theta) \tag{3.4}$$

where θ is a tree graph, *i.e.* it is a "decorated" tree with
(1) k branches λ of equal length oriented towards the "root" r,
(2) each node as well as the root (not considered a node) carries a label $j_v \in \{1, \ldots, n\}$
(3) at each node v enter k_v branches $\lambda_1 \equiv vv_1, \ldots, \lambda_{k_v} \equiv vv_{k_v}$ where

$v_1, \ldots, v_{k_v}$ are the k_v nodes preceding v (it is $\sum_{v<r} k_v = k$). and
(4) the node v symbolizes the tensor $\partial^{k_v}_{x_{j_{v_1}}, x_{j_{v_2}}, \ldots x_{j_{v_{k_v}}}} \varphi_{j_v}(\alpha)$

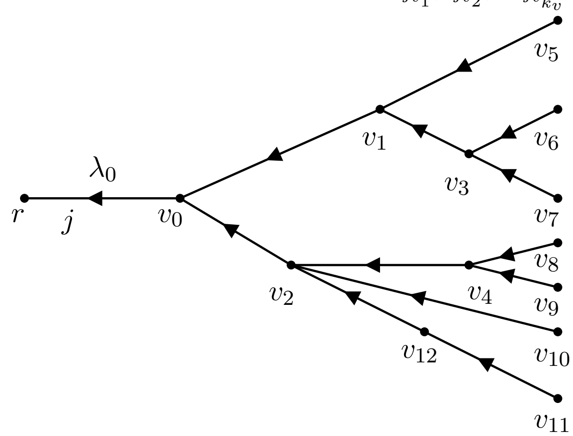

Fig.2

Fig.2: A decorated tree. Labels $j_v, v < r$ not marked and intended as contracted.

(5) Two such decorated trees are considered equivalent if superposable by pivoting the branches around the nodes without permitting overlapping.
(6) The *value*, $\mathrm{Val}\,(\theta)$ is defined as

$$\mathrm{Val}\,(\theta) = \prod_{v\in\theta} \Big(\frac{1}{k_v!}\partial^{k_v}_{j_{v_1},\ldots,j_{v_{k_v}}} \varphi_{j_v}(\alpha)\Big) \tag{3.5}$$

with sum over node labels, *except the root label* $j = j_r$, is understood.

It can be said that also the determination and theory of the functions B follows a path that can be traced to Lagrange's theory of "litteral equations".

The analysis in the φ^4-field theory context of the just described method of getting to the running couplings expansion can be found in [4]. Lagrange made use of Eq.(3.4) in his works on celestial mechanics, relying on his study of Kepler's equation giving eccentric anomaly ξ or true anomaly u in terms of average anomaly ℓ:

$$\ell = \xi + e\sin\xi, \qquad \xi = \ell + \sum_{k=1}^{\infty} \frac{e^k}{k!}\partial_\ell^{k-1}(\sin^k \ell) \tag{3.6}$$

The coefficients can be readily evaluated if $\sin^k \xi$ is expressed in terms of angles multiples of ℓ and since $\frac{du}{d\xi} = \frac{(1-e^2)^{\frac12}}{1+e\cos\xi}$ the expansion of the true anomaly is also computed applying Eq.(3.2) with $\varphi(\ell) = e\sin\ell$ and $\psi(\ell) = \frac{\sqrt{1-e^2}}{1+e\cos\ell}$, [1, T.III, p.114].

Several applications of perturbation theory have been considered by Lagrange often leading, as a byproduct, to key discoveries like the determination of the five Lagrangian points, [1, T.VI, p.280], found while attempting to find approximate solutions to the three body problem. And the many works on the secular variations of the planetary nodes, the determination of the orbit of comets, the Moon librations.

4 Overview on Mechanics

Lagrange's contribution to the formulation and application of the principles of Mechanics is so well known and modern that it is continuously used today. The 1788 *Traité de Mécanique Analytique*, whose second edition is in [1, T.XI] and [1, T.XII], is entirely based on a consistent reduction to first principles of every problem considered.

This means constantly returning to applying the least action principle in the form of the combination the the principle of virtual works with the principle of D'Alembert, [6, p.51]:

Décomposés les Mouvemens $a, b, c,$ &tc imprimés à chaque Corps, chacun en deux autres $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ &tc qui soient tels, que si l'on n'eût imprimé aux Corps que les Mouvemens $\alpha, \beta, \gamma,$ &tc ils eussent pû conserver ces Mouvemens sans se nuire réciproquement; et que si on ne leur eût imprimé que les Mouvemens $\alpha', \beta', \gamma',$ &tc le système fut demeuré en repos; il est clair que $\alpha, \beta, \gamma,$ &tc seront les Mouvemens que ces Corps prendront en vertu de leur action. Ce Q.F. trouver.[2]

Lagrange's addition is clearly explicitly stated in the second work on the Moon librations where he says, about his method:

... Elle n'est autre chose que le principe de Dynamique de M. d'Alembert, réduit en formule au moyen du principe de l'équilibre appelé communément loi des vitesses virtuelles. Mais la combinaison de ces deux principes est un pas qui n'avait pas été fait, et c'est peut-être le seul degré de perfection qui, après la découverte de M. d'Alembert, manquait encore à la Théorie de la

[2] The last words, "What was to be found", are there because the principle is "derived" as an answer to a problem posed in the previous page. Here $a, b, c,$ &tc are the impressed forces $\mathbf{f}_i$, and $\alpha, \beta, \gamma,$ &tc are $m_i\mathbf{a}_i$ while $\alpha', \beta', \gamma',$ &tc are the constraint reactions $-\mathbf{R}_i$: *i.e.* $\mathbf{f}_i = m_i\mathbf{a}_i - \mathbf{R}_i$; in modern form, [7, §3.18,Vol.2], the $\mathbf{f}_i - m_i\mathbf{a}_i$ may be called the "lost forces" and the principle is *During the motion of a system of point masses, however constrained and subject to forces, at each instant the lost forces are equilibrated by virtue of the constraints.*

Dynamique., [1, V.,p11].[3]

The equvalent action principle is used in the form

Dans le mouvement d'un système quelconque de corps animés par des forces mutuelles d'attraction, ou tendantes à des centres fixes, et prortionnelles à des fonctions quelconques des distances, les courbes décrites par les différents corps, et leurs vitesses, sont nécessairement telles que la somme des produits de chaque masse par l'intégrale de la vitesse multipliée par l'élèment de la courbe est un maximum ou un minimum, pourvu que l'on regarde les premiers et les derniers points de chaque courbe comme donnés, en sorte que les variations des coordonnées répondantes ces points soient nulles.

C'est le théorème dont nous avons parlé à la fin de la première Section, sous le nom de Principe de la moindre action, [1, T.XI, p. 318].

He arrives at the *Traité de Mécanique Analytique*, which "brought Rational Mechanics to a state of perfection", [8, p.221], as the completion of a long series of remarkable applications. Namely
(1) the theory of sound (*i.e.* the vibrating string) and the theory of the librations of the Moon, 1763, [1, T.VI, p.8] (revisited in [1, T.V, p.5], 1781), which is perhaps the first example of development and systematic application of pertubation theory based on analytical mechanics, *i.e.* on the action principle and the Euler-Lagrange equations.
(2) the theory of rigid body, which he undertakes curiously insisting in avoiding starting from the proper axes of rotation, [1, T.III, p.579]:

... ce qui exige la résolution d'une équation cubique. Cependant, à considérer le Problème en lui-mme, il semble qu'on devrait pouvoir le résoudre directement et indépendamment des propriétés des axes de rotation, propriétés dont la démonstration est assez difficile, et qui devraient d'ailleurs être plutôt des conséquences de la solution même que les fondements de cette solution,

he rederives the theory in an original way, certainly very involved for today eyes (used to Euler's equations as beginning point [9, p. 402], [10]), obtaining, as a byproduct, the theorem about the reality of the eigenvalues of a 3×3 symmetric matrix, [1, T.III, p. 605], and (later, in the *Traité*) the motion of the "Lagrange's top", [1, T.XII, p. 253]

Ce cas est celui où l'axe des coordonnées c, c'est--dire la droite qui passe par le point de suspension et par le centre de gravité, est un axe naturel de rotation, et où les moments d'inertie autour des deux autres axes sont égaux

[3] *i.e.* $\sum_i (\mathbf{f}_i - m_i \mathbf{a}_i) \cdot \delta \mathbf{x}_i = 0$.

(art. 32), ce qui a lieu en général dans tous les solides de révolution, lorsque le point fixe est pris dans l'axe de révolution. La solution de ce cas est facile, d'aprs les trois intégrales qu'on vient de trouver,

a remarkable integrability property that Poisson rediscovered (in fact he does not quote Lagrange, see the footnote by J. Bertrand at the *loc.cit.*).
(3) the theory of the secular variations of the planetary elements and several other celestial questions (the fifth volume of the collected paper is entirely dedicated to celestial mechanics, [1, T.V]) opening the way to Laplace's celestial mechanics and slightly preceding the completion and publication of the Mécanique. This was the first time analytical mechanics was used to attempt *long time* orbital predictions.
(4) the contributions are not only directed towards the principles but concentrate on concrete problems for instance the integration by separation of variables, *i.e.* by quadratures, of several dynamical systems: *e.g.* the two centers of gravitational attraction in the three dimensional case, [1, T.II, p.67], the mentioned symmetric top.
(5) The *Traité* also reflects the attitude of Lagrange regarding matter as constituted by particles: *i.e.* his consistently kept atomistic view, see for instance [1, T.XI, p.189]:

Quoique nous ignorions la constitution intérieure des fluides, nous ne pouvons douter que les particules qui les composent ne soient matérielles, et que par cette raison les lois générales de l'équilibre ne leur conviennent comme aux corps solides,

his conception of Mechanics as based on a variational principle was adopted universally by the physicists that developed further the atomistic theories: in particular Clausius and Boltzmann make constant use of the action principle using also Lagrange's notations, [11, p. 25], [12], and of the well known commutation rule $\delta d = d\delta$ introduced by Lagrange and become universal, [1, T.I, p.337].
(6) Very often the "Lagrange multipliers", that he introduced to simplify in various cases calculus of variations problems, are incorporated in the variational equation of Euler-Lagrange, [1, T.XI, p.340].
(7) It is important to comment also on the style of all his papers which continues to inform the *Traité*: at a time when quoting other works was not very usual all papers of Lagrange start with a dense summary of the previous works on the subject from which he draws the foundations of his contributions and gives an important help to the historians of science.
(8) The terminology used by Lagrange has become the adopted terminology

in most cases: at times, however, the reader might be confused by the use of terms which no longer have the same meaning and sorting things out requires very careful reading: an example is in [1, T.XI, p.244] where the footnote added by J. Bertrand provides an essential help to the reader who has not read enough in detail the earlier pages and tomes.
(9) In the *Traité* the theory of the small oscillations in systems of several degrees of freedom is treated as a perturbation theory problem and applied to the rigid mody motions and to celestial mechanics: it was employed extensively in the 1800-ths, starting with Laplace, until Poincaré made clear the need of substantial improvements on the method by pointing out the existence of motions that could not be reduced to simple "quasi periodic" ones. New ideas came much later with the work of Siegel, Kolmogorov, Arnold, Moser, Eliasson: Lagrange's imprint on the problem however remains not only through the secular perturbations treatment but also through another of his contributions, namely the theory of the implict functions, Eq.(3.2), as outlined in the next section.

5 KAM

The problem of stability of quasi periodic motions is closely related to Lagrange's inversion formula Eq.(3.2) and to its combinatorial version (3.4). It can be formulated as an implicit functions problem for a function $\mathbf{h}(\alpha)$ defined on the torus T^ℓ satisfying, in the simplest non trivial cases, the *Hamilton–Jacobi* equation

$$\mathcal{K}\mathbf{h}(\boldsymbol{\alpha}) \stackrel{def}{=} \mathbf{h}(\boldsymbol{\alpha}) + (\boldsymbol{\omega}\cdot\boldsymbol{\partial})^{-2}\Big(\varepsilon\boldsymbol{\partial} f(\boldsymbol{\alpha}+\mathbf{h}(\boldsymbol{\alpha}))\Big) = \mathbf{0} \tag{5.1}$$

where $\boldsymbol{\omega}\in R^\ell$ is a vector with Diophantine property: $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|\ge C_0^{-1}|\nu|^{-\tau}$ for some $C_0,\tau>0$ and $\boldsymbol{\partial} f$ is the gradient of an analytic even (for simplicity) function on T^ℓ, $(\boldsymbol{\omega}\cdot\boldsymbol{\partial})^{-2}$ is the linear (pseudo)differential operator on the functions analytic and odd on T^ℓ applied to the function of $\boldsymbol{\alpha}$ $\boldsymbol{\alpha}\to\varepsilon\boldsymbol{\partial} f(\boldsymbol{\alpha}+\mathbf{h}(\boldsymbol{\alpha}))$ and $\mathbf{h}(\boldsymbol{\alpha})$ is to be determined, odd in $\boldsymbol{\alpha}\in T^\ell$.

This is leads to consider an infinite dimensional version of Lagrange's inversion: it can be solved in exactly the same way writing $\mathbf{A}=\mathbf{h}+\mathcal{K}\mathbf{h}$ and solving for $\mathbf{h}$ when $\mathbf{A}=\mathbf{0}$, as in Eq.(3.2). This is more easily done if the Eq.(5.1) is considered in Fourier's transfom. Writing $\mathbf{h}(\boldsymbol{\alpha})=\sum_{k=1}^\infty\varepsilon^k\mathbf{h}^{[k]}(\boldsymbol{\alpha})$ and denoting $\mathbf{h}^{(k)}_{\boldsymbol{\nu},j}$ the Fourier transform of $\mathbf{h}^{(k)}_j$, $\boldsymbol{\nu}\in Z^\ell$ an expression for j-th component $h^{[k]}_{\boldsymbol{\nu},j}$ is given via trees with k root-oriented branches as:

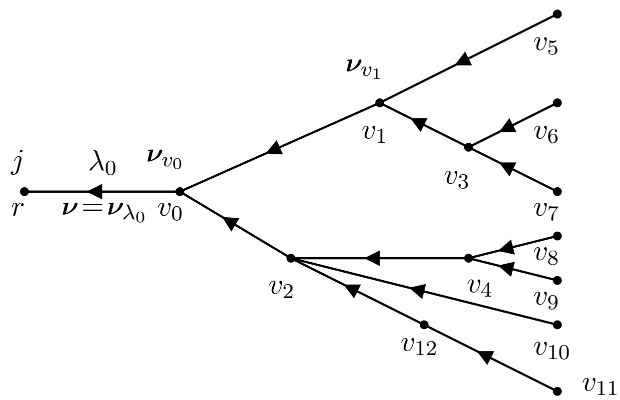

Fig.3

Fig.3: A tree θ con $k_{v_0}=2, k_{v_1}=2, \ldots$ and $k=13$, with a few decorations

In this case on each node there is an extra label $\boldsymbol{\nu}_v$ (marked in Fig.3 only on v_0 and v_1) and on each line there is an extra label $\boldsymbol{\nu}(\lambda)=\sum_{w\le v}\boldsymbol{\nu}_w$; also in this case the labels $j_v, j_{v_1}, \ldots, j_{v_{k_v}}$ associated with the nodes have to be contracted when appearing twice (*i.e.* unless $v=r$ as $j_r=j$ appears only once). The trees will be identified if reducible to each other by pivoting as in the simple scalar case of Sec.3. After some algebra it appears that

$$\mathbf{h}^{[k]}_{\boldsymbol{\nu}}=\sum_\theta \mathrm{Val}(\theta),\qquad \mathrm{Val}(\theta)=\Big(\prod_v \frac{f_{\boldsymbol{\nu}_v}}{k_v!}\Big)\Big(\prod_\lambda \frac{\boldsymbol{\nu}_v\cdot\boldsymbol{\nu}_{v'}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu}(\lambda))^2}\Big) \tag{5.2}$$

where the sum is over all trees with k branches.

Eq.(5.2) was developed in the context of celestial mechanics by Lindstedt and Newcomb. It turns the proof of the KAM theorem into a simple algebraic check in which the main difficulty of the *small divisors*, which appears because a naive estimate of $\mathbf{h}^{[k]}_{\boldsymbol{\nu}}$ has size $O(k!^\tau)$ (making the formula illusory, because apparently divergent), can be solved by checking that the values of the trees which are too large have competing signs and almost cancel between themselves leading to an estimate $|\mathbf{h}^{[k]}_{\boldsymbol{\nu}}|\le c^k e^{-\kappa|\boldsymbol{\nu}|}$ for suitable $c,\kappa>0$.

The tree representation is particularly apt to exhibit the cancellations that occur: the consequent proof of the KAM theorem, [13], is not the classical one and it is often considered too complicated. In this respect a comment of Lagrange is relevant:

D'ailleurs mes recherches n'ont rien de commun avec le leurs que le problème qui en fait l'object; et c'est toujours contribuer à l'avancement des Mathématiques que de montrer comment on peut résoudre les même questions et parvenir au même resultats par des voies très-différentes; les méthodes se *prètent par ce moyen un jour mutuel et en acquièrent souvent un plus grand degré d'évidence et de généralité.*

[1, T.6, p.280].

Going back to Kepler's problem the work of Carlini and of Levi-Civita (independent, later) made clear that Lagrange's series, Eq.(3.6), can be resummed into a power series in the parameter $\eta = \frac{e \exp \sqrt{1-e^2}}{1+\sqrt{1-e^2}}$ with radius of convergence 1, [14, Appendice, p.44],[15], thus redetermining the D'Alembert's radius of convergence $r^* = 0.6627434...$ of the power series in e (called "Laplace's limit")[4] as the closest point to 0 of the curve $|\eta| = 1$. Furthermore for e real and $e < 1$ it is $\eta < 1$ and an expansion for the eccentric anomaly is obtained by a series convergent for all eccentricities $e < 1$.

Recently the same formula has been used to study resonant quasi periodic motions in integrable systems subject to a perturbing potential εV with ε small: the resulting tree expansion allowed the study of the series in cases in which it is likely to be not convergent: in spite of this it has been shown that a resummation is possible and leads to a representation of the invariant torus which is analytic in ε in a region of the form, [20],

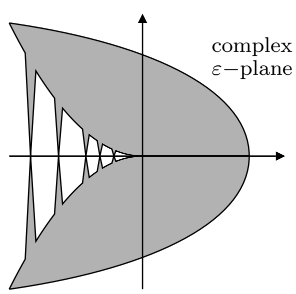

Fig.4

where the contact points of the holomorphy region and the negative real axis have a Lebesgue density point at 0; and in other cases to a fractional power series representation, [21], or also to a Borel summability in a region with $\varepsilon = 0$ on its boundary, [22].

The resummation leading to the above result is a typical resummation that appears in the eighteenth century analysis and is precisely the one that is used against the objections of D'Alembert to the vibrating string solution:

Or je demande si, toutes les fois que dans une formule algébrique il se trou-

[4] This is not the only resummation of Lagrange's series: the most famous is perhaps Carlini's resummation in terms of the Bessel functions $J_n(z)$, namely $\xi = \ell + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne) \sin n\ell$, found at the same time by Bessel, [16], [14], [17], [18], [19].

vera par exemple une série géométrique infinie, telle que $1-x+x^2-x^3+\ldots$, *on ne sera pas en droit d'y substituer* $\frac{1}{1+x}$ *quoique cette quantité ne soit réellement égale à la somme de la série proposée qu'en supposant le dernier terme* x^∞ *nul. Il me semble qu'on ne saurait contester l'exactitude d'une telle substitution sans renverser les principes les plus communs de l'analyse.*

[1, T.I, p.323]: the modernity of this defying viewpoint will not escape the readers' attention.

6 Comments

(1) The work of Lagrange has raised a large number of comments and deep critical analysis, here I mention a few: [23], [24], [25], [26].
(2) Although born and raised in Torino there are very few notes written in Italian: all of them seem to be in mail exchanges; a remarkable one has been inserted in the Tome VII of the collected works, [1, T.VII, p. 583]. It is a very deferent letter in which he explains a somewhat unusual algorithm to evaluate derivatives and integrals. Consider the series

$$[xy]^m \stackrel{def}{=} \sum_{k=0}^{\infty} \binom{m}{k} [x]^{m-k} [y]^k \tag{6.1}$$

(which the Lagrange writes without the brackets, added here for clarity). The positive powers are interpreted as derivatives (or more properly as infinitesimal increments with respect to the variation of an *unspecified* variable) and the negative as integrals provided at least x or y is an infinitesimal increment. Thus for $m > 0$ and integer the expression is a finite sum and yields the Leibnitz differentiation rule $d(xy)^m = \sum_{k=0}^{\infty} \binom{m}{k} (dx)^{m-k} (dy)^k$. If $m = -1-k < 0$ then $[dx]^{-1-k}$ is interpreted as $k+1$ iteration of indefinite integration over an infinitesimal interval dx; this means $dx^{-1} = x, dx^{-2} = \frac{x^2}{2dx}, \ldots, [dx]^{-1-k} = \frac{x^{k+1}}{(k+1)!\, dx^k}$.

For $m = -1$ Lagrange gives the example $\int y dx$: from Eq. (6.1)

$$\begin{aligned} [dx\, y]^{-1} &= \sum_{k=0}^{\infty} (-1)^k [dx]^{-1-k} d^k y \\ \int y dx &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{(k+1)! dx^k} d^k y = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{(k+1)!} \frac{d^k y}{dx^k} \end{aligned} \tag{6.2}$$

(a relation that the 18 years old Lagrange attributes to "Giovanni Bernoullio", 1694). A further example is worked out:

$$[dxdy]^{-2} = \sum_{k=0}^{\infty} \binom{-2}{k} [dx]^{-2-k} d^{k+1}y = \sum_{k=0}^{\infty} (-1)^k \frac{k+1}{(k+2)!} x^{k+2} \frac{d^{k+1}y}{dx^{k+1}} \quad (6.3)$$

which therefore yields the indefinite integral $\int \int dy\, dx$ which is shown to be identical to $\int ydx$ simply by differentiating the *r.h.s.* of (6.3) and checking its identity with the *r.h.s.* of (6.2) (Lagrange suggests, equivalently, to differentiate both equations twice).

The letter is signed *Luigi De La Grange*, and addressed to "Illustrissimo Signor Da Fagnano", well known mathematician, who would soon help Lagrange to get his first paper published.

(3) Consideration of friction is not frequent in the works of Lagrange, it is mentioned in a remark on the vibrating strings theory, [1, T.I, p.109, 241], and the analysis on the tautochrone curves in T.II,III.

It is also considered in the astronomical problems to examine the consequences, on the variations of the planetary elements, of a small rarefied medium filling the solar system (if any) in the Tomes VI and VII. Or in the influence of friction on the oscillations of a pendulum in the Traité (XII) and in fluid mechanics problems.

Although his attention to applications has been constant (for instance studying the best shape to give to a column to strengthen it) friction enters only very marginally in the remarkable theory of the anchor escapement, [1, T.IV, p.341]: this is surprising because it is an essential feature controlling the precision of the clocks and the very possibility of building them, [4, Ch.1,Sec.2.17].

(4) Among other applications discussed by Lagrange are problems in Optics, again referring to the variational properties of light paths, and in Probability theory.

(5) Infinitesimals, in the sense of Leibnitz, are pervasive in his work (a nice example is the 1754 letter to G.C. Fagnano quoted above which shows that Lagrange learnt very early their use and their formidable power of easing the task of long algebraic steps (at the time there were already some objections to their use): this makes reading his papers easy and pleasant; at the time there were several objections to their use which eventually lead to the rigorous refoundation of analysis.

However Lagrange himself seems to have realized that something ought to be done in systematizing the foundations of analysis: the tellingly long title of his lecture notes *Théorie des Fonctions analytiques, contenants les principes du calcul differentiel dégagé de toutes considérations d'infiniment* *petits ou d'evanouissants, de limites ou de fluxions et réduits à l'analyse algébrique de quantités finies*, [27], and the very first page of the *Leçons sur le calcul des fonctions*, [28], are a clear sign of his hidden qualms on the matter.

The fact that they developed at a late stage of his life, while he wholeheartedly adopted the Leibnitz methods in his youth, show that the problem of mathematical rigor had grown, even for the great scientists, to a point that it was necessary to work more on it.

(6) An informative history of his life can be found in the "Notices sur la vie et les ouvrages" in the preface by M. Delambre of the Tome I of the collected works, [1] and in the commemoration by P. Cossali at the University of Padova, [29]. If abstraction is made of the bombastic rethoric celebrating, in the first (quite a) few pages, Eugene Napoleon and, in the last (quite a) few pages, Napoleon I himself the remaining about 130 pages of Cossali's *Elogio* contain a useful and detailed summary and evaluation of all the works of Lagrange (also exposed in a circumvoluted rethorical style).

References

[1] J.L. Lagrange. *Oeuvres.* Gauthiers-Villars, Paris, 1867-1892.

[2] G. Gallavotti. *The Elements of Mechanics (II edition);.* http://ipparco.roma1.-infn.it, Roma, 2008 [I edition was Springer 1984].

[3] E. Presutti. *Scaling limits in Statistical Mechanics and Microstructures in Continuum Mechanics.* Springer, Berlin, 2009.

[4] G. Gallavotti. Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods. *Reviews of Modern Physics*, 57:471–562, 1985.

[5] F. Chiò. *Ricerche sopra la serie di Lagrangia, Memoria I.* Giuseppe Pomba e C. Editori, Torino, 1846.

[6] J.B. D'Alembert. *Traité de Dynamique.* David l'ainé, Paris, 1743.

[7] T. Levi-Civita and U. Amaldi. *Lezioni di Meccanica Razionale.* Zanichelli, Bologna, 1927, ristampe 1974, 2013.

[8] A. Lichnerowicz. La Mécanique à l'Académie. In Academie des Sciences, editor, *Troisième Centenaire*, pages 213–238. Gauthier-Villars, Paris, 1967.

[9] L. Euler. *Mechanik.* (ed. J.Ph.Wolfers), C. A. Koch's Verlagshandlung, Greifswald, 1853.

[10] J.P. Françoise, P. Garrido, and G. Gallavotti. Rigid motions: action-angles, relative cohomology and polynomials with roots on the unit circle. *Journal of Mathematical Physics*, 54:032901, 2013.

[11] L. Boltzmann. *Über die mechanische Bedeutung des zweiten Hauptsatzes der Wärmetheorie*, volume 1, p.9 of *Wissenschaftliche Abhandlungen, ed. F. Hasenöhrl*. Chelsea, New York, 1968.

[12] R. Clausius. Ueber die Zurückführung des zweites Hauptsatzes der mechanischen Wärmetheorie und allgemeine mechanische Prinzipien. *Annalen der Physik*, 142:433–461, 1871.

[13] G. Gallavotti. Twistless KAM tori. *Communications in Mathematical Physics*, 164:145–156, 1994.

[14] F. Carlini. Ricerche sulla convergenza della serie che serve alla soluzione del problema di Keplero. *Effemeridi Astronomiche di Milano, German translation by Jacobi as DOI: 10.1002/asna.18500301502, p.213-228*, 10:2–48, 1818.

[15] T. Levi-Civita. Sopra l'equazione di Kepler. *Rendiconti dell'Accademia dei Lincei*, XIII, also in Opere, T.2:260–268, 1904.

[16] F. Carlini. Ricerche sulla convergenza della serie che serve alla soluzione del problema di Keplero. *Giornale di Fisica, Chimica e Storia Naturale German translation by Jacobi as DOI: 10.1002/asna.18500301402, p.197-212*, 10:458–460, 1817.

[17] F.W. Bessel. *Briefwechsel zwischen W. Olbers und F.W. Bessel*, volume T.2. Avenarius und Mendelsohnn, 1818.

[18] C.G.J. Jacobi. Untersuchungen die Convergenz der Reihe durch welche das Kepler'sche Problem gelost wird. *Astronomische Nachrichten*, 50:197–244, 1850.

[19] P. Colwell. Bessel Functions and Kepler's Equation. *The Anerican Mathematical Monthly*, pages 45–48, 1992.

[20] G. Gallavotti and G. Gentile. Degenerate elliptic resonances. *Communications in Mathematical Physics*, 257:319–362, 2005.

[21] G. Gallavotti, G. Gentile, and A. Giuliani. Fractional Lindstedt series. *Journal of Mathematical Physics*, 47:012702 (+33), 2006.

[22] O. Costin, G. Gallavotti, G. Giuliani, and G. Gentile. Borel summability and Lindstedt series. *Communications in Mathematical Physics*, 269:175–193, 2006.

[23] C. Frazer. J. L. Lagrange's Early Contributions to the Principles and Methods of Mechanics. *Archive for History of Exact Sciences*, 28:197–241, 1983.

[24] C. Frazer. J. L. Lagrange's Changing Approach to the Foundations of the Calculus of Variations. *Archive for History of Exact Sciences*, 32:151–191, 1985.

[25] M. Panza. The origins of analytic mechanics in the 18th century. *in A history of analysis, ed. H.N. Jahnke, AMS*, pages 137–153, 2003.

[26] T. Archibald, C. Frazer, and I. Grattan-Guinness. *The History of Differential Equations, 1670-1950*, volume 51/2004. Reports of the Mathematisches Forschungsinstitut Oberwolfach, 2004.

[27] J.L. Lagrange. *Théorie des Fonctions analytiques*. Journal de l'École Polytechnique, Imprimerie de la Republique, Paris, 1797 (V).

[28] J.L. Lagrange. *Leçons sur le calcul des fonctions*. Courcier, Paris, 1806.

[29] P. Cossali. *Elogio di Luigi Lagrange*. Tipografia Bettoni, Padova, 1813.

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