# Effective temperature for black holes

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**Abstract**

The physical interpretation of black hole's quasinormal modes is fundamental for realizing unitary quantum gravity theory as black holes are considered theoretical laboratories for testing models of such an ultimate theory and their quasinormal modes are natural candidates for an interpretation in terms of quantum levels.

The spectrum of black hole's quasinormal modes can be re-analysed by introducing a black hole's *effective temperature* which takes into account the fact that, as shown by Parikh and Wilczek, the radiation spectrum cannot be strictly thermal. This issue changes in a fundamental way the physical understanding of such a spectrum and enables a re-examination of various results in the literature which realizes important modifies on quantum physics of black holes. In particular, the formula of the horizon's area quantization and the number of quanta of area result modified becoming functions of the quantum "overtone" number $n$. Consequently, the famous formula of Bekenstein-Hawking entropy, its sub-leading corrections and the number of microstates are also modified. Black hole's entropy results a function of the quantum overtone number too.

We emphasize that this is the first time that black hole's entropy is directly connected with a quantum number.

Previous results in the literature are re-obtained in the limit $n \to \infty$.

**Keywords:** Black hole, entropy, Hawking radiation, effective temperature.

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Hawking radiation [1] can be visualized as particles that have tunnelled across the black hole's horizon [2, 3]. In [2, 3] Parikh and Wilczek showed that the barrier depends on the tunnelling particle itself. Parikh released an intriguing physical interpretation of this fundamental issue by discussing the existence of a tunnel through the black hole's horizon [2]. By implementing energy conservation, the black hole contracts during the process of radiation. Thus, the horizon recedes from its original radius to a new, smaller radius [2]. Hence, the radiation spectrum cannot be strictly thermal. The correction to the thermal spectrum has profound implications for realizing unitary quantum gravity theory and for the black hole's information puzzle [2, 3]. In fact, black holes are considered theoretical laboratories for developing unitary quantum gravity theory and their quasinormal modes are the best candidates for an interpretation in terms of quantum levels.

In this work we re-analyse the spectrum of black hole's quasinormal modes by taking into account the issue that the radiation spectrum is not strictly thermal.

Working with $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$ (Planck units) the probability of tunnelling takes the form [1, 2, 3]

$$\Gamma \sim \exp(-\frac{\omega}{T_H}), \tag{1}$$

where $T_H \equiv \frac{1}{8\pi M}$ is the Hawking temperature and $\omega$ the energy-frequency of the emitted radiation.

The remarkable correction by Parikh and Wilczek, due by an exact calculation of the action for a tunnelling spherically symmetric particle, yields [2, 3]

$$\Gamma \sim \exp[-\frac{\omega}{T_H}(1 - \frac{\omega}{2M})]. \tag{2}$$

This result has also taken into account the conservation of energy and this enables a correction, the additional term $\frac{\omega}{2M}$ [2]. If we introduce the *effective temperature* (which depends from the energy-frequency of the emitted radiation)

$$T_E(\omega) \equiv \frac{2M}{2M - \omega} T_H = \frac{1}{4\pi(2M - \omega)}, \tag{3}$$

Eq. (2) can be rewritten in Boltzmann-like form

$$\Gamma \sim \exp[-\beta_E(\omega)\omega] = \exp(-\frac{\omega}{T_E(\omega)}), \tag{4}$$

where $\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}$ and $\exp[-\beta_E(\omega)\omega]$ is the *effective Boltzmann factor* appropriate for an object with inverse effective temperature $T_E(\omega)$. The ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M-\omega}$ represents the deviation of the radiation spectrum of a black hole from the strictly thermal feature. In other terms, as the correction in [2, 3] implies that a black hole does not strictly emit like a black body, the effective temperature represents the temperature of a black body that would emit the same total amount of radiation.

Now, we apply the introduction of the effective temperature $T_E(\omega)$ to the analysis of the spectrum of black hole's quasinormal modes.

For Schwarzschild black holes, the quasinormal mode frequencies are usually labelled as $\omega_{nl}$, where $l$ is the angular momentum quantum number [4, 5]. For each $l$ ($l \geq 2$ for gravitational perturbations), we have a countable infinity of quasinormal modes, labelled by the "overtone" number $n$ ($n = 1, 2, ...$) [5]. For large $n$ the frequencies of quasinormal modes for the Schwarzschild black hole become independent of $l$ having the structure [4, 5]

$$\begin{array}{c} \omega_n = \ln 3 \times T_H + 2\pi i(n + \frac{1}{2}) \times T_H + \mathcal{O}(n^{-\frac{1}{2}}) = \\ = \frac{\ln 3}{8\pi M} + \frac{2\pi i}{8\pi M}(n + \frac{1}{2}) + \mathcal{O}(n^{-\frac{1}{2}}). \end{array} \tag{5}$$

This result was originally obtained numerically in [6, 7], while an analytic proof was given in [8, 9].

In any case, Eq. (5) is an approximation as it has been derived with the assumption that the black hole radiation spectrum is strictly thermal. To take into due account the deviation from the thermal spectrum in [2, 3] one has to substitute the Hawking temperature $T_H$ with the effective temperature $T_E$ in Eq. (5). In this way, the correct expression for the frequencies of quasinormal modes for the Schwarzschild black hole, which takes into account the important issue that the radiation spectrum is not strictly thermal, is

$$\begin{array}{c} \omega_n = \ln 3 \times T_E(\omega_n) + 2\pi i(n + \frac{1}{2}) \times T_E(\omega_n) + \mathcal{O}(n^{-\frac{1}{2}}) = \\ = \frac{\ln 3}{4\pi(2M - \omega_n)} + \frac{2\pi i}{4\pi(2M - \omega_n)}(n + \frac{1}{2}) + \mathcal{O}(n^{-\frac{1}{2}}). \end{array} \tag{6}$$

Let us explain this key point. The imaginary part of (5) is simple to understand [9]. The quasinormal modes determine the position of poles of a Green's function on the given background, and the Euclidean black hole solution converges to a thermal circle at infinity with the inverse temperature $\beta_H = \frac{1}{T_H}$ [9]. Hence, it is not surprising that the spacing of the poles in (5) coincides with the spacing $2\pi i T_H$ expected for a thermal Green's function [9]. But, if we want to consider the deviation from the thermal spectrum which has been found in [2, 3] it is natural to assume that the Euclidean black hole solution converges to a *non-thermal* circle at infinity. Therefore, it is straightforward the substitution

$$\beta_H = \frac{1}{T_H} \to \beta_E(\omega) = \frac{1}{T_E(\omega)}, \tag{7}$$

which takes into account the deviation of the radiation spectrum of a black hole from the strictly thermal feature. In this way, the spacing of the poles in (6) coincides with the spacing

$$2\pi i T_E(\omega) = 2\pi i T_H(\frac{2M}{2M - \omega}), \tag{8}$$

expected for a non-thermal Green's function (a dependence from the frequency is present).

On the other hand, one could be not satisfied of a similar classical intuitive explanation to substitute Eq. (6) for Eq. (5). Hence, we further release a rigorous argument. We recall that quasinormal modes are frequencies of the radial spin-j perturbations $\phi$ of the four-dimensional Schwarzschild background which are governed by the following differential equation [8, 9]

$$\left(-\frac{\partial^2}{\partial x^2}+V(x)-\omega^2\right)\phi. \tag{9}$$

This equation is treated as a Schrodinger equation with the Regge-Wheeler potential ($j=2$ for gravitational perturbations) [8, 9]

$$V(x)=V\left[x(r)\right]=\left(1-\frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2}-\frac{6M}{r^3}\right). \tag{10}$$

The Regge-Wheeler "tortoise" coordinate $x$ is related to the radial coordinate $r$ by [8, 9]

$$\begin{aligned} x &= r+2M\ln\left(\tfrac{r}{2M}-1\right)\\ \tfrac{\partial}{\partial x} &= \left(1-\tfrac{2M}{r}\right)\tfrac{\partial}{\partial r}. \end{aligned} \tag{11}$$

By realizing a rigorous analytical calculation, in [8] Motl derived Eq. (5) starting from Eqs. (9) and (10) and satisfying purely outgoing boundary conditions both at the horizon ($r=2M$) and in the asymptotic region ($r=\infty$). But, if we want to take into due account the conservation of energy, we have to substitute the original black hole's mass $M$ in Eqs. (9) and (10) with an *effective mass* of the contracting black hole. In other words, if $M$ is the initial mass of the black hole *before* the emission, and $M-\omega$ is the final mass of the hole *after* the emission [3], Eqs. (2) and (3) enable the introduction of the effective mass

$$M_E\equiv M-\frac{\omega}{2}$$

of the black hole *during* the emission of the particle, i.e. *during* the contraction's phase of the black hole. Notice that the introduced effective mass is a perfect average of the initial and final masses. Then, Eqs. (10) and (11) have to be substituted with the *effective equations*

$$V(x)=V\left[x(r)\right]=\left(1-\frac{2M_E}{r}\right)\left(\frac{l(l+1)}{r^2}-\frac{6M_E}{r^3}\right) \tag{12}$$

and

$$\begin{aligned} x &= r+2M_E\ln\left(\tfrac{r}{2M_E}-1\right)\\ \tfrac{\partial}{\partial x} &= \left(1-\tfrac{2M_E}{r}\right)\tfrac{\partial}{\partial r}. \end{aligned} \tag{13}$$

If one realizes step by step the same rigorous analytical calculation in [8], but starting from Eqs. (9) and (12) and satisfying purely outgoing boundary conditions both at the *effective horizon* ($r=2M_E$) and in the asymptotic region ($r=\infty$), the final result will be, obviously and rigorously, Eq. (6).

Now, we can proceed with our analysis.

Notice that in Eq. (6) the frequency $\omega_n$ is present in both of the left hand side and the right hand side. One could solve this equation and write down an analytic form for $\omega_n$ but we will see that this is not essential for our goals.

In [5] the spectrum of black hole's quasinormal modes has been analysed in terms of superposition of damped oscillations, of the form

$$\exp(-i\omega_I t)[a \sin \omega_R t + b \cos \omega_R t] \tag{14}$$

with a spectrum of complex frequencies $\omega = \omega_R + i\omega_I$. A damped harmonic oscillator $\mu(t)$ is governed by the equation [5]

$$\ddot{\mu} + K\dot{\mu} + \omega_0^2 \mu = F(t), \tag{15}$$

where $K$ is the damping constant, $\omega_0$ the proper frequency of the harmonic oscillator, and $F(t)$ an external force per unit mass. If $F(t) \sim \delta(t)$, i.e. considering the response to a Dirac delta function, the result for $\mu(t)$ is a superposition of a term oscillating as $\exp(i\omega t)$ and of a term oscillating as $\exp(-i\omega t)$, see [5] for details. Then, the behavior (14) is reproduced by a damped harmonic oscillator, through the identifications [5]

$$\tfrac{K}{2} = \omega_I, \qquad \sqrt{\omega_0^2 - \tfrac{K}{4}^2} = \omega_R, \tag{16}$$

which gives

$$\omega_0 = \sqrt{\omega_R^2 + \omega_I^2}. \tag{17}$$

An important point emphasized in [5] is that identification $\omega_0 = \omega_R$ is correct only in the approximation $\frac{K}{2} \ll \omega_0$, i.e. only for very long-lived modes. For a lot of black hole's quasinormal modes, for example for highly excited modes, the opposite limit can be correct. In [5] this observation has been used to re-examine some aspects of quantum physics of black holes that were discussed in previous literature assuming that the relevant frequencies were $(\omega_R)_n$ than $(\omega_0)_n$. Here, we further improve the analysis by taking into account the important issue that the radiation spectrum is not strictly thermal. Let us modify the analysis in [5]. By using the new expression (6) for the frequencies of quasinormal modes, we define

$$m_0 \equiv \tfrac{\ln 3}{4\pi[2M-(\omega_0)_n]}, \qquad p_n \equiv \tfrac{2\pi}{4\pi[2M-(\omega_0)_n]}(n + \tfrac{1}{2}). \tag{18}$$

Then, Eq. (17) can be rewritten in the enlightening form

$$(\omega_0)_n = \sqrt{m_0^2 + p_n^2}. \tag{19}$$

These results improve Eqs. (8) and (9) in [5] as the new expression (6) for the frequencies of quasinormal modes takes into account that the radiation spectrum is not strictly thermal. For highly excited modes

$$(\omega_0)_n \approx p_n = \frac{2\pi}{4\pi[2M - (\omega_0)_n]}(n + \frac{1}{2}). \tag{20}$$

Thus, differently from [5], levels are *not* equally spaced even for highly excited modes. Indeed, there are deviations due to the non-strictly thermal behavior of the spectrum (black hole's effective temperature depends on the energy level).

Using Eq. (18), Eq. (19) can be rewritten like

$$(\omega_0)_n = \frac{1}{4\pi[2M - (\omega_0)_n]}\sqrt{(\ln 3)^2 + 4\pi^2(n+\frac{1}{2})^2}, \tag{21}$$

which is easily solved giving

$$(\omega_0)_n = M \pm \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(n+\frac{1}{2})^2}}. \tag{22}$$

Clearly, only the solution $(\omega_0)_n \ll M$ has physical meaning, i.e. the one with the sign minus in the right hand side

$$(\omega_0)_n = M - \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(n+\frac{1}{2})^2}}. \tag{23}$$

The interpretation is of a particle quantized with anti-periodic boundary conditions on a circle of length

$$L = \frac{1}{T_E(\omega_n)} = 4\pi\left(M + \sqrt{M^2 - \frac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(n+\frac{1}{2})^2}}\right), \tag{24}$$

i.e. the length of the circle depends from the overtone number $n$. In [5] Maggiore found a particle quantized with anti-periodic boundary conditions on a circle of length $L = 8\pi M$. Our correction takes into account the conservation of energy, i.e. the additional term $\frac{\omega}{2M}$ in Eq. (2).

Now, let us see various important consequences of the above approach on the quantum physics of black holes starting by the *area quantization.*

Bekenstein [10] showed that the area quantum of the Schwarzschild black hole is $\triangle A = 8\pi$ (we recall that the *Planck distance* $l_p = 1.616 \times 10^{-33}\ cm$ is equal to one in Planck units). By using properties of the spectrum of Schwarzschild black hole quasinormal modes a different numerical coefficient has been found by Hod in [11]. The analysis in [11] started by the observation that, as for the Schwarzschild black hole the *horizon area* $A$ is related to the mass through the relation $A = 16\pi M^2$, a variation $\triangle M$ in the mass generates a variation

$$\triangle A = 32\pi M \triangle M \tag{25}$$

in the area. By considering a transition from an unexcited black hole to a black hole with very large $n$, Hod assumed *Bohr's correspondence principle* to be valid for large $n$ and enabled a semiclassical description even in absence of a full unitary quantum gravity theory. Thus, from Eq. (5), the minimum quantum which can be absorbed in the transition is $\triangle M = \omega = \frac{\ln 3}{8\pi M}$. This gives

$\triangle A = 4\ln 3$. The presence of the numerical factor $4\ln 3$ stimulated possible connections with loop quantum gravity [12]. By using Eq. (6) than Eq. (5), Hod's result can be improved. We get

$$\triangle M = \omega = \frac{\ln 3}{4\pi(2M - \omega)}, \tag{26}$$

which is easily solved giving

$$\triangle M = M \pm \sqrt{M^2 - \frac{1}{4\pi}\ln 3}. \tag{27}$$

Even in this case, only the solution $\triangle M \ll M$ has physical meaning, i.e. the one with the sign minus in the right hand side,

$$\triangle M = M - \sqrt{M^2 - \frac{1}{4\pi}\ln 3}. \tag{28}$$

Again, the modify takes into account the conservation of energy, i.e. the additional term $\frac{\omega}{2M}$ in Eq. (2) that represents the deviation of the radiation spectrum of a black hole from the strictly thermal feature. By using Eq. (25) we get

$$\triangle A = 32\pi M(M - \sqrt{M^2 - \frac{1}{4\pi}\ln 3}). \tag{29}$$

Criticism on Hod's conjecture were discussed in [5]. The main point is that Bohr's correspondence principle strictly holds only for transitions from $n$ to $n'$ where both $n, n' \gg 1$. Thus, Maggiore [5] suggested that $(\omega_0)_n$ should be used than $(\omega_R)_n$, by obtaining the original Bekenstein's result, i.e. $\triangle A = 8\pi$. In any case, the result in [5] can be improved too, by taking into account the deviation from the strictly thermal feature in Eq. (2), i.e. by using Eq. (6) than Eq. (5). Assuming a transition $n \to n-1$ Eq. (23) gives an absorbed energy

$$\triangle M = (\omega_0)_n - (\omega_0)_{n-1} = f(M, n) \tag{30}$$

where we have defined

$$f(M, n) \equiv$$

$$\equiv \sqrt{M^2 - \tfrac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(n - \tfrac{1}{2})^2}} - \sqrt{M^2 - \tfrac{1}{4\pi}\sqrt{(\ln 3)^2 + 4\pi^2(n + \tfrac{1}{2})^2}}. \tag{31}$$

Therefore

$$\triangle A = 32\pi M \triangle M = 32\pi M \times f(M, n) \tag{32}$$

For very large $n$ one gets

$$f(M,n)\approx$$
$$\approx\sqrt{M^2-\tfrac{1}{2}(n-\tfrac{1}{2})}-\sqrt{M^2-\tfrac{1}{2}(n+\tfrac{1}{2})}\approx\tfrac{1}{4M}, \tag{33}$$

and Eq. (32) becomes $\triangle A\approx 8\pi$ which is the original result of Bekenstein for the area quantization [10]. Then, only in the limit $n\to\infty$ the levels are equally spaced. Indeed, for finite $n$ there are deviations, see Eq. (20).

This analysis will have important consequences on entropy and microstates.

Assuming that, for large $n$, the horizon area is quantized [5] with a quantum $\triangle A=\alpha$, where $\alpha=32\pi M\cdot f(M,n)$ for us, $\alpha=8\pi$ for Bekenstein [10] and Maggiore [5], $\alpha=4\ln 3$ for Hod [11], the total horizon area must be $A=N\triangle A=N\alpha$ (notice that the number of quanta of area, the integer $N$, is *not* the overtone number $n$). Our approach gives:

$$N=\frac{A}{\triangle A}=\frac{16\pi M^2}{\alpha}=\frac{16\pi M^2}{32\pi M\cdot f(M,n)}=\frac{M}{2f(M,n)}. \tag{34}$$

Hawking radiation and black hole's entropy are the two most important predictions of a yet unknown unitary quantum theory of gravity. The famous formula of Bekenstein-Hawking entropy [1, 13, 14] now reads

$$S_{BH}=\frac{A}{4}=8\pi NM\triangle M=8\pi NM\cdot f(M,n), \tag{35}$$

becoming a function of the overtone number $n$.

In the limit $n\to\infty$ $f(M,n)\to\frac{1}{4M}$ and we re-obtain the standard result [5, 15, 16, 17]

$$S_{BH}\to 2\pi N. \tag{36}$$

In any case, it is a general belief that here is no reason to expect that Bekenstein-Hawking entropy to be the whole answer for a correct theory of quantum gravity [18]. In order to have a better understanding of black hole's entropy, it is imperative to go beyond Bekenstein-Hawking entropy and identify the sub-leading corrections [18]. In [19] Zhang used the quantum tunnelling approach in [2, 3] to obtain the sub-leading corrections to the second order approximation. In that approach, the black hole's entropy contains three parts: the usual Bekenstein-Hawking entropy, the logarithmic term and the inverse area term [19]

$$S_{total}=S_{BH}-\ln S_{BH}+\frac{3}{2A}. \tag{37}$$

The logarithmic and inverse area terms are the consequence of requesting to satisfying the unitary quantum gravity theory [19]. Apart from a coefficient, this correction to the black hole's entropy is consistent with the one of loop quantum gravity [19]. In fact, in loop quantum gravity the coefficient of the logarithmic term has been rigorously fixed at $\frac{1}{2}$ [19, 20]. By using the correction (35) to Bekenstein-Hawking entropy Eq. (37) can be rewritten as

$$S_{total} = 8\pi NM \cdot f(M,n) - \ln 8\pi NM \cdot f(M,n) + \frac{3}{64\pi NM \cdot f(M,n)} \tag{38}$$

that in the limit $n \to \infty$ becomes

$$S_{total} \to 2\pi N - \ln 2\pi N + \frac{3}{16\pi N}. \tag{39}$$

Our results imply that at level $N$ the black hole has a number of microstates

$$g(N) \propto \exp\left[8\pi NM \cdot f(M,n) - \ln 8\pi NM \cdot f(M,n) + \frac{3}{64\pi NM \cdot f(M,n)}\right], \tag{40}$$

that in the limit $n \to \infty$ reads

$$g(N) \propto \exp(2\pi N - \ln 2\pi N + \frac{3}{16\pi N}). \tag{41}$$

In summary, in this work the spectrum of black hole's quasinormal modes has been re-analysed by taking into account, through the introduction of an effective temperature, the correction in [2, 3] which shows that the radiation spectrum cannot be strictly thermal. This important issue modifies in an fundamental way the physical interpretation of the black hole's spectrum, enabling a re-examination of various results in the literature. In particular, the formula of the horizon's area quantization and the number of quanta of area result modified becoming functions of the quantum overtone number $n$. Hence, the famous formula of Bekenstein-Hawking entropy its sub-leading corrections and the number of microstates are also modified. Black hole's entropy becomes a function of the quantum overtone number too.

The presented results are fundamental for realizing unitary quantum gravity theory. In fact, Hawking radiation and black hole's entropy are the two fundamental predictions of such a definitive theory and black holes are considered theoretical laboratories for testing models of it. Thus, black hole's quasinormal modes are the best candidates for an interpretation in terms of quantum levels. We emphasize that this is the first time that black hole's entropy has been directly connected with a quantum number.

Notice that previous results in the literature are re-obtained in the limit $n \to \infty$. This point confirms the correctness of the analysis in this work which improves previous approximations.

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