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THE YAMABE EQUATION IN A NON-LOCAL SETTING

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Abstract. Aim of this paper is to study the following elliptic equation driven by a general non-local integrodifferential operator $\mathcal{L}_K$

$$\begin{cases} \mathcal{L}_K u + \lambda u + |u|^{2^*-2}u = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

where $s \in (0,1)$, Ω is an open bounded set of $\mathbb{R}^n$, $n > 2s$, with Lipschitz boundary, λ is a positive real parameter, $2^* = 2n/(n-2s)$ is a fractional critical Sobolev exponent, while $\mathcal{L}_K$ is the non-local integrodifferential operator

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n}\Big(u(x+y) + u(x-y) - 2u(x)\Big)K(y)\,dy\,, \quad x \in \mathbb{R}^n\,.$$

As a concrete example, we consider the case when $K(x) = |x|^{-(n+2s)}$, which gives rise to the fractional Laplace operator $-(-\Delta)^s$. In this framework, in the existence result proved along the paper, we show that our problem admits a non-trivial solution for any $\lambda > 0$, provided $n \geqslant 4s$ and λ is different from the eigenvalues of $(-\Delta)^s$. This result may be read as the non-local fractional counterpart of the one obtained by Capozzi, Fortunato and Palmieri in [8] (see also [3]) and by Gazzola and Ruf in [11] for the classical Laplace equation with critical nonlinearities.

In this sense the present work may be seen as the extension of some classical results for the Laplacian to the case of non-local fractional operators.

Contents

1. Introduction

Key words and phrases. Mountain Pass Theorem, Linking Theorem, critical nonlinearities, best fractional critical Sobolev constant, Palais–Smale condition, variational techniques, integrodifferential operators, fractional Laplacian.

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1.1. **Critical non-local integrodifferential equations.** In the literature there are many papers related to the study of the critical elliptic equation

$$\begin{cases} -\Delta u - \lambda u = |u|^{2_*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \quad 2_* = 2n/(n-2)\,, n > 2\,, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n > 2$, is an open, bounded set and λ is a positive parameter (see, for instance, [1, 19, 21] and references therein).

Equation (1.1) is particularly relevant for its relations with problems arising in differential geometry and in physics, where a lack of compactness occurs. One of the most important problem which gives rise to equation (1.1) is the Yamabe problem: given a n dimensional compact Riemannian manifold $(\mathcal{M}, g)$, $n > 2$, with scalar curvature $k = k(x)$, find a metric $\tilde{g}$ conformal to g with constant scalar curvature $\tilde{k}$. If we put

$$\tilde{g} = u^{4/(n-2)}g\,,$$

where $u > 0$ is the conformal factor, then the Yamabe problem can be formulated as follows: find $u > 0$ satisfying the equation

$$-4\frac{n-1}{n-2}\Delta_{\mathcal{M}}u = \tilde{k}u^{2^*-1} - k(x)u \quad \text{in } \mathcal{M}\,.$$

Here $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$ with respect to the metric g.

Motivated by the interest shown in the literature for non-local operators of elliptic type, in this paper we study the non-local counterpart of problem (1.1), that is we consider critical problems modeled by

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.2}$$

where $s \in (0, 1)$ is fixed and $(-\Delta)^s$ is the fractional Laplace operator defined, up to normalization factors, as

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\,dy\,, \quad x \in \mathbb{R}^n \tag{1.3}$$

(see [10] and references therein for further details on the fractional Laplacian), while $\Omega \subset \mathbb{R}^n$, $n > 2s$, is open, bounded and with Lipschitz boundary, $\lambda > 0$ and

$$2^* = 2n/(n-2s) \tag{1.4}$$

is the fractional critical Sobolev exponent.

In [15, 16] we studied problems of the type

$$\begin{cases} (-\Delta)^s u - \lambda u = |u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.5}$$

where $q \in (2, 2^*)$, that is we considered non-local equations with subcritical growth, and in this setting we extended the validity of some existence results known in the classical subcritical case of the Laplacian to the non-local framework. Also, in [17] we considered the critical equation (1.2) in the case when the parameter $\lambda \in (0, \lambda_{1,s})$, where $\lambda_{1,s}$ denotes the first eigenvalue of the fractional Laplace operator $(-\Delta)^s$ with homogeneous Dirichlet boundary conditions. In this framework we proved a non-local Brezis-Nirenberg type result, i.e. we showed that problem (1.2) admits a non-trivial solution for any $\lambda \in (0, \lambda_{1,s})$, provided $n \geqslant 4s$ (see [7, 17]). The case when $n < 4s$ was treated in the paper [18], where we proved the existence of non-trivial weak solution for the equation (1.2) for any value of the parameter $\lambda > 0$ different from the eigenvalues of $(-\Delta)^s$.

Aim of this paper is to complete the study of problem (1.2), by showing that the existence result obtained in [17, Theorem 4] holds true for any $\lambda > 0$ different from the eigenvalues of $(-\Delta)^s$.

More precisely, along this paper we consider the following general non-local equation

$$\begin{cases} \mathcal{L}_K u + \lambda u + |u|^{2^*-2}u = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.6}$$

where $\mathcal{L}_K$ is the non-local operator defined as

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \Big(u(x+y) + u(x-y) - 2u(x)\Big)K(y)\,dy\,, \quad x \in \mathbb{R}^n\,, \tag{1.7}$$

with the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ such that

$$mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^2, 1\}\,; \tag{1.8}$$

$$\text{there exists } \theta > 0 \text{ such that } K(x) \geqslant \theta |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,; \tag{1.9}$$

$$K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.10}$$

As a model for K we can take the singular kernel $K(x) = |x|^{-(n+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$ defined in (1.3).

Note that in problem (1.6) the Dirichlet datum is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, consistently with the non-local character of the operator $\mathcal{L}_K$.

Along the paper we will be interested in the weak formulation of (1.6) given by the following problem (for this, it is convenient to assume (1.10))

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy - \lambda \int_\Omega u(x)\varphi(x)\,dx \\ \qquad\qquad\qquad\qquad\qquad = \displaystyle\int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)dx \quad \forall\, \varphi \in X_0 \\ u \in X_0\,. \end{cases} \tag{1.11}$$

In order to study problem (1.2) (and, in general, (1.6)) the usual fractional Sobolev spaces are not adequate. Hence, to overcome this difficulty in [15] (see also [14]) we introduced a new functional space X_0, which, in our opinion, is the suitable space in which working in. The definition of X_0 will be given in Subsection 2.1. In the model case $\mathcal{L}_K = -(-\Delta)^s$ the space X_0 consists of all the functions of the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ which vanish a.e. outside Ω (see [17, Lemma 7]) .

1.2. **Main theorems of the paper.** It is easily seen that problem (1.6) admits the trivial function $u \equiv 0$ as a solution. The aim of this paper is to find non-trivial weak solutions for (1.6) , that is non-trivial solutions of (1.11) . For this, we will use a variational approach adapting the techniques of [7, 8] (see also [1, 19, 21] and references therein) to our non-local setting.

Problem (1.11) is the Euler–Lagrange equation of the functional $\mathcal{J}_{K,\lambda} : X_0 \to \mathbb{R}$ defined as follows

$$\mathcal{J}_{K,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx\,. \tag{1.12}$$

Thus, our goal will be finding critical points of $\mathcal{J}_{K,\lambda}$ and, for this, we will use some variants of the classical Mountain Pass and Linking Theorems due to Ambrosetti and Rabinowitz (see [2, 13]), which take into account that, since the embedding $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n)$ is not compact (see Lemma 3), the functional $\mathcal{J}_{K,\lambda}$ does not verify the Palais–Smale condition globally, but only in a suitable range related to the best fractional critical Sobolev constant defined as

$$S_K := \inf_{v \in X_0 \setminus \{0\}} S_K(v)\,, \tag{1.13}$$

where

$$X_0 \setminus \{0\} \ni v \mapsto S_K(v) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy}{\displaystyle\Big(\int_\Omega |v(x)|^{2^*}dx\Big)^{2/2^*}}\,. \tag{1.14}$$

The constant S_K is well defined, as it can be seen by Lemma 3 below, and it is strictly positive.

Along the paper we also need the following function

$$X_0 \setminus \{0\} \ni v \mapsto S_{K,\lambda}(v) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda \int_\Omega |v(x)|^2\,dx}{\displaystyle\Big(\int_\Omega |v(x)|^{2^*}dx\Big)^{2/2^*}}\,. \tag{1.15}$$

Note that, since $v \in X_0$, the integrals over Ω in (1.14) and in (1.15) can be extended to all $\mathbb{R}^n$, that is

$$S_K(v) = \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2^*}dx\Big)^{2/2^*}}$$

and

$$S_{K,\lambda}(v) = \frac{\displaystyle\int_{\mathbb{R}^{2n}} |v(x)-v(y)|^2 K(x-y)\,dx\,dy - \lambda \int_{\mathbb{R}^n} |v(x)|^2\,dx}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2^*}dx\Big)^{2/2^*}}\,.$$

Hence, $S_K(\cdot)$ and $S_{K,\lambda}(\cdot)$ do not depend on the domain Ω, while S_K does, since X_0 depends on Ω.

In the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ the counterparts of S_K and $S_{K,\lambda}(\cdot)$ are given, respectively, by the constant S_s defined as follows (see also Lemma 4 in the sequel)

$$\begin{aligned} S_s &:= \inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} S_s(v)\,, \\ H^s(\mathbb{R}^n)\setminus\{0\} \ni v \mapsto S_s(v) &:= \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2^*}dx\Big)^{2/2^*}}\,, \end{aligned} \tag{1.16}$$

and by the function

$$H^s(\mathbb{R}^n)\setminus\{0\} \ni v \mapsto S_{s,\lambda}(v) := \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_{\mathbb{R}^n}|v(x)|^2\,dx}{\displaystyle\Big(\int_{\mathbb{R}^n} |v(x)|^{2^*}dx\Big)^{2/2^*}}\,. \tag{1.17}$$

Note that for any $\lambda > 0$

$$S_{K,\lambda} := \inf_{v\in X_0\setminus\{0\}} S_{K,\lambda}(v) \leqslant S_K$$

and

$$S_{s,\lambda} := \inf_{v\in H^s(\mathbb{R}^n)\setminus\{0\}} S_{s,\lambda}(v) \leqslant S_s\,.$$

The interesting case is when the strict inequality holds true. Indeed, in this setting a suitable compactness propriety holds true for the functional $\mathcal{J}_{K,\lambda}$. This fact allows us to prove our main existence result, by suitably using critical points theorems.

In the sequel $\{\lambda_k\}_{k\in\mathbb{N}}$ will denote the sequence of the eigenvalues of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data with

$$0 < \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \ldots$$

and

$$\lambda_k \to +\infty \text{ as } k \to +\infty\,.$$

Moreover, $\{e_k\}_{k\in\mathbb{N}}$ will be the sequence of eigenfunctions corresponding to λ_k. We recall that this sequence is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of X_0. For a complete study of the spectrum of the integrodifferential operator $-\mathcal{L}_K$ we refer to [16, Proposition 9 and Appendix A].

The main result of the present paper is the following existence theorem:

Theorem 1. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying conditions* (1.8)–(1.10)*. Furthermore, assume that*

$$\begin{gathered}\textit{there exists } u_0 \in X_0 \setminus \{0\} \textit{ with } u_0 \geqslant 0 \textit{ a.e. in } \mathbb{R}^n \textit{ such that}\\ S_{K,\lambda}(u) < S_K \textit{ for any } u \in \mathcal{U},\\ \textit{where } \mathcal{U} = \begin{cases} \mathit{span}\{u_0\} & \textit{if } \lambda \in (0,\, \lambda_1)\\ \mathit{span}\{e_1, \ldots, e_k, u_0\} & \textit{if } \lambda \in [\lambda_k,\, \lambda_{k+1})\,,\ k \in \mathbb{N}\,.\end{cases}\end{gathered} \tag{1.18}$$

Then, for any $\lambda > 0$ problem (1.11) *admits a solution $u \in X_0$, which is not identically zero.*

Note that when $\lambda \in (0,\, \lambda_1)$ condition (1.18) is exactly

$$S_{K,\lambda} < S_K\,, \tag{1.19}$$

so that Theorem 1 reduces to [17, Theorem 1]. Also, a concrete case in which condition (1.18) is satisfied is presented in the forthcoming Theorem 2.

As a matter of fact, when dealing with the model kernel $K(x) = |x|^{-(n+2s)}$, problem (1.6) reduces to (1.2), whose weak formulation is given by

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda \int_\Omega u(x)\varphi(x)\,dx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \displaystyle = \int_\Omega |u(x)|^{2^*-2} u(x)\varphi(x)dx \\ \qquad\qquad\qquad\qquad\qquad \forall\, \varphi \in H^s(\mathbb{R}^n) \text{ with } \varphi = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \\ u \in H^s(\mathbb{R}^n) \text{ with } u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{1.20}$$

In the fractional Laplace setting the counterpart of Theorem 1 is given by the following result:

Theorem 2. *Let $s \in (0,1)$, $n \geqslant 4s$, Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and λ be a positive parameter.*

Then, problem (1.20) *admits a solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero and such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, provided λ is not an eigenvalue of $(-\Delta)^s$ with homogeneous Dirichlet boundary data.*

This existence theorem may be seen as the natural extension to the non-local framework of a well-know result obtained by Capozzi, Fortunato and Palmieri in [8], where the classical critical Laplace equation (1.1) was studied (for more general critical nonlinearities see [11]). Indeed, when $s = 1$, Theorem 2 reads as [8, Theorem 0.1] (see also [11, Corollary 1] and [21, Theorem 2.24]).

The proof of Theorem 2 is based on the fact that, in the fractional Laplace setting, condition (1.18) is related to the strict inequality $S_{s,\lambda} < S_s$, which holds true in any dimension n provided $n \geqslant 4s$. For more details, see Section 7.

In general the main existence results proved in the present paper are obtained through variational methods (precisely applying classical minimax theorems to $\mathcal{J}_{K,\lambda}$). The main difficulties are related to the encoding the Dirichlet boundary datum in the variational formulation. Working in the space X_0 allows us to overcome this difficulty. Moreover, since equations (1.2) and (1.6) have a critical growth, another problem is related to the lack of compactness in the embedding $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n)$ (or, in the case of the fractional Laplacian, $H^s(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$). In order to overcome these difficulties, first we will show that the functional $\mathcal{J}_{K,\lambda}$ satisfies the Palais–Smale condition at any level smaller than a certain threshold related to the best fractional critical Sobolev constant. Then, after showing that $\mathcal{J}_{K,\lambda}$ has the right geometric structure, we will prove that the minimax critical level of $\mathcal{J}_{K,\lambda}$ lies below such a threshold.

When dealing with a non-local setting, in the pure power case all the results presented here extend the existence theorems obtained in [15, 16, 17] to the critical case. Furthermore, the present paper extends some classical results for the Laplacian to the case of non-local fractional operators (see, for instance, [3, 7, 8, 11]).

We also would like to recall the recent papers [4, 20], where the authors studied a non-local version of the Brezis-Nirenberg equation for an operator conceptually different from the one treated in the present work.

The paper is organized as follows. Section 2 is devoted to some preliminary results: since the functional analytic space we will work is not a standard space, first we introduce it and then we recall some of its embeddings into the usual Lebesgue spaces. In Section 3 we illustrate the strategy, based on a variational approach, we will use in order to prove the main result of the paper. In Section 4 we show that the Euler–Lagrange functional associated with problem (1.11) satisfies a suitable compactness condition, while Section 5 is devoted to the study of the geometric structure of the same functional. In Section 6 we prove Theorem 1 via variational techniques, while in Section 7 we discuss the case of the fractional Laplacian operator and we prove Theorem 2.

2. Some preliminary results

In this section we recall some preliminary results which will be useful along the paper.

As we said in the Introduction, in our setting the fractional Sobolev space is not enough in order to study problems (1.2) and (1.6). For this reason in [15] (see also [14]) we introduced a new functional analytic space X_0 inspired by (but not equivalent to) the fractional Sobolev spaces which seems to be the appropriate space in which work in. Since this space is not a standard one, here we recall its definition and some of its properties which we will use along the paper. The reader familiar with this topic may skip it.

2.1. **The functional setting.** In our framework the functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y) \mapsto (g(x)-g(y))\sqrt{K(x-y)} \text{ is in } L^2\big(\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy\big),$$

where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$, while

$$X_0 = \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

We note that X and X_0 are non-empty, since $C^2_0(\Omega) \subseteq X_0$ by [14, Lemma 11] (for this we need condition (1.8)).

The space X is endowed with the norm defined as

$$\|g\|_X = \|g\|_{L^2(\Omega)} + \Big(\int_Q |g(x)-g(y)|^2 K(x-y)dx\,dy\Big)^{1/2}, \tag{2.1}$$

where $Q = \mathbb{R}^{2n} \setminus \mathcal{O}$ and $\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^{2n}$. It is easily seen that $\|\cdot\|_X$ is a norm on X (see, for instance, [15] for a proof).

In the following we denote by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \Big)^{1/2} . \tag{2.2}$$

We remark that, even in the model case in which $K(x) = |x|^{-(n+2s)}$, the norms in (2.1) and (2.2) are not the same, because $\Omega \times \Omega$ is strictly contained in Q (this makes the classical fractional Sobolev space approach not sufficient for studying the problem).

For further details on the fractional Sobolev spaces we refer to [10] and to the references therein.

By [15, Lemmas 6 and 7] in the sequel we can take the function

$$X_0 \ni v \mapsto \|v\|_{X_0} = \left(\int_Q |v(x)-v(y)|^2 K(x-y)\, dx\, dy \right)^{1/2} \tag{2.3}$$

as norm on X_0. Also $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space, with scalar product

$$\langle u, v\rangle_{X_0} = \int_Q \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\, K(x-y)\, dx\, dy\, . \tag{2.4}$$

Note that in (2.3) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^{2n}$, since $v \in X_0$ (and so $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$).

Along the paper we need some embeddings of the spaces X_0 and $H^s(\mathbb{R}^n)$ into the usual Lebesgue spaces. With respect to these embeddings, the space X_0 behaves like $H^1_0(\Omega)$, while the fractional Sobolev space $H^s(\mathbb{R}^n)$ as the usual Sobolev space $H^1(\mathbb{R}^n)$: this is due to the fact that the functions $v \in X_0$ are such that $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ and so X_0 may be seen as a space of functions defined in the bounded set Ω . Indeed, the following results, proved in [15, Lemma 8] and in [17, Lemma 9], hold true:

Lemma 3. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions* (1.8)–(1.10) *. Then, the following assertions hold true:*

a) if Ω has a Lipschitz boundary, then the embedding $X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is compact for any $\nu \in [1, 2^*)$;

b) the embedding $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n)$ is continuous.

The counterpart of Lemma 3 in the usual fractional Sobolev spaces is given by the following one proved in [10, Theorem 6.5]:

Lemma 4. *The embedding $H^s(\mathbb{R}^n) \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [2, 2^*]$.*

2.2. **A variational characterization of the eigenvalues of $-\mathcal{L}_K$.** In this subsection we will give a variational characterization of the eigenvalues of the integrodifferential operator $-\mathcal{L}_K$, which will be useful in the sequel. A complete study of the spectrum of the $-\mathcal{L}_K$ can be found in [16, Proposition 9 and Appendix A] .

Proposition 5. *Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the sequence of the eigenvalues of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data and*

$$0 < \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \ldots$$

and let $\{e_k\}_{k\in\mathbb{N}}$ be the sequence of eigenfunctions corresponding to λ_k .

Then, for any $k \in \mathbb{N}$ the eigenvalues can be characterized as follows:

$$\lambda_k = \max_{u \in span\{e_1,\ldots,e_k\}\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y) dx\, dy}{\displaystyle\int_\Omega |u(x)|^2\, dx} .$$

Proof. Let $k\in\mathbb{N}$. Since λ_k is the eigenvalue corresponding to the eigenfunction e_k, we have that

$$\begin{aligned}\lambda_k &= \frac{\displaystyle\int_{\mathbb{R}^{2n}}|e_k(x)-e_k(y)|^2K(x-y)dx\,dy}{\displaystyle\int_\Omega |e_k(x)|^2\,dx}\\ &\leqslant \max_{u\in \mathrm{span}\{e_1,\dots,e_k\}\setminus\{0\}}\frac{\displaystyle\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)dx\,dy}{\displaystyle\int_\Omega |u(x)|^2\,dx}.\end{aligned} \tag{2.5}$$

Furthermore, let $u\in \mathrm{span}\{e_1,\dots,e_k\}\setminus\{0\}$. Then

$$u=\sum_{i=1}^k u_ie_i \quad\text{with}\quad \sum_{i=1}^k u_i^2\neq 0$$

and

$$\frac{\displaystyle\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)dx\,dy}{\displaystyle\int_\Omega |u(x)|^2\,dx}=\frac{\displaystyle\sum_{i=1}^k u_i^2\|e_i\|_{X_0}}{\displaystyle\sum_{i=1}^k u_i^2}=\frac{\displaystyle\sum_{i=1}^k u_i^2\lambda_i}{\displaystyle\sum_{i=1}^k u_i^2}\leqslant\lambda_k\,,$$

since $\big\{e_1,\dots,e_k\big\}$ are orthonormal in $L^2(\Omega)$ and orthogonal in X_0 (see [16, Proposition 9–f)]) and $\lambda_k\geqslant\lambda_i$ for any $i=1,\dots,k$. As a consequence, passing to the maximum over $u\in\mathrm{span}\{e_1,\dots,e_k\}\setminus\{0\}$ we get

$$\max_{u\in \mathrm{span}\{e_1,\dots,e_k\}\setminus\{0\}}\frac{\displaystyle\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)dx\,dy}{\displaystyle\int_\Omega |u(x)|^2\,dx}\leqslant\lambda_k\,,$$

which together with (2.5) gives the assertion. □

In the sequel it will be useful also the following regularity result for the eigenvalues of $-\mathcal{L}_K$, whose proof can be found in [18, Proposition 4] in the setting of the fractional Laplacian (in the general case we can proceed in the same way):

Proposition 6. *Let $e\in X_0$ and $\lambda>0$ be such that*

$$\langle e,\varphi\rangle_{X_0}=\lambda\int_\Omega e(x)\varphi(x)\,dx \tag{2.6}$$

for any $\varphi\in X_0$. Then $e\in L^\infty(\Omega)$.

3. Strategy for proving Theorem 1

This section is devoted to the study of problem (1.11) which is the Euler–Lagrange equation of the functional $\mathcal{J}_{K,\lambda}$ defined in (1.12). Notice that this functional is well defined thanks to the definition of X_0 and Lemma 3-b). Moreover, $\mathcal{J}_{K,\lambda}$ is Fréchet differentiable in $u\in X_0$ and for any $\varphi\in X_0$

$$\begin{aligned}\langle \mathcal{J}'_{K,\lambda}(u),\varphi\rangle=\int_{\mathbb{R}^{2n}}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy-\lambda\int_\Omega u(x)\varphi(x)\,dx\\ -\int_\Omega |u(x)|^{2^*-2}(x)u(x)\varphi(x)\,dx\,.\end{aligned}$$

In order to prove the existence of solutions for problem (1.11), we will apply some variants of the Mountain Pass Theorem and of the Linking Theorem (see, for instance, [2, 5, 12, 13]) which take into account the fact that in the critical setting there is a lack of compactness in the embedding $X_0 \hookrightarrow L^{2^*}(\mathbb{R}^n)$ (see Lemma 3-*b*)). As a consequence of this, as it happens in classical case, also in the non-local critical setting the functional $\mathcal{J}_{K,\lambda}$ does not verify the Palais–Smale condition globally, but only in an energy range determined by the best fractional critical Sobolev constant S_K given in formula (1.13).

In what follows, λ_1 will be the first eigenvalue of $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data. In the case when $\lambda \in (0, \lambda_1)$, problem (1.11) was studied in [17] in a setting more general than the power function. Hence, here we consider only the case when $\lambda \geqslant \lambda_1$ and, in order to get our goal, we will use appropriately the Linking Theorem of Rabinowitz (see [13, Theorem 5.3]).

Since $\lambda \geqslant \lambda_1$, we can suppose that

$$\lambda \in [\lambda_k, \lambda_{k+1}) \text{ for some } k \in \mathbb{N},$$

where λ_k is the k-th eigenvalue of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary conditions. We recall that, in what follows, e_k will be the k–th eigenfunction corresponding to the eigenvalue λ_k of $-\mathcal{L}_K$, and

$$\mathbb{P}_{k+1} := \left\{ u \in X_0 \text{ s.t. } \langle u, e_j \rangle_{X_0} = 0 \quad \forall j = 1, \dots, k \right\}, \tag{3.1}$$

while $\operatorname{span}\{e_1, \dots, e_k\}$ will denote the linear subspace generated by the first k eigenfunctions of $-\mathcal{L}_K$ for any $k \in \mathbb{N}$. Finally, we recall that $\mathcal{J}_{K,\lambda}$ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ ($(PS)_c$-condition for short) if any sequence u_j in X_0 such that

$$\mathcal{J}_{K,\lambda}(u_j) \to c$$

and

$$\sup\Big\{ \big| \langle\, \mathcal{J}'_{K,\lambda}(u_j), \varphi \,\rangle \big| \,:\, \varphi \in X_0\,, \|\varphi\|_{X_0} = 1 \Big\} \to 0$$

as $j \to +\infty$ admits a subsequence strongly convergent in X_0.

In order to prove Theorem 1, our strategy consists in adapting the techniques used in the classical case of the Laplacian in the critical setting (see [1, 19, 21] and references therein) to the non-local critical framework. Roughly, in order to apply [13, Theorem 5.3] we will prove these three facts:

- compactness conditions for $\mathcal{J}_{K,\lambda}$: the functional $\mathcal{J}_{K,\lambda}$ satisfies the $(PS)_c$-condition at any level c smaller than a certain threshold related to the best fractional critical Sobolev constant, i.e., to be precise, for any c such that $c < \frac{s}{n} S_K^{n/(2s)}$;
- geometric structure of $\mathcal{J}_{K,\lambda}$: the functional $\mathcal{J}_{K,\lambda}$ has the geometry required by the Linking Theorem;
- estimate of the critical level of $\mathcal{J}_{K,\lambda}$: the Linking critical level of $\mathcal{J}_{K,\lambda}$ lies below the threshold where the $(PS)_c$-condition holds true. For this we will use assumption (1.18).

4. A local Palais–Smale condition for the functional $\mathcal{J}_{K,\lambda}$

We start by proving that the functional $\mathcal{J}_{K,\lambda}$ satisfies the Palais–Smale condition in a suitable energy range involving the best fractional critical Sobolev constant S_K given in (1.13).

Proposition 7. *Let $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$ and let $c \in \mathbb{R}$ be such that*

$$c < \frac{s}{n} S_K^{n/(2s)}. \tag{4.1}$$

Let u_j be a sequence in X_0 such that

$$\mathcal{J}_{K,\lambda}(u_j) \to c \tag{4.2}$$

and

$$\sup\Big\{ \big| \langle\, \mathcal{J}'_{K,\lambda}(u_j), \varphi \,\rangle \big| \,:\, \varphi \in X_0\,, \|\varphi\|_{X_0} = 1 \Big\} \to 0 \tag{4.3}$$

as $j \to +\infty$.

Then, there exists $u_\infty \in X_0$ *such that, up to a subsequence,* $\|u_j - u_\infty\|_{X_0} \to 0$ *as* $j \to +\infty$.

Proof. We proceed by steps.

Step 1. *The sequence* u_j *is bounded in* X_0.

Proof. The proof of Step 1 is quite standard (see, for instance, [17]), but we prefer to repeat it for reader's convenience. For any $j \in \mathbb{N}$ by (4.2) and (4.3) it easily follows that there exists $\kappa > 0$ such that

$$|\mathcal{J}_{K,\lambda}(u_j)| \leqslant \kappa, \tag{4.4}$$

and

$$\Big|\langle \mathcal{J}'_{K,\lambda}(u_j), \frac{u_j}{\|u_j\|_{X_0}}\rangle\Big| \leqslant \kappa. \tag{4.5}$$

As a consequence of (4.4) and (4.5) we have

$$\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{2}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle \leqslant \kappa\,(1 + \|u_j\|_{X_0}). \tag{4.6}$$

Moreover, it is easy to see that

$$\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{2}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle = -\left(\frac{1}{2^*} - \frac{1}{2}\right)\|u_j\|^{2^*}_{L^{2^*}(\Omega)} = \frac{s}{n}\,\|u_j\|^{2^*}_{L^{2^*}(\Omega)},$$

so that, thanks to (4.6), we get that for any $j \in \mathbb{N}$

$$\|u_j\|^{2^*}_{L^{2^*}(\Omega)} \leqslant \kappa_*\,(1 + \|u_j\|_{X_0}) \tag{4.7}$$

for a suitable positive constant κ_*.

Consequently, recalling that $2^* > 2$, the use of the Hölder inequality yields

$$\|u_j\|^2_{L^2(\Omega)} \leqslant |\Omega|^{2s/n}\|u_j\|^2_{L^{2^*}(\Omega)} \leqslant \kappa_*^{2/2^*}\,|\Omega|^{2s/n}\,(1 + \|u_j\|_{X_0})^{2/2^*}.$$

That is, using that $2/2^* < 1$,

$$\|u_j\|^2_{L^2(\Omega)} \leqslant \widetilde{\kappa}\,(1 + \|u_j\|_{X_0}), \tag{4.8}$$

for a suitable $\widetilde{\kappa} > 0$ that does not depend on j. By (4.4), (4.7) and (4.8), we conclude that

$$\begin{aligned}\kappa \geqslant \mathcal{J}_{K,\lambda}(u_j) &= \frac{1}{2}\|u_j\|^2_{X_0} - \frac{\lambda}{2}\|u_j\|^2_{L^2(\Omega)} - \frac{1}{2^*}\|u_j\|^{2^*}_{L^{2^*}(\Omega)} \\ &\geqslant \frac{1}{2}\|u_j\|^2_{X_0} - \underline{\kappa}\,(1 + \|u_j\|_{X_0}),\end{aligned}$$

with $\underline{\kappa} > 0$ independent of j. Hence, the proof of Step 1 is complete. □

Step 2. *Problem* (1.11) *admits a solution* $u_\infty \in X_0$.

Proof. Since u_j is bounded in X_0 and X_0 is a reflexive space (being a Hilbert space, by [15, Lemma 7]), up to a subsequence, still denoted by u_j, there exists $u_\infty \in X_0$ such that $u_j \to u_\infty$ weakly in X_0, that is

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} \big(u_j(x) - u_j(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy \to \\ &\qquad \int_{\mathbb{R}^{2n}} \big(u_\infty(x) - u_\infty(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy \quad \text{for any } \varphi \in X_0\end{aligned} \tag{4.9}$$

as $j \to +\infty$. Moreover, by Step 1, (4.7), Lemma 3-*b*) and the fact that $L^{2^*}(\mathbb{R}^n)$ is a reflexive space we have that, up to a subsequence

$$u_j \to u_\infty \quad \text{weakly in } L^{2^*}(\mathbb{R}^n) \tag{4.10}$$

as $j \to +\infty$, while by Lemma 3-*a*), up to a subsequence,

$$u_j \to u_\infty \quad \text{in } L^\nu(\mathbb{R}^n) \tag{4.11}$$

$$u_j \to u_\infty \quad \text{a.e. in } \mathbb{R}^n \tag{4.12}$$

as $j \to +\infty$ for any $\nu \in [1, 2^*)$.

As a consequence of (4.7) it is easy to see that $|u_j|^{2^*-2}u_j$ is bounded in $L^{2^*/(2^*-1)}(\Omega)$ uniformly in $j \in \mathbb{N}$. Hence, by this and (4.10) we get that

$$|u_j|^{2^*-2}u_j \to |u_\infty|^{2^*-2}u_\infty \quad \text{weakly in } L^{2^*/(2^*-1)}(\Omega) \tag{4.13}$$

as $j \to +\infty$.

Since $\big(2^*/(2^*-1)\big)' = 2^*$, by (4.13) it is easily seen that

$$\int_\Omega |u_j(x)|^{2^*-2}u_j(x)\varphi(x)\,dx \to \int_\Omega |u_\infty(x)|^{2^*-2}u_\infty(x)\varphi(x)\,dx \quad \text{for any} \quad \varphi \in L^{2^*}(\Omega)\,,$$

and so, in particular,

$$\int_\Omega |u_j(x)|^{2^*-2}u_j(x)\varphi(x)\,dx \to \int_\Omega |u_\infty(x)|^{2^*-2}u_\infty(x)\varphi(x)\,dx \quad \text{for any} \quad \varphi \in X_0 \tag{4.14}$$

as $j \to +\infty$ (here we use Lemma 3-b)).

By (4.3), for any $\varphi \in X_0$

$$\begin{aligned} 0 \leftarrow \langle \mathcal{J}'_{K,\lambda}(u_j), \varphi\rangle = \int_{\mathbb{R}^{2n}} \big(u_j(x)-u_j(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy - \lambda\int_\Omega u_j(x)\varphi(x)\,dx \\ - \int_\Omega |u_j(x)|^{2^*-2}u_j(x)\varphi(x)\,dx\,, \end{aligned}$$

so that, passing to the limit in this expression as $j \to +\infty$ and taking into account (4.9), (4.11) and (4.14) we get

$$\begin{aligned} \int_{\mathbb{R}^{2n}} \big(u_\infty(x)-u_\infty(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy - \lambda\int_\Omega u_\infty(x)\varphi(x)\,dx \\ - \int_\Omega |u_\infty(x)|^{2^*-2}u_\infty(x)\varphi(x)\,dx = 0 \end{aligned}$$

for any $\varphi \in X_0$, that is u_∞ is a solution of problem (1.11) and Step 2 follows. □

Step 3. *The following relation holds true:*

$$\mathcal{J}_{K,\lambda}(u_\infty) = \frac{s}{n}\int_\Omega |u_\infty(x)|^{2^*}\,dx \geqslant 0\,.$$

Proof. By Step 2, taking $\varphi = u_\infty \in X_0$ as a test function in (1.11), we get

$$\int_{\mathbb{R}^{2n}} |u_\infty(x)-u_\infty(y)|^2K(x-y)dx\,dy - \lambda\int_\Omega |u_\infty(x)|^2\,dx = \int_\Omega |u_\infty(x)|^{2^*}\,dx\,,$$

so that

$$\mathcal{J}_{K,\lambda}(u_\infty) = \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_\Omega |u_\infty(x)|^{2^*}\,dx = \frac{s}{n}\int_\Omega |u_\infty(x)|^{2^*}\,dx \geqslant 0\,.$$

Hence, Step 3 is proved. □

Step 4. *The following equality holds true:*

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u_j) = \mathcal{J}_{K,\lambda}(u_\infty) + \frac{1}{2}\int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2K(x-y)dx\,dy \\ - \frac{1}{2^*}\int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx + o(1) \end{aligned}$$

as $j \to +\infty$.

Proof. First of all, we observe that by Step 1 and Lemma 3-*b*), the sequence u_j is bounded in X_0 and in $L^{2^*}(\Omega)$. Hence, since (4.12) holds true, by the Brezis-Lieb Lemma (see [6, Theorem 1]), we get

$$\begin{aligned}\int_{\mathbb{R}^{2n}} |u_j(x)-u_j(y)|^2 K(x-y)\,dx\,dy \\ = \int_{\mathbb{R}^{2n}} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy \\ + \int_{\mathbb{R}^{2n}} |u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy + o(1)\end{aligned} \tag{4.15}$$

and

$$\int_\Omega |u_j(x)|^{2^*}\,dx = \int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx + \int_\Omega |u_\infty(x)|^{2^*}\,dx + o(1) \tag{4.16}$$

as $j\to+\infty$.

Therefore, using also the definition of $\mathcal{J}_{K,\lambda}$, by (4.11), (4.15) and (4.16) we deduce that

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u_j) &= \frac12\int_{\mathbb{R}^{2n}} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy \\ &\quad + \frac12\int_{\mathbb{R}^{2n}} |u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u_\infty(x)|^2dx \\ &\quad - \frac{1}{2^*}\int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx - \frac{1}{2^*}\int_\Omega |u_\infty(x)|^{2^*}\,dx + o(1) \\ &= \mathcal{J}_{K,\lambda}(u_\infty) + \frac12\int_{\mathbb{R}^{2n}} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)\,dx\,dy \\ &\quad - \frac{1}{2^*}\int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx + o(1)\end{aligned}$$

as $j\to+\infty$, which gives the desired assertion. □

Step 5. *The following equality holds true:*

$$\int_{\mathbb{R}^{2n}} |u_j(x)-u_\infty(x)-u_j(y)+u_\infty(y)|^2K(x-y)dx\,dy = \int_\Omega |u_j(x)-u_\infty(x)|^{2^*}dx + o(1)$$

as $j\to+\infty$.

Proof. First of all, note that, as a consequence of (4.10), (4.13) and (4.16), we get

$$\begin{aligned}\int_\Omega \Big(|u_j(x)|^{2^*-2}u_j(x) &- |u_\infty(x)|^{2^*-2}u_\infty(x)\Big)\Big(u_j(x)-u_\infty(x)\Big)\,dx \\ &= \int_\Omega |u_j(x)|^{2^*}\,dx - \int_\Omega |u_\infty(x)|^{2^*-1}u_\infty(x)u_j(x)\,dx \\ &\quad - \int_\Omega |u_j(x)|^{2^*-2}u_j(x)u_\infty(x)\,dx + \int_\Omega |u_\infty(x)|^{2^*}\,dx \\ &= \int_\Omega |u_j(x)|^{2^*}\,dx - \int_\Omega |u_\infty(x)|^{2^*}\,dx + o(1) \\ &= \int_\Omega |u_j(x)-u_\infty(x)|^{2^*}\,dx + o(1)\end{aligned} \tag{4.17}$$

as $j\to+\infty$.

By (4.3) and Steps 1 and 2, it is easily seen that

$$o(1) = \langle \mathcal{J}'_{K,\lambda}(u_j), u_j-u_\infty\rangle = \langle \mathcal{J}'_{K,\lambda}(u_j)-\mathcal{J}'_{K,\lambda}(u_\infty), u_j-u_\infty\rangle \tag{4.18}$$

as $j \to +\infty$. Moreover, by (4.11) and (4.17)

$$
\begin{aligned}
\langle \mathcal{J}'_{K,\lambda}(u_j)-&\mathcal{J}'_{K,\lambda}(u_\infty), u_j - u_\infty\rangle \\
&= \int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy \\
&\qquad - \lambda \int_\Omega |u_j(x) - u_\infty(x)|^2\,dx \\
&\qquad - \int_\Omega \Big(|u_j(x)|^{2^*-2}u_j(x) - |u_\infty(x)|^{2^*-2}u_\infty(x)\Big)\,(u_j(x) - u_\infty(x))\,dx \\
&= \int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy \\
&\qquad - \int_\Omega |u_j(x) - u_\infty(x)|^{2^*}\,dx + o(1)
\end{aligned}
\tag{4.19}
$$

as $j \to +\infty$. Hence, the assertion of Step 5 comes from (4.18) and (4.19). $\square$

Now, we can conclude the proof of Proposition 7.

By Step 5 it is easy to see that

$$
\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^{2n}} |u_j(x)-&u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy - \frac{1}{2^*}\int_\Omega |u_j(x) - u_\infty(x)|^{2^*}\,dx \\
&= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy + o(1) \\
&= \frac{s}{n}\int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy + o(1)\,,
\end{aligned}
$$

and so, as a consequence of this and Step 4,

$$
\begin{aligned}
\mathcal{J}_{K,\lambda}(u_\infty) + \frac{s}{n}&\int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)dx\,dy \\
&= \mathcal{J}_{K,\lambda}(u_j) + o(1) \\
&= c + o(1)\,,
\end{aligned}
\tag{4.20}
$$

as $j \to +\infty$, thanks to (4.2). Now, by Step 1 the sequence $\|u_j\|_{X_0}$ is bounded in $\mathbb{R}$. Hence, up to a subsequence, if necessary, we can assume that

$$
\|u_j - u_\infty\|^2_{X_0} = \int_{\mathbb{R}^{2n}} |u_j(x) - u_\infty(x) - u_j(y) + u_\infty(y)|^2 K(x-y)\,dx\,dy \to L \tag{4.21}
$$

and so, again as a consequence of Step 5,

$$
\int_\Omega |u_j(x) - u_\infty(x)|^{2^*}\,dx \to L
$$

as $j \to +\infty$. Of course $L \in [0, +\infty)$ and, by definition of S_K, it holds true

$$
L \geqslant L^{2/2^*} S_K\,,
$$

so that

$$
L = 0 \quad \text{or} \quad L \geqslant S_K^{n/(2s)}\,.
$$

The case $L \geqslant S_K^{n/(2s)}$ can not occur. Otherwise, by (4.20), (4.21) and Step 3, we would get

$$
c = \mathcal{J}_{K,\lambda}(u_\infty) + \frac{s}{n}\,L \geqslant \frac{s}{n}\,L \geqslant \frac{s}{n}\,S_K^{n/(2s)}\,,
$$

which contradicts (4.1). Thus $L = 0$ and so, by (4.21), we obtain that

$$
\|u_j - u_\infty\|_{X_0} \to 0
$$

as $j \to +\infty$. This ends the proof of Proposition 7. $\square$

5. The geometry of the functional $\mathcal{J}_{K,\lambda}$

Along this section we show that the functional $\mathcal{J}_{K,\lambda}$ has the geometric structure required by [13, Theorem 5.3]. In a non-local setting for nonlinearities with subcritical growth a proof of this is given in [16]. Essentially, here we adapt the proof given in [16, Subsection 4.2] to the critical case. With respect to the classical case of the Laplacian some technical differences arise.

Proposition 8. *Let $\lambda \in [\lambda_k, \lambda_{k+1})$ and let $\mathbb{P}_{k+1}$ be as in (3.1) for some $k \in \mathbb{N}$. Then, there exist $\rho > 0$ and $\beta > 0$ such that for any $u \in \mathbb{P}_{k+1}$ with $\|u\|_{X_0} = \rho$ it results that $\mathcal{J}_{K,\lambda}(u) \geqslant \beta$.*

Proof. Let u be a function in $\mathbb{P}_{k+1}$. Thanks to the choice of λ and to the variational characterization of λ_{k+1} (see [16, Proposition 9-*d*)]) we get

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u) &= \frac{1}{2}\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}dx\\ &\geqslant \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)dx\,dy - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}dx\,.\end{aligned} \tag{5.1}$$

Using [15, Lemma 6] and (1.9), from (5.1) we deduce that

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u) &\geqslant \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)\,dx\,dy\\ &\quad - \frac{c^{2^*/2}}{2^*}\left(\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{2^*/2}\\ &\geqslant \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)\,dx\,dy\\ &\quad - \frac{1}{2^*}\left(\frac{c}{\theta}\right)^{2^*/2}\left(\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)\,dx\,dy\right)^{2^*/2}\\ &= \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\|u\|^2_{X_0} - \frac{1}{2^*}\left(\frac{c}{\theta}\right)^{2^*/2}\|u\|^{2^*}_{X_0}\,.\end{aligned} \tag{5.2}$$

Hence, for a suitable positive constant κ we have

$$\mathcal{J}_{K,\lambda}(u) \geqslant \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\|u\|^2_{X_0}\left(1-\kappa\|u\|^{2^*-2}_{X_0}\right)\,.$$

Now, let $u \in \mathbb{P}_{k+1}$ be such that $\|u\|_{X_0} = \rho > 0$. Since $2^* > 2$, choosing ρ sufficiently small (i.e. ρ such that $1-\kappa\rho^{2^*-2} > 0$), we get

$$\inf_{\substack{u\in\mathbb{P}_{k+1}\\ \|u\|_{X_0}=\rho}} \mathcal{J}_{K,\lambda}(u) \geqslant \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\rho^2(1-\kappa\rho^{2^*-2}) =: \beta > 0\,,$$

and this ends the proof of Proposition 8. □

Proposition 9. *Let $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$. Then, $\mathcal{J}_{K,\lambda}(u) \leqslant 0$ for any $u \in$ span$\{e_1, \dots, e_k\}$.*

Proof. Let $u \in \mathrm{span}\{e_1, \dots, e_k\}$. Then

$$u(x) = \sum_{i=1}^k u_i e_i(x)\,,$$

with $u_i \in \mathbb{R}$, $i = 1, \dots, k$. Since $\{e_1, \dots, e_k, \dots\}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal one of X_0 by [16, Proposition 9–*f*)], we get

$$\int_\Omega |u(x)|^2\,dx = \sum_{i=1}^k u_i^2 \tag{5.3}$$

and

$$\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy = \sum_{i=1}^{k} u_i^2 \|e_i\|_{X_0}^2\,. \tag{5.4}$$

Then, by (5.3) and (5.4) and using [16, Proposition 9-b) and e)] (here we also use the fact that $e_i \in X_0$ for $i=1,\dots,k$, see [16, Proposition 9–f)]) we get

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u) &= \frac{1}{2}\sum_{i=1}^{k} u_i^2\Big(\|e_i\|_{X_0}^2-\lambda\Big) - \frac{1}{2^*}\|u\|_{L^{2^*}(\Omega)}^{2^*}\\ &\leqslant \frac{1}{2}\sum_{i=1}^{k} u_i^2\big(\lambda_i-\lambda\big) \leqslant 0\,,\end{aligned}$$

since $\lambda_i \leqslant \lambda_k \leqslant \lambda$ for any $i=1,\dots,k$. Hence, Proposition 9 follows. □

Proposition 10. *Let $\lambda \geqslant 0$ and let $\mathbb{F}$ be a finite dimensional subspace of X_0. Then, there exists $R>\rho$ such that $\mathcal{J}_{K,\lambda}(u)\leqslant 0$ for any $u \in \mathbb{F}$ with $\|u\|_{X_0}\geqslant R$, where ρ is given in Proposition 8.*

Proof. Let $u \in \mathbb{F}$. Then, the non-negativity of λ gives

$$\begin{aligned}\mathcal{J}_{K,\lambda}(u) &= \frac{1}{2}\|u\|_{X_0}^2 - \frac{\lambda}{2}\|u\|_{L^2(\Omega)}^2 - \frac{1}{2^*}\|u\|_{L^{2^*}(\Omega)}^{2^*}\\ &\leqslant \frac{1}{2}\|u\|_{X_0}^2 - \frac{1}{2^*}\|u\|_{L^{2^*}(\Omega)}^{2^*}\\ &\leqslant \frac{1}{2}\|u\|_{X_0}^2 - \frac{\kappa}{2^*}\|u\|_{X_0}^{2^*}\,,\end{aligned}$$

for some positive constant κ, thanks to the fact that in any finite dimensional space all the norms are equivalent.

Hence, if $\|u\|_{X_0}\to+\infty$, then

$$\mathcal{J}_{K,\lambda}(u)\to-\infty\,,$$

since $2^*>2$ by assumption, and so the assertion of Proposition 10 follows. □

6. Proof of Theorem 1

As we said at the beginning of Section 3, in the case when $\lambda \in (0,\lambda_1)$ our thesis follows by [17, Theorem 2], where more general critical nonlinearities than the power (critical) function were studied.

Now, let us consider the case when $\lambda \in [\lambda_k,\lambda_{k+1})$ for some $k\in\mathbb{N}$. Let u_0 be the function given in condition (1.18). Note that we can choose $u_0 \geqslant 0$ a.e. in $\mathbb{R}^n$, since if $v \in X_0$, then its positive part v^+ and its negative part v^- belong to X_0 by [14, Lemma 12].

Starting from u_0 we construct the functions $z,\tilde z \in \mathbb{P}_{k+1}$ as follows

$$z = u_0 - \sum_{i=1}^{k}\Big(\int_\Omega u_0(x)e_i(x)\,dx\Big)e_i\,, \quad \tilde z = z/\|z\|_{X_0}\,. \tag{6.1}$$

Moreover, let $V=\operatorname{span}\{e_1,\dots,e_k\}$ and $W=\mathbb{P}_{k+1}$. It is easily seen that V is finite dimensional and $V\oplus W = X_0$.

By Propositions 8–10 we get that $\mathcal{J}_{K,\lambda}$ has the geometric structure required by the Linking Theorem (see assumptions (I_1') and (I_5) of [13, Theorem 5.3] and also [13, Remark 5.5-(iii)]), that is

$$\inf_{\substack{u\in W\\ \|u\|_{X_0}=\rho}} \mathcal{J}_{K,\lambda}(u) \geqslant \beta > 0\,,$$

$$\sup_{u\in V}\mathcal{J}_{K,\lambda}(u)\leqslant 0$$

and

$$\sup_{\substack{u\in \mathbb{F}\\ \|u\|_{X_0}\geqslant R}} \mathcal{J}_{K,\lambda}(u)\leqslant 0\,, \quad \mathbb{F}=V\oplus \mathrm{span}\{\tilde z\}\,,$$

where β and R are as in Propositions 8 and 10, respectively, and $\tilde z$ is as in (6.1).

Now, let c_L be the Linking critical level of $\mathcal{J}_{K,\lambda}$, i.e.

$$c_L=\inf_{h\in\Gamma}\max_{u\in Q}\mathcal{J}_{K,\lambda}(h(u))\,,$$

where

$$\Gamma=\big\{h\in C(\overline{Q};X_0): h=id \text{ on } \partial Q\big\}$$

and

$$Q=\Big(\overline{B}_R\cap V\Big)\oplus\big\{r\tilde z: r\in(0,R)\big\}\,.$$

Our goal is to show that c_L is smaller than $\frac{s}{n}\,S_K^{n/(2s)}$, that is proving that the Linking critical level of $\mathcal{J}_{K,\lambda}$ lies below the threshold where the $(PS)_c$–condition holds true. For this, first of all we note that for any $u\in X_0\setminus\{0\}$

$$\max_{\zeta\geqslant 0}\mathcal{J}_{K,\lambda}(\zeta u)=\frac{s}{n}\,S_{K,\lambda}^{n/(2s)}(u) \tag{6.2}$$

(see, for instance, [17, Proposition 20]).

By definition of c_L we get that for any $h\in\Gamma$

$$c_L\leqslant \max_{u\in Q}\mathcal{J}_{K,\lambda}(h(u))$$

and so, in particular, taking $h=id$ on $\overline{Q}$

$$c_L\leqslant \max_{u\in Q}\mathcal{J}_{K,\lambda}(u)\leqslant \max_{u\in\mathbb{F}}\mathcal{J}_{K,\lambda}(u)\,, \tag{6.3}$$

being $Q\subset\mathbb{F}$. Since $\mathbb{F}$ is a linear space,

$$\begin{aligned}\max_{u\in\mathbb{F}}\mathcal{J}_{K,\lambda}(u)&=\max_{\substack{u\in\mathbb{F}\\ \zeta\neq 0}}\mathcal{J}_{K,\lambda}(|\zeta|\cdot\frac{u}{|\zeta|})\\ &=\max_{\substack{u\in\mathbb{F}\\ \zeta> 0}}\mathcal{J}_{K,\lambda}(\zeta u)\\ &\leqslant\max_{\substack{u\in\mathbb{F}\\ \zeta\geqslant 0}}\mathcal{J}_{K,\lambda}(\zeta u)\,.\end{aligned} \tag{6.4}$$

Hence, (6.2)–(6.4) and assumption (1.18) yield

$$c_L\leqslant \max_{\substack{u\in\mathbb{F}\\ \zeta\geqslant 0}}\mathcal{J}_{K,\lambda}(\zeta u)=\frac{s}{n}\max_{u\in\mathbb{F}}S_{K,\lambda}^{n/(2s)}(u)<\frac{s}{n}\,S_K^{n/(2s)}\,, \tag{6.5}$$

since $\mathbb{F}=V\oplus\mathrm{span}\{\tilde z\}=\mathrm{span}\{e_1,\dots,e_k,u_0\}$ by construction of $\tilde z$.

Therefore, the Linking Theorem [13, Theorem 5.3] yields that problem (1.11) admits a solution $u\in X_0$ with critical value $\mathcal{J}_{K,\lambda}(u)=c_L\geqslant\beta$. Since $\beta>0=\mathcal{J}_{K,\lambda}(0)$, we deduce that u is not identically zero and this concludes the proof of Theorem 1.

7. The fractional Laplace case: proof of Theorem 2

In this section, as a concrete application, we consider the model fractional kernel

$$K(x)=|x|^{-(n+2s)},$$

which gives rise to the fractional Laplace operator $-(-\Delta)^s$ defined is (1.3). Hence, here we study problem (1.2) or, to be more precise, its weak formulation, that is problem (1.20).

In order to prove Theorem 2 we will apply Theorem 1. For doing this, it is enough to show that condition (1.18) is satisfied in the fractional Laplace setting. First of all, we have to construct the function u_0 and the linear space $\mathcal{U}$ given in (1.18) in a suitable way (i.e. in such a way that (1.18) is verified).

7.1. **Construction of the function u_0 and of the linear space $\mathcal{U}$.** In order to construct such u_0, we can proceed as in [17, Section 4]. To illustrate the procedure we need the following result proved in [9, Theorem 1.1]:

Theorem 11. *The infimum in formula* (1.16) *is attained, that is*

$$S_s = S_s(\tilde{u}),$$

where

$$\tilde{u}(x) = \kappa\Big(\mu^2 + |x - x_0|^2\Big)^{-(n-2s)/2}, \quad x \in \mathbb{R}^n \tag{7.1}$$

with $\kappa \in \mathbb{R} \setminus \{0\}$, $\mu > 0$ and $x_0 \in \mathbb{R}^n$ fixed constants.

Equivalently, the function $\bar{u}$ defined as

$$\bar{u}(x) = \frac{\tilde{u}(x)}{\|\tilde{u}\|_{L^{2^*}(\mathbb{R}^n)}} \tag{7.2}$$

is such that

$$S_s = \inf_{\substack{v \in H^s(\mathbb{R}^n) \\ \|v\|_{L^{2^*}(\mathbb{R}^n)}=1}} \int_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^{2n}} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x - y|^{n+2s}}\, dx\, dy. \tag{7.3}$$

In what follows we suppose that, up to a translation, $x_0 = 0$ in (7.1). As in the classical case of the Laplacian, following [17, Section 4], we can construct an explicit solution of the limiting critical problem

$$(-\Delta)^s u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^n \tag{7.4}$$

as follows:

$$u^*(x) = \bar{u}\left(\frac{x}{S_s^{1/(2s)}}\right), \qquad x \in \mathbb{R}^n. \tag{7.5}$$

Note that the function $u^* \in H^s(\mathbb{R}^n)$ is a solution of problem (7.4) satisfying the property

$$\|u^*\|^{2^*}_{L^{2^*}(\mathbb{R}^n)} = S_s^{n/(2s)}. \tag{7.6}$$

Now, starting from u^* we can define the family of functions U_ε as

$$U_\varepsilon(x) = \varepsilon^{-(n-2s)/2}\, u^*(x/\varepsilon), \qquad x \in \mathbb{R}^n \tag{7.7}$$

for any $\varepsilon > 0$.

Finally, we need to put appropriately U_ε to zero outside Ω. For this let us fix $\delta > 0$ such that

$$B_{4\delta} \subset \Omega$$

and let us consider the cut-off function $\eta \in C^\infty(\mathbb{R}^n)$ such that $0 \leqslant \eta \leqslant 1$ in $\mathbb{R}^n$, $\eta \equiv 1$ in B_δ and $\eta \equiv 0$ in $\mathcal{C}B_{2\delta}$, where $B_\delta = B(0, \delta)$ and $\mathcal{C}B_\delta = \mathbb{R}^n \setminus B_\delta$. For every $\varepsilon > 0$ we denote by u_ε the following family of functions

$$u_\varepsilon(x) = \eta(x) U_\varepsilon(x), \quad x \in \mathbb{R}^n, \tag{7.8}$$

where U_ε is given in (7.7).

Of course $u_\varepsilon \in H^s(\mathbb{R}^n)$, $u_\varepsilon \geqslant 0$ a.e. in $\mathbb{R}^n$ and $u_\varepsilon = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$. Hence, by [17, Lemma 7] it is easily seen that $u_\varepsilon \in X_0$. Moreover, the function u_ε satisfies the following crucial estimates:

Proposition 12. *Let $s \in (0, 1)$ and $n > 2s$. Then, the following estimates hold true*

$$\int_{\mathbb{R}^{2n}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x - y|^{n+2s}}\, dx\, dy \leqslant S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n-2s}),$$

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|^2\,dx \geqslant \begin{cases} C_s\varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}) & \textit{if} \quad n > 4s \\ C_s\varepsilon^{2s}\big|\log\varepsilon\big| + \mathcal{O}(\varepsilon^{2s}) & \textit{if} \quad n = 4s \\ C_s\varepsilon^{n-2s} + \mathcal{O}(\varepsilon^{2s}) & \textit{if} \quad n < 4s\,, \end{cases}$$

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*}\,dx = S_s^{n/(2s)} + \mathcal{O}(\varepsilon^n\,)\,,$$

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*-1}\,dx = \mathcal{O}(\varepsilon^{(n-2s)/2}\,)\,,$$

and

$$\int_{\mathbb{R}^n} |u_\varepsilon(x)|\,dx = \mathcal{O}(\varepsilon^{(n-2s)/2}\,)$$

as $\varepsilon \to 0$ *, for some positive constant* C_s *depending on* s *.*

As a consequence, if $\varepsilon > 0$ *is sufficiently small, then for any* $\lambda > 0$

$$S_{s,\,\lambda}(u_\varepsilon) < S_s\,, \tag{7.9}$$

provided $n \geqslant 4s$ *.*

Proof. The first three estimates were proved in [17, Propositions 21 and 22 and Subsection 4.2.1] . It remains to show the last two ones. Let us start by proving the L^{2^*-1}-estimate. By (7.7) and (7.8) we have that

$$\begin{aligned}\int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*-1}\,dx &\leqslant \int_{B_{2\delta}} |U_\varepsilon(x)|^{2^*-1}\,dx \\ &= \kappa^*\Big(\varepsilon^{-(n-2s)/2}\Big)^{2^*-1}\int_{B_{2\delta}}\Big[\Big(\mu^2+|x/\varepsilon|^2\Big)^{-(n-2s)/2}\Big]^{2^*-1}\,dx \\ &= \kappa^*\varepsilon^{(n-2s)/2}\int_0^{2\delta/\varepsilon}\frac{r^{n-1}}{\Big(\mu^2+r^2\Big)^{(n+2s)/2}}\,dr \\ &\cong \tilde{\kappa}\varepsilon^{(n-2s)/2}\left(1+\int_1^{2\delta/\varepsilon} r^{-2s-1}\,dr\right) \\ &= \mathcal{O}(\varepsilon^{(n-2s)/2})\end{aligned}$$

as $\varepsilon \to 0$.

In a similar way we get

$$\begin{aligned}\int_{\mathbb{R}^n} |u_\varepsilon(x)|\,dx &\leqslant \int_{B_{2\delta}} |U_\varepsilon(x)|\,dx \\ &= \kappa^*\varepsilon^{-(n-2s)/2}\int_{B_{2\delta}}\Big(\mu^2+|x/\varepsilon|^2\Big)^{-(n-2s)/2}\,dx \\ &= \kappa^*\varepsilon^{(n+2s)/2}\int_0^{2\delta/\varepsilon}\frac{r^{n-1}}{\Big(\mu^2+r^2\Big)^{(n-2s)/2}}\,dr \\ &\cong \tilde{\kappa}\varepsilon^{(n+2s)/2}\left(1+\int_1^{2\delta/\varepsilon} r^{2s-1}\right) \\ &= \mathcal{O}(\varepsilon^{(n-2s)/2})\end{aligned}$$

as $\varepsilon \to 0$. In both cases, here κ^* and $\tilde{\kappa}$ are positive constants.

Finally, let us show (7.9). If $n > 4s$, by definition of $S_{s,\lambda}(\cdot)$ and the previous estimates we get

$$\begin{aligned} S_{s,\lambda}(u_\varepsilon) &= \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega |u_\varepsilon(x)|^2\,dx}{\Big(\displaystyle\int_\Omega |u_\varepsilon(x)|^{2^*}\,dx\Big)^{2/2^*}} \\ &\leqslant \frac{S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n-2s}) - \lambda C_s \varepsilon^{2s}}{\Big(S_s^{n/(2s)} + \mathcal{O}(\varepsilon^n)\Big)^{2/2^*}} \\ &\leqslant S_s + \varepsilon^{2s}\Big(\mathcal{O}(\varepsilon^{n-4s}) - \lambda C_s\Big) < S_s\,, \end{aligned}$$

for any $\lambda > 0$, provided $\varepsilon > 0$ is sufficiently small.

If $n = 4s$, similarly we get

$$\begin{aligned} S_{s,\lambda}(u_\varepsilon) &\leqslant \frac{S_s^{n/(2s)} + \mathcal{O}(\varepsilon^{n-2s}) - \lambda C_s \varepsilon^{2s}|\log\varepsilon| + \mathcal{O}(\varepsilon^{2s})}{\Big(S_s^{n/(2s)} + \mathcal{O}(\varepsilon^n)\Big)^{2/2^*}} \\ &\leqslant S_s + \mathcal{O}(\varepsilon^{2s}) - \lambda C_s\varepsilon^{2s}|\log\varepsilon| \\ &\leqslant S_s + \varepsilon^{2s}|\log\varepsilon|\Big(|\log\varepsilon|^{-1} - \lambda C_s\Big) < S_s\,, \end{aligned}$$

for any $\lambda > 0$, provided $\varepsilon > 0$ is sufficiently small. This concludes the proof of Proposition 12. □

Now, we construct the function u_0 given in (1.18) as follows:

$$u_0 = u_\varepsilon \ \text{ for } \varepsilon \text{ sufficiently small}\,, \tag{7.10}$$

and the linear space $\mathcal{U}$ as

$$\mathcal{U} = \mathcal{U}_\varepsilon := \begin{cases} \mathrm{span}\{u_\varepsilon\} & \text{if } \lambda \in (0,\lambda_{1,s}) \\ \mathrm{span}\{e_{1,s},\dots,e_{k,s},u_\varepsilon\} & \text{if } \lambda \in [\lambda_{k,s}\,\lambda_{k+1,s})\,,\ k\in\mathbb{N}\,, \end{cases} \tag{7.11}$$

where the functions $e_{k,s}$, are the eigenfunctions of $(-\Delta)^s$ corresponding to $\lambda_{k,s}$ for any $k \in \mathbb{N}$ (see [16, Proposition 9] for a detailed spectral theory for the fractional Laplacian).

In the next subsection we will show that, with this choice of u_0 and $\mathcal{U}$, condition (1.18) is satisfied.

7.2. **A non-degeneracy estimate.** Note that, if $\lambda \in (0,\lambda_{1,s})$, then condition (1.18) reduces to

$$S_{s,\lambda}(u_0) = S_{s,\lambda}(u_\varepsilon) < S_s\,, \tag{7.12}$$

which is always satisfied, provided ε is sufficiently small, thanks to (7.9).

Now, it remains to consider the case when $\lambda \in [\lambda_{k,s}\,\lambda_{k+1,s})$ for some $k \in \mathbb{N}$. For this we need the following result:

Proposition 13. *Let $s \in (0,1)$ and $n > 2s$. Let $\lambda \in [\lambda_{k,s},\lambda_{k+1,s})$ for some $k \in \mathbb{N}$ and let $\mathcal{U}_\varepsilon$ be the linear space defined in (7.11). Finally, let*

$$M_\varepsilon := \max_{\substack{u\in\mathcal{U}_\varepsilon \\ \|u\|_{L^{2^*}(\Omega)}=1}} S_{s,\lambda}(u)\,, \tag{7.13}$$

where the function $H^s(\mathbb{R}^n)\setminus\{0\} \ni v \mapsto S_{s,\lambda}(v)$ is defined as in (1.17). Then

a) M_ε is achieved in $u_M \in \mathcal{U}_\varepsilon$ and u_M can be written as follows[1]

$$u_M = v + tu_\varepsilon\,, \ \textit{ with } v \in \textit{span}\{e_{1,s},\dots,e_{k,s}\} \textit{ and } t \geqslant 0\,; \tag{7.14}$$

[1]Beware that u_M, v and t (and also $\tilde{v}$ in (7.16) below) depend on ε. For simplicity we omit this dependence in the notation.

b) the following estimate holds true

$$M_\varepsilon \leqslant \begin{cases} \big(\lambda_{k,\,s}-\lambda\big)\|v\|^2_{L^2(\Omega)} & \text{if } t=0\\ \big(\lambda_{k,\,s}-\lambda\big)\|v\|^2_{L^2(\Omega)}+S_{s,\,\lambda}(u_\varepsilon)\Big(1+\mathcal{O}(\varepsilon^{(n-2s)/2)})\|v\|_{L^2(\Omega)}\Big) & \\ \qquad\qquad +\,\mathcal{O}(\varepsilon^{(n-2s)/2)})\,\|v\|_{L^2(\Omega)} & \text{if } t>0 \end{cases}$$

as $\varepsilon\to 0$, *where* v *is given in* (7.14).

Proof. First of all, note that $u_\varepsilon\in X_0$ and $e_{i,\,s}\in X_0$ for any $i=1,\dots,k$, thanks to [16, Proposition 9-*b*) and *e*)] and to [17, Lemma 7]. As a consequence of this $\mathcal{U}_\varepsilon\subset X_0$ and, by definition of $S_{s,\,\lambda}$,

$$M_\varepsilon := \max_{\substack{u\in\mathcal{U}_\varepsilon\\ \|u\|_{L^{2^*}(\Omega)}=1}}\left(\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy-\lambda\int_\Omega |u(x)|^2\,dx\right).$$

Let us start by proving part *a*). The maximum in formula (7.13) is achieved thanks to the Weierstrass Theorem (applied here in a finite dimensional space). Thus, let $u_M\in\mathcal{U}_\varepsilon$ be the function such that

$$M_\varepsilon=\int_{\mathbb{R}^{2n}}\frac{|u_M(x)-u_M(y)|^2}{|x-y|^{n+2s}}\,dx\,dy-\lambda\int_\Omega|u_M(x)|^2\,dx \quad\text{and}\quad \|u_M\|_{L^{2^*}(\Omega)}=1\,.$$

Note that $u_M\not\equiv 0$. Moreover, since $u_M\in\mathcal{U}_\varepsilon$, by definition of $\mathcal{U}_\varepsilon$ we have that

$$u_M=v+tu_\varepsilon\,,$$

for some $v\in\operatorname{span}\{e_{1,\,s},\dots,e_{k,\,s}\}$ and $t\in\mathbb{R}$. We can suppose that $t\geqslant 0$ (otherwise, if $t\leqslant 0$ we can replace u_M with $-u_M$). This concludes the proof of assertion *a*).

Now, let us show *b*). First of all, note that, if $t=0$ in formula (7.14), then $u_M=v\in\operatorname{span}\{e_{1,\,s},\dots,e_{k,\,s}\}$ and

$$\begin{aligned} M_\varepsilon &= \int_{\mathbb{R}^{2n}}\frac{|u_M(x)-u_M(y)|^2}{|x-y|^{n+2s}}\,dx\,dy-\lambda\int_\Omega|u_M(x)|^2\,dx\\ &= \int_{\mathbb{R}^{2n}}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy-\lambda\int_\Omega|v(x)|^2\,dx\\ &\leqslant (\lambda_{k,\,s}-\lambda)\|v\|^2_{L^2(\Omega)}\,, \end{aligned}$$

by Proposition 5. Thus, assertion *b*) holds true when $t=0$.

Now, let us consider the case when $t>0$ in (7.14). By the Hőlder inequality and the properties of u_M, we can bound u_M as follows

$$\|u_M\|^2_{L^2(\Omega)}\leqslant|\Omega|^{n/(2s)}\|u_M\|^2_{L^{2^*}(\Omega)}=|\Omega|^{n/(2s)}. \tag{7.15}$$

Also, since $u_M\in\mathcal{U}_\varepsilon$, by definition of $\mathcal{U}_\varepsilon$ we have that

$$u_M=\tilde v+tz_\varepsilon\,, \tag{7.16}$$

where

$$\tilde v=v+t\sum_{i=1}^k\Big(\int_\Omega u_\varepsilon(x)e_{i,\,s}(x)\,dx\Big)e_{i,\,s}\in\operatorname{span}\{e_{1,\,s},\dots,e_{k,\,s}\}$$

and

$$z_\varepsilon=u_\varepsilon-\sum_{i=1}^k\Big(\int_\Omega u_\varepsilon(x)e_{i,\,s}(x)\,dx\Big)e_{i,\,s}\,, \tag{7.17}$$

so that $\tilde v$ and z_ε are orthogonal in $L^2(\Omega)$ (see [16, Proposition 9–*f*)]). As a consequence of this, we get that

$$\|u_M\|^2_{L^2(\Omega)}=\|\tilde v\|^2_{L^2(\Omega)}+t^2\|z_\varepsilon\|^2_{L^2(\Omega)}\geqslant\|\tilde v\|^2_{L^2(\Omega)}. \tag{7.18}$$

By (7.15) and (7.18), we have obtained that $\|u_M\|_{L^2(\Omega)}$, and $\|\tilde{v}\|_{L^2(\Omega)}$ are all bounded uniformly in ε by a suitable $\tilde{c} > 0$.

Now, we claim that, up to renaming such a constant,

$$t \leqslant \tilde{c}\,. \tag{7.19}$$

For this note that by Proposition 6 we have that $e_{i,\,s} \in L^\infty(\Omega)$ for any $i \in \mathbb{N}$, so that also $\tilde{v} \in \operatorname{span}\{e_{1,\,s}, \dots, e_{k,\,s}\}$ does. Hence, $\tilde{v} \in L^{2^*}(\Omega)$, since Ω is bounded. Moreover, by the equivalence of the norms in a finite dimensional space, we also have

$$\|\tilde{v}\|_{L^{2^*}(\Omega)} \leqslant \tilde{c}\,. \tag{7.20}$$

By Propositions 6 and 12 we have

$$\left|\int_\Omega u_\varepsilon(x) e_{i,s}(x)\,dx\right| \leqslant \|u_\varepsilon\|_{L^1(\Omega)}\|e_{i,s}\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^{(n-2s)/2})$$

as $\varepsilon \to 0$. As a consequence, using (7.17) and again Proposition 12,

$$\begin{aligned}
\|z_\varepsilon\|_{L^{2^*}(\Omega)} &\geqslant \|u_\varepsilon\|_{L^{2^*}(\Omega)} - \sum_{i=1}^k \left|\int_\Omega u_\varepsilon(x) e_{i,s}(x)\,dx\right| \|e_{i,s}\|_{L^{2^*}(\Omega)} \\
&= S_s^{(n-2s)/(4s)} + \mathcal{O}(\varepsilon^{(n-2s)/2}) \\
&\geqslant \frac{S_s^{(n-2s)/(4s)}}{2}
\end{aligned}$$

for ε sufficiently small. So, by (7.16), the fact that $t \geqslant 0$ and (7.20) we have

$$\frac{S_s^{(n-2s)/(4s)}\,t}{2} \leqslant t\|z_\varepsilon\|_{L^{2^*}(\Omega)} \leqslant \|u_M\|_{L^{2^*}(\Omega)} + \|\tilde{v}\|_{L^{2^*}(\Omega)} \leqslant 1 + \tilde{c}\,,$$

thanks to the properties of u_M. Therefore the claim in (7.20) follows.

Note that by Proposition 6 also $v \in \operatorname{span}\{e_{1,\,s}, \dots, e_{k,\,s}\}$ belongs to $L^\infty(\Omega)$.

Finally, the convexity[2], the monotonicity properties of the integrals, (7.19) and the fact that in $\operatorname{span}\{e_{1,\,s}, \dots, e_{k,\,s}\}$ all the norms are equivalent yield

$$\begin{aligned}
1 = \|u_M\|^{2^*}_{L^{2^*}(\Omega)} &= \int_\Omega |u_M(x)|^{2^*}\,dx \\
&\geqslant \int_\Omega |tu_\varepsilon(x)|^{2^*}\,dx + 2^* \int_\Omega \big(tu_\varepsilon(x)\big)^{2^*-1} v(x)\,dx \\
&\geqslant \|tu_\varepsilon\|^{2^*}_{L^{2^*}(\Omega)} - 2^*\tilde{c}^{\,2^*-1}\|u_\varepsilon\|^{2^*-1}_{L^{2^*-1}(\Omega)}\|v\|_{L^\infty(\Omega)} \\
&\geqslant \|tu_\varepsilon\|^{2^*}_{L^{2^*}(\Omega)} - \hat{c}\,\|u_\varepsilon\|^{2^*-1}_{L^{2^*-1}(\Omega)}\|v\|_{L^2(\Omega)}\,,
\end{aligned} \tag{7.21}$$

for some positive constant $\hat{c}$, so that

$$\|tu_\varepsilon\|^{2^*}_{L^{2^*}(\Omega)} \leqslant 1 + \hat{c}\,\|u_\varepsilon\|^{2^*-1}_{L^{2^*-1}(\Omega)}\|v\|_{L^2(\Omega)}\,. \tag{7.22}$$

[2]If f is a differentiable convex function, then $f(y) \geqslant f(x) + f'(x)(y-x)$. Here we take $f(s) = s^{2^*}$, $x = tu_\varepsilon$ and $y = u_M = v + tu_\varepsilon$.

Now, we are ready to prove the inequality in part $b)$. By (7.14) we get

$$
\begin{aligned}
M_\varepsilon &= \int_{\mathbb{R}^{2n}} \frac{|u_M(x)-u_M(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega |u_M(x)|^2\,dx \\
&= \int_{\mathbb{R}^{2n}} \frac{|v(x)+tu_\varepsilon(x)-v(y)-tu_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega |v(x)+tu_\varepsilon(x)|^2\,dx \\
&= \int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + t^2\int_{\mathbb{R}^{2n}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\qquad + 2t\int_{\mathbb{R}^{2n}} \frac{\big(u_\varepsilon(x)-u_\varepsilon(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dxdy \\
&\qquad - \lambda\int_\Omega |v(x)|^2\,dx - \lambda t^2\int_\Omega |u_\varepsilon(x)|^2\,dx - 2\lambda t\int_\Omega u_\varepsilon(x)v(x)\,dx \\
&\leqslant \big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)} + S_{s,\lambda}(u_\varepsilon)\|tu_\varepsilon\|^2_{L^{2^*}(\Omega)} \\
&\qquad + 2t\int_{\mathbb{R}^{2n}} \frac{\big(u_\varepsilon(x)-u_\varepsilon(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}} - 2\lambda t\int_\Omega u_\varepsilon(x)v(x)\,dx\,,
\end{aligned} \tag{7.23}
$$

thanks to Proposition 5 and to the definition of $S_{s,\lambda}(\cdot)$.

Now we write

$$
v = \sum_{i=1}^k v_i e_{i,s}
$$

for some $v_i \in \mathbb{R}$, so that

$$
\|v\|^2_{L^2(\Omega)} = \sum_{i=1}^k v_i^2\,.
$$

Also (see (2.4)),

$$
\langle u_\varepsilon, v\rangle_{X_0} = \sum_{i=1}^k v_i\langle u_\varepsilon, e_{i,s}\rangle_{X_0} = \sum_{i=1}^k \lambda_{i,s}v_i\int_\Omega u_\varepsilon(x)e_{i,s}(x)\,dx.
$$

So, by the Hőlder inequality,

$$
\begin{aligned}
|\langle u_\varepsilon, v\rangle_{X_0}| &\leqslant \sum_{i=1}^k \lambda_{i,s}|v_i|\,\|u_\varepsilon\|_{L^1(\Omega)}\|e_{i,s}\|_{L^\infty(\Omega)} \\
&\leqslant \lambda_{k+1,s}\|u_\varepsilon\|_{L^1(\Omega)}\|v\|_{L^\infty(\Omega)} \\
&\leqslant \bar\kappa\|u_\varepsilon\|_{L^1(\Omega)}\|v\|_{L^2(\Omega)}\,,
\end{aligned}
$$

for a suitable $\bar\kappa > 0$ possibly depending on k. More explicitly,

$$
\left|\int_{\mathbb{R}^{2n}} \frac{\big(u_\varepsilon(x)-u_\varepsilon(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dxdy\right| \leqslant \bar\kappa\|v\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^1(\Omega)}. \tag{7.24}
$$

Gathering the results in (7.23) and (7.24), and using again the Hőlder inequality we get

$$
\begin{aligned}
M_\varepsilon &\leqslant \big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)} + S_{s,\lambda}(u_\varepsilon)\|tu_\varepsilon\|^2_{L^{2^*}(\Omega)} + 2t\bar\kappa\|v\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^1(\Omega)} \\
&\qquad + 2\lambda t\|u_\varepsilon\|_{L^1(\Omega)}\|v\|_{L^\infty(\Omega)} \\
&\leqslant \big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)} + S_{s,\lambda}(u_\varepsilon)\|tu_\varepsilon\|^2_{L^{2^*}(\Omega)} + \kappa\|u_\varepsilon\|_{L^1(\Omega)}\|v\|_{L^2(\Omega)}\,,
\end{aligned}
$$

for some positive κ, thanks to (7.19) and the equivalence of the norms in the finite dimensional space $\mathrm{span}\{e_{1,s},\dots,e_{k,s}\}$.

This and (7.22) yield

$$
M_\varepsilon \leqslant \big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)} + S_{s,\lambda}(u_\varepsilon)\Big(1+\hat c\,\|u_\varepsilon\|^{2^*-1}_{L^{2^*-1}(\Omega)}\|v\|_{L^2(\Omega)}\Big)^{2/2^*} + \kappa\|u_\varepsilon\|_{L^1(\Omega)}\|v\|_{L^2(\Omega)}\,,
$$

so that, by Proposition 12

$$M_\varepsilon \leqslant \big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)} + S_{s,\lambda}(u_\varepsilon)\Big(1+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\Big)^{2/2^*} + \mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}$$
$$\leqslant \big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)} + S_{s,\lambda}(u_\varepsilon)\Big(1+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\Big) + \mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\,,$$

being $2<2^*$ (since $s>0$) . Then, assertion b) is proved.

This ends the proof of Proposition 13 . □

Now, we are ready to prove Theorem 2 .

7.3. **End of the proof of Theorem 2.** When $\lambda\in(0,\lambda_{1,s})$ condition (1.18) reduces to (7.12). Hence, Theorem 2 comes from [17, Theorem 4].

Let us consider the case when $\lambda>\lambda_{1,s}$. As we said at the beginning of Section 7, in order to prove Theorem 2, the idea consists of applying Theorem 1 (for this we have to show that condition (1.18) is satisfied). If we assume that $\lambda\in(\lambda_{k,s}\,\lambda_{k+1,s})$ for some $k\in\mathbb{N}$, in order to prove condition (1.18) it is enough to show that

$$M_\varepsilon := \max_{\substack{u\in\mathcal{U}_\varepsilon\\ \|u\|_{L^{2^*}(\Omega)}=1}} S_{s,\lambda}(u) < S_s\,, \tag{7.25}$$

taking into account the scale invariance of the function $S_{s,\lambda}(\cdot)$.

By Proposition 13-a)

$$M_\varepsilon = \int_{\mathbb{R}^{2n}} \frac{|u_M(x)-u_M(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega |u_M(x)|^2\,dx\,,$$

with

$$u_M = v+tu_\varepsilon\,,$$

for some $v\in\operatorname{span}\{e_{1,s},\dots,e_{k,s}\}$ and $t\geqslant 0$.

If $t=0$, by Proposition 13-b) and the choice of $\lambda>\lambda_{k,s}$, we get that

$$M_\varepsilon \leqslant \big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)} < 0 < S_s\,,$$

and so (7.25) is proved.

Now, suppose that $t>0$. Again by Proposition 13-b) we deduce that for ε sufficiently small

$$\begin{aligned} M_\varepsilon &\leqslant \big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)} + S_{s,\lambda}(u_\varepsilon)\Big(1+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\Big)\\ &\qquad + \mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\\ &= \mathcal{O}(\varepsilon^{n-2s}) + S_{s,\lambda}(u_\varepsilon)\Big(1+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\Big)\,,\end{aligned} \tag{7.26}$$

since the parabola $\big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)} + \mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}$ stays always below its vertex, that is

$$\big(\lambda_{k,s}-\lambda\big)\|v\|^2_{L^2(\Omega)} + \mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)} \leqslant \frac{1}{4(\lambda-\lambda_{k,s})}\mathcal{O}(\varepsilon^{n-2s}) = \mathcal{O}(\varepsilon^{n-2s})\,.$$

We have to distinguish the cases $n>4s$ and $n=4s$. Let us start with the first one, i.e. $n>4s$. Thus, by (7.26) and Proposition 12

$$\begin{aligned} M_\varepsilon &\leqslant \mathcal{O}(\varepsilon^{n-2s}) + S_{s,\lambda}(u_\varepsilon)\Big(1+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\Big)\\ &\leqslant \mathcal{O}(\varepsilon^{n-2s}) + \Big(S_s+\mathcal{O}(\varepsilon^{n-2s})-\lambda C_s\varepsilon^{2s}\Big)\Big(1+\mathcal{O}(\varepsilon^{(n-2s)/2})\|v\|_{L^2(\Omega)}\Big)\\ &= S_s+\mathcal{O}(\varepsilon^{n-2s})-\lambda C_s\varepsilon^{2s}\\ &= S_s+\varepsilon^{2s}\Big(\mathcal{O}(\varepsilon^{n-4s})-\lambda C_s\Big)\\ &< S_s\end{aligned}$$

as $\varepsilon \to 0$. Here we use also the fact that λ is positive (being $\lambda > \lambda_{k,s} > 0$). This ends the proof of (7.25) when $n > 4s$.

Now, let $n = 4s$. Arguing as above, by (7.26) and Proposition 12 we get

$$\begin{aligned} M_\varepsilon &\leqslant \mathcal{O}(\varepsilon^{2s}) + S_{s,\lambda}(u_\varepsilon)\Big(1 + \mathcal{O}(\varepsilon^s)\|v\|_{L^2(\Omega)}\Big) \\ &\leqslant S_s + \mathcal{O}(\varepsilon^{2s}) - \lambda C_s \varepsilon^{2s}|\log \varepsilon| \\ &= S_s + \varepsilon^{2s}|\log\varepsilon|\Big(|\log\varepsilon|^{-1} - \lambda C_s\Big) \\ &< S_s \end{aligned}$$

as $\varepsilon \to 0$. This conclude the proof in the case when $n = 4$.

Hence, condition (1.18) is satisfied when $n \geqslant 4s$ for any $\lambda > 0$ different from the eigenvalues of $(-\Delta)^s$. Then, Theorem 1 provides a solution $u \in X_0 \setminus \{0\}$ of problem (1.20). Since $X_0 \subseteq H^s(\mathbb{R}^n)$ by [15, Lemma 5-b)] (see also [17, Lemma 7]), the proof of Theorem 2 is complete.

Note that, of course we can prove Theorem 2 directly, by applying the Mountain Pass and the Linking Theorem to the Euler–Lagrange functional

$$\mathcal{J}_{s,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \frac{\lambda}{2}\int_\Omega |u(x)|^2\,dx - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}\,dx$$

associated with problem (1.20). In this framework condition (1.18) (or, to be precise, (7.12) when $\lambda \in (0, \lambda_{1,s})$ and (7.25) when $\lambda \in (\lambda_{k,s}\, \lambda_{k+1,s})$, $k \in \mathbb{N}$) says that the Mountain Pass or the Linking critical level of $\mathcal{J}_{s,\lambda}$ stays below the threshold where the (PS)–condition holds true.

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