ON THE STRUCTURE OF CRITICAL ENERGY LEVELS FOR THE CUBIC FOCUSING NLS ON STAR GRAPHS

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Abstract. We provide information on a non trivial structure of phase space of the cubic NLS on a three-edge star graph. We prove that, contrarily to the case of the standard NLS on the line, the energy associated to the cubic focusing Schrödinger equation on the three-edge star graph with a free (Kirchhoff) vertex does not attain a minimum value on any sphere of constant L^2-norm. We moreover show that the only stationary state with prescribed L^2-norm is indeed a saddle point.

1. Introduction

A major issue in nonlinear dynamics consists in the search for stationary solutions and in the study of their stability properties. In the case of hamiltonian systems, a first picture of the phase space can be drawn by identifying critical points of the energy and their nature. In particular it is important to know if a *ground state* exists, where a ground state is defined as the minimizer of the energy functional, possibly restricted to suitable submanifolds. In the context of the nonlinear Schrödinger (NLS) equation, one typically restricts the energy functional to a manifold on which a second conserved quantity (sometimes called mass, or charge) is constant. From a physical point of view, such a restriction is meaningful, as the extra conserved quantity often represents a physical characteristic of the system (e.g., the mass, or the number of particles). In the one dimensional case, the ground states of the NLS with power nonlinearity on the line are well known and completely described in the classical paper [3], where more general nonlinearities are also treated. It turns out that on the line, the NLS energy constrained to the manifold of the states of constant mass, attains its minimum value in correspondence of a unique (up to translation) positive, symmetric and decreasing (for $x > 0$) function. No other critical points of the energy exist.

In this paper we are interested in the case of the focusing nonlinear cubic Schrödinger equation on a three-edge star graph, sometimes called in the physical literature a Y junction. To put the issue in a physical context, we begin by recalling that the *linear* Schrödinger equation on a graph is a well developed subject, as an effective description of dynamics of many mesoscopic systems such as, for example, quantum nanowires (see [5, 8, 9] and references therein). It is of interest to extend the analysis to the nonlinear wave propagation on networks. In particular, it is well known that the NLS appears as an effective equation in several different areas: the description of Bose condensates, the propagation of electromagnetic pulses in nonlinear (Kerr) media and Langmuir waves in plasma physics. In many situations it seems of interest to treat the propagation of NLS solutions associated to such phenomena in one dimensional ramified structures, the prototype of which

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is a three-edge star graph, or Y-junction. The subject is at its beginnings and for some preliminary experimental, numerical and analytical works see [6, 7, 12, 10, 11, 13]. A first rigorous analysis of nonlinear stationary (or bound) states for NLS on a star graphs is outlined in [2], where solitary waves for a star graph with delta vertex are constructed, including the case of a free or Kirchhoff vertex. The Kirchhoff vertex is the closest analogue to the free particle on the line, to which it reduces when the line is considered as a graph with two edges. In the linear case the Kirchhoff laplacian on a graph, in analogy with the laplacian on the line, has only absolutely continuous spectrum, and so only scattering states are possible for the Schrödinger dynamics. However, in the presence of a nonlinearity a three-edge star graph with Kirchhoff conditions at the vertex, admits a unique stationary state, which is quite simply described: it is the state on the graph which coincides on every edge with half a soliton. One could suspect that this stationary state is the ground state. On the contrary, we show that, perhaps unexpectedly, this is not the case.

This fact highlights that the NLS on graphs, even on the simplest one, exhibits remarkable differences w.r.t. the same evolution equation on the line.

In this letter we consider the case of the cubic focusing NLS and analyze the energy constrained to a fixed sphere in L^2, i.e. the set of states of prescribed mass. We show that the constrained energy is bounded from below (a fact that can be shown similarly to the case of $\mathbb{R}^n$ and subcritical nonlinearity, see [4] and [2]), but it approaches the infimum without attaining a minimum value. The more, the only nonlinear stationary state is a saddle point of the constrained energy functional. The existence of a saddle point of the energy is a remarkable feature, inducing on the phase space of the system stable and unstable manifolds which, in the absence of other critical points, are the main structural properties of the dynamics. The consequences of this fact will be investigated in a subsequent paper.

2. Preliminaries

Before giving the main results, we start by fixing the framework, the notation, and recalling some basic results.

1. A three-edge star graph $\mathcal{G}$ can be thought of as composed by three halflines with a common origin, called *vertex*. A *state* or *wavefunction* on the graph is an element of the Hilbert space $L^2(\mathcal{G}) = \bigoplus_{i=1}^3 L^2(\mathbb{R}^+; dx_i)$, and can be represented as a column vector, namely

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \qquad \Psi_i \in L^2(\mathbb{R}^+).$$

The space $L^2(\mathcal{G})$ is naturally endowed with the hermitian product

$$(\Phi, \Psi)_{L^2(\mathcal{G})} = \sum_{i=1}^{3} \int_0^{+\infty} \overline{\Phi_i(x_i)} \Psi_i(x_i)\, dx_i.$$

Sobolev and L^p-spaces on $\mathcal{G}$ are defined analogously, namely

$$H^s(\mathcal{G}) \;=\; \bigoplus_{i=1}^{3} H^s(\mathbb{R}^+), \qquad L^p(\mathcal{G}) \;=\; \bigoplus_{i=1}^{3} L^p(\mathbb{R}^+).$$

In the following, we denote by $\|\Psi\|_p$ the norm of the function Ψ in the space $L^p(\mathcal{G})$. When $p = 2$ we shall simply write $\|\Psi\|$.

2. The dynamics of the system is described by the Schrödinger equation

$$i\partial_t \Psi(t) \;=\; -\Delta\Psi(t) - |\Psi(t)|^2\Psi(t), \tag{2.1}$$

where:

- The operator $-\Delta$ acts on the domain

$$D(-\Delta) \;:=\; \{\Psi \in H^2(\mathcal{G}),\ \Psi_1(0) = \Psi_2(0) = \Psi_3(0),\ \Psi_1'(0) + \Psi_2'(0) + \Psi_3'(0) = 0\}$$

  and its action reads

$$-(\Delta\Psi)_i \;=\; -\Psi_i'', \qquad i = 1, 2, 3.$$

  The condition at the vertex is usually referred to as the *Kirchhoff's boundary condition*. The operator $-\Delta$ is selfadjoint on $L^2(\mathcal{G})$. Notice that, on the graph, the laplacian with Kirchhoff boundary conditions is the natural generalization of the free laplacian on the line, as it is easily seen by considering the line as a two-edge star graph, and noticing that the boundary conditions reduces to continuity of the wavefunction and continuity of the derivative at the vertex, i.e. $\Psi \in H^2(\mathbb{R})$.
- The nonlinear term in (2.1) is defined componentwise, namely

$$(|\Psi|^2\Psi)_i := |\Psi_i|^2\Psi_i.$$

3. The problem (2.1) is globally well-posed in $H^1(\mathcal{G})$ (see [1]). The L^2-norm and the energy

$$E(\Psi) \;=\; \frac{1}{2}\|\Psi'\|^2 - \frac{1}{4}\|\Psi\|_4^4 \;=\; \sum_{i=1}^{3}\left(\frac{1}{2}\|\Psi_i'\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{4}\|\Psi_i\|^4_{L^4(\mathbb{R}^+)}\right) \tag{2.2}$$

are conserved by time evolution.

In the sequel we use the following notation:

$$E_1(\psi) \;=\; \frac{1}{2}\|\psi'\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{4}\|\psi\|^4_{L^4(\mathbb{R}^+)}, \qquad \psi \in H^1(\mathbb{R}^+) \tag{2.3}$$

and

$$E_2(\psi) \;=\; \frac{1}{2}\|\psi'\|^2_{L^2(\mathbb{R})} - \frac{1}{4}\|\psi\|^4_{L^4(\mathbb{R})}, \qquad \psi \in H^1(\mathbb{R}). \tag{2.4}$$

4. Let us recall two well-known results on minimization for the cubic NLS on one- and two-edge star graphs (i.e., on the halfline and on the line):

(1) The minimum of the functional E_1 on the functions in $H^1(\mathbb{R}^+)$ with squared L^2-norm equal to $m > 0$ is achieved on the function (up to a phase factor)

$$\phi_m(x) \;=\; \frac{m}{\sqrt{2}}\cosh^{-1}\left(\frac{m}{2}x\right), \qquad x \geq 0,$$

and gives

$$E_1(\phi_m) \;=\; -\frac{m^3}{24}. \tag{2.5}$$

(2) The minimum of the functional E_2 on the functions in $H^1(\mathbb{R})$ with squared L^2-norm equal to $m > 0$ is achieved on the functions (up to a phase factor)

$$\varphi_m^y(x) \;=\; \frac{m}{2\sqrt{2}}\cosh^{-1}\Big(\frac{m}{4}(x-y)\Big), \qquad x \in \mathbb{R} \tag{2.6}$$

and gives

$$E_2(\varphi_m^y) \;=\; -\frac{m^3}{96}. \tag{2.7}$$

3. Results

As recalled in the introduction the energy functional (2.2) at fixed L^2 norm is bounded from below for subcritical nonlinearity, which is our case.
Let us introduce the following family of states:

Definition 3.1. We call *sesquisoliton* (i.e. "one and half" soliton) any function of the form

$$\Phi^x_{m_1,m_2}(x_1,x_2,x_3) := \begin{pmatrix} \frac{m_1}{\sqrt{2}}\cosh^{-1}\left(\frac{m_1}{2}x_1\right) \\ \frac{m_2}{2\sqrt{2}}\cosh^{-1}\left(\frac{m_2}{4}(x_2-x)\right) \\ \frac{m_2}{2\sqrt{2}}\cosh^{-1}\left(\frac{m_2}{4}(x_3+x)\right) \end{pmatrix} \tag{3.1}$$

where $0 < m_1 \le m_2$, $x \ge 0$, and the following condition of "continuity at the vertex" holds:

$$m_1 \;=\; \frac{m_2}{2}\cosh^{-1}\Big(\frac{m_2}{4}x\Big). \tag{3.2}$$

Notice that the sesquisolitons are obviously elements of $H^1(\mathcal{G})$. As a matter of fact they also belong to $D(-\Delta)$.
Moreover, in the case $x = 0$ one has $m_1 = \frac{m_2}{2}$, and one obtains a symmetric configuration with three half solitons concurring at the vertex. In [2] it is shown that this is a standing wave for NLS equation (2.1).
Now, we define the manifold of the sequisolitons with fixed L^2-norm as follows:

$$\mathcal{S}_M := \{\Phi^x_{m_1,m_2},\ \|\Phi^x_{m_1,m_2}\|^2 = M\}.$$

Theorem 3.2. *For any Ψ such that $\|\Psi\|^2 = M$, the following chain holds:*

$$E(\Psi) > \inf_{\|\Psi\|^2=M} E(\Psi) = \inf_{\Psi\in\mathcal{S}_M} E(\Psi) = -\frac{M^3}{96}. \tag{3.3}$$

Proof. Given $\Psi \in L^2(\mathcal{G})$, it is possible to construct a sesquisoliton with the same L^2-norm but with lower energy. We proceed as follows. Let us suppose that

$$\|\Psi_1\| \le \min(\|\Psi_2\|, \|\Psi_3\|). \tag{3.4}$$

Then, consider the sesquisoliton (3.1) with $m_1 = \|\Psi_1\|^2, m_2 = \|\Psi_2\|^2 + \|\Psi_3\|^2$, and $x \ge 0$ chosen in order to satisfy the condition (3.2). Notice that such a choice is always possible since $2m_1 \le m_2$, and is unique.

FIGURE 1. A sesquisoliton on the three-edge star graph.

If the condition (3.4) is not fulfilled, then one first relabels the edges in order to have the minimal mass on the first one, and thus proceeds as before.

It is immediately seen that $\|\Psi_1\|^2 = \|(\Phi^x_{m_1,m_2})_1\|^2 = m_1$ and $\|\Psi_2\|^2 + \|\Psi_3\|^2 = \|(\Phi^x_{m_1,m_2})_2\|^2 + \|(\Phi^x_{m_1,m_2})_3\|^2 = m_2$, thus $\|\Phi^x_{m_1,m_2}\|_2^2 = M$.

Let us define the following function on the real line:

$$\psi(\xi) \ := \left\{ \begin{array}{ll} \Psi_2(-\xi), & \quad \xi < 0 \\ \Psi_3(\xi), & \quad \xi > 0 \end{array} \right. \tag{3.5}$$

and notice that, by (2.6) and (3.1),

$$\varphi^{-x}_{m_2}(\xi) \ := \left\{ \begin{array}{ll} (\Phi^x_{m_1,m_2})_2(-\xi), & \quad \xi < 0 \\ (\Phi^x_{m_1,m_2})_3(\xi), & \quad \xi > 0 \end{array} \right. \tag{3.6}$$

Furthermore, by (2.5), (2.7), (2.2), (2.3), (2.4), (3.5), and (3.6) one immediately has the following chain of inequalities

$$E(\Psi) \ = \ E_1(\Psi_1) + E_2(\psi) \ \geq \ E_1(\phi_{m_1}) + E_2(\varphi^{-x}_{m_2}) \ = \ E(\Phi^x_{m_1,m_2})$$

so the first of the two identities in (3.3) is proven. To prove the second, we use (2.5) and (2.7) and obtain

$$E(\Phi^x_{m_1,m_2}) \;=\; -\frac{m_1^3}{24} - \frac{m_2^3}{96},$$

so, noting that $M = m_1 + m_2$, we obtain

$$E(\Phi^x_{m_1,m_2}) \;=\; -\frac{m_1^3}{24} + \frac{(m_1 - M)^3}{96}, \tag{3.7}$$

where m_1 plays the role of a parameter. We stress that, due to the constraint of the mass of the soliton, m_1 can vary in the interval $(0, M/3]$. Differentiating (3.7), one immediately has that $E(\Phi^x_{m_1,m_2})$ is monotonically increasing in such interval, so

$$\inf_{\Psi \in \mathcal{S}_M} E(\Psi) \;=\; \lim_{m_1 \to 0+} E(\Phi^x_{m_1,m_2}) \;=\; -\frac{M^3}{96}.$$

To complete the proof, we must show that, for any Ψ in $H^1(\mathcal{G})$, $E(\Psi)$ is strictly larger than $-M^3/96$. To this aim, we notice that such an infimum cannot be achieved, as for $m_1 = 0$ the condition (3.2) does not correspond to an admissible sesquisoliton, so it cannot be fulfilled. □

Corollary 3.3. The sesquisoliton $\Phi^0_{M/3,2M/3}$ is a saddle point for the energy functional.

Proof. First, we notice that $\Phi^0_{M/3,2M/3}$ is a critical point. Indeed, it satisfies the Euler-Lagrange equation for the energy functional constrained on the manifold $\|\Psi\|^2 = M$, namely

$$-\Delta\Psi - |\Psi|^2\Psi + \omega\Psi \;=\; 0,$$

where ω is a Lagrange multiplier, coinciding with $\frac{M^2}{36}$.

In order to prove that $\Phi^0_{M/3,2M/3}$ is a saddle point, it is sufficient to show that it maximizes the energy restricted to a curve to which it belongs in the constraint manifold, and minimizes the energy when restricted to a different curve.

By the proof of theorem (3.2) we have a curve on which $\Phi^0_{M/3,2M/3}$ is a maximum of the energy, that is, the curve made of sesquisolitons parametrized by m_1. On the other hand, by (2.5) we know that $\Phi^0_{M/3,2M/3}$ minimizes the energy at fixed mass on any edge. Then, it is a minimum of the energy restricted to the submanifold of function with $\Psi_1 = \Psi_2 = \Psi_3$. □

The previous result can be extended to all star graphs with a similar construction, and a more systematic analysis of the character of stationary states on star graphs will be given in a future work.

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Figure 2. The unique stationary state of the cubic NLS on the three-edge star graph.

References

[1] Adami R., Cacciapuoti C., Finco D., Noja D.: Fast solitons on star graphs, Rev Math. Phys. **23**, 4, 409-451 (2011)

[2] Adami, R. Cacciapuoti C., Finco D., Noja D.: Stationary states of NLS on star graphs arXiv:1104.3839v2 (2011)

[3] Berestycki H., Lions P.L.: Nonlinear scalar field equations I and II, Arch. Rat. Mech. Anal. **82**, 313-375 (1983)

[4] Cazenave T.: Semilinear Schrödinger equations, AMS (2003)

[5] Exner P., Keating J.P., Kuchment P., Sunada T., and Teplyaev A.: Analysis on graphs and its applications, AMS (2008).

[6] Gnutzmann, S. Smilansky, U., Derevyanko S.: Stationary scattering from a nonlinear network, Phys. Rev. A **83**, 033831 (2011)

[7] Kevrekidis P.G., Frantzeskakis D.J., Theocharis G., Kevrekidis I.G.: Guidance of matter waves through Y-junctions, Phys. Lett. A **317**, 51322 (2003)

[8] Kostrykin V., Schrader R.: Kirchhoff's rule for quantum wires, J.Phys. A: Math. Gen. **32** no. 4, 595-630 (1999)

[9] Kuchment P.: Graph models for waves in thin structures, Waves Random Media, **12** (4): R1-R24 (2002)

[10] Sobirov Z., Matrasulov D., Sabirov K., Sawada S., Nakamura K.: Integrable nonlinear Schrödinger equation on simple networks: Connection formula at vertices, Phys. Rev. E **81**, 066602 (2010)

[11] Tokuno A., Oshikawa M., Demler E.: Dynamics of one dimesional Bose liquids: Andreev-like reflection at Y junctions and the absence of the Aharonov-Bohm effect, Phys. Rev. Lett. **100**, 140402 (2008)

[12] Miroshnichenko A. E., Molina M.I., Kivshar Y.S.: Localized modes and bistable scattering in nonlinear network junctions, Phys. Rev. E **75**, 046602 (2007)

[13] Wan, W. Muenzel S. Fleischer, W.: Wave Tunneling and Hysteresis in Nonlinear Junctions, Phys. Rev. Lett. **104**, 073903 (2010)

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