WEAK AND VISCOSITY SOLUTIONS OF THE FRACTIONAL LAPLACE EQUATION

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ABSTRACT. Aim of this paper is to show that weak solutions of the following fractional Laplacian equation

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

are also continuous solutions of this problem in the viscosity sense.

Here $s \in (0,1)$ is a fixed parameter, Ω is a bounded, open subset of $\mathbb{R}^n$, $n > 2s$, with C^2-boundary, and $(-\Delta)^s$ is the fractional Laplacian operator, that may be defined as

$$(-\Delta)^s u(x) := c(n,s) \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \, dy,$$

for a suitable positive normalizing constant $c(n,s)$, depending only on n and s.

Contents

1. Introduction

In the literature, different notions of solutions are taken into account when dealing with elliptic equations, such as, for instance, the *weak* (also called distributional, or variational) solutions (i.e. the solutions that belong to a suitable Sobolev space and satisfy the equation in a distributional sense, when integrated against a suitable set of test functions) and the *viscosity* solutions (i.e. all the smooth functions that touch either from above or below the continuous solution are required to be either subsolutions or supersolutions).

In this note, we point out that, for the fractional Laplace equation, the notion of weak solution implies the one of viscosity solution.

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In what follows we take Ω to be a bounded, open subset of $\mathbb{R}^n$ with C^2-boundary, with $n > 2s$ and $s \in (0,1)$ and we consider the fractional boundary value problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega., \end{cases} \tag{1.1}$$

where f and g are sufficiently smooth. The operator $(-\Delta)^s$ is the fractional Laplacian, that may be defined as

$$(-\Delta)^s u(x) := c(n,s) \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy, \tag{1.2}$$

for a suitable normalizing constant $c(n,s) > 0$ (see, e.g., [5, 8, 13] for the basic theory of the fractional Laplacian and for the associated fractional Sobolev spaces) . Of course, a more general framework could be considered, but we focus on problem (1.1) under the more basic assumptions for the sake of concreteness.

The definition in (1.2) is well posed if u is smooth enough and it does not grow too much at infinity (for instance if $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$). On the other hand, it is convenient to consider weak formulations of (1.1), both to construct solutions using variational methods and to develop a regularity theory based on comparison principles.

These two different scopes indeed reflect two different attitudes and methodologies (namely test functions and energy estimates versus touching the solution with a barrier), and they are reminiscent of the dichotomy between operators in divergence and non-divergence form in the classical elliptic framework. These two different points of view naturally translate into two different notions of solutions, i.e. weak versus viscosity ones.

As a matter of fact, in the recent literature, some interest has been developed for weak solutions of (1.1), that is for functions $u \in \mathbb{H}^s(\Omega) \cap L^\infty(\mathbb{R}^n)$ that satisfy

$$\begin{cases} c(n,s) \displaystyle\int_{\mathbb{R}^{2n}} \frac{\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } \ \varphi \in \mathbb{H}^s_0(\Omega), \\ u = g \ \text{ a.e. in } \ \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{1.3}$$

Here we denote by $\mathbb{H}^s(\Omega)$ the set of all measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that $u \in L^2(\Omega)$ and

$$\int_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\setminus\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy < +\infty\,.$$

We also denoted by $\mathbb{H}^s_0(\Omega)$ the functions in $\mathbb{H}^s(\Omega)$ which vanish outside Ω.

The interest of these weak solutions mainly comes from the calculus of variations, since the expression in (1.3) arises from the Euler-Lagrange equation of a fractional type energy functional (see, e.g., [11, 12] and references therein).

On the other hand, when dealing with comparison principles, it is also natural to consider viscosity solutions of (1.1) (see, e.g., [13]). For this we recall that $u \in C(\mathbb{R}^n)$ is a viscosity subsolution for (1.1) if for any open set $U \subset \Omega$, any $x_o \in U$ and any $\phi \in C^2(U)$ such that $\phi(x_o) = u(x_o)$ and $\phi \geqslant u$ in U, if we let

$$v(x) := \begin{cases} \phi(x) & \text{in } U \\ u(x) & \text{outside } U, \end{cases} \tag{1.4}$$

we have that $-(-\Delta)^s v(x_o) \geqslant f(x_o)$. Similarly, we say that $u \in C(\mathbb{R}^n)$ is a viscosity supersolution for (1.1) if for any $\psi \in C^2(U)$ such that $\psi(x_o) = u(x_o)$ and $\psi \leqslant u$ in U, if we let

$$w(x) := \begin{cases} \psi(x) & \text{in } U \\ u(x) & \text{outside } U, \end{cases}$$

we have that $-(-\Delta)^s w(x_o) \leqslant f(x_o)$. Then, u is a viscosity solution of problem (1.1) if it is both a viscosity subsolution and a viscosity supersolution.

Aim of this paper is to prove a relation between weak and viscosity solutions of problem (1.1). With respect to this, the main observation that we make in this note is the following one:

Theorem 1. *Let $f \in C(\mathbb{R}^n)$ and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let $u \in \mathbb{H}^s(\Omega) \cap L^\infty(\mathbb{R}^n)$ be a weak solution of* (1.1). *Then, u is a viscosity solution of* (1.1).

The converse of Theorem 1 holds true if f is smooth enough thanks to the regularity theory for viscosity solutions (see, e.g., [13, Propositions 2.1.10 and 2.1.11]). Related results have been recently announced[1] in [9].

In order to prove Theorem 1 we first need to show that every weak solution of (1.1) is continuous in $\mathbb{R}^n$, as stated in the following theorem:

Theorem 2. *Let $f \in L^\infty(\Omega)$ and let $u \in \mathbb{H}^s(\Omega) \cap L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n \setminus \Omega)$ be a weak solution of*

$$(-\Delta)^s u = f \quad in \ \ \Omega\,.$$

Then, $u \in C(\mathbb{R}^n)$.

For this, we first give an interior regularity result for weak solutions (see Proposition 5 for the details) and, later, we show that the weak solutions of our problem are continuous up to the boundary (hence in the whole $\mathbb{R}^n$), if so are outside the domain where the problem is set.

We would like to note that in the definition of weak solution and in all the previous results we assume that the weak solutions of problem (1.1) are in $L^\infty(\Omega)$. We do this only for the sake of clarity and concreteness. In fact, this condition can be removed. Indeed, in Proposition 7 we show that every weak solution of (1.1) is in $L^\infty(\Omega)$, provided the data are smooth enough.

The paper is organized as follows. In Section 2 we recall the Perron method and the existence theory of viscosity solutions for fractional boundary value problems. Then, in Section 3 we develop the regularity theory for weak solutions, by establishing continuity up to the boundary, and so proving Theorem 2: with this we will be able to complete the proof of Theorem 1 in Section 4. Finally, in Section 5 we point out that weak solutions are automatically bounded.

2. Perron method and existence theory for viscosity solutions

Here we recall that the classical Perron method (see e.g. [7]) can be suitably adapted to fractional problems, giving an existence theory in the viscosity setting. For instance, as a particular case of [1, Theorem 1] we have the following result:

Proposition 3. *Let Ω be open, bounded and with C^2-boundary and let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, there exists a viscosity solution $u \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ of the problem*

$$\begin{cases} (-\Delta)^s u = 0 & in\ \Omega, \\ \quad u = g & in\ \mathbb{R}^n \setminus \Omega. \end{cases}$$

As a consequence of Proposition 3 we have the following existence result:

Corollary 4. *Let Ω be open, bounded and with C^2-boundary. Let $f \in C^2(\mathbb{R}^n)$ and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, there exists a viscosity solution $u \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ of the problem*

$$\begin{cases} (-\Delta)^s u = f & in\ \Omega, \\ \quad u = g & in\ \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.1}$$

[1] After a first draft of this note was completed, we have received the very nice paper [10], which contains, among others, a result closely related to Theorems 1 and 2 obtained with different proofs.

Proof. We may modify f outside Ω in such a way that f is compactly supported in some B_R and therefore bounded and uniformly continuous. We take $\Gamma(x) := c(n,s)\,|x|^{2s-n}$ the fundamental solution of the fractional Laplacian (see, e.g., [8, page 44] and [2, Subsection 2.2] for the basic properties of fractional fundamental solutions), that is $(-\Delta)^s\Gamma = \delta_0$, where δ_0 is the Dirac delta function in 0.

Let $\tilde g := g - \Gamma * f$. Notice that

$$\Gamma * f(x) = c(n,s) \int\limits_{B_R(x)} |y|^{2s-n} f(x-y)\,dy,$$

thanks to our further assumptions on f. Since

$$\int\limits_{B_R(x)} |y|^{2s-n}\,dy < +\infty$$

we obtain that $\Gamma * f \in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, again by the assumptions on f. Accordingly, $\tilde g \in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, so we may apply Proposition 3 and obtain a viscosity solution $\tilde u$ of $(-\Delta)^s\tilde u = 0$ in Ω and $\tilde u = \tilde g$ outside Ω.

Also $\Gamma * f \in C^2(\mathbb{R}^n)$ and it solves

$$(-\Delta)^s(\Gamma * f) = \big((-\Delta)^s\Gamma\big) * f = f.$$

Let $u := \tilde u + \Gamma * f$. Then u solves (2.1) and this ends the proof of Corollary 4. □

3. Regularity theory for weak solutions

In this section we give some regularity results for weak solutions of problem (1.1). First of all, next proposition deals with the interior regularity theory for weak solutions:

Proposition 5. *Let $f \in L^\infty(B_1)$ and let $u \in \mathbb{H}^s(B_3)\cap L^\infty(\mathbb{R}^n)$ be a weak solution of*

$$(-\Delta)^s u = f \quad in \quad B_1\,. \tag{3.1}$$

Then, $u \in C^\alpha(B_{1/4})$ for any $0 < \alpha < \min\{2s, 1\}$ and

$$\|u\|_{C^\alpha(B_{1/4})} \leqslant C\Big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1)}\Big),$$

for a suitable $C > 0$ depending on n, s and α.

Proof. The case of smooth solutions was considered in [6], so it is enough to reduce to this case by a standard convolution argument. Namely, we take $\rho \in C_0^\infty(B_1;[0,1])$ and we consider the mollifier $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(x/\varepsilon)$, $\varepsilon > 0$. We define $u_\varepsilon := u * \rho_\varepsilon$ and $f_\varepsilon := f * \rho_\varepsilon$.

Of course, u_ε is a smooth function, by construction. Let us show that u_ε solves the following equation

$$(-\Delta)^s u_\varepsilon = f_\varepsilon \quad \text{in} \quad B_{1/2}\,. \tag{3.2}$$

For this, first we observe that for any test function ϕ supported in $B_{1/2}$

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \left| \frac{(u_\varepsilon(x) - u_\varepsilon(y))\,(\phi(x) - \phi(y))}{|x-y|^{n+2s}} \right| dx\,dy \\
&\quad = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^{2n}} \frac{|u(x+z) - u(y+z)|\,|\phi(x) - \phi(y)|\,\rho_\varepsilon(z)}{|x-y|^{n+2s}}\,dx\,dy \right] dz \\
&\quad = \int_{B_\varepsilon} \left[\int_{Q(B_1)} \frac{|u(x+z) - u(y+z)|\,|\phi(x) - \phi(y)|\,\rho_\varepsilon(z)}{|x-y|^{n+2s}}\,dx\,dy \right] dz \\
&\quad \leqslant \frac{\varepsilon^{-n}}{2} \int_{B_\varepsilon} \left[\int_{Q(B_2)} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_{Q(B_1)} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \right] dz \\
&\quad < +\infty\,,
\end{aligned}$$

since $u \in \mathbb{H}^s(B_3) \cap L^\infty(\mathbb{R}^n)$. Here we used the notation

$$Q(B_r) := \mathbb{R}^{2n} \setminus \Big((\mathbb{R}^n \setminus B_r) \times (\mathbb{R}^n \setminus B_r) \Big), \quad r > 0\,.$$

Therefore, we may use Fubini's Theorem and obtain that for any $\phi \in C_0^\infty(B_{1/2})$

$$\begin{aligned}
&\int_{\mathbb{R}^n} f_\varepsilon(x)\phi(x)\,dx \\
&\quad = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(x+z)\phi(x)\rho_\varepsilon(z)\,dx \right] dz \\
&\quad = \int_{B_\varepsilon} \left[\int_{\mathbb{R}^n} f(\tilde{x})\phi(\tilde{x} - z)\rho_\varepsilon(z)\,d\tilde{x} \right] dz \\
&\quad = \int_{B_\varepsilon} \left[\int_{\mathbb{R}^{2n}} \frac{(u(\tilde{x}) - u(\tilde{y}))\,(\phi(\tilde{x} - z) - \phi(\tilde{y} - z))\,\rho_\varepsilon(z)}{|\tilde{x} - \tilde{y}|^{n+2s}}\,d\tilde{x}\,d\tilde{y} \right] dz \\
&\quad = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^{2n}} \frac{(u(x+z) - u(y+z))\,(\phi(x) - \phi(y))\,\rho_\varepsilon(z)}{|x-y|^{n+2s}}\,dx\,dy \right] dz \\
&\quad = \int_{\mathbb{R}^{2n}} \left[\int_{\mathbb{R}^n} \frac{(u(x+z) - u(y+z))\,(\phi(x) - \phi(y))\,\rho_\varepsilon(z)}{|x-y|^{n+2s}}\,dz \right] dx\,dy \\
&\quad = \int_{\mathbb{R}^{2n}} \frac{(u_\varepsilon(x) - u_\varepsilon(y))\,(\phi(x) - \phi(y))}{|x-y|^{n+2s}}\,dx\,dy\,,
\end{aligned}$$

thanks to the fact that u is a weak solution of (3.1) . This means that u_ε is a smooth solution of (3.2) .

As a consequence of this, we may apply [6, Corollary 4.3] and obtain that $u \in C^\alpha(B_{1/4})$ for any $0 < \alpha < \min\{2s, 1\}$ and

$$\|u_\varepsilon\|_{C^\alpha(B_{1/4})} \leqslant C\Big(\|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} + \|f_\varepsilon\|_{L^\infty(B_1)} \Big)\,.$$

Hence, the desired result follows by sending $\varepsilon \searrow 0$. □

Now we are ready to show that the weak solutions of the fractional Laplace equation are continuous up to the boundary (hence in the whole of the space), if so are outside the domain where the problem is set. With respect to this our result is stated in Theorem 2. Here below we provide a proof of it.

3.1. **Proof of Theorem 2.** The idea of the proof is to consider a smooth neighborhood Ω_ε of Ω and a smooth function β_ε that agrees with u outside Ω_ε and with $(-\Delta)^s\beta_\varepsilon$ very large (by the smoothness of β_ε in Ω_ε, the notion of viscosity and weak solution are the same for it). By possibly translating β_ε a little bit up, one checks that β_ε must be above u. Then, the continuity of β_ε gives a control from above of the values of u near $\partial\Omega$. By a reverse argument, one obtains a control from below, and hence the continuity of u near $\partial\Omega$. Below are the details of this argument.

We argue by contradiction, by supposing that there exist $x_o \in \mathbb{R}^n$ and a sequence x_j in $\mathbb{R}^n$ such that

$$\lim_{j\to+\infty} x_j = x_o \quad \text{but} \quad |u(x_j) - u(x_o)| \geqslant a$$

for some $a > 0$. For definiteness, we suppose that

$$u(x_j) - u(x_o) \geqslant a \tag{3.3}$$

since the case $u(x_o) - u(x_j) \geqslant a$ can be treated in a similar way.

First of all, we claim that

$$x_o \in \partial\Omega\,. \tag{3.4}$$

Indeed, $x_o \in \overline{\Omega}$, since u is continuous in the open set $\mathbb{R}^n \setminus \overline{\Omega}$, by assumption. Moreover, it cannot be that x_o lies in the interior of Ω, because u is also continuous in any domain compactly contained in Ω, thanks to the local regularity theory stated in Proposition 5. This proves (3.4).

Now let $\varepsilon \in (0, 2]$ and let Ω_ε be a smooth ε-neighborhood of Ω, that is a set with C^2-boundary and such that

$$\bigcup_{x\in\Omega} B_{\varepsilon/2}(x) \subseteq \Omega_\varepsilon \subseteq \bigcup_{x\in\Omega} B_\varepsilon(x)\,.$$

Moreover, let β_ε be the viscosity solution of

$$\begin{cases} (-\Delta)^s\beta_\varepsilon = M & \text{in } \Omega_\varepsilon \\ \beta_\varepsilon = u & \text{in } \mathbb{R}^n \setminus \Omega_\varepsilon\,, \end{cases} \tag{3.5}$$

where $M := 2 + \|f\|_{L^\infty(\Omega)}$. The existence of such a viscosity solution is warranted by Corollary 4 and by the regularity assumptions[2] on u.

Note that $\beta_\varepsilon \in C(\overline{\Omega}_\varepsilon)$ (again by Corollary 4) with modulus of continuity of β_ε in Ω_ε independent of ε: see for instance [4, Theorem 32] for the continuity up to the boundary.

As a matter of fact, arguing as in Proposition 5, and by possibly replacing M with $M-1 = 1 + \|f\|_{L^\infty(\Omega)}$, by convolution (here we denote the convolution of β_ε again by β_ε) for fixed $\varepsilon > 0$ we may suppose that[3]

$$\begin{gathered} (-\Delta)^s\beta_\varepsilon \geqslant M \text{ in } \Omega_\varepsilon \text{ in the viscosity sense,} \\ \beta_\varepsilon = u \text{ in } \mathbb{R}^n \setminus \Omega_\varepsilon, \text{ and} \\ \beta_\varepsilon \in C^2(\overline{\Omega}) \cap C(\overline{\Omega}_\varepsilon) \end{gathered} \tag{3.6}$$

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[2]To be precise, in order to use Corollary 4 to obtain the desired solution β_ε in (3.5), one needs to extend the datum u continuously inside Ω_ε. This can be accomplished by taking $\phi_\varepsilon \in C^\infty(\mathbb{R}^n)$ such that $\phi_\varepsilon = 1$ in $\mathbb{R}^n \setminus \Omega_{\varepsilon/2}$ and $\phi_\varepsilon = 0$ in $\Omega_{\varepsilon/4}$. Then one defines $g_\varepsilon := \phi_\varepsilon u + (1 - \phi_\varepsilon)$. By construction $g_\varepsilon \in C(\mathbb{R}^n)$, so Corollary 4 gives the existence of a viscosity solution β_ε of $(-\Delta)^s\beta_\varepsilon = M$ in Ω_ε, with $\beta_\varepsilon = g_\varepsilon = u$ in $\mathbb{R}^n \setminus \Omega_\varepsilon$.

[3]In full detail: one may take a mollifier $\rho_{\varepsilon'}$ with ε' small (possibly with respect to ε too) and a cut-off $\phi_\varepsilon \in C^\infty(\mathbb{R}^n)$ such that $\phi_\varepsilon = 1$ in $\mathbb{R}^n \setminus \Omega_{\varepsilon/2}$ and $\phi_\varepsilon = 0$ in $\Omega_{\varepsilon/4}$. Then we replace the original β_ε in (3.5) with $\phi_\varepsilon\beta_\varepsilon + (1 - \phi_\varepsilon)(\beta_\varepsilon * \rho_{\varepsilon'})$, that is smooth inside $\Omega_{\varepsilon/4}$. Notice that, thanks to the cut-off, this procedure does not change either the values outside Ω_ε or the modulus of continuity inside Ω.

with modulus of continuity ρ of β_ε in Ω_ε, defined as

$$\rho(t) = \sup_{\substack{x,y\in\Omega_\varepsilon \\ |x-y|<t}} |\beta_\varepsilon(x) - \beta_\varepsilon(y)|\,,$$

that is independent of ε (while $\|\beta_\varepsilon\|_{C^2(\overline{\Omega})}$ may depend on ε).

Now, let also σ be the modulus of continuity of u in the compact[4] set $\overline{\Omega}_2 \setminus \Omega \subset \mathbb{R}^n \setminus \Omega$, that is

$$\sigma(t) = \sup_{\substack{x,y\in\overline{\Omega}_2\setminus\Omega \\ |x-y|<t}} |u(x) - u(y)|\,.$$

We set

$$\eta_\varepsilon := \varepsilon + \rho(\varepsilon) + \sigma(\varepsilon)\,. \tag{3.7}$$

First of all, we point out that

$$\lim_{\varepsilon\searrow 0} \eta_\varepsilon \;=\; 0\,. \tag{3.8}$$

Moreover, we claim that

$$\beta_\varepsilon + \eta_\varepsilon > u \text{ in } \mathbb{R}^n \setminus \Omega\,. \tag{3.9}$$

To check this, let $x \in \mathbb{R}^n \setminus \Omega$. First, if $x \in \mathbb{R}^n \setminus \Omega_\varepsilon$, then $\beta_\varepsilon(x) = u(x)$, due to (3.6), and so (3.9) holds true. If, on the other hand, $x \in \Omega_\varepsilon\setminus\Omega$ then there exists $y \in \partial\Omega_\varepsilon \subseteq \mathbb{R}^n\setminus\Omega_\varepsilon$ such that $|x-y| \leqslant \varepsilon$. Therefore $\beta_\varepsilon(y) = u(y)$, $|\beta_\varepsilon(x) - \beta_\varepsilon(y)| \leqslant \rho(\varepsilon)$ and $|u(x) - u(y)| \leqslant \sigma(\varepsilon)$. As a consequence,

$$\beta_\varepsilon(x) - u(x) \geqslant \beta_\varepsilon(y) - u(y) - \big(\rho(\varepsilon) + \sigma(\varepsilon)\big) = -\big(\rho(\varepsilon) + \sigma(\varepsilon)\big) > -\eta_\varepsilon\,,$$

which proves (3.9).

Now, let $v_\varepsilon := u - \beta_\varepsilon - \eta_\varepsilon$. We claim that $v_\varepsilon \leqslant 0$ in $\mathbb{R}^n$. For this, note that, by (3.6) and the assumptions on u, the function v_ε is a weak solution of

$$(-\Delta)^s v_\varepsilon \leqslant f - M \text{ in } \Omega\,. \tag{3.10}$$

Moreover, by (3.9), we know that $v_\varepsilon^+ = 0$ in $\mathbb{R}^n\setminus\Omega$. Also, $v_\varepsilon \in \mathbb{H}^s(\Omega)$ since so are u and β_ε, by (3.6).

Accordingly, we can use v_ε^+ as a test function in the weak formulation of (3.10). Hence, setting

$$\langle\phi,\psi\rangle_s := c(n,s)\int_{\mathbb{R}^{2n}} \frac{\big(\phi(x)-\phi(y)\big)\big(\psi(x)-\psi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy$$

$$\text{and} \qquad \|\phi\|_s := \sqrt{\langle\phi,\phi\rangle_s}\,,$$

from (3.10) we obtain that

$$\langle v_\varepsilon^+, v_\varepsilon\rangle_s \leqslant \int_\Omega v_\varepsilon^+(x)(f(x) - M)\,dx \leqslant 0\,,$$

thanks to the choice of M.

[4] Of course Ω_2 is simply Ω_ε for $\varepsilon = 2$.

As a consequence of this, writing $v_\varepsilon^- := v_\varepsilon^+ - v_\varepsilon \geqslant 0$, and using that $v_\varepsilon^+ v_\varepsilon^- = 0$, we get

$$\begin{aligned} 0 \geqslant \langle v_\varepsilon^+, v_\varepsilon\rangle_s &= \langle v_\varepsilon^+, v_\varepsilon^+ - v_\varepsilon^-\rangle_s \\ &= \|v_\varepsilon^+\|_s^2 - \langle v_\varepsilon^+, v_\varepsilon^-\rangle_s \\ &= \|v_\varepsilon^+\|_s^2 - \int_{\mathbb{R}^{2n}} \frac{\big(v_\varepsilon^+(x) - v_\varepsilon^+(y)\big)\big(v_\varepsilon^-(x) - v_\varepsilon^-(y)\big)}{|x-y|^{n+2s}}\, dx\, dy \\ &= \|v_\varepsilon^+\|_s^2 + \int_{\mathbb{R}^{2n}} \frac{\big(v_\varepsilon^+(x) v_\varepsilon^-(y) + v_\varepsilon^-(x) v_\varepsilon^+(y)\big)}{|x-y|^{n+2s}}\, dx\, dy \\ &\geqslant \|v_\varepsilon^+\|_s^2 . \end{aligned}$$

Therefore, $v_\varepsilon^+ = 0$ in $\mathbb{R}^n$ and so the claim is proved, that is $u \leqslant \beta_\varepsilon + \eta_\varepsilon$ in $\mathbb{R}^n$ by the definition of v_ε.

Thus, from this and (3.3), we see that, fixed $\varepsilon \in (0, 2]$, for large j,

$$\begin{aligned} a + u(x_o) &\leqslant u(x_j) \\ &\leqslant \beta_\varepsilon(x_j) + \eta_\varepsilon \\ &\leqslant \beta_\varepsilon(x_o) + \rho(|x_j - x_o|) + \eta_\varepsilon \\ &\leqslant \beta_\varepsilon(x_o) + 2\eta_\varepsilon , \end{aligned} \tag{3.11}$$

since $\rho(|x_j - x_o|) = \eta_{|x_j - x_o|} - \sigma(|x_j - x_o|) < \eta_{|x_j - x_o|} < \eta_\varepsilon$ by (3.7) .

Moreover, recalling (3.4), we know that there exists $y_\varepsilon \in \mathbb{R}^n \setminus \Omega_\varepsilon$ such that $|y_\varepsilon - x_o| \leqslant \varepsilon$. Notice that $\beta_\varepsilon(y_\varepsilon) = u(y_\varepsilon)$ by (3.6). Therefore, (3.11) gives

$$\begin{aligned} a + u(x_o) &\leqslant \beta_\varepsilon(x_o) + 2\eta_\varepsilon \\ &= u(y_\varepsilon) + \beta_\varepsilon(x_o) - \beta_\varepsilon(y_\varepsilon) + 2\eta_\varepsilon \\ &\leqslant u(y_\varepsilon) + 3\eta_\varepsilon . \end{aligned} \tag{3.12}$$

Also, y_ε, $x_o \in \Omega_2 \setminus \Omega$, and so, being $u \in C(\mathbb{R}^n \setminus \Omega)$

$$u(y_\varepsilon) - u(x_o) \leqslant \sigma(|y_\varepsilon - x_o|) \leqslant \sigma(\varepsilon) \leqslant \eta_\varepsilon ,$$

thanks to the fact that $\sigma(\varepsilon) = \eta_\varepsilon - \rho(\varepsilon) < \eta_\varepsilon$. Thus, (3.12) implies that

$$a + u(x_o) \leqslant u(y_\varepsilon) + 3\eta_\varepsilon \leqslant u(y_\varepsilon) - u(x_o) + u(x_o) + 3\eta_\varepsilon \leqslant u(x_o) + 4\eta_\varepsilon ,$$

that is

$$a \leqslant 4\eta_\varepsilon .$$

By taking the limit as $\varepsilon \searrow 0$ and recalling (3.8) we obtain $a \leqslant 0$, which is against our assumption. Thus, $u \in C(\mathbb{R}^n)$ and this concludes the proof of Theorem 2 .

4. Proof of Theorem 1

By extending f in a bounded way outside Ω, we may suppose that $f \in L^\infty(\mathbb{R}^n)$. Let u be a weak solution of problem (1.1) and fix $x_o \in \Omega$. Then, by Theorem 2

$$u \in C(\mathbb{R}^n). \tag{4.1}$$

By convolution (as done in Proposition 5) and recalling the stability of viscosity solutions (see, e.g., [3, Lemma 4.5]) we may suppose that u is also smooth in a small neighborhood Ω'

of x_o and, from (1.3), for any $\varphi \in C_0^\infty(\Omega')$

$$\begin{aligned} c(n,s) \int\limits_{\mathbb{R}^{2n}} & \frac{\big(2u(x) - u(x+Y) - u(x-Y)\big)\,\varphi(x)}{|Y|^{n+2s}}\,dx\,dY \\ &= c(n,s) \int\limits_{\mathbb{R}^{2n}} \frac{\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\ &= \int\limits_{\mathbb{R}^n} f(x)\varphi(x)\,dx\,. \end{aligned}$$

Therefore,

$$c(n,s) \int\limits_{\mathbb{R}^n} \frac{\big(2u(x) - u(x+Y) - u(x-Y)\big)}{|Y|^{n+2s}}\,dY = f(x)$$

for almost any $x \in \Omega'$ and, in fact, for any $x \in \Omega'$, due to (4.1). In particular

$$c(n,s) \int\limits_{\mathbb{R}^n} \frac{\big(2u(x_o) - u(x_o+Y) - u(x_o-Y)\big)}{|Y|^{n+2s}}\,dY = f(x_o)\,.$$

Now, let us take v as in (1.4): since $v \geqslant u$ and $v(x_o) = u(x_o)$, we obtain

$$\begin{aligned} (-\Delta)^s v(x_o) &= c(n,s) \int\limits_{\mathbb{R}^n} \frac{\big(2v(x_o) - v(x_o+Y) - v(x_o-Y)\big)}{|Y|^{n+2s}}\,dY \\ &\leqslant c(n,s) \int\limits_{\mathbb{R}^n} \frac{\big(2u(x_o) - u(x_o+Y) - u(x_o-Y)\big)}{|Y|^{n+2s}}\,dY \\ &= f(x_o). \end{aligned}$$

This shows that u is a viscosity subsolution of (1.1), and one can prove similarly that u is a viscosity supersolution. This ends the proof of Theorem 1. □

Remark 6. The result of Theorem 1 also holds when $f = f(x,u)$ for $f \in C(\mathbb{R}^{n+1})$. Indeed, one sets $\tilde{f}(x) := f(x,u(x)) \in L^\infty(\mathbb{R}^{n+1})$ and applies Theorem 2 to obtain that u is continuous and therefore so is $\tilde{f}$. Then one may apply Theorem 1 (with $\tilde{f}$ in the place of f) and obtain that u is a viscosity solution of (1.1).

5. On the boundedness of weak solutions

In this section we discuss the boundedness of weak solutions of problem (1.1).

In all the previous results of the present paper, for the sake of simplicity, we have focused on the case of bounded solutions of (1.1), i.e. on solutions $u \in L^\infty(\mathbb{R}^n)$. Here we show that, in fact, this assumption can be removed. Indeed, the following result holds true:

Proposition 7. *Let $u : \mathbb{R}^N \to \mathbb{R}$, $u \in \mathbb{H}^s(\Omega)$ be a weak solution of problem* (1.1) *(i.e., suppose that* (1.3) *holds), with $f \in C(\mathbb{R}^n)$ and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, $u \in L^\infty(\mathbb{R}^n)$.*

Proof. For the proof we use the same arguments used in [12, Proposition 4] in order to prove a L^∞–regularity result for the eigenfunctions of the operator $(-\Delta)^s$.

By extending f in a bounded way outside Ω, we may suppose that $f \in L^\infty(\mathbb{R}^n)$ and so $f \in L^2(\Omega)$. Moreover, we may assume that u does not vanish identically (otherwise there is nothing to prove). Let $\delta > 0$, to be taken appropriately small in what follows (the choice of δ will be done on (5.12) below). Up to multiplying u by a small constant, we may and do assume that

$$\|g\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{1}{2} \tag{5.1}$$

and

$$\|u\|_{L^2(\Omega)} = \sqrt{\delta}\,. \tag{5.2}$$

First of all, we claim that for any $v \in \mathbb{H}^s(\mathbb{R}^n)$

$$\big(v(x)-v(y)\big)\big(v^+(x)-v^+(y)\big) \geqslant \big|v^+(x)-v^+(y)\big|^2 \tag{5.3}$$

for any x, $y \in \mathbb{R}^n$. To check (5.3), since the role of x and y is symmetric, we can always suppose $v(x) \geqslant v(y)$. Also (5.3) is clearly an identity when x, $y \in \{v \geqslant 0\}$ and when x, $y \in \{v < 0\}$. So it only remains to check (5.3) when $x \in \{v \geqslant 0\}$ and $y \in \{v < 0\}$. In this case

$$v^+(x) - v^+(y) = v(x) < v(x) + |v(y)| = v(x) - v(y)\,,$$

and so, if we multiply by $v^+(x) - v^+(y) \geqslant 0$ both sides, we obtain (5.3).

Now, for any $k \in \mathbb{N}$, we let $C_k := 1 - 2^{-k}$, $v_k := u - C_k$, $w_k := v_k^+$ and $U_k := \|w_k\|^2_{L^2(\Omega)}$. Notice that $w_k \in \mathbb{H}^s(\Omega) \cap L^\infty(\mathbb{R}^n \setminus \Omega)$, since $u \in \mathbb{H}^s(\Omega)$ and $u = g$ a.e. in $\mathbb{R}^n \setminus \Omega$ and $g \in L^\infty(\mathbb{R}^n \setminus \Omega)$. Moreover,

$$0 \leqslant w_k \leqslant |u| + |C_k| \leqslant |u| + 1 \in L^2(\Omega)\,,$$

being Ω bounded, and

$$w_k \to (u-1)^+ \quad \text{a.e. in } \mathbb{R}^n$$

as $k \to +\infty$. Therefore, by the Dominated Convergence Theorem applied in $L^2(\Omega)$, we get

$$U_k := \|w_k\|^2_{L^2(\Omega)} \to \|(u-1)^+\|^2_{L^2(\Omega)} \tag{5.4}$$

as $k \to +\infty$.

Moreover, for any $k \in \mathbb{N}$ we have that $C_{k+1} > C_k$ and so $v_{k+1} < v_k$ a.e. in $\mathbb{R}^n$ from which we deduce that

$$w_{k+1} \leqslant w_k \quad \text{a.e. in } \mathbb{R}^n\,. \tag{5.5}$$

Furthermore, by (5.1), in $\mathbb{R}^n \setminus \Omega$ we have that

$$v_{k+1} = g - 1 + 2^{-k-1} \leqslant -\frac{1}{2} + 2^{-k-1} \leqslant 0,$$

hence

$$v^+_{k+1} = 0 \text{ outside } \Omega\,. \tag{5.6}$$

Also note that

$$v_{k+1}(x) - v_{k+1}(y) = u(x) - u(y)$$

for any x, $y \in \mathbb{R}^n$. Using this, the fact that u is a weak solution of (1.1), (5.3), (5.6) and the fact that $v^+_{k+1} = w_{k+1} \in \mathbb{H}^s(\Omega)$, we obtain

$$\begin{aligned}
\int_{\Omega\times\Omega} \frac{\big|w_{k+1}(x)-w_{k+1}(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy &= \int_{\Omega\times\Omega} \frac{\big|v^+_{k+1}(x)-v^+_{k+1}(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\leqslant \int_{\mathbb{R}^{2n}} \frac{\big(v_{k+1}(x)-v_{k+1}(y)\big)\big(v^+_{k+1}(x)-v^+_{k+1}(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\
&= \int_{\mathbb{R}^{2n}} \frac{\big(u(x)-u(y)\big)\big(v^+_{k+1}(x)-v^+_{k+1}(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\
&= \int_{\Omega} f(x)\,w_{k+1}(x)\,dx\,.
\end{aligned}$$

As a consequence of this,

$$
\begin{aligned}
\int\limits_{\Omega\times\Omega} \frac{\big|w_{k+1}(x)-w_{k+1}(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy &\leqslant \int\limits_{\Omega\cap\{w_{k+1}>0\}} |f(x)|\,w_{k+1}(x)\,dx \\
&\leqslant \|f\|_{L^\infty(\mathbb{R}^n)} \int\limits_{\Omega\cap\{w_{k+1}>0\}} w_{k+1}(x)\,dx \\
&\leqslant \|f\|_{L^\infty(\mathbb{R}^n)} \int\limits_{\Omega\cap\{w_{k+1}>0\}} w_{k}(x)\,dx \\
&\leqslant \|f\|_{L^\infty(\mathbb{R}^n)} \big|\Omega\cap\{w_{k+1}>0\}\big|^{1/2}\,\|w_k\|_{L^2(\Omega)} \\
&\leqslant \|f\|_{L^\infty(\mathbb{R}^n)} \big|\Omega\cap\{w_{k+1}>0\}\big|^{1/2}\,U_k^{1/2}\,,
\end{aligned}
\tag{5.7}
$$

thanks (5.5), the regularity of f and Hölder inequality.

Now we claim that

$$
\{w_{k+1}>0\}\subseteq\{w_k>2^{-(k+1)}\}. \tag{5.8}
$$

To establish this, we observe that if $x\in\{w_{k+1}>0\}$ then

$$
0<v_{k+1}^+(x)=\max\{u(x)-C_{k+1},0\}\,.
$$

Hence $u(x)-C_{k+1}>0$. Accordingly

$$
v_k(x)=u(x)-C_k>C_{k+1}-C_k=2^{-(k+1)}\,,
$$

so that, as a consequence, $w_k(x):=v_k^+(x)=v_k(x)>2^{-(k+1)}$, which proves (5.8).

As a consequence of (5.8), we obtain

$$
\begin{aligned}
U_k &= \|w_k\|^2_{L^2(\Omega)} \\
&\geqslant \int\limits_{\Omega\cap\{w_k>2^{-(k+1)}\}} w_k^2(x)\,dx \\
&\geqslant 2^{-2(k+1)}\big|\Omega\cap\{w_k\geqslant 2^{-(k+1)}\}\big| \\
&\geqslant 2^{-2(k+1)}\big|\Omega\cap\{w_{k+1}>0\}\big|\,,
\end{aligned}
$$

so that

$$
\big|\Omega\cap\{w_{k+1}>0\}\big|\leqslant 2^{2(k+1)}\,U_k\,. \tag{5.9}
$$

Then by (5.7) and (5.9) we get that

$$
\int\limits_{\Omega\times\Omega} \frac{\big|w_{k+1}(x)-w_{k+1}(y)\big|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant 2^{k+1}\,\|f\|_{L^\infty(\mathbb{R}^n)}\,U_k\,. \tag{5.10}
$$

Now we use the Hölder inequality (with exponents $2^*/2$ and $n/(2s)$) and the fractional Sobolev inequality (see, e.g., Theorem 6.7 in [5]) to get that

$$
\begin{aligned}
U_{k+1} &\leqslant \left(\int\limits_{\Omega} |w_{k+1}(x)|^{2^*}\,dx\right)^{2/2^*} \big|\Omega\cap\{w_{k+1}>0\}\big|^{2s/n} \\
&\leqslant \tilde{c}\int\limits_{\Omega\times\Omega} \frac{|w_{k+1}(x)-w_{k+1}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\,\big|\Omega\cap\{w_{k+1}>0\}\big|^{2s/n}\,,
\end{aligned}
$$

for some positive constant $\tilde{c}$ depending only on n and s. Consequently, by (5.9) and (5.10), we see that

$$\begin{aligned}U_{k+1} &\leqslant \tilde{c}\, 2^{k+1}\, \|f\|_{L^\infty(\mathbb{R}^n)}\, U_k\, (2^{2(k+1)} U_k)^{2s/n} \\ &= \tilde{c}\, \|f\|_{L^\infty(\mathbb{R}^n)}\, 2^{1+4s/n} \big(2^{1+4s/n}\big)^k U_k^{1+2s/n} \\ &\leqslant \Big(1+\tilde{c}\, \|f\|_{L^\infty(\mathbb{R}^n)}\, 2^{1+4s/n}\Big)\big(2^{1+4s/n}\big)^k U_k^{1+2s/n} \\ &\leqslant \Big[\Big(1+\tilde{c}\, \|f\|_{L^\infty(\mathbb{R}^n)}\, 2^{1+4s/n}\Big)2^{1+4s/n}\Big]^k U_k^{1+2s/n} \\ &= C^k U_k^\beta\,,\end{aligned} \tag{5.11}$$

where $\beta := 1 + (2s/n) > 1$ and $C > 1$ only depends on n, s and f.

Now, we are ready to perform our choice of δ: namely we assume that $\delta > 0$ is so small that

$$\delta^{\beta-1} < \frac{1}{C^{1/(\beta-1)}} \tag{5.12}$$

We also fix

$$\eta \in \left(\delta^{\beta-1}, \frac{1}{C^{1/(\beta-1)}}\right).$$

Notice that, since $C > 1$ and $\beta > 1$

$$\eta \in (0, 1)\,. \tag{5.13}$$

Moreover

$$\delta^{\beta-1} \leqslant \eta \quad\text{and}\quad C\eta^{\beta-1} \leqslant 1\,. \tag{5.14}$$

We claim that

$$U_k \leqslant \delta\eta^k\,. \tag{5.15}$$

The proof is by induction. First of all

$$U_0 := \|w_0\|^2_{L^2(\Omega)} = \|u^+\|^2_{L^2(\Omega)} \leqslant \|u\|^2_{L^2(\Omega)} = \delta\,,$$

which is (5.15) when $k = 0$. Now, let us now suppose that (5.15) holds true for k and let us prove it for $k + 1$. For this, we use (5.11) and (5.14):

$$U_{k+1} \leqslant C^k U_k^\beta \leqslant C^k(\delta\eta^k)^\beta = \delta(C\eta^{\beta-1})^k \delta^{\beta-1}\eta^k \leqslant \delta\eta^{k+1}\,.$$

This proves (5.15). Then, by (5.13) and (5.15), we conclude that

$$\lim_{k\to+\infty} U_k = 0\,.$$

Hence, by (5.4), $(u - 1)^+ = 0$ a.e. in Ω, that is $u \leqslant 1$ a.e. in Ω. By replacing u with $-u$, we obtain $\|u\|_{L^\infty(\Omega)} \leqslant 1$. □

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