Black hole's quantum levels

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Abstract

By introducing a black hole's *effective temperature,* which takes into account both of the non-strictly thermal and non-strictly continuous characters of Hawking radiation, we recently re-analyzed black hole's quasi-normal modes and interpreted them naturally in terms of quantum levels for emissions of particles. After a careful review of previous results, in this work we improve such an analysis by removing an approximation that we implicitly used in our previous work and by obtaining the corrected expressions for the formulas of the horizon's area quantization and the number of quanta of area and hence also for Bekenstein-Hawking entropy, its sub-leading corrections and the number of micro-states, i.e. quantities which are fundamental to realize unitary quantum gravity theory, like functions of the quantum "overtone" number e (*emission*) and, in turn, of the black hole's quantum excited level. Another approximation concerning the maximum value of e is also corrected.

We also consider quasi-normal modes in terms of quantum levels for *absorptions* too, in addition to our previous analysis which considered quasi-normal modes naturally associated to Hawking radiation and hence to emissions only. In that case, the above cited quantities result to be functions of the quantum "overtone" number a (*absorption*).

In this way, the whole black hole's quantum spectrum, for both of absorption and emission is obtained.

Previous results in the literature are re-obtained in the limit of very large "overtone" numbers e and a and of very small quasi-normal mode's frequency.

The results of this paper are very important on the route to realize the underlying unitary quantum gravity theory. In fact, black holes are considered natural theoretical laboratories to test such a theory which has to match the semi-classical results of the present paper.

1 Introduction

Parikh and Wilczek [1, 2] have shown that Hawking radiation's spectrum [3] cannot be strictly thermal. Such a non-strictly thermal character implies that the spectrum is also not strictly continuous and thus generates a natural correspondence between Hawking radiation and black hole's quasi-normal modes [4]. This issue endorses the idea that, in an underlying unitary quantum gravity theory, black holes result in highly excited states.

We recently used this key point to re-analyze the spectrum of black hole's quasi-normal modes by introducing a black hole's *effective temperature* [4, 5]. Our analysis changed the physical understanding of such a spectrum and enabled a re-examination of various results in the literature which realized important modifies on quantum physics of black holes [4, 5]. In particular, the formula of the horizon's area quantization and the number of quanta of area were modified becoming functions of the quantum emission's "overtone" number e [4, 5]. Consequently, Bekenstein-Hawking entropy, its sub-leading corrections and the number of micro-states, i.e. quantities which are fundamental to realize the underlying unitary quantum gravity theory, were also modified [4, 5]. They became functions of the emission's quantum "overtone" number e too, while previous results in the literature were re-obtained in the very large e limit [4, 5].

The analysis in [4, 5] permitted to naturally interpret quasi-normal modes in terms of quantum levels in the case of *emission* (Hawking radiation).

Here we remove an approximation that was implicitly used in our previous works [4, 5], by obtaining the corrected expressions for the cited formulas like functions of e. Another approximation concerning the maximum value of e is also corrected.

The analysis is further improved by considering quasi-normal modes in terms of quantum levels for *absorption* too. In this way, the whole black hole's quantum spectrum, for both of absorptions and emissions are obtained.

In the case of absorptions, Bekenstein-Hawking entropy, its sub-leading corrections and the number of micro-states result to be functions of the quantum "overtone" number a (*absorption*).

Previous results in the literature are now re-obtained in the limit of very large quantum "overtone" numbers e and a, and very small quasi-normal mode's frequency.

2 Effective temperature, Hawking radiation and quasi-normal modes: a review

Analyzing Hawking radiation [3] as tunneling, Parikh and Wilczek showed that the radiation spectrum cannot be strictly thermal [1, 2]. Parikh released an intriguing physical interpretation of this fundamental issue by discussing the existence of a secret tunnel through the black hole's horizon [1]. The energy conservation implies that the black hole contracts during the process of radiation [1, 2]. Thus, the horizon recedes from its original radius to a new, smaller radius

[1, 2]. The consequence is that black holes cannot strictly emit thermally [1, 2]. This is consistent with unitarity [1] and has profound implications for the black hole information puzzle because arguments that information is lost during black hole's evaporation rely in part on the assumption of strict thermal behavior of the spectrum [1, 2, 6].

Working with $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$ (Planck units), the probability of emission is [1, 2, 3]

$$\Gamma \sim \exp(-\frac{\omega}{T_H}), \tag{1}$$

where $T_H \equiv \frac{1}{8\pi M}$ is the Hawking temperature and ω the energy-frequency of the emitted radiation.

Parikh and Wilczek released a remarkable correction, due to an exact calculation of the action for a tunneling spherically symmetric particle, which yields [1, 2]

$$\Gamma \sim \exp[-\frac{\omega}{T_H}(1 - \frac{\omega}{2M})]. \tag{2}$$

This important result, which takes into account the conservation of energy, enables a correction, the additional term $\frac{\omega}{2M}$ [1, 2].

In various frameworks of physics and astrophysics the deviation from the thermal spectrum of an emitting body is taken into account by introducing an *effective temperature* which represents the temperature of a black body that would emit the same total amount of radiation [4, 5]. The effective temperature can be introduced for black holes too [4, 5]. It depends from the energy-frequency of the emitted radiation and is defined as [4, 5]

$$T_E(\omega) \equiv \frac{2M}{2M - \omega} T_H = \frac{1}{4\pi(2M - \omega)}. \tag{3}$$

Then, eq. (2) can be rewritten in Boltzmann-like form [4, 5]

$$\Gamma \sim \exp[-\beta_E(\omega)\omega] = \exp(-\frac{\omega}{T_E(\omega)}), \tag{4}$$

where $\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}$ and $\exp[-\beta_E(\omega)\omega]$ is the *effective Boltzmann factor* appropriate for an object with inverse effective temperature $T_E(\omega)$ [4, 5]. The ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M-\omega}$ represents the deviation of the radiation spectrum of a black hole from the strictly thermal feature [4, 5]. If M is the initial mass of the black hole *before* the emission, and $M - \omega$ is the final mass of the hole *after* the emission [2, 4, 5], eqs. (2) and (3) enable the introduction of the *effective mass* and of the *effective horizon* [4, 5]

$$M_E \equiv M - \frac{\omega}{2}, \; r_E \equiv 2M_E \tag{5}$$

of the black hole *during* the emission of the particle, i.e. *during* the contraction's phase of the black hole [4, 5]. The *effective quantities* T_E, M_E and r_E are average quantities. M_E is the average of the initial and final masses, r_E is the average of the initial and final horizons and T_E is the inverse of the average value of the inverses of the initial and final Hawking temperatures (*before* the emission $T_{H \text{ initial}} = \frac{1}{8\pi M}$, *after* the emission $T_{H \text{ final}} = \frac{1}{8\pi(M-\omega)}$) [4]. Notice that the analyzed process is *discrete* rather than *continuous* [4]. In fact, the black hole's state before the emission of the particle and the black hole's state after the emission of the particle are different countable black hole's physical states separated by an *effective state* which is characterized by the effective quantities [4]. Hence, the emission of the particle can be interpreted like a *quantum transition* of frequency ω between the two discrete states [4]. The tunneling visualization is that whenever a tunneling event works, two separated classical turning points are joined by a trajectory in imaginary or complex time [1, 4].

In [4, 5] we have shown that the correction to the thermal spectrum is also very important for the physical interpretation of black hole's quasi-normal modes, which, in turn, results very important to realize unitary quantum gravity theory as black holes are considered theoretical laboratories for developing such an ultimate theory and their quasi-normal modes are the best and most natural candidates for an interpretation in terms of quantum levels [4, 5, 7].

The intriguing idea that black hole's quasi-normal modes carry important information about black hole's area quantization is due to the remarkable works by Hod [8, 9]. Hod's original proposal found various objections over the years [7, 10] which have been answered in a good way by Maggiore [7], who refined Hod's conjecture. Quasi-normal modes are also believed to probe the small scale structure of the spacetime [11].

The quasi-normal frequencies are usually labelled as ω_{nl}, where l is the angular momentum quantum number [4, 5, 7, 12]. For each l ($l\geq$ 2 for gravitational perturbations), there is a countable sequence of quasi-normal modes, labelled by the "overtone" number n ($n = 1, 2, ...$) [4, 5, 7]. For large n the quasi-normal frequencies of the Schwarzschild black hole become independent of l having the structure [4, 5, 7, 12]

$$\begin{array}{c}
\omega_n = \ln 3 \times T_H + 2\pi i(n + \frac{1}{2}) \times T_H + \mathcal{O}(n^{-\frac{1}{2}}) = \\
= \frac{\ln 3}{8\pi M} + \frac{2\pi i}{8\pi M}(n + \frac{1}{2}) + \mathcal{O}(n^{-\frac{1}{2}}).
\end{array} \tag{6}$$

This result was originally obtained numerically in [13, 14], while an analytic proof was given later in [15, 16].

The spectrum of black hole's quasi-normal modes can be analysed in terms of superposition of damped oscillations, of the form [4, 5, 7]

$$\exp(-i\omega_I t)[a \sin \omega_R t + b \cos \omega_R t] \tag{7}$$

with a spectrum of complex frequencies $\omega = \omega_R + i\omega_I$. A damped harmonic oscillator $\mu(t)$ is governed by the equation [4, 5, 7]

$$\ddot{\mu} + K\dot{\mu} + \omega_0^2 \mu = F(t), \tag{8}$$

where K is the damping constant, ω_0 the proper frequency of the harmonic oscillator, and $F(t)$ an external force per unit mass. If $F(t) \sim \delta(t)$, i.e. considering the response to a Dirac delta function, the result for $\mu(t)$ is a superposition of a term oscillating as $\exp(i\omega t)$ and of a term oscillating as $\exp(-i\omega t)$, see [7] for details. Then, the behavior (7) is reproduced by a damped harmonic oscillator, through the identifications [4, 5, 7]

$$\tfrac{K}{2} = \omega_I, \qquad \sqrt{\omega_0^2 - \tfrac{K}{4}^2} = \omega_R, \tag{9}$$

which gives

$$\omega_0 = \sqrt{\omega_R^2 + \omega_I^2}. \tag{10}$$

In [7] it has been emphasized that the identification $\omega_0 = \omega_R$ is correct only in the approximation $\frac{K}{2} \ll \omega_0$, i.e. only for very long-lived modes. For a lot of black hole's quasi-normal modes, for example for highly excited modes, the opposite limit can be correct. Maggiore [7] used this observation to re-examine some aspects of quantum physics of black holes that were discussed in previous literature assuming that the relevant frequencies were $(\omega_R)_n$ rather than $(\omega_0)_n$. A problem concerning attempts to associate quasi-normal modes to Hawking radiation was that ideas on the continuous character of Hawking radiation did not agree with attempts to interpret the frequency of the quasi-normal modes [15]. In fact, the discrete character of the energy spectrum (6) should be incompatible with the spectrum of Hawking radiation whose energies are of the same order but continuous [15]. Actually, the issue that Hawking radiation is not strictly thermal and, as we have shown, it has discrete rather than continuous character, removes the above difficulty [4]. In other words, the discrete character of Hawking radiation permits to interpret the quasi-normal frequencies ω_{nl} in terms of energies of physical Hawking quanta too [4]. In fact, quasi-normal modes are damped oscillations representing the reaction of a black hole to small, discrete perturbations [4, 5, 7, 8, 9]. A discrete perturbation can be the capture of a particle which causes an increase in the horizon area [7, 8, 9]. Hence, if the emission of a particle which causes a decrease in the horizon area is a discrete rather than continuous process, it is quite natural to assume that it is also a perturbation which generates a reaction in terms of countable quasi-normal modes [4]. This natural correspondence between Hawking radiation and black hole's quasi-normal modes permits to consider quasi-normal modes in terms of quantum levels not only for absorbed energies like in [7, 8, 9], but also for emitted energies like in [4, 5]. This issue endorses the idea that, in an underlying unitary quantum gravity theory, black holes can be considered highly excited states [4, 5, 7].

The introduction of the effective temperature $T_E(\omega)$ can be applied to the analysis of the spectrum of black hole's quasi-normal modes [4, 5]. Another key point is that eq. (6) is an approximation as it has been derived with the assumption that the black hole's radiation spectrum is strictly thermal. To take into due account the deviation from the thermal spectrum in eq. (2) one has to substitute the Hawking temperature T_H with the effective temperature T_E in eq. (6) [4, 5]. Therefore, the correct expression for the quasi-normal frequencies of the Schwarzschild black hole, which takes into account the non-strictly thermal behavior of the radiation spectrum is [4, 5]

$$\begin{aligned}\omega_e &= \ln 3 \times T_E(\omega_e) + 2\pi i (e + \tfrac{1}{2}) \times T_E(\omega_e) + \mathcal{O}(e^{-\frac{1}{2}}) = \\ &= \tfrac{\ln 3}{4\pi[2M-(\omega_0)_e]} + \tfrac{2\pi i}{4\pi[2M-(\omega_0)_e]}(e + \tfrac{1}{2}) + \mathcal{O}(e^{-\frac{1}{2}}).\end{aligned} \tag{11}$$

Notice that in eq. (11) and in the following the quantum "overtone" number for the emission's levels is labelled e (*emission*) differently from [4, 5] where it was labelled n. This is because here we will introduce a second quantum "overtone" number a (*absorption*) for the absorption's levels.

The important result (11) can be explained as follows [4, 5]. Quasi-normal modes are frequencies of the radial spin-j perturbations ϕ of the four-dimensional Schwarzschild background which are governed by the following master differential equation [15, 16]

$$\left(-\frac{\partial^2}{\partial x^2} + V(x) - \omega^2\right)\phi. \tag{12}$$

By introducing the Regge-Wheeler potential ($j = 2$ for gravitational perturbations) eq. (12) is treated as a Schrodinger equation [15, 16]

$$V(x) = V\left[x(r)\right] = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right). \tag{13}$$

The relation between the Regge-Wheeler "tortoise" coordinate x and the radial coordinate r is [15, 16]

$$\begin{aligned} x &= r + 2M \ln\left(\tfrac{r}{2M} - 1\right) \\ \tfrac{\partial}{\partial x} &= \left(1 - \tfrac{2M}{r}\right)\tfrac{\partial}{\partial r}. \end{aligned} \tag{14}$$

In [15], Motl derived eq. (6) with a rigorous analytical calculation which starts from eqs. (12) and (13) and satisfies purely outgoing boundary conditions both at the horizon ($r = 2M$) and in the asymptotic region ($r = \infty$). In order to take into due account the conservation of energy, one has to substitute the original black hole's mass M in eqs. (12) and (13) with the effective mass of the contracting black hole defined in eq. (5) [4, 5].

Hence, eqs. (13) and (14) are replaced by the *effective equations* [4, 5]

$$V(x) = V\left[x(r)\right] = \left(1 - \frac{2M_E}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{6M_E}{r^3}\right) \tag{15}$$

and

$$\begin{aligned} x &= r + 2M_E \ln\left(\tfrac{r}{2M_E} - 1\right) \\ \tfrac{\partial}{\partial x} &= \left(1 - \tfrac{2M_E}{r}\right)\tfrac{\partial}{\partial r}. \end{aligned} \tag{16}$$

By realizing step by step the same rigorous analytical calculation in [13], but starting from eqs. (12) and (15) and satisfying purely outgoing boundary conditions both at the effective horizon ($r_E = 2M_E$) and in the asymptotic region ($r = \infty$), the final result will be, obviously and rigorously, eq. (11) [4, 5].

An intuitive, elegant interpretation is the following [4, 5]. The imaginary part of (6) can be easily understood [16]. The quasi-normal frequencies determine the position of poles of a Green's function on the given background, and the

Euclidean black hole solution converges to a thermal circle at infinity with the inverse temperature $\beta_H = \frac{1}{T_H}$ [16]. Thus, the spacing of the poles in eq. (6) coincides with the spacing $2\pi i T_H$ expected for a thermal Green's function [16]. But, if one considers the deviation from the thermal spectrum it is natural to assume that the Euclidean black hole solution converges to a *non-thermal* circle at infinity [4, 5]. Therefore, it is straightforward the replacement [4, 5]

$$\beta_H = \frac{1}{T_H} \to \beta_E(\omega) = \frac{1}{T_E(\omega)}, \tag{17}$$

which takes into account the deviation of the radiation spectrum of a black hole from the strictly thermal feature. In this way, the spacing of the poles in eq. (11) coincides with the spacing [4, 5]

$$2\pi i T_E(\omega) = 2\pi i T_H(\frac{2M}{2M-\omega}), \tag{18}$$

expected for a *non-thermal* Green's function (a dependence on the frequency is present) [4, 5].
By using the new expression (11) for the frequencies of quasi-normal modes, one defines [4, 5]

$$m_0 \equiv \frac{\ln 3}{4\pi[2M-(\omega_0)_e]}, \qquad p_e \equiv \frac{2\pi}{4\pi[2M-(\omega_0)_e]}(e+\frac{1}{2}). \tag{19}$$

Then, eq. (10) is re-written in the enlightening form [4, 5]

$$(\omega_0)_e = \sqrt{m_0^2 + p_e^2}. \tag{20}$$

These results improve eqs. (8) and (9) in [7] as the new expression (11) for the frequencies of quasi-normal modes takes into account that the radiation spectrum is not strictly thermal. For highly excited modes one gets [4, 5]

$$(\omega_0)_e \approx p_e = \frac{2\pi}{4\pi[2M-(\omega_0)_e]}(e+\frac{1}{2}). \tag{21}$$

Thus, differently from [7], levels are *not* equally spaced even for highly excited modes [4, 5]. Indeed, there are deviations due to the non-strictly thermal behavior of the spectrum (black hole's effective temperature depends on the energy level).
Using eq. (19), one can re-write eq. (20) as [4, 5]

$$(\omega_0)_e = \frac{1}{4\pi[2M-(\omega_0)_e]}\sqrt{(\ln 3)^2 + 4\pi^2(e+\frac{1}{2})^2}, \tag{22}$$

which is easily solved giving [4, 5]

$$(\omega_0)_e = M \pm \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e+\frac{1}{2})^2}}. \tag{23}$$

As a black hole cannot emit more energy than its total mass, the physical solution is the one obeying $(\omega_0)_e < M$ [4, 5]

$$(\omega_0)_e = M - \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e+\frac{1}{2})^2}}. \tag{24}$$

The interpretation is of a particle quantized with anti-periodic boundary conditions on a circle of length [4, 5]

$$L = \frac{1}{T_E(\omega_0)_e} = 4\pi \left(M + \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e+\frac{1}{2})^2}} \right), \tag{25}$$

i.e. the length of the circle depends on the overtone number e. Maggiore [7] found a particle quantized with anti-periodic boundary conditions on a circle of length $L = 8\pi M$. Our correction takes into account the conservation of energy, i.e. the additional term $\frac{\omega}{2M}$ in Eq. (2) [4, 5].

As $(\omega_0)_e$ has to be a real number (an emitted energy), we need also

$$M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e+\frac{1}{2})^2} \geq 0 \tag{26}$$

in eq. (24). The expression (26) is solved giving a maximum value for the emission's overtone number e [4]

$$e \leq e_{max} = 2\pi^2 \left(\sqrt{16M^4 - (\frac{\ln 3}{\pi})^2} - 1 \right), \tag{27}$$

corresponding to $(\omega_0)_{e_{max}} = M$. Again, a black hole cannot emit more energy than its total mass. Thus, the countable sequence of quasi-normal modes for emitted energies is not infinity although e can be very large [4]. By restoring ordinary units in eq. (27), one gets, for example, $e_{max} \sim 10^{79}$ for a a black hole's mass of order 10 solar masses [4].

Various important consequences on the quantum physics of black holes arise from the above approach [4, 5]. Let us start with the *area quantization*.

Bekenstein [17] showed that the area quantum of the Schwarzschild black hole is $\triangle A = 8\pi$ (notice that the *Planck length* $l_p = 1.616 \times 10^{-33}$ cm is equal to one in Planck units). By using properties of the spectrum of Schwarzschild black hole's quasi-normal modes, Hod found a different numerical coefficient [8, 9]. Hod's analysis started by the observation that, as for the Schwarzschild black hole the *horizon area* A is related to the mass through the relation $A = 16\pi M^2$, a variation $\triangle M$ in the mass generates a variation

$$\triangle A = 32\pi M \triangle M \tag{28}$$

in the area.

An important criticism by Maggiore [7] on Hod's conjecture is that only transitions from the ground state (i.e. a black hole which is not excited) to a state with large a (or vice versa) have been considered by Hod (notice that Maggiore and Hod considered absorptions rather than emissions [7, 8, 9], hence here we re-label the quantum "overtone" number a (*absorption*)). Actually, Bohr's correspondence principle strictly holds only for transitions from a to a' where both $a, a' \gg 1$ [7] and it is also legitimate to consider such transitions [7]. Thus, Maggiore suggested that $(\omega_0)_a$ should be used rather than $(\omega_R)_a$ [7], re-obtaining the original Bekenstein's result, i.e. $\triangle A = 8\pi$. In any case, Maggiore's result can be also improved if one takes into account the deviation from the strictly thermal feature in eq. (2), i.e. by using eq. (11) rather than eq. (6) [4, 5]. From eq. (24) one sees that an emission involving e and $e-1$ gives a variation of energy

$$\triangle M_e = (\omega_0)_{e-1} - (\omega_0)_e = -f_e(M,e) \tag{29}$$

where we have defined [4, 5]

$$f_e(M,e) \equiv$$

$$\equiv \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e-\frac{1}{2})^2}} - \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e+\frac{1}{2})^2}}. \tag{30}$$

The sign in (29), i.e. negative, is the same of eq. (29) of [4] and it is different with respect to the correspondent eq. (30) in [5] because here and in [4] we considered an emission while in [5] we considered an absorption.
By considering an absorption which generates a transition from an unexcited black hole to a black hole with very large a, Hod assumed *Bohr's correspondence principle* to be valid for large a and enabled a semi-classical description even in absence of a full unitary quantum gravity theory [8, 9]. Thus, from eq. (6), the minimum quantum which can be involved in the transition is $\triangle M_a = \omega = \frac{\ln 3}{8\pi M}$. This gives $\triangle A = 4\ln 3$. The presence of the numerical factor $4\ln 3$ stimulated possible connections with loop quantum gravity [18].
Combining eqs. (28) and (29) one gets [4, 5]

$$\triangle A = 32\pi M \triangle M_e = -32\pi M \times f_e(M,e). \tag{31}$$

For very large e (but we recall that $e \leq e_{max}$, see eq. (27)) one obtains [4, 5]

$$f_e(M,e) \approx$$

$$\approx \sqrt{M^2 - \frac{1}{2}(e-\frac{1}{2})} - \sqrt{M^2 - \frac{1}{2}(e+\frac{1}{2})} \approx \frac{1}{4M}, \tag{32}$$

and eq. (31) becomes $\triangle A \approx -8\pi$ which is the original result of Bekenstein for the area quantization (a part a sign because we consider an emission rather than an absorption). Then, only in the very large e limit the levels are approximately equally spaced [4, 5]. Indeed, for smaller e there are deviations, see eq. (21).
Important consequences on entropy and micro-states arise from the above analysis [4, 5].
Let us assume that, for large e, the horizon area is quantized [4, 5, 7] with a quantum $|\triangle A| = \alpha$, where $\alpha = 32\pi M \cdot f_e(M,e)$ for us [4], $\alpha = 8\pi$ for Bekenstein [17] and Maggiore [7], $\alpha = 4\ln 3$ for Hod [8, 9] and Dreyer [18]. The total horizon area must be $A = N|\triangle A| = N\alpha$ where the integer N is the number of quanta of area. Our approach gives [4, 5]

$$N = \frac{A}{|\triangle A|} = \frac{16\pi M^2}{\alpha} = \frac{16\pi M^2}{32\pi M \cdot f_e(M,e)} = \frac{M}{2f_e(M,e)}. \tag{33}$$

The famous formula of Bekenstein-Hawking entropy [3, 19, 20] now becomes [4, 5]

$$S_{BH} = \frac{A}{4} = 8\pi N M |\triangle M_e| = 8\pi N M \cdot f_e(M,e). \tag{34}$$

Thus, we get the important result that Bekenstein-Hawking entropy is a function of the quantum overtone number e.
In the very large e limit eq. (32) gives $f_e(M,e) \rightarrow \frac{1}{4M}$ and the standard result [7, 21, 22, 23]

$$S_{BH} \rightarrow 2\pi N \tag{35}$$

is re-obtained [4, 5].
On the other hand, it is a common and general belief that there is no reason to expect that Bekenstein-Hawking entropy will be the whole answer for a correct unitary quantum gravity theory [4, 5, 24]. For a better understanding of black hole's entropy one needs to go beyond Bekenstein-Hawking entropy and identify the sub-leading corrections [4, 5, 24]. The quantum tunnelling approach can be used to obtain the sub-leading corrections to the second order approximation [25]. One gets that the black hole's entropy contains three parts: the usual Bekenstein-Hawking entropy, the logarithmic term and the inverse area term [25]

$$S_{total} = S_{BH} - \ln S_{BH} + \frac{3}{2A}. \tag{36}$$

In fact, if one wants to satisfy the unitary quantum gravity theory the logarithmic and inverse area terms are requested [25]. Apart from a coefficient, this correction to the black hole's entropy is consistent with the one of loop quantum gravity [25], where the coefficient of the logarithmic term has been rigorously fixed at $\frac{1}{2}$ [25, 26]. The correction (34) to Bekenstein-Hawking entropy permits to re-write eq. (36) as [4, 5]

$$S_{total} = 8\pi NM \cdot f_e(M,e) - \ln\left[8\pi NM \cdot f_e(M,e)\right] + \frac{3}{64\pi NM \cdot f_e(M,e)} \tag{37}$$

that in the very large e limit becomes [4, 5]

$$S_{total} \rightarrow 2\pi N - \ln 2\pi N + \frac{3}{16\pi N}. \tag{38}$$

These results imply that at level N the black hole has a number of micro-states [4, 5]

$$g(N) \propto \exp\left\{8\pi NM \cdot f_e(M,e) - \ln\left[8\pi NM \cdot f_e(M,e)\right] + \frac{3}{64\pi NM \cdot f_e(M,e)}\right\}, \tag{39}$$

that in the very large e limit reads [4, 5]

$$g(N) \propto \exp\left[2\pi N - \ln\left(2\pi N\right) + \frac{3}{16\pi N}\right]. \tag{40}$$

3 Removing some implicit simplifications

3.1 Varying mass of the black hole

Actually, in previous Section, and hence also in [4, 5], we used an implicit simplification. In this Subsection we improve the analysis by removing such a simplification and by releasing the correct results.

In fact, we note that in an emission from the ground state to a state with large $e-1$ the mass of the black hole changes from M to

$$M_{e-1} \equiv M - (\omega_0)_{e-1} \tag{41}$$

and in the transition from state with $e-1$ to the state with e it again changes from M_{e-1} to

$$M_e \equiv M - (\omega_0)_{e-1} + \triangle M_e, \tag{42}$$

which, by using eq. (29), becomes

$$\begin{array}{c} M_e = M - (\omega_0)_{e-1} - f_e(M, e) = \\ \\ = M - (\omega_0)_{e-1} + (\omega_0)_{e-1} - (\omega_0)_e = M - (\omega_0)_e. \end{array} \tag{43}$$

By considering eq. (24), eqs. (41) and (43) read

$$M_{e-1} = \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e - \frac{1}{2})^2}} \tag{44}$$

and

$$M_e = \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e + \frac{1}{2})^2}}. \tag{45}$$

This implies that, if one uses eqs. (24) and eq. (44), eq. (31) has to be correctly re-written as

$$\begin{array}{c} \triangle A_{e-1} \equiv 32\pi M_{e-1} \triangle M_e = -32\pi M_{e-1} \times f_e(M, e) = \\ \\ 32\pi \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e - \frac{1}{2})^2}} \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e + \frac{1}{2})^2}} + \\ \\ -32\pi \left(M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e - \frac{1}{2})^2} \right). \end{array} \tag{46}$$

This equation is very important as it gives the area quantum of an excited black hole for an emission from the level $e-1$ to the level e in function of the quantum number e and of the initial black hole's mass.

In previous Section and in [4, 5], we implicitly used the simplification $(\omega_0)_{e-1} \ll M$, i.e. the energy associated to the quasi-normal frequency is much less than the mass-energy of the black hole. Clearly, in that case the correction given by eq. (46) results nonessential as one can neglect the difference between the initial mass M and the mass of the excited black hole M_{e-1}, but it becomes very important when $(\omega_0)_{e-1} \lesssim M$ i.e. in the last stages of the black hole's evaporation. In that case, further corrections on formulas in previous Section and in [4, 5] are needed.

Putting $A_{e-1} \equiv 16\pi M_{e-1}^2$, the formulas of the number of quanta of area and of the Bekenstein-Hawking entropy become

$$N_{e-1} \equiv \frac{A_{e-1}}{|\triangle A_{e-1}|} = \frac{16\pi M_{e-1}^2}{\alpha} = \frac{16\pi M_{e-1}^2}{32\pi M_{e-1}\cdot f_e(M,e)} = \frac{M_{e-1}}{2f_e(M,e)} =$$

$$= \frac{\sqrt{M^2-\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(e-\frac{1}{2})^2}}}{2\left(\sqrt{M^2-\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(e-\frac{1}{2})^2}}-\sqrt{M^2-\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(e+\frac{1}{2})^2}}\right)} \tag{47}$$

and

$$(S_{BH})_{e-1} \equiv \frac{A_{e-1}}{4} = 8\pi N_{e-1}M_{e-1}|\triangle M_e| = 8\pi N_{e-1}M_{e-1}\cdot f_e(M,e) =$$

$$= 4\pi\left(M^2-\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(e-\frac{1}{2})^2}\right). \tag{48}$$

respectively.

The formula of the total entropy that takes into account the sub-leading corrections to Bekenstein-Hawking entropy becomes

$$(S_{total})_{e-1} = 8\pi N_{e-1}M_{e-1}\cdot f_e(M,e) - \ln\left[8\pi N_{e-1}M_{e-1}\cdot f_e(M,e)\right] + \frac{3}{64\pi N_{e-1}M_{e-1}\cdot f_e(M,e)} =$$

$$= 4\pi\left(M^2-\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(e-\frac{1}{2})^2}\right)+$$

$$-\ln\left[4\pi\left(M^2-\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(e-\frac{1}{2})^2}\right)\right]+\frac{3}{32\pi\left(M^2-\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(e-\frac{1}{2})^2}\right)} \tag{49}$$

which, in turn, implies that at level N_{e-1} the black hole has a number of micro-states

$$g(N_{e-1}) \propto \exp\left\{8\pi N_{e-1}M_{e-1}\cdot f_e(M,e) - \ln\left[8\pi N_{e-1}M_{e-1}\cdot f_e(M,e)\right] + \frac{3}{64\pi N_{e-1}M_{e-1}\cdot f_e(M,e)}\right\} =$$

$$= \exp\{4\pi\left[M^2-\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(e-\frac{1}{2})^2}\right]+$$

$$-\ln\left[4\pi\left(M^2-\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(e-\frac{1}{2})^2}\right)\right]+\frac{3}{32\pi\left(M^2-\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(e-\frac{1}{2})^2}\right)}\}. \tag{50}$$

Again, all these corrections, which represent the correct formulas of an excited black hole for an emission from the level $e-1$ to the level e in function of the quantum number e and of the initial black hole's mass, result very important in the last stages of the black hole's evaporation, when $(\omega_0)_{e-1} \lesssim M$.

Indeed, when $(\omega_0)_{e-1} \ll M$ formulas (31), (33), (37) and (39), are a good approximation. When both of the constrains $e \gg 1$ and $(\omega_0)_e \ll M$ hold, formulas (35), (38) and (40) can be used. They are in full agreement with previous results in the literature [7, 24, 25, 26].

3.2 A note on black hole's remnants

We recall that, by using the Generalized Uncertainty Principle, Adler, Chen and Santiago [27] have shown that the total evaporation of a black hole is prevented in exactly the same way that the Uncertainty Principle prevents the hydrogen atom from total collapse. In fact, the collapse is prevented, not by symmetry, but by dynamics, as the *Planck distance* and the *Planck mas*s are approached [27]. That important result implies that eq. (26) has to be slightly modified, becoming (the Planck mass is equal to 1 in Planck units)

$$M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(e+\frac{1}{2})^2} \geq 1. \tag{51}$$

By solving eq. (51) one gets a different value of the maximum value for the emission's overtone number e

$$e \leq e_{max} = 2\pi^2\left(\sqrt{16(M^2-1)^2 - (\frac{\ln 3}{\pi})^2} - 1\right). \tag{52}$$

By restoring ordinary units in eq. (52), one gets again $e_{max} \sim 10^{79}$ for a a black hole's mass of order 10 solar masses, i.e. the order of magnitude of e_{max} remains the same. The difference becomes important only for very micro black holes, i.e. when the original black hole's mass is of the order of the Planck mass.

4 The absorption spectrum

Now, we further improve the analysis by computing the absorption spectrum of the Schwarzschild black hole.

In case of an absorption the expression for the quasi-normal frequencies of the black hole results slightly modified with respect to eq. (11), which concerns the emission's frequencies, becoming

$$\begin{array}{c} \omega_a = \ln 3 \times T_E(\omega_a) + 2\pi i(a+\frac{1}{2}) \times T_E(\omega_a) + \mathcal{O}(a^{-\frac{1}{2}}) = \\ = \frac{\ln 3}{4\pi[2M+(\omega_0)_a]} + \frac{2\pi i}{4\pi[2M+(\omega_0)_a]}(a+\frac{1}{2}) + \mathcal{O}(a^{-\frac{1}{2}}), \end{array} \tag{53}$$

where now a is the quantum "overtone" number for the *absorption's levels*.

In fact, in case of an absorption the effective temperature (3) can be easily re-defined as

$$T_E(\omega) \equiv \frac{2M}{2M+\omega}T_H = \frac{1}{4\pi(2M+\omega)}. \tag{54}$$

In order to proof the correctness of eq. (53) one can use the same discussion in Section 2 by using eqs. from (12) to (16), but re-defining the effective mass and of the effective horizon as

$$M_E \equiv M + \frac{\omega}{2}, \; r_E \equiv 2M_E. \tag{55}$$

Again, if one realizes step by step the same rigorous analytical calculation in [13], but starting from eqs. (12) and (15) and satisfying purely outgoing boundary conditions both at the effective horizon ($r_E = 2M_E$ with M_E given now by eq. (55)) and in the asymptotic region ($r = \infty$), the final result will be, obviously and rigorously, eq. (53). Clearly M_E is again the average of the initial and final masses, r_E is again the average of the initial and final horizons and T_E is again the inverse of the average value of the inverses of the initial and final Hawking temperatures. Now it is $T_{H \text{ initial}} = \frac{1}{8\pi M}$ *before* the absorption and $T_{H \text{ final}} = \frac{1}{8\pi(M+\omega)}$ *after* the absorption. In other words, M_E and r_E are the effective mass and of the effective horizon of the black hole *during* the absorption of the particle, i.e. *during* the expansion's phase of the black hole.

The intuitive interpretation is the same of the quasi-normal frequencies in case of emissions. The deviation of the spectrum of black hole's quasi-normal modes from the strictly thermal feature implies that the spacing of the poles in eq. (53) coincides with the spacing

$$2\pi i T_E(\omega) = 2\pi i T_H(\frac{2M}{2M+\omega}), \tag{56}$$

expected for a *non-thermal* Green's function. In fact, a dependence on the frequency is present in eq. (56) too.
By using the expression (53) for the absorption's frequencies of quasi-normal modes, one re-writes eqs. (19) and (20) as

$$m_0 \equiv \frac{\ln 3}{4\pi[2M+(\omega_0)_a]}, \qquad p_a \equiv \frac{2\pi}{4\pi[2M+(\omega_0)_a]}(a+\frac{1}{2}). \tag{57}$$

and

$$(\omega_0)_a = \sqrt{m_0^2 + p_a^2}. \tag{58}$$

respectively. For highly excited modes we find now

$$(\omega_0)_a \approx p_a = \frac{2\pi}{4\pi[2M+(\omega_0)_a]}(a+\frac{1}{2}). \tag{59}$$

Hence, levels are not equally spaced even for absorption's highly excited modes but there are deviations due to the non-strictly thermal behavior of the spectrum. In other words, the energy's conservation is taken into due account for absorptions too.
Using eq. (57), one can re-write eq. (58) as

$$(\omega_0)_a = \frac{1}{4\pi[2M+(\omega_0)_a]}\sqrt{(\ln 3)^2 + 4\pi^2(a+\frac{1}{2})^2}. \tag{60}$$

The solution of eq. (60) is

$$(\omega_0)_a = -M \pm \sqrt{M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(a+\frac{1}{2})^2}}. \tag{61}$$

Clearly, a black hole cannot absorb negative energies which are greater than its mass in absolute value. Hence, the physical solution is the one obeying $(\omega_0)_a < |M|$, i.e.

$$(\omega_0)_a = \sqrt{M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(a+\frac{1}{2})^2}} - M. \tag{62}$$

The interpretation is of a particle quantized with anti-periodic boundary conditions on a circle of length

$$L=\frac{1}{T_E(\omega_0)_a}=4\pi\left(M+\sqrt{M^2+\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(a+\frac{1}{2})^2}}\right), \tag{63}$$

i.e. the length of the circle depends on the absorption's "overtone" number a. From eq. (62) one sees that an absorption involving a and $a-1$ gives a variation of energy

$$\triangle M_a=(\omega_0)_a-(\omega_0)_{a-1}=f_a(M,a) \tag{64}$$

where we have defined

$$f_a(M,a)\equiv$$

$$\equiv\sqrt{M^2+\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(a+\frac{1}{2})^2}}-\sqrt{M^2+\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(a-\frac{1}{2})^2}}. \tag{65}$$

We note that in an absorption from the ground state to a state with large $a-1$ the mass of the black hole changes from M to

$$M_{a-1}\equiv M+(\omega_0)_{a-1} \tag{66}$$

and in the absorption from state with $a-1$ to the state with a it again changes from M_{a-1} to

$$M_a\equiv M+(\omega_0)_{a-1}+\triangle M_a, \tag{67}$$

which, by using eq. (64), becomes

$$\begin{array}{c} M_a=M+(\omega_0)_{a-1}+f_a(M,a)= \\ \\ =M+(\omega_0)_{a-1}-(\omega_0)_{a-1}+(\omega_0)_a=M+(\omega_0)_a. \end{array} \tag{68}$$

By considering eq. (62), eqs. (66) and (68) read

$$M_{a-1}=\sqrt{M^2+\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(a-\frac{1}{2})^2}} \tag{69}$$

and

$$M_a=\sqrt{M^2+\frac{1}{4\pi}\sqrt{(\ln 3)^2+4\pi^2(a+\frac{1}{2})^2}}. \tag{70}$$

By using eqs. (62) and eq. (69), the equation of the area quantum (28) becomes

$$\triangle A_{a-1} \equiv 32\pi M_{a-1} \triangle M_a = 32\pi M_{a-1} \times f_a(M,a) =$$

$$32\pi \sqrt{M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a-\frac{1}{2})^2}} \sqrt{M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a+\frac{1}{2})^2}} +$$

$$-32\pi \left(M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a-\frac{1}{2})^2} \right). \tag{71}$$

Eq. (71) gives the area quantum of an excited black hole for an absorption from the level $a-1$ to the level a in function of the quantum number a and of the initial black hole's mass.

Putting $A_{a-1} \equiv 16\pi M_{a-1}^2$, the formula of the number of quanta of area is

$$N_{a-1} \equiv \frac{A_{a-1}}{\triangle A_a} = \frac{16\pi M_{a-1}^2}{\alpha} = \frac{16\pi M_{a-1}^2}{32\pi M_{a-1} \cdot f_a(M,e)} = \frac{M_{a-1}}{2 f_a(M,a)} =$$

$$= \frac{\sqrt{M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a-\frac{1}{2})^2}}}{2\left(\sqrt{M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a+\frac{1}{2})^2}} - \sqrt{M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a-\frac{1}{2})^2}} \right)}. \tag{72}$$

Hence, we obtain the formula of the Bekenstein-Hawking entropy for quantum levels in case of absorption

$$(S_{BH})_{a-1} \equiv \frac{A_{a-1}}{4} = 8\pi N_{a-1} M_{a-1} \triangle M_a = 8\pi N_{a-1} M_{a-1} \cdot f_a(M,a) =$$

$$= 4\pi \left(M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a-\frac{1}{2})^2} \right). \tag{73}$$

The formula of the total entropy that takes into account the sub-leading corrections to Bekenstein-Hawking entropy becomes

$$(S_{total})_{a-1} = 8\pi N_{a-1} M_{a-1} \cdot f_a(M,a) - \ln\left[8\pi N_{a-1} M_{a-1} \cdot f_a(M,a)\right] + \frac{3}{64\pi N_{a-1} M_{a-1} \cdot f_a(M,a)} =$$

$$= 4\pi \left(M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a-\frac{1}{2})^2} \right) +$$

$$-\ln\left[4\pi \left(M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a-\frac{1}{2})^2} \right) \right] + \frac{3}{32\pi \left(M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a-\frac{1}{2})^2} \right)} \tag{74}$$

which, in turn, implies that at level N_{a-1} in case of absorption the black hole has a number of micro-states

$$g(N_{a-1}) \propto \exp\left\{ 8\pi N_{a-1} M_{a-1} \cdot f_a(M,a) - \ln\left[8\pi N_{a-1} M_{a-1} \cdot f_a(M,a)\right] + \frac{3}{64\pi N_{a-1} M_{a-1} \cdot f_a(M,a)} \right\} =$$

$$= \exp\{ 4\pi \left(M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a-\frac{1}{2})^2} \right) +$$

$$-\ln\left[4\pi \left(M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a-\frac{1}{2})^2} \right) \right] + \frac{3}{32\pi \left(M^2 + \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2 (a-\frac{1}{2})^2} \right)} \}. \tag{75}$$

In the limit $(\omega_0)_{a-1} \ll M$ (the energy associated to the quasi-normal mode is much less than the mass-energy of the black hole) we get $M_{a-1} \approx M$, $A_{a-1} \approx A$, $N_{a-1} \approx N$, and eqs. (73), (74) and (75) read

$$(S_{BH})_{a-1} \approx \frac{A}{4} = 8\pi NM\triangle M_a = 8\pi NM \cdot f_a(M,a), \tag{76}$$

$$(S_{total})_{a-1} \approx 8\pi NM \cdot f_a(M,a) - \ln\left[8\pi NM \cdot f_a(M,a)\right] + \frac{3}{64\pi NM \cdot f_a(M,a)}, \tag{77}$$

and

$$g(N_{a-1}) \approx \exp\left\{8\pi NM \cdot f_a(M,a) - \ln\left[8\pi NM \cdot f_a(M,a)\right] + \frac{3}{64\pi NM \cdot f_a(M,a)}\right\}, \tag{78}$$

respectively. On the other hand, if a is enough large we get

$$\begin{gathered} f_a(M,a) \approx \\ \approx \sqrt{M^2 + \tfrac{1}{2}(a+\tfrac{1}{2})} - \sqrt{M^2 + \tfrac{1}{2}(a-\tfrac{1}{2})} \approx \tfrac{1}{4M} \end{gathered} \tag{79}$$

and eqs. (76), (77) and (78) reduce to

$$(S_{BH})_{a-1} \approx 2\pi N, \tag{80}$$

$$(S_{total})_{a-1} \approx 2\pi N - \ln(2\pi N) + \frac{3}{16\pi N} \tag{81}$$

and

$$g(N_{a-1}) \approx \exp\left[2\pi N - \ln(2\pi N) + \frac{3}{16\pi N}\right], \tag{82}$$

respectively, which, in turn, are in perfect agreement with (35), (38) and (40) and with previous results in the literature [7, 24, 25, 26].

5 Conclusion remarks

In this paper we analyzed black hole's quasi-normal modes in terms of quantum levels following the idea that, in an underlying unitary quantum gravity theory, black holes result in highly excited states. By using the concept of effective temperature, we took into account the important issue that quasi-normal modes' spectrum is not strictly thermal in both of emission and absorption. The obtained results look particularly intriguing as important modifies on quantum physics of black holes have been realized. In fact, the formula of the horizon's area quantization and the number of quanta of area result functions of the quantum "overtone" number e in case of emission and of the quantum "overtone" number a in case of absorption. Consequently, Bekenstein-Hawking entropy, its sub-leading corrections and the number of micro-states, i.e. quantities which are fundamental to realize the underlying unitary quantum gravity theory, become functions of the quantum "overtone" numbers too, e for emission and a for absorption. In other words, the cited important quantities result to depend on the excited quantum state of the black hole in both of the cases of emission and absorption.

An approximation concerning the maximum value of e has been also corrected.

It is important to emphasize that previous results in the literature were re-obtained in the very large e, a limits when the energy associated to the quasinormal frequency is much less than the mass-energy of the black hole. This issue confirms the correctness of the analysis in this paper which improves previous approximations.

References

[1] M. K. Parikh, Gen. Rel. Grav. 36, 2419 (2004, First Award at Gravity Research Foundation).

[2] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000).

[3] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).

[4] C. Corda, arXiv:1205.5251 (2012, Honorable Mention at Gravity Research Foundation, to appear in a Special Issue of Int. Journ. Mod. Phys. D).

[5] C. Corda, JHEP 1108, 101 (2011).

[6] S. W. Hawking, Phys. Rev. D 14, 2460 (1976).

[7] M. Maggiore, Phys. Rev. Lett. 100, 141301 (2008).

[8] S. Hod, Phys. Rev. Lett. 81 4293 (1998).

[9] S. Hod, Gen. Rel. Grav. 31, 1639 (1999, Fifth Award at Gravity Research Foundation).

[10] R. A. Konoplya and A. Zhidenko, Rev. Mod. Phys. 83, 793-836 (2011).

[11] M. Casals, Ad. C. Ottewill, arXiv:1112.2695v2 [gr-qc] (2011).

[12] T. Padmanabhan, Class. Quant. Grav. 21, L1 (2004).

[13] H. P. Nollert, Phys. Rev. D 47, 5253 (1993).

[14] N. Andersson, Class. Quant. Grav. 10, L61 (1993).

[15] L. Motl, Adv. Theor. Math. Phys. 6, 1135 (2003).

[16] L. Motl and A. Neitzke, Adv. Theor. Math. Phys. 7, 307 (2003).

[17] J. D. Bekenstein, Lett. Nuovo Cim. 11, 467 (1974).

[18] O. Dreyer, Phys. Rev. Lett. 90, 081301 (2003).

[19] J. D. Bekenstein, Nuovo Cim. Lett. 4, 737 (1972).

[20] J. D. Bekenstein, Phys. Rev. D7, 2333 (1973).

[21] A. Barvinsky and G. Kunstatter, arXiv:gr-qc/9607030v1 (1996).

[22] A. Barvinsky S. Das and G. Kunstatter, Class. Quant. Grav. 18, 4845 (2001).

[23] D. Kothawala, T. Padmanabhan, S. Sarkar, Phys. Rev. D 78, 104018 (2008).

[24] S. Shankaranarayanan, Mod. Phys. Lett. A 23, 1975-1980 (2008).

[25] J. Zhang, Phys. Lett. B 668, 353-356 (2008).

[26] A. Ghosh, P. Mitra, Phys. Rev. D 71, 027502 (2005).

[27] R. J. Adler, P. Chen and D. I. Santiago, Gen. Rel. Grav. 3, 2101-2108 (2001, Third Award at Gravity Research Foundation).