A CRITICAL FRACTIONAL LAPLACE EQUATION IN THE RESONANT CASE

RAFFAELLA SERVADEI

Abstract. In this paper we complete the study of the following non-local fractional equation involving critical nonlinearities

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

started in the recent papers [11, 15, 16, 17].

Here $s \in (0,1)$ is a fixed parameter, $(-\Delta)^s$ is the fractional Laplace operator, λ is a positive constant, $2^* = 2n/(n-2s)$ is the fractional critical Sobolev exponent and Ω is an open bounded subset of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary.

Aim of this paper is to study this critical problem in the special case when $n \neq 4s$ and λ is an eigenvalue of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary datum. In this setting we prove that this problem admits a non-trivial solution, so that with the results obtained in [11, 15, 16, 17], we are able to show that this critical problem admits a nontrivial solution provided

- $n > 4s$ and $\lambda > 0$
- $n = 4s$ and $\lambda > 0$ is different from the eigenvalues of $(-\Delta)^s$
- $2s < n < 4s$ and $\lambda > 0$ is sufficiently large.

In this way we extend completely the famous result of Brezis and Nirenberg (see [3, 4, 7]) for the critical Laplace equation to the non-local setting of the fractional Laplace equation.

Contents

1. Introduction

The Yamabe problem for the Laplace operator (or, more generally, for uniformly elliptic operators) was widely studied in the literature (see, [8, 18, 20] and references therein). Recently, also in the non-local framework many papers concerning critical equations has appeared, both for pure mathematical interest and for the various applications in many fields (such as, e.g., optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves). In the non-local

Key words and phrases. Critical nonlinearities, best critical Sobolev constant, variational techniques, Linking Theorem, integrodifferential operators, fractional Laplacian.

2010 AMS Subject Classification: Primary: 49J35, 35A15, 35S15; Secondary: 47G20, 45G05.

Supported by the MIUR National Research Project *Variational and Topological Methods in the Study of Nonlinear Phenomena* and by the ERC grant ϵ (*Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities*).

framework, for instance, in [2, 19] the authors study a critical problem driven by a non-local operator defined as the power of the Laplacian, while in [11, 15, 16, 17] critical problems driven by the fractional Laplace operator $(-\Delta)^s$ are considered.

Aim of this paper is to complete the study carried on in [11, 15, 16, 17], where the existence of a nontrivial solution for the problem

$$
\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{1.1}
$$

was established in the special case when the parameter $\lambda > 0$ is different from the eigenvalues of $(-\Delta)^s$.

Here $s \in (0,1)$ is fixed and $-(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as

$$
-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy\,, \quad x \in \mathbb{R}^n\,, \tag{1.2}
$$

the set $\Omega \subset \mathbb{R}^n$, $n > 2s$, is open, bounded and with Lipschitz boundary, $\lambda > 0$ and $2^* = 2n/(n-2s)$ is the fractional critical Sobolev exponent (notice that when $s = 1$ the above exponent reduces to the classical critical Sobolev exponent). The homogeneous Dirichlet datum in (1.1) is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, as it happens in the classical case of the Laplacian, consistently with the non-local nature of the operator $(-\Delta)^s$.

In the recent works [11, 15, 16, 17] we proved that the famous result by Brezis and Nirenberg (see [3, 4, 7]) for the Laplace equation continues to hold also in the nonlocal setting of (1.1), provided λ is not an eigenvalue of $(-\Delta)^s$. With respect to the classical Brezis-Nirenberg result in these papers it remains open the resonant case in dimension different from $4s$, that is the case when $n \neq 4s$ and λ is an eigenvalue of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary data. Aim of this paper is to consider (1.1) in this setting, since we though that it is interesting to check what happens in this case in order to verify if the classical result known for the Laplacian can be extended to the non-local fractional framework. In this way the study of the critical fractional Laplace problem is completed.

The main result we obtain in the present paper is the following:

Theorem 1. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Moreover, let λ be an eigenvalue of $(-\Delta)^s$ with homogeneous Dirichlet boundary data.*

Then, problem (1.1) admits a weak solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero, and such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, provided that either

- *$n > 4s$, or*
- *$2s < n < 4s$ and λ is sufficiently large.*

As a consequence of Theorem 1 and of [11, Theorem 2], [15, Theorem 4] and [16, Theorem 1] we get the following existence result, which extends completely to the non-local fractional framework the well-known Brezis-Nirenberg type results given in [3, 4, 7] for the Laplace equation:

Theorem 2. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary.*

Then, problem (1.1) admits a weak solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero, and such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, provided that either

- *$n > 4s$ and $\lambda > 0$, or*
- *$n = 4s$ and $\lambda > 0$ is different from the eigenvalues of $(-\Delta)^s$, or*
- *$n < 4s$ and $\lambda > 0$ is sufficiently large.*

Roughly speaking, Theorem 2 says that what happens in the non-local framework is exactly what we know in the classing setting (see [3, 4, 7] and also [8, 18, 20] and references therein; of course the extension from the local setting to the non-local one is not straightforward as we will see in the course of the proofs).

In order to prove Theorem 1 we mainly use the fact that problem (1.1) is variational in nature and its weak formulation is given by

$$
\begin{cases}
\displaystyle\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\
\qquad\qquad\qquad\qquad\qquad\qquad\displaystyle = \int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)dx \\
\qquad\qquad\qquad\qquad\qquad\qquad\forall\,\varphi\in H^s(\mathbb{R}^n) \text{ with } \varphi=0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega \\
u\in H^s(\mathbb{R}^n) \text{ with } u=0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega\,.
\end{cases}
\tag{1.3}
$$

Problem (1.3) represents the Euler-Lagrange of the functional $\mathcal{J}_{s,\lambda}: X_0\to\mathbb{R}$ defined as

$$
\mathcal{J}_{s,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx\,, \tag{1.4}
$$

where the functional space X_0 is the Hilbert space defined as

$$
X_0=\big\{u\in H^s(\mathbb{R}^n): u=0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega\big\}
$$

and endowed with the norm given by

$$
X_0\ni v\mapsto \|v\|_{X_0} = \Big(\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\,. \tag{1.5}
$$

For this see [13, Lemma 7], while for a general definition of X_0 and its properties we refer to [11, 12, 13, 14, 15, 17]. Here $H^s(\mathbb{R}^n)$ is the usual fractional Sobolev space (for this see, for instance, [6]), endowed with the so-called *Gagliardo norm*

$$
\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \Big(\int_{\mathbb{R}^{2n}} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\,. \tag{1.6}
$$

Thus, in looking for weak solutions of (1.1) (that is solutions of problem (1.3)) we study the critical points of the functional $\mathcal{J}_{s,\lambda}$ using classical minimax theorems, namely the Linking Theorem (see, e.g., [1, 9, 10]). In order to apply such critical points theorem to $\mathcal{J}_{s,\lambda}$ we argue as in [11, 15, 16]. In particular, since the functional $\mathcal{J}_{s,\lambda}$ is not compact (due to the fact that $H^s(\mathbb{R}^n)$ is not compactly embedded into the critical Lebesgue space $L^{2^*}(\mathbb{R}^n)$), the Palais-Smale condition does not hold globally, but only in a suitable range related to the best constant in the embedding $H^s(\mathbb{R}^n)\hookrightarrow L^{2^*}(\mathbb{R}^n)$. For this reason, in order to apply the minimax theorem, we need to estimate the critical level of $\mathcal{J}_{s,\lambda}$ and, for proving such a estimate, we need to construct an explicit solution of the following limiting problem

$$
(-\Delta)^s u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^n\,.
$$

For more details on this see Subsection 2.1.

The paper is organized as follows. In Section 2 we will deal with the variational formulation of the problem. After introducing some notations and recalling some preliminary results, we will discuss the geometric and compactness properties of the functional $\mathcal{J}_{s,\lambda}$. Section 3 will be devoted to the estimate of the minimax critical level of $\mathcal{J}_{s,\lambda}$ and to the proof of Theorem 1.

2. The variational formulation of the problem

As we said in the Introduction, problem (1.1) has a variational structure. Hence, in order to look for weak solutions of problem (1.1) we study the critical points of the functional $\mathcal{J}_{s,\lambda}$ using classical minimax theorems.

First of all, note that this functional is well defined thanks to [13, Lemma 8] and [15, Lemma 9]. Moreover, $\mathcal{J}_{s,\lambda}$ is Fréchet differentiable in $u \in X_0$ and for any $\varphi \in X_0$

$$\begin{aligned}\langle \mathcal{J}'_{s,\lambda}(u), \varphi\rangle = \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega u(x)\varphi(x)\,dx \\ - \int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)\,dx\,.\end{aligned}$$

Before proving Theorem 1 we need some notation and some preliminary results.

2.1. Notations and preliminary results.

In the sequel we will denote by $\big(\lambda_{k,s}\big)_{k\in\mathbb{N}}$ the sequence of eigenvalues of the operator $(-\Delta)^s$, with homogeneous Dirichlet boundary data, such that

$$0 < \lambda_{1,s} < \lambda_{2,s} \leqslant \ldots \leqslant \lambda_{k,s} \leqslant \lambda_{k+1,s} \leqslant \ldots$$

and

$$\lambda_{k,s} \to +\infty \quad \text{as} \quad k \to +\infty\,.$$

Moreover, $e_{k,s}$ will be the eigenfunction corresponding to $\lambda_{k,s}$ for any $k \in \mathbb{N}$ and

$$\mathbb{P}_{k+1,s} := \big\{u \in X_0 \text{ s.t. } \langle u, e_{j,s}\rangle_{X_0} = 0 \quad \forall j = 1,\ldots,k\big\}\,,$$

where $\langle\cdot,\cdot\rangle_{X_0}$ is the scalar product on X_0 defined as

$$\langle u,v\rangle_{X_0} = \int_{\mathbb{R}^{2n}} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy\,. \tag{2.1}$$

For a complete study of the eigenvalues and the eigenfunctions of $(-\Delta)^s$ (and, more generally, of non-local integrodifferential operators) we refer to [14, Proposition 9 and Appendix A].

Here we also need to introduce the best fractional critical Sobolev constant S_s for the embedding of $H^s(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$ defined as

$$\begin{aligned} S_s &:= \inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} S_s(v)\,, \\ H^s(\mathbb{R}^n)\setminus\{0\} \ni v \mapsto S_s(v) &:= \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\Big(\displaystyle\int_{\mathbb{R}^n} |v(x)|^{2^*}dx\Big)^{2/2^*}}\,. \end{aligned} \tag{2.2}$$

With this, we can recall the following result obtained in [5, Theorem 1.1]:

Proposition 3. *The infimum in formula* (2.2) *is attained, that is*

$$S_s = S_s(\tilde u)\,,$$

where

$$\tilde u(x) = \kappa\Big(\mu^2 + |x-x_0|^2\Big)^{-(n-2s)/2}\,, \quad x \in \mathbb{R}^n \tag{2.3}$$

with $\kappa \in \mathbb{R}\setminus\{0\}$, $\mu > 0$ *and* $x_0 \in \mathbb{R}^n$ *fixed constants.*

Equivalently, the function $\bar u$ *defined as*

$$\bar u(x) = \frac{\tilde u(x)}{\|\tilde u\|_{L^{2^*}(\mathbb{R}^n)}} \tag{2.4}$$

is such that

$$S_s = \inf_{\substack{v\in H^s(\mathbb{R}^n)\\ \|v\|_{L^{2^*}(\mathbb{R}^n)}=1}} \int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \int_{\mathbb{R}^{2n}} \frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\,. \tag{2.5}$$

In the sequel we suppose that, up to a translation, $x_0 = 0$ in (2.3). Arguing as in [15, Section 4], we consider the function

$$u^*(x) = \bar{u}\left(\frac{x}{S_s^{1/(2s)}}\right), \qquad x\in\mathbb{R}^n\,, \tag{2.6}$$

which is an explicit solution of the limiting problem

$$(-\Delta)^s u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^n \tag{2.7}$$

such that

$$\|u^*\|^{2^*}_{L^{2^*}(\mathbb{R}^n)} = S_s^{n/(2s)}\,, \tag{2.8}$$

and also the functions

$$U_\varepsilon(x) = \varepsilon^{-(n-2s)/2}\,u^*(x/\varepsilon)\,, \qquad x\in\mathbb{R}^n \tag{2.9}$$

and

$$u_\varepsilon(x) = \eta(x)U_\varepsilon(x)\,, \quad x\in\mathbb{R}^n\,, \tag{2.10}$$

for any $\varepsilon > 0$. Here $\eta\in C^\infty(\mathbb{R}^n)$ is a function such that $0\leqslant\eta\leqslant 1$ in $\mathbb{R}^n$, $\eta\equiv 1$ in B_δ and $\eta\equiv 0$ in $\mathcal{C}B_{2\delta}$, where $B_\delta = B(0,\delta)$ and $\mathcal{C}B_\delta = \mathbb{R}^n\setminus B_\delta$, with $\delta>0$ fixed so that

$$B_{4\delta}\subset\Omega\,.$$

Note that $u_\varepsilon\in X_0$ and $u_\varepsilon = 0$ a.e. in $\mathbb{R}^n\setminus\Omega$. What is important about u_ε is that this function satisfies some crucial estimates that we recall here below (for a proof see [15, Propositions 21 and 22 and Subsection 4.2.1] and [11, Proposition 12]):

Proposition 4. *Let $s\in(0,1)$ and $n>2s$. Then, the following estimates hold true*

$$\int_{\mathbb{R}^{2n}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n-2s})\,,$$

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|^2\,dx \geqslant \begin{cases} C_s\varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}) & \text{if} \quad n>4s\\ C_s\varepsilon^{2s}\big|\log\varepsilon\big| + \mathcal{O}(\varepsilon^{2s}) & \text{if} \quad n=4s\\ C_s\varepsilon^{n-2s} + \mathcal{O}(\varepsilon^{2s}) & \text{if} \quad n<4s\,, \end{cases}\,,$$

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*}\,dx = S_s^{n/(2s)} + \mathcal{O}(\varepsilon^n)\,,$$

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*-1}\,dx = \mathcal{O}(\varepsilon^{(n-2s)/2})\,,$$

and

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|\,dx = \mathcal{O}(\varepsilon^{(n-2s)/2})$$

as $\varepsilon\to 0$, for some positive constant C_s depending on s.

2.2. **Geometric and compactness condition of the functional $\mathcal{J}_{s,\lambda}$.** In the present paper we are interested in problem (1.1), when the parameter λ is an eigenvalue of $(-\Delta)^s$, say when $\lambda = \lambda_{k,s}$ for some $k \in \mathbb{N}$. Without loss of generality, we can assume that $\lambda_{k,s}$ has multiplicity $h \in \mathbb{N}$ and that

$$\lambda_{k-1,s} < \lambda_{k,s} = \lambda_{k+1,s} = \cdots = \lambda_{k+h-1,s} < \lambda_{k+h,s}\,. \tag{2.11}$$

In order to prove Theorem 1 our strategy will consist in applying the Linking Theorem (see [9, 10]) to the functional $\mathcal{J}_{s,\lambda_{k,s}}$, that is $\mathcal{J}_{s,\lambda}$ with $\lambda = \lambda_{k,s}$.

As it is well-known, the main ingredients of this minimax theorem are a suitable geometric structure and a compactness condition, namely the Palais-Smale condition at level $c \in \mathbb{R}$, given by

every Palais–Smale sequence $(u_j)_{j\in\mathbb{N}}$ in X_0 at level $c \in \mathbb{R}$
admits a subsequence strongly convergent in X_0.

We say that $(u_j)_{j\in\mathbb{N}}$ in X_0 is a Palais-Smale sequence for $\mathcal{J}_{s,\lambda}$ at level $c \in \mathbb{R}$ if

$$\mathcal{J}_{s,\lambda}(u_j) \to c \quad\text{and}\quad \sup\Big\{\big|\langle\, \mathcal{J}'_{s,\lambda}(u_j), \varphi\,\rangle\big| \,:\, \varphi \in X_0\,, \|\varphi\|_{X_0} = 1\Big\} \to 0 \quad\text{as}\quad j \to +\infty\,.$$

By [16, Proposition 10] we know that the functional $\mathcal{J}_{s,\lambda}$ satisfies the geometric features required by the Linking Theorem, when $\lambda \geqslant \lambda_{1,s}$. Hence, in particular, $\mathcal{J}_{s,\lambda_{k,s}}$ has the geometric structure of the Linking Theorem, that is the following proposition is valid

Proposition 5. *There exist $\rho > 0$ and $\beta > 0$ such that*

a) for any $u \in \mathbb{P}_{k+1,s}$ with $\|u\|_{X_0} = \rho$ it results that $\mathcal{J}_{s,\lambda_{k,s}}(u) \geqslant \beta$;

b) $\mathcal{J}_{s,\lambda_{k,s}}(u) \leqslant 0$ for any $u \in span\{e_{1,s}, \dots, e_{k,s}\}$;

c) for any finite dimensional subspace $\mathbb{F}$ of X_0, there exists $R > \rho$ such that

$$\sup_{\substack{u\in\mathbb{F}\\ \|u\|_{X_0}\geqslant R}} \mathcal{J}_{s,\lambda_{k,s}}(u) \leqslant 0\,.$$

In particular we can construct $\mathbb{F}$ as follows

$$\mathbb{F} = \mathbb{F}_\varepsilon := span\{e_{1,s}, \dots, e_{k,s}\} \oplus span\{\tilde{z}_\varepsilon\}\,, \tag{2.12}$$

with $\tilde{z}_\varepsilon = z_\varepsilon/\|z_\varepsilon\|_{X_0}$, $z_\varepsilon = u_\varepsilon - \sum_{i=1}^k \Big(\int_\Omega u_\varepsilon(x) e_{i,s}(x)\,dx\Big) e_{i,s}$ and u_ε as in (2.10), $\varepsilon > 0$.

Moreover, by [15, Proposition 5] (which holds true for any $\lambda > 0$) and the fact that $\lambda_{k,s} \geqslant \lambda_{1,s} > 0$ (see [14, Proposition 9–*a*)]) the functional $\mathcal{J}_{s,\lambda_{k,s}}$ also verifies the Palais-Smale condition up to a suitable threshold, i.e. the following result holds true

Proposition 6. *Let $c \in \mathbb{R}$ be such that*

$$c < \frac{s}{n} S_s^{n/(2s)}\,, \tag{2.13}$$

where S_s is the constant defined in (2.2).

Then, the functional $\mathcal{J}_{s,\lambda_{k,s}}$ satisfies the Palais-Smale condition at level c.

Roughly speaking, Proposition 6 says that the functional $\mathcal{J}_{s,\lambda_{k,s}}$ satisfies the Palais-Smale condition only below a suitable threshold related to the fractional critical Sobolev constant. Since this condition does not hold globally, in order to apply the Linking Theorem to $\mathcal{J}_{s,\lambda_{k,s}}$, we need to estimate the critical level of this functional and show that it stays below the threshold where the Palais-Smale condition is satisfied. This will be done in the next section.

3. Estimate of the minimax critical level

This section is devoted to the estimate of the minimax critical level of the functional $\mathcal{J}_{s,\lambda_{k,s}}$. Here, in some sense, we argue as in [11, Subsection 7.2] and in [16, Section 7] even if, with respect to these papers, some differences arise, due to the fact that here we consider the resonant case, that is the case when λ is an eigenvalue of $(-\Delta)^s$. In fact, here the estimates are more delicate: on this we will be more precise in the sequel.

The Linking critical level of $\mathcal{J}_{s,\lambda_{k,s}}$ is given by

$$c_{\varepsilon,\lambda_{k,s}} = \inf_{h\in\Gamma}\max_{u\in Q}\mathcal{J}_{s,\lambda_{k,s}}(h(u)),$$

where

$$\begin{aligned}\Gamma &= \big\{h\in C(\overline{Q};X_0) : h = id \text{ on } \partial Q\big\},\\ Q &= \Big(\overline{B}_R\cap\operatorname{span}\{e_{1,s},\dots,e_{k,s}\}\Big)\oplus\big\{r\tilde{z}_\varepsilon : r\in(0,R)\big\},\end{aligned}\tag{3.1}$$

and R and $\tilde{z}_\varepsilon$ are as in Proposition 5. In order to estimate $c_{\varepsilon,\lambda_{k,s}}$, we will use the particular choice of $\mathbb{F}=\mathbb{F}_\varepsilon$ given in (2.12) (note that $Q\subset\mathbb{F}_\varepsilon$).

We want to show the following result

Proposition 7. *Let S_s be as in (2.2). Then,*

$$c_{\varepsilon,\lambda_{k,s}} < \frac{s}{n}S_s^{n/(2s)}$$

for ε sufficiently small, provided $n>4s$ or $n<4s$ and $\lambda_{k,s}$ is large enough.

Proof. First of all, note that by definition of $c_{\varepsilon,\lambda_{k,s}}$ for any $h\in\Gamma$

$$c_{\varepsilon,\lambda_{k,s}} \leqslant \max_{u\in Q}\mathcal{J}_{s,\lambda_{k,s}}(h(u))\tag{3.2}$$

and so, in particular, taking $h=id$ and using the fact that $Q\subset\mathbb{F}_\varepsilon$, we have

$$c_{\varepsilon,\lambda_{k,s}} \leqslant \max_{u\in Q}\mathcal{J}_{s,\lambda_{k,s}}(u) \leqslant \max_{u\in\mathbb{F}_\varepsilon}\mathcal{J}_{s,\lambda_{k,s}}(u).\tag{3.3}$$

Since $\mathbb{F}_\varepsilon$ is a linear space,

$$\begin{aligned}\max_{u\in\mathbb{F}_\varepsilon}\mathcal{J}_{s,\lambda_{k,s}}(u) &= \max_{\substack{u\in\mathbb{F}_\varepsilon\\ \zeta\neq0}}\mathcal{J}_{s,\lambda_{k,s}}\Big(|\zeta|\cdot\frac{u}{|\zeta|}\Big)\\ &= \max_{\substack{u\in\mathbb{F}_\varepsilon\\ \zeta>0}}\mathcal{J}_{s,\lambda_{k,s}}(\zeta u)\\ &\leqslant \max_{\substack{u\in\mathbb{F}_\varepsilon\\ \zeta\geqslant0}}\mathcal{J}_{s,\lambda_{k,s}}(\zeta u).\end{aligned}\tag{3.4}$$

Hence, (3.2)–(3.4) yield that

$$c_{\varepsilon,\lambda_{k,s}} \leqslant \max_{\substack{u\in\mathbb{F}_\varepsilon\\ \zeta\geqslant0}}\mathcal{J}_{s,\lambda_{k,s}}(\zeta u).\tag{3.5}$$

Then, in order to prove Proposition 7, by (3.5) it is enough to show that

$$\max_{\substack{u\in\mathbb{F}_\varepsilon\\ \zeta\geqslant0}}\mathcal{J}_{s,\lambda_{k,s}}(\zeta u) < \frac{s}{n}S_s^{n/(2s)}.\tag{3.6}$$

Note that, by [15, Proposition 20], for any $u\in X_0\setminus\{0\}$

$$\max_{\zeta\geqslant0}\mathcal{J}_{s,\lambda_{k,s}}(\zeta u) = \frac{s}{n}\left(\frac{\|u\|_{X_0}-\lambda_{k,s}\|u\|^2_{L^2(\Omega)}}{\|u\|^{2^*}_{L^{2^*}(\Omega)}}\right)^{n/(2s)},\tag{3.7}$$

and that the right-hand side in (3.7) is scale invariant. Hence, as a consequence of (3.7), relation (3.6) is equivalent to

$$M_\varepsilon := \max_{\substack{u\in\mathbb{F}_\varepsilon\\ \|u\|_{L^{2^*}(\Omega)}=1}}\Big(\|u\|^2_{X_0}-\lambda_{k,s}\|u\|^2_{L^2(\Omega)}\Big) < S_s,\tag{3.8}$$

so that, in order to prove Proposition 7, it is enough to show that (3.8) holds true.

Let us prove inequality (3.8) . At this purpose, let us recall that, by [11, Proposition 13-a)] M_ε is achieved in some $u_M \in \mathbb{F}_\varepsilon$, which can be written as follows [1]

$$u_M = \tilde{v} + t z_\varepsilon\,, \tag{3.9}$$

where $\tilde{v} \in \mathrm{span}\{e_{1,\,s}, \dots, e_{k,\,s}\}$, $t \geqslant 0$ and z_ε is as in Proposition 5, and such that

$$\|u_M\|_{L^{2^*}(\Omega)} = 1\,. \tag{3.10}$$

For now on we proceed by steps.

Claim 1. *There exists a positive constant $\bar{c}$ such that $t = t_\varepsilon \leqslant \bar{c}$ for $\varepsilon > 0$ small enough.*

Proof. First of all, note that, by the Hőlder inequality and the properties of u_M, we can bound u_M as follows

$$\|u_M\|^2_{L^2(\Omega)} \leqslant |\Omega|^{n/(2s)} \|u_M\|^2_{L^{2^*}(\Omega)} = |\Omega|^{n/(2s)}. \tag{3.11}$$

Moreover, $\tilde{v}$ and z_ε are orthogonal in $L^2(\Omega)$ and in X_0, since the sequence $\big(e_{k,\,s}\big)_{k\in\mathbb{N}}$ of eigenfunctions corresponding to λ_k is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of X_0 (see [14, Proposition 9–f)]) . As a consequence of this, we get that

$$\|u_M\|^2_{L^2(\Omega)} = \|\tilde{v}\|^2_{L^2(\Omega)} + t^2_\varepsilon \|z_\varepsilon\|^2_{L^2(\Omega)} \geqslant \|\tilde{v}\|^2_{L^2(\Omega)}. \tag{3.12}$$

By (3.11) and (3.12), we easily get that both $\|u_M\|_{L^2(\Omega)}$ and $\|\tilde{v}\|_{L^2(\Omega)}$ are bounded uniformly in ϵ by a suitable $\tilde{c} > 0$.

Furthermore, by [16, Proposition 4] we know that $e_{i,\,s} \in L^\infty(\Omega)$ for any $i \in \mathbb{N}$, so that also $\tilde{v} \in \mathrm{span}\{e_{1,\,s}, \dots, e_{k,\,s}\}$ does. Hence, $\tilde{v} \in L^{2^*}(\Omega)$, since Ω is bounded and, by the equivalence of the norms in a finite dimensional space, we also have

$$\|\tilde{v}\|_{L^{2^*}(\Omega)} \leqslant \tilde{c}\,. \tag{3.13}$$

Also, by Proposition 4 and again [16, Proposition 4] we have

$$\left| \int_\Omega u_\varepsilon(x) e_{i,s}(x)\,dx \right| \leqslant \|u_\varepsilon\|_{L^1(\Omega)} \|e_{i,s}\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^{(n-2s)/2})$$

as $\varepsilon \to 0$. As a consequence, using the definition of z_ε and again Proposition 4, we get

$$\begin{aligned} \|z_\varepsilon\|_{L^{2^*}(\Omega)} &\geqslant \|u_\varepsilon\|_{L^{2^*}(\Omega)} - \sum_{i=1}^k \left| \int_\Omega u_\varepsilon(x) e_{i,s}(x)\,dx \right| \|e_{i,s}\|_{L^{2^*}(\Omega)} \\ &= S_s^{(n-2s)/(4s)} + \mathcal{O}(\varepsilon^{(n-2s)/2}) \\ &\geqslant \frac{S_s^{(n-2s)/(4s)}}{2} \end{aligned}$$

for ε sufficiently small.

Then, by (3.9), the fact that $t = t_\varepsilon \geqslant 0$, (3.10) and (3.13), we have

$$\frac{S_s^{(n-2s)/(4s)}\, t_\varepsilon}{2} \leqslant t_\varepsilon \|z_\varepsilon\|_{L^{2^*}(\Omega)} \leqslant \|u_M\|_{L^{2^*}(\Omega)} + \|\tilde{v}\|_{L^{2^*}(\Omega)} \leqslant 1 + \tilde{c}\,,$$

for ε small enough . Hence, t_ε is bounded for ε sufficiently small and this ends the proof of Claim 1. □

Claim 2. *The function $u_M \in \mathbb{F}_\varepsilon$ can be written as*

$$u_M = v + P_k \tilde{v} + t \tilde{u}_\varepsilon\,,$$

[1] Beware that u_M, $\tilde{v}$ and t (and also v in Claim 2 below) depend on ε . For simplicity we omit this dependence in the notation.

where $t \geqslant 0$ *,*

$$v = \sum_{i=1}^{k-1} \Big(\int_\Omega (\tilde{v}(x) - tu_\varepsilon(x)) e_{i,\,s}(x)\, dx \Big) e_{i,\,s} \in span\{e_{1,\,s}, \dots, e_{k-1,\,s}\}\,, \tag{3.14}$$

$$\tilde{u}_\varepsilon = u_\varepsilon - P_k u_\varepsilon \tag{3.15}$$

and the map $X_0 \ni w \mapsto P_k w$ *denotes the projection of* w *on the direction* $e_{k,\,s}$*, that is*

$$P_k w = \Big(\int_\Omega w(x) e_{k,\,s}(x)\, dx \Big) e_{k,\,s}\,.$$

Moreover, there exists a positive constant $\bar{\kappa}$*, independent of* ε*, such that*

$$\Big| \int_\Omega \tilde{u}_\varepsilon(x)\, v(x)\, dx \Big| = \Big| \int_\Omega u_\varepsilon(x)\, v(x)\, dx \Big| \leqslant \bar{\kappa} \|v\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^1(\Omega)}\,, \tag{3.16}$$

and

$$\Big| \int_{\mathbb{R}^{2n}} \frac{\big(\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)\big)\big(v(x) - v(y)\big)}{|x-y|^{n+2s}}\, dxdy \Big| \leqslant \bar{\kappa} \|v\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^1(\Omega)} \tag{3.17}$$

for any $\varepsilon > 0$ *.*

Proof. By (3.9) and the definition of z_ε (as given in Proposition 5), it is easily seen that

$$\begin{aligned} u_M &= \sum_{i=1}^{k} \Big(\int_\Omega \tilde{v}(x) e_{i,\,s}(x)\, dx \Big) e_{i,\,s} + t\Big(u_\varepsilon - \sum_{i=1}^{k} \Big(\int_\Omega u_\varepsilon(x) e_{i,\,s}(x)\, dx \Big) e_{i,\,s} \Big) \\ &= \sum_{i=1}^{k-1} \Big(\int_\Omega (\tilde{v}(x) - tu_\varepsilon(x)) e_{i,\,s}(x)\, dx \Big) e_{i,\,s} + P_k \tilde{v} + t(u_\varepsilon - P_k u_\varepsilon) \\ &= v + P_k \tilde{v} + t\tilde{u}_\varepsilon\,, \end{aligned}$$

with v and $\tilde{u}_\varepsilon$ as in (3.14) and (3.15), respectively.

Let us start showing that (3.16) holds true. For this, note that v and $P_k u_\varepsilon$ are orthogonal in $L^2(\Omega)$, so that

$$\int_\Omega \tilde{u}_\varepsilon(x)\, v(x)\, dx = \int_\Omega \Big(u_\varepsilon(x) - P_k u_\varepsilon \Big)\, v(x)\, dx = \int_\Omega u_\varepsilon(x)\, v(x)\, dx\,,$$

while the Hőlder inequality and the equivalence of the norm in a finite dimensional space give

$$\Big| \int_\Omega \tilde{u}_\varepsilon(x)\, v(x)\, dx \Big| = \Big| \int_\Omega u_\varepsilon(x)\, v(x)\, dx \Big| \leqslant \|u_\varepsilon\|_{L^1(\Omega)} \|v\|_{L^\infty(\Omega)} \leqslant \bar{\kappa} \|u_\varepsilon\|_{L^1(\Omega)} \|v\|_{L^2(\Omega)}$$

for a suitable $\bar{\kappa} > 0$, independent of ε . Thus, (3.16) is proved.

Now, let us show (3.17) . At this purpose, we write

$$v = \sum_{i=1}^{k-1} v_i e_{i,s} \tag{3.18}$$

for some $v_i \in \mathbb{R}$, so that, again by [14, Proposition 9–f)],

$$\|v\|^2_{L^2(\Omega)} = \sum_{i=1}^{k-1} v_i^2\,.$$

By (3.18), the fact that $e_{i,\,s}$ is an eigenfunction of $(-\Delta)^s$ with eigenvalue $\lambda_{i,\,s}$ and the definition of scalar product in X_0 (see, (2.1)), we have

$$\langle \tilde{u}_\varepsilon, v \rangle_{X_0} = \sum_{i=1}^{k-1} v_i \langle \tilde{u}_\varepsilon, e_{i,s} \rangle_{X_0} = \sum_{i=1}^{k-1} \lambda_{i,s} v_i \int_\Omega \tilde{u}_\varepsilon(x) e_{i,s}(x)\, dx = \sum_{i=1}^{k-1} \lambda_{i,s} v_i \int_\Omega u_\varepsilon(x) e_{i,s}(x)\, dx\,,$$

also thanks to the definition of $\tilde{u}_\varepsilon$ and the orthogonality properties of $e_{i,s}$. So, by this and again the Hőlder inequality, we get

$$\begin{aligned}|\langle \tilde{u}_\varepsilon, v\rangle_{X_0}| &\leqslant \sum_{i=1}^{k-1} \lambda_{i,s}|v_i|\, \|u_\varepsilon\|_{L^1(\Omega)}\|e_{i,s}\|_{L^\infty(\Omega)}\\ &\leqslant \bar{\kappa}\|u_\varepsilon\|_{L^1(\Omega)}\|v\|_{L^2(\Omega)}\,,\end{aligned}$$

that is

$$\left|\int_{\mathbb{R}^{2n}} \frac{\big(\tilde{u}_\varepsilon(x)-\tilde{u}_\varepsilon(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dxdy\right| \leqslant \bar{\kappa}\|v\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^1(\Omega)}$$

for a suitable $\bar{\kappa}>0$ possibly depending on k, but independent of ε. Hence, (3.17) is proved and this ends the proof of Claim 2. □

Now, we are ready to show the validity of (3.8) , that is

$$M_\varepsilon = \int_{\mathbb{R}^{2n}} \frac{|u_M(x)-u_M(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda_{k,\,s}\int_\Omega |u_M(x)|^2\,dx < S_s\,. \tag{3.19}$$

In doing this, we have to take into account that $u_M = v + P_k\tilde{v} + t\tilde{u}_\varepsilon$ by Claim 2 and that, in particular, we have to estimate three different contributions coming from v, $P_k\tilde{v}$ and $\tilde{u}_\varepsilon$. With respect to similar calculations carried on in [11, Subsection 7.2] and [16, Section 7], here we have to pay attention to the contribution coming from v, due to the resonance occurring in this case.

Let us show (3.19) . By Claim 2 we have that

$$\begin{aligned}M_\varepsilon &= \int_{\mathbb{R}^{2n}} \frac{|u_M(x)-u_M(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda_{k,\,s}\int_\Omega |u_M(x)|^2\,dx\\
&= \int_{\mathbb{R}^{2n}} \frac{|v(x)+P_k\tilde{v}(x)+t\tilde{u}_\varepsilon(x)-v(y)-P_k\tilde{v}(y)-t\tilde{u}_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&\qquad -\lambda_{k,\,s}\int_\Omega |v(x)+P_k\tilde{v}(x)+t\tilde{u}_\varepsilon(x)|^2\,dx\\
&= \|v\|^2_{X_0}+\|P_k\tilde{v}\|^2_{X_0}+t^2\|\tilde{u}_\varepsilon\|^2_{X_0}+2t\int_{\mathbb{R}^{2n}} \frac{\big(\tilde{u}_\varepsilon(x)-\tilde{u}_\varepsilon(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dxdy \qquad (3.20)\\
&\quad -\lambda_{k,\,s}\Big(\|v\|^2_{L^2(\Omega)}+\|P_k\tilde{v}\|^2_{L^2(\Omega)}+t^2\|\tilde{u}_\varepsilon\|^2_{L^2(\Omega)}\Big)-2\lambda_{k,\,s}t\int_\Omega \tilde{u}_\varepsilon(x)\,v(x)\,dx\\
&= \|v\|^2_{X_0}+t^2\|\tilde{u}_\varepsilon\|^2_{X_0}+2t\int_{\mathbb{R}^{2n}} \frac{\big(\tilde{u}_\varepsilon(x)-\tilde{u}_\varepsilon(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dxdy\\
&\quad -\lambda_{k,\,s}\Big(\|v\|^2_{L^2(\Omega)}+t^2\|\tilde{u}_\varepsilon\|^2_{L^2(\Omega)}\Big)-2\lambda_{k,\,s}t\int_\Omega \tilde{u}_\varepsilon(x)\,v(x)\,dx\,,\end{aligned}$$

thanks to the orthogonality properties of v, $P_k\tilde{v}$ and $\tilde{u}_\varepsilon$ and also to the definition of $\lambda_{k,\,s}$.

Now, note that by (3.15)

$$\begin{aligned}
\|\tilde{u}_\varepsilon\|^2_{X_0} - \lambda_{k,s}\|\tilde{u}_\varepsilon\|^2_{L^2(\Omega)} &= \|u_\varepsilon - P_k u_\varepsilon\|^2_{X_0} - \lambda_{k,s}\|u_\varepsilon - P_k u_\varepsilon\|^2_{L^2(\Omega)} \\
&= \|u_\varepsilon\|^2_{X_0} + \|P_k u_\varepsilon\|^2_{X_0} \\
&\quad - 2\int_{\mathbb{R}^{2n}} \frac{\big(u_\varepsilon(x) - u_\varepsilon(y)\big)\big(P_k u_\varepsilon(x) - P_k u_\varepsilon(y)\big)}{|x-y|^{n+2s}}\,dxdy \\
&\quad - \lambda_{k,s}\Big(\|u_\varepsilon\|^2_{L^2(\Omega)} + \|P_k u_\varepsilon\|^2_{L^2(\Omega)}\Big) \\
&\quad + 2\lambda_{k,s}\int_\Omega u_\varepsilon(x)\,P_k u_\varepsilon(x)\,dx \\
&= \|u_\varepsilon\|^2_{X_0} - \lambda_{k,s}\|u_\varepsilon\|^2_{L^2(\Omega)} \\
&\quad - 2\int_{\mathbb{R}^{2n}} \frac{\big(u_\varepsilon(x) - u_\varepsilon(y)\big)\big(P_k u_\varepsilon(x) - P_k u_\varepsilon(y)\big)}{|x-y|^{n+2s}}\,dxdy \\
&\quad + 2\lambda_{k,s}\int_\Omega u_\varepsilon(x)\,P_k u_\varepsilon(x)\,dx \\
&= \|u_\varepsilon\|^2_{X_0} - \lambda_{k,s}\,\|u_\varepsilon\|^2_{L^2(\Omega)} \\
&\quad - 2\Big(\|P_k u_\varepsilon\|^2_{X_0} - \lambda_{k,s}\|P_k u_\varepsilon\|^2_{L^2(\Omega)}\Big) \\
&= \|u_\varepsilon\|^2_{X_0} - \lambda_{k,s}\,\|u_\varepsilon\|^2_{L^2(\Omega)}\,,
\end{aligned} \tag{3.21}$$

thanks to the definition of P_k .

Then, combining (3.16), (3.17), (3.20), Claim 1 and (3.21), we get

$$\begin{aligned}
M_\varepsilon &= \|v\|^2_{X_0} + t^2\|\tilde{u}_\varepsilon\|^2_{X_0} + 2t\int_{\mathbb{R}^{2n}} \frac{\big(\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)\big)\big(v(x) - v(y)\big)}{|x-y|^{n+2s}}\,dxdy \\
&\qquad -\lambda_{k,s}\Big(\|v\|^2_{L^2(\Omega)} + t^2\|\tilde{u}_\varepsilon\|^2_{L^2(\Omega)}\Big) - 2\lambda_{k,s}t\int_\Omega \tilde{u}_\varepsilon(x)\,v(x)\,dx \\
&= \|v\|^2_{X_0} + t^2\|u_\varepsilon\|^2_{X_0} + 2t\int_{\mathbb{R}^{2n}} \frac{\big(\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(y)\big)\big(v(x) - v(y)\big)}{|x-y|^{n+2s}}\,dxdy \\
&\qquad -\lambda_{k,s}\Big(\|v\|^2_{L^2(\Omega)} + t^2\|u_\varepsilon\|^2_{L^2(\Omega)}\Big) - 2\lambda_{k,s}t\int_\Omega \tilde{u}_\varepsilon(x)\,v(x)\,dx \\
&\leqslant \|v\|^2_{X_0} - \lambda_{k,s}\|v\|^2_{L^2(\Omega)} + t^2\Big(\|u_\varepsilon\|^2_{X_0} - \lambda_{k,s}\|u_\varepsilon\|^2_{L^2(\Omega)}\Big) + \tilde{\kappa}\|v\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^1(\Omega)}\,,
\end{aligned} \tag{3.22}$$

provided $\varepsilon > 0$ is sufficiently small and for some $\tilde{\kappa} > 0$, independent of ε .

Since $v \in \operatorname{span}\{e_{1,s}, \dots, e_{k-1,s}\}$ and (2.11) holds true, by [11, Proposition 5] and (3.22), we have

$$\begin{aligned}
M_\varepsilon &\leqslant \|v\|^2_{X_0} - \lambda_{k,s}\|v\|^2_{L^2(\Omega)} + t^2\Big(\|u_\varepsilon\|^2_{X_0} - \lambda_{k,s}\|u_\varepsilon\|^2_{L^2(\Omega)}\Big) + \tilde{\kappa}\|v\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^1(\Omega)} \\
&\leqslant \Big(\lambda_{k-1,s} - \lambda_{k,s}\Big)\|v\|^2_{L^2(\Omega)} + t^2\Big(\|u_\varepsilon\|^2_{X_0} - \lambda_{k,s}\|u_\varepsilon\|^2_{L^2(\Omega)}\Big) + \tilde{\kappa}\|v\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^1(\Omega)} \\
&= \Big(\lambda_{k-1,s} - \lambda_{k,s}\Big)\|v\|^2_{L^2(\Omega)} + S_{s,\lambda_{k,s}}(u_\varepsilon)\|tu_\varepsilon\|^2_{L^{2^*}(\Omega)} + \tilde{\kappa}\|v\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^1(\Omega)}\,,
\end{aligned} \tag{3.23}$$

if ε is small enough, where $S_{s,\lambda_{k,s}}(\cdot)$ is the function defined as

$$H^s(\mathbb{R}^n)\setminus\{0\} \ni u \mapsto S_{s,\lambda_{k,s}}(u) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda_{k,s}\int_{\mathbb{R}^n}|u(x)|^2\,dx}{\displaystyle\Big(\int_{\mathbb{R}^n}|u(x)|^{2^*}\,dx\Big)^{2/2^*}}\,. \tag{3.24}$$

Now, note that by the convexity[2], the monotonicity properties of the integrals, (3.9), Claim 1 and the fact that in $\mathrm{span}\{e_{1,s},\dots,e_{k,s}\}$ all the norms are equivalent, we have

$$\begin{aligned}1=\|u_M\|^{2^*}_{L^{2^*}(\Omega)}&=\int_\Omega |u_M(x)|^{2^*}\,dx\\
&\geqslant \int_\Omega |tu_\varepsilon(x)|^{2^*}\,dx+2^*\int_\Omega \big(tu_\varepsilon(x)\big)^{2^*-1}\tilde v(x)\,dx\\
&\geqslant \|tu_\varepsilon\|^{2^*}_{L^{2^*}(\Omega)}-2^*\bar c^{\,2^*-1}\|u_\varepsilon\|^{2^*-1}_{L^{2^*-1}(\Omega)}\|\tilde v\|_{L^\infty(\Omega)}\\
&\geqslant \|tu_\varepsilon\|^{2^*}_{L^{2^*}(\Omega)}-\hat c\,\|u_\varepsilon\|^{2^*-1}_{L^{2^*-1}(\Omega)}\|\tilde v\|_{L^2(\Omega)}\,,\end{aligned}\tag{3.25}$$

for some positive constant $\hat c$, so that

$$\|tu_\varepsilon\|^{2^*}_{L^{2^*}(\Omega)}\leqslant 1+\hat c\,\|u_\varepsilon\|^{2^*-1}_{L^{2^*-1}(\Omega)}\|\tilde v\|_{L^2(\Omega)}\tag{3.26}$$

for ε sufficiently small.

Moreover, by the Young inequality for any $\sigma>0$ we have

$$\tilde\kappa\|v\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^1(\Omega)}\leqslant \sigma\|v\|^2_{L^2(\Omega)}+\frac{\tilde\kappa^2}{4\sigma}\,\|u_\varepsilon\|^2_{L^1(\Omega)}\tag{3.27}$$

for any $\varepsilon>0$.

Hence, (3.23), (3.25) and (3.27) give, for ε small enough

$$\begin{aligned}M_\varepsilon&=\Big(\lambda_{k-1,s}-\lambda_{k,s}\Big)\|v\|^2_{L^2(\Omega)}+S_{s,\lambda_{k,s}}(u_\varepsilon)\|tu_\varepsilon\|^2_{L^{2^*}(\Omega)}+\tilde\kappa\|v\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^1(\Omega)}\\
&\leqslant \Big(\lambda_{k-1,s}-\lambda_{k,s}\Big)\|v\|^2_{L^2(\Omega)}+S_{s,\lambda_{k,s}}(u_\varepsilon)\Big(1+\hat c\,\|u_\varepsilon\|^{2^*-1}_{L^{2^*-1}(\Omega)}\|\tilde v\|_{L^2(\Omega)}\Big)\\
&\qquad+\sigma\|v\|^2_{L^2(\Omega)}+\frac{\tilde\kappa^2}{4\sigma}\,\|u_\varepsilon\|^2_{L^1(\Omega)}\\
&=\Big(\lambda_{k-1,s}-\lambda_{k,s}+\sigma\Big)\|v\|^2_{L^2(\Omega)}+S_{s,\lambda_{k,s}}(u_\varepsilon)\Big(1+\mathcal O(\varepsilon^{(n-2s)/2})\Big)+\mathcal O(\varepsilon^{n-2s})\,,\end{aligned}\tag{3.28}$$

thanks to Proposition 4.

Now, let us choose $\sigma>0$ be such that $\sigma<\lambda_{k,s}-\lambda_{k-1,s}$ (this choice is admissible since $\lambda_{k,s}-\lambda_{k-1,s}>0$ by (2.11)). Then, (3.28) yields

$$M_\varepsilon\leqslant S_{s,\lambda_{k,s}}(u_\varepsilon)\Big(1+\mathcal O(\varepsilon^{(n-2s)/2})\Big)+\mathcal O(\varepsilon^{n-2s})\tag{3.29}$$

as $\varepsilon\to 0$.

Now, let us distinguish the two different cases $n>4s$ and $n<4s$.

Case 1: $n>4s$**.** By Proposition 4 and by definition of $S_{s,\lambda_{k,s}}(\cdot)$ (see (3.24)) we get

$$\begin{aligned}S_{s,\lambda_{k,s}}(u_\varepsilon)&=\frac{\displaystyle\int_{\mathbb R^{2n}}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy-\lambda_{k,s}\int_\Omega |u_\varepsilon(x)|^2\,dx}{\displaystyle\Big(\int_\Omega |u_\varepsilon(x)|^{2^*}\,dx\Big)^{2/2^*}}\\
&\leqslant \frac{S_s^{n/(2s)}+\mathcal O(\varepsilon^{n-2s})-\lambda_{k,s}C_s\varepsilon^{2s}}{\Big(S_s^{n/(2s)}+\mathcal O(\varepsilon^n)\Big)^{2/2^*}}\\
&\leqslant S_s+\mathcal O(\varepsilon^{n-2s})-\lambda_{k,s}C_s\varepsilon^2\,,\end{aligned}$$

as $\varepsilon\to 0$.

[2]If f is a differentiable convex function, then $f(y)\geqslant f(x)+f'(x)(y-x)$. Here we take $f(s)=s^{2^*}$, $x=tu_\varepsilon$ and $y=u_M=v+tu_\varepsilon$.

Thus, as a consequence of this and of (3.29) we deduce

$$\begin{aligned} M_\varepsilon &\leqslant \Big(S_s + \mathcal{O}(\varepsilon^{n-2s}) - \lambda_{k,\,s} C_s \varepsilon^2\Big)\Big(1 + \mathcal{O}(\varepsilon^{(n-2s)/2})\Big) + \mathcal{O}(\varepsilon^{n-2s}) \\ &= S_s + \mathcal{O}(\varepsilon^{n-2s}) - \lambda_{k,\,s} C_s \varepsilon^2 \\ &= S_s + \varepsilon^{2s}\Big(\mathcal{O}(\varepsilon^{n-4s}) - \lambda_{k,\,s} C_s\Big) < S_s\,, \end{aligned}$$

provided $\varepsilon > 0$ is sufficiently small. Hence, (3.19) holds true. This concludes the proof of Proposition 7 in the case when $n > 4s$.

Case 2: $n < 4s$**.** Again by Proposition 4 and by definition of $S_{s,\,\lambda_{k,\,s}}(\cdot)$ we have

$$\begin{aligned} S_{s,\,\lambda_{k,\,s}}(u_\varepsilon) &\leqslant \frac{S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n-2s}) - \lambda_{k,\,s} C_s \varepsilon^{n-2s} + \mathcal{O}(\varepsilon^{2s})}{\Big(S_s^{n/(2s)} + \mathcal{O}(\varepsilon^n)\Big)^{2/2^*}} \\ &\leqslant S_s + \mathcal{O}(\varepsilon^{n-2s}) - \lambda_{k,\,s} \widetilde{C}_s \varepsilon^{n-2s} + \mathcal{O}(\varepsilon^{2s})\,, \end{aligned}$$

as $\varepsilon \to 0$.

Thus, by this and (3.29) we deduce

$$\begin{aligned} M_\varepsilon &\leqslant \Big(S_s + \mathcal{O}(\varepsilon^{n-2s}) - \lambda_{k,\,s} \widetilde{C}_s \varepsilon^{n-2s} + \mathcal{O}(\varepsilon^{2s})\Big)\Big(1 + \mathcal{O}(\varepsilon^{(n-2s)/2})\Big) + \mathcal{O}(\varepsilon^{n-2s}) \\ &= S_s + \mathcal{O}(\varepsilon^{n-2s}) - \lambda_{k,\,s} \widetilde{C}_s \varepsilon^{n-2s} + \mathcal{O}(\varepsilon^{2s}) \\ &= S_s + \varepsilon^{n-2s}\Big(\mathcal{O}(1) - \lambda_{k,\,s} \widetilde{C}_s\Big) + \mathcal{O}(\varepsilon^{2s}) < S_s\,, \end{aligned}$$

if $\lambda_{k,\,s}$ is large enough, say $\lambda_{k,\,s} > \lambda_s > 0$ and provided $\varepsilon > 0$ is sufficiently small. Thus, (3.19) is verified.

Ultimately, the proof of Proposition 7 is concluded. □

Finally, we can prove Theorem 1 as an application of classical minimax theorems.

3.1. Proof of Theorem 1.

Now, Theorem 1 easily follows from the Linking Theorem (see [9, 10]), thanks to Propositions 5, 6 and 7.

Acknowledgements. The author would like to thank Professor Marco Degiovanni for interesting and useful discussions.

References

[1] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., 14, 349–381 (1973).

[2] B. Barrios, E. Colorado, A. De Pablo and U. Sanchez, *On some critical problems for the fractional Laplacian operator*, to appear in J. Differential Equations, available online at http://arxiv.org/abs/1106.6081

[3] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., 36, no. 4, 437–477 (1983).

[4] A. Capozzi, D. Fortunato and G. Palmieri, *An existence result for nonlinear elliptic problems involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 2, no. 6, 463–470 (1985).

[5] A. Cotsiolis and N. Tavoularis, *Best constants for Sobolev inequalities for higher order fractional derivatives*, J. Math. Anal. Appl., 295, 225–236 (2004).

[6] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136, no. 5, 521–573 (2012).

[7] F. Gazzola and B. Ruf, *Lower-order perturbations of critical growth nonlinearities in semilinear elliptic equations*, Adv. Differential Equations, 2, no. 4, 555–572 (1997).

[8] O. Kavian, Introduction à la théorie des points critiques et applications aux problèmes elliptiques, *Mathématiques & Applications*, Springer-Verlag, Paris (1993).

[9] P.H. Rabinowitz, *Some critical point theorems and applications to semilinear elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 5, 215–223 (1978).

[10] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Reg. Conf. Ser. Math.*, 65, *American Mathematical Society*, Providence, RI (1986).

[11] R. Servadei, *The Yamabe equation in a non-local setting*, preprint, available at http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=12-40 .

[12] R. Servadei and E. Valdinoci, *Lewy-Stampacchia type estimates for variational inequalities driven by (non)local operators*, preprint, available at http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=11-103 .

[13] R. Servadei and E. Valdinoci, *Mountain Pass solutions for non-local elliptic operators*, J. Math. Anal. Appl., 389, 887–898 (2012).

[14] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, preprint, available at http://www.math.utexas.edu/mp_arc-bin/mpa?yn=11-131.

[15] R. Servadei and E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, to appear in Trans. Amer. Math. Soc.

[16] R. Servadei and E. Valdinoci, *A Brezis-Nirenberg result for non-local critical equations in low dimension*, preprint, available at http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=12-41.

[17] R. Servadei and E. Valdinoci, *Fractional Laplacian equations with critical Sobolev exponent*, preprint, available at http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=12-58.

[18] M. Struwe, Variational methods, Applications to nonlinear partial differential equations and Hamiltonian systems, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3, *Springer Verlag*, Berlin–Heidelberg (1990).

[19] J. Tan, *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differential Equations, 36, no. 1-2, 21–41 (2011).

[20] M. Willem, Minimax theorems, *Progress in Nonlinear Differential Equations and their Applications*, 24, Birkhäuser, Boston (1996).

Dipartimento di Matematica, Università della Calabria, Ponte Pietro Bucci 31 B, 87036 Arcavacata di Rende (Cosenza), Italy

E-mail address: servadei@mat.unical.it