THE DIRAC-HARDY AND DIRAC-SOBOLEV INEQUALITIES IN L^1

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Abstract. Dirac-Sobolev and Dirac-Hardy inequalities in L^1 are established in which the L^p spaces which feature in the classical Sobolev and Hardy inequalities are replaced by weak L^p spaces. Counter examples to the analogues of the classical inequalities are shown to be provided by zero modes for appropriate Pauli operators constructed by Loss and Yau.

Key words: Dirac-Sobolev inequalities, Dirac-Hardy inequalities, zero modes, Sobolev inequalities, Hardy inequalities

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1. Introduction

Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ be the triple of 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.1}$$

and set

$$\mathbf{p} := -i\boldsymbol{\nabla}, \quad \boldsymbol{\sigma} \cdot \mathbf{p} = -i \sum_{j=1}^{3} \sigma_j \frac{\partial}{\partial x_j}.$$

By the Dirac-Sobolev inequality we mean the following: that when $1 \le p < 3$, $p^* = 3p/(3-p)$, and for all $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$, the space of $\mathbb{C}^2$-valued functions whose components lie in $C_0^\infty(\mathbb{R}^3)$,

$$\left(\int_{\mathbb{R}^3} |f(\mathbf{x})|_{p^*}^{p^*} d\mathbf{x} \right)^{1/p^*} \le C(p) \left(\int_{\mathbb{R}^3} |(\boldsymbol{\sigma} \cdot \mathbf{p}) f(\mathbf{x})|_p^p d\mathbf{x} \right)^{1/p} \tag{1.2}$$

where for $\mathbf{a} = (a_1, a_2) \in \mathbb{C}^2$, $|\mathbf{a}|_p^p = |a_1|^p + |a_2|^p$. It is shown by Ichinose and Saitō in [3] (see "Note added in proof" at end of paper) that for $1 < p < \infty$, there are positive constants $c_1(p), c_2(p)$ which are such that

$$c_1(p)\int_{\mathbb{R}^3}|\mathbf{p}f(\mathbf{x})|_p^p d\mathbf{x} \le \int_{\mathbb{R}^3}|(\boldsymbol{\sigma}\cdot\mathbf{p})f(\mathbf{x})|_p^p d\mathbf{x} \le c_2(p)\int_{R^3}|\mathbf{p}f(\mathbf{x})|_p^p d\mathbf{x}, \quad (1.3)$$

and hence for $1<p<3$, (1.2) is a consequence of the Sobolev inequality

$$\left(\int_{\mathbb{R}^3}|f(\mathbf{x})|_{p^*}^{p^*}d\mathbf{x}\right)^{1/p^*} \le \tilde{C}(p)\left(\int_{\mathbb{R}^3}|\mathbf{p}f(\mathbf{x})|_p^p d\mathbf{x}\right)^{1/p}. \quad (1.4)$$

On defining the Dirac-Sobolev space $H_{D,0}^{1,p}(\mathbb{R}^3,\mathbb{C}^2)$ to be the completion of $C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)$ with respect to the norm

$$\|f\|_{D,1,p} := \left\{\int_{\mathbb{R}^3}[|f(\mathbf{x})|_p^p + |(\boldsymbol{\sigma}\cdot\mathbf{p})f(\mathbf{x})|_p^p]d\mathbf{x}\right\}^{1/p},$$

(1.3) proves that $H_{D,0}^{1,p}(\mathbb{R}^3,\mathbb{C}^2)$ is isomorphic to $H_0^{1,p}(\mathbb{R}^3,\mathbb{C}^2)$ if $1<p<\infty$, where $H_0^{1,p}(\mathbb{R}^3,\mathbb{C}^2)$ denotes the Sobolev space defined to be the completion of $C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)$ with respect to the norm

$$\|f\|_{S,1,p} := \left\{\int_{\mathbb{R}^3}[|f(\mathbf{x})|_p^p + |\mathbf{p}f(\mathbf{x})|_p^p]d\mathbf{x}\right\}^{1/p}.$$

However, as $p\to 1$, $c_1(p)\to 0$ and so (1.3) only implies that $H_0^{1,1}(\mathbb{R}^3,\mathbb{C}^2)$ is continuously embedded in $H_{D,0}^{1,1}(\mathbb{R}^3,\mathbb{C}^2)$. In fact Ichinose and Saitō go on to prove that the embedding $H_0^{1,1}(\mathbb{R}^3,\mathbb{C}^2)\hookrightarrow H_{D,0}^{1,1}(\mathbb{R}^3,\mathbb{C}^2)$ is indeed strict. Hence, in the case $p=1$, (1.2) is not a consequence of the analogous Sobolev inequality. We prove that the $p=1$ case of (1.2) is untrue. We demonstrate this with a function used by Loss and Yau in [5] to prove the existence of zero modes of a Pauli operator $\{\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})\}^2$ (or equivalently, of the Weyl-Dirac operator $\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})$) with some appropriate magnetic potential $\mathbf{A}$. A result of Saitō and Umeda in [6] on the growth properties of zero modes of Pauli operators indicates that zero modes have quite generally the properties we need of the counter-example. We prove in Theorem 2.1

$$\|f\|_{3/2,\infty} \le C_1\int_{\mathbb{R}^3}|(\boldsymbol{\sigma}\cdot\mathbf{p})f(\mathbf{x})|d\mathbf{x}, \quad (1.5)$$

where $|\cdot| = |\cdot|_1$ and for any $q>0$,

$$\|f\|_{q,\infty}^q := \sup_{t>0}\ t^q\mu(\{\mathbf{x}\in\mathbb{R}^3 : |f(\mathbf{x})|>t\}), \quad (1.6)$$

μ denoting Lebesgue measure. We recall that $\|f\|_{q,\infty}<\infty$ if and only if f belongs to the weak-L^q space $L^{q,\infty}(\mathbb{R}^3,\mathbb{C}^2)$. Moreover, $\|\cdot\|_{q,\infty}$ is not a norm on $L^{q,\infty}(\mathbb{R}^3,\mathbb{C}^2)$ but for $q>1$ it is equivalent to a norm; see [2], Section 3.4.

Analogous questions arise for the Dirac-Hardy inequality

$$\int_{\mathbb{R}^3} \frac{|f(\mathbf{x})|_p^p}{|\mathbf{x}|^p} d\mathbf{x} \leq C(p) \int_{\mathbb{R}^3} |(\boldsymbol{\sigma}\cdot\mathbf{p})f(\mathbf{x})|_p^p d\mathbf{x} \tag{1.7}$$

and similar answers are obtained. The inequality is true for $1 < p < \infty$ by (1.3), but not for $p = 1$ in which case we prove that

$$\big\| |f|/|\cdot| \big\|_{1,\infty} \leq C_2 \int_{\mathbb{R}^3} |(\boldsymbol{\sigma}\cdot\mathbf{p})f(\mathbf{x})| d\mathbf{x}. \tag{1.8}$$

The plan of the paper is as follows. In Section 2 we shall prove the results concerning the Dirac-Sobolev and Dirac-Hardy inequalities discussed above. We shall give estimates of the optimal constant $C(p)$ in the Dirac-Sobolev inequality (1.2) for $1 < p < 3$ in Section 3 and show that $C(p) \to \infty$ as $p \downarrow 1$. In order to check if the results in Section 2 are dimension related, we investigate higher dimensional analogues in Section 4. A weak Hölder-type inequality is given in an Appendix.

2. The weak Dirac-Sobolev and Dirac-Hardy inequalities

To show that the inequality (1.2) does not hold, we shall prove that a counter-example is provided by a zero mode for an appropriate Pauli (or Weyl-Dirac) operator constructed by Loss-Yau in [5]. This is the $\mathbb{C}^2$-valued function

$$\psi(\mathbf{x}) = \frac{1}{(1+r^2)^{3/2}}(I + i\mathbf{x}\cdot\boldsymbol{\sigma}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r = |\mathbf{x}|, \tag{2.1}$$

where I is the 2×2 identity matrix. In view of the anti-commutation relation $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}I$, it follows that

$$|\psi(\mathbf{x})| = \frac{1}{1+r^2}. \tag{2.2}$$

Also, ψ satisfies the Loss-Yau equation

$$(\boldsymbol{\sigma}\cdot\mathbf{p})\psi(\mathbf{x}) = \frac{3}{1+r^2}\psi(\mathbf{x}). \tag{2.3}$$

Let $\chi_n \in C_0^\infty(\mathbb{R})$ be such that

$$\chi_n(r) = \begin{cases} 1, & r \leq n \\ 0, & r \geq n+2, \end{cases} \quad |\chi_n'(r)| \leq 1. \tag{2.4}$$

Then $\psi_n := \chi_n \psi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$ and we see that

$$\begin{aligned} \|(\boldsymbol{\sigma}\cdot\mathbf{p})\psi_n\|_{L^1(\mathbb{R}^3,\mathbb{C}^2)} &= \|\chi_n(\boldsymbol{\sigma}\cdot\mathbf{p})\psi - i\,\chi_n'(\boldsymbol{\sigma}\cdot\frac{\mathbf{x}}{r})\psi\|_{L^1(\mathbb{R}^3,\mathbb{C}^2)} \\ &\le 4\pi\Big(\int_0^{n+2}\frac{3}{1+r^2}\,dr + \int_n^{n+2}dr\Big) \\ &\le C_0, \end{aligned} \tag{2.5}$$

for some constant positive C_0, independent of n.

Now suppose that the case $p = 1$ of the inequality (1.2) is true. Then it would follow from (2.5) that

$$\begin{aligned} C_0 &\ge \|\psi_n\|_{L^{3/2}(\mathbb{R}^3,\mathbb{C}^2)} \\ &\ge \Big(\int_{|\mathbf{x}|\le n}|\psi(\mathbf{x})|^{3/2}\,d\mathbf{x}\Big)^{2/3} \\ &\ge \text{const.}\,(\log n)^{2/3}. \end{aligned} \tag{2.6}$$

and hence a contradiction.

The properties of the zero mode ψ, defined by (2.1), which lead to the inequality (1.2) being contradicted when $p = 1$ are that $(\boldsymbol{\sigma}\cdot\mathbf{p})\psi \in L^1(\mathbb{R}^3, \mathbb{C}^2)$ and $\psi(\mathbf{x}) \asymp r^{-2}$ at infinity (*i.e.*, $r^2\psi(\mathbf{x})$ goes to a constant vector in $\mathbb{C}^2$ as $r \to \infty$). It was shown in Saitō-Umeda [6] that these two properties are satisfied by the zero modes of any Weyl-Dirac operator

$$\mathbb{D}_A = \boldsymbol{\sigma}\cdot\big(\mathbf{p} + \mathbf{A}(\mathbf{x})\big) \tag{2.7}$$

whose magnetic potential $\mathbf{A} = (A_1, A_2, A_3)$ is such that

$$A_j \ \text{ is measurable}, \quad |A_j(\mathbf{x})| \le C(1+r)^{-\rho}, \quad \rho > 1 \tag{2.8}$$

for $j = 1, 2, 3$.

As was mentioned in the Introduction, what is true is the following

Theorem 2.1. *There exists a positive constant C_1 such that*

$$\|f\|_{L^{3/2,\infty}(\mathbb{R}^3,\mathbb{C}^2)} \le C_1\|(\sigma\cdot\mathbf{p})f\|_{L^1(\mathbb{R}^3,\mathbb{C}^2)} \tag{2.9}$$

for all $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$.

Proof. Let $g = (\sigma\cdot\mathbf{p})f$. Since $(\boldsymbol{\sigma}\cdot\mathbf{p})^2 = -\Delta$ and the fundamental solution of $-\Delta$ in $\mathbb{R}^3$ is convolution with $(1/4\pi|\cdot|)$, it follows that $(\boldsymbol{\sigma}\cdot\mathbf{p})$ has fundamental solution with kernel $(\boldsymbol{\sigma}\cdot\mathbf{p})(1/4\pi|\cdot|)$ and hence that

$$\begin{aligned} f(\mathbf{x}) &= \frac{-i}{4\pi}\int_{\mathbb{R}^3}[(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})|\mathbf{x}-\mathbf{y}|^{-1}]g(\mathbf{y})\,d\mathbf{y} \\ &= \frac{i}{4\pi}\int_{\mathbb{R}^3}\frac{\boldsymbol{\sigma}\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}g(\mathbf{y})d\mathbf{y}. \end{aligned} \tag{2.10}$$

Note that this also follows from the more general result in Saitō-Umeda [7, Theorem 4.2]. Consequently

$$|f(\mathbf{x})| \le \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|\mathbf{x}-\mathbf{y}|^2}|g(\mathbf{y})|\,d\mathbf{y} =: \frac{1}{4\pi}I_1(|g|)(\mathbf{x}), \tag{2.11}$$

where $I_1(|g|)$ is the 3-dimensional Riesz potential of $|g|$; see Edmunds and Evans [2, Section 3.5] for the terminology and properties we need. In view of [2, Remark 3.5.7(i)], we see that the Riesz potential I_1 is of weak type $(1, 3/2; 3, \infty)$. In particular, I_1 is of weak type $(1, 3/2)$ (cf. [2, Theorem 3.5.13], [8, Theorem 1, pp.119 - 120]), which means that there exists a positive constant C such that for all $u \in L^1(\mathbb{R}^3)$

$$\|I_1(u)\|_{L^{3/2,\infty}(\mathbb{R}^3)} \le C\|u\|_{L^1(\mathbb{R}^3)}. \tag{2.12}$$

The inequality (2.9) follows. □

It is evident that the two properties of the zero mode ψ defined by (2.1) also leads to a contradiction of the inequality (1.7). What is now true is the following:

Theorem 2.2. *For all $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$*

$$\|f/|\cdot|\|_{L^{1,\infty}(\mathbb{R}^3,\mathbb{C}^2)} \le C_2\|(\boldsymbol{\sigma}\cdot\mathbf{p})f\|_{L^1(\mathbb{R}^3,\mathbb{C}^2)}, \tag{2.13}$$

where $C_2 \le (9\pi)^{1/3}C_1$ and C_1 is the optimal constant in (2.9).

Proof. On applying the weak Hölder inequality in the Appendix with $p = 3/2$ and $q = 3$, and noting that $\|1/|\cdot|\|_{3,\infty} = (4\pi/3)^{1/3}$, we get

$$\|f/|\cdot|\|_{1,\infty} \le 3^{2/3}\pi^{1/3}\|f\|_{3/2,\infty}. \tag{2.14}$$

Hence the theorem follows from (2.9). □

3. Estimate of the optimal constants

In this section, we estimate the optimal constant $C(p)$ in the inequality (1.2) for $1 < p < 3$, and show that $C(p) \to \infty$ as $p \downarrow 1$.

Let ψ be the Loss-Yau zero mode defined by (2.1). It does not lie in $C_0^\infty(\mathbb{R}^3,\mathbb{C}^2)$ but is in $H^{1,p}_{D,0}(\mathbb{R}^3,\mathbb{C}^2)$. Hence the optimal constant $C(p)$ must satisfy the inequality

$$C(p) \geq \|\psi\|_{L^{p^*}(\mathbb{R}^3,\mathbb{C}^2)} \big/ \|(\boldsymbol{\sigma}\cdot\mathbf{p})\psi\|_{L^p(\mathbb{R}^3,\mathbb{C}^2)}, \tag{3.1}$$

where $p^* = 3p/(3-p)$. On passing to polar coordinates, we have

$$\begin{aligned} \|\psi\|_{p^*}^{p^*} &= 4\pi \int_0^\infty (1+r^2)^{-p^*} r^2 dr \\ &\geq 4\pi \Big\{ \int_0^1 2^{-p^*} r^2\, dr + \int_1^\infty (2r^2)^{-p^*} r^2 dr \Big\} \\ &= 4\pi\, 2^{-p^*}\, 3^{-2} \frac{2p}{p-1}. \end{aligned} \tag{3.2}$$

On the other hand, by (2.3), we see that

$$\begin{aligned} \|(\boldsymbol{\sigma}\cdot\mathbf{p})\psi\|_p^p &= \int_{\mathbb{R}^3} \frac{3^p}{(1+r^2)^{2p}}\, dx \\ &= 4\pi\, 3^p \int_0^\infty (1+r^2)^{-2p} r^2 dr \\ &\leq 4\pi\, 3^p \Big\{ \int_0^1 r^2\, dr + \int_1^\infty r^{-4p+2} dr \Big\} \\ &= \pi\, 2^4 3^{p-1} \frac{p}{4p-3}. \end{aligned} \tag{3.3}$$

Combining (3.1) with (3.2) and (3.3), we obtain

$$C(p) \geq \pi^{-1/3}\, 2^{-2-1/p}\, 3^{-1/3-1/p}\, \frac{p^{-1/3}(4p-3)^{1/p}}{(p-1)^{1/p-1/3}}. \tag{3.4}$$

It is evident that the right hand side of (3.4) goes to ∞ as $p \downarrow 1$.

We recall that for $p>1$, the optimal constant $\tilde{C}(p)$ in the Sobolev inequality (1.4) is

$$\tilde{C}(p) = \pi^{-1/2} 3^{-1/p} \left(\frac{p-1}{3-p} \right)^{(p-1)/p} \left\{ \frac{\Gamma(5/2)\Gamma(3)}{\Gamma(3/p)\Gamma(4-3/p)} \right\}^{1/3}$$

which tends to $\tilde{C}(1)$, the optimal constant in the case $p=1$, as $p \to 1$.

4. The weak Dirac-Sobolev and weak Dirac-Hardy inequalities in m dimensions

Let $\gamma_1, \cdots, \gamma_m$ be Hermitian $\ell \times \ell$ matrices satisfying the anti-commutation relations

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk} I, \tag{4.1}$$

where I denotes the $\ell \times \ell$ identity matrix. For example, we can take $\ell = 2^{m-2}$ and construct the matrices by the following iterative procedure. To indicate the dependence on m, write the matrices as $\gamma_1^{(m)}, \cdots, \gamma_m^{(m)}$. For $m = 3$, $\ell = 2$ and they are given by the Pauli matrices in (1.1). Given matrices $\gamma_1^{(m)}, \cdots$, $\gamma_m^{(m)}$ we define

$$\gamma_j^{(m+1)} = \begin{pmatrix} 0 & \gamma_j^{(m)} \\ \gamma_j^{(m)} & 0 \end{pmatrix}, \ j = 1, \cdots, m, \quad \gamma_{m+1}^{(m+1)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{4.2}$$

The m-dimensional analogue of the inequality (1.2) for $p = 1$ is

$$\left(\int_{\mathbb{R}^m} |f(\mathbf{x})|^{m/(m-1)} d\mathbf{x} \right)^{(m-1)/m} \leq C \int_{\mathbb{R}^m} |(\boldsymbol{\gamma} \cdot \mathbf{p}) f(\mathbf{x})| d\mathbf{x} \tag{4.3}$$

for $f \in C_0^\infty(\mathbb{R}^m, \mathbb{C}^\ell)$, where

$$\boldsymbol{\gamma} \cdot \mathbf{p} = -i \sum_{j=1}^m \gamma_j \frac{\partial}{\partial x_j}, \quad \mathbf{p} = -i \boldsymbol{\nabla}.$$

To show that (4.3) does not hold we introduce an m-dimensional analogue of the Loss-Yau zero mode, namely

$$\psi(\mathbf{x}) = \frac{1}{(1+r^2)^{m/2}} (I + i \mathbf{x} \cdot \boldsymbol{\gamma}) \phi_0, \quad r = |\mathbf{x}|, \tag{4.4}$$

where $\phi_0 = {}^t(1, 0, \cdots, 0) \in \mathbb{C}^\ell$. It follows from the anti-commutation relations (4.1) that

$$|\psi(\mathbf{x})| = \frac{1}{(1+r^2)^{(m-1)/2}}, \tag{4.5}$$

and that ψ satisfies the m-dimensional analogue of the Loss-Yau equation (2.3), namely,

$$(\boldsymbol{\gamma} \cdot \mathbf{p}) \psi(\mathbf{x}) = \frac{m}{1+r^2} \psi(\mathbf{x}). \tag{4.6}$$

Let $\chi_n \in C_0^\infty(\mathbb{R})$ be the same function as in (2.4), and put $\psi_n := \chi_n \psi \in C_0^\infty(\mathbb{R}^m, \mathbb{C}^\ell)$. As in to (2.5), we see that

$$\begin{aligned} \|(\boldsymbol{\gamma} \cdot \mathbf{p}) \psi_n\|_{L^1(\mathbb{R}^m, \mathbb{C}^\ell)} &= \|\chi_n (\boldsymbol{\gamma} \cdot \mathbf{p}) \psi - i \, \chi_n' (\boldsymbol{\gamma} \cdot \frac{\mathbf{x}}{r}) \psi\|_{L^1(\mathbb{R}^m, \mathbb{C}^\ell)} \\ &\leq S_m \Big(\int_0^{n+2} \frac{m}{1+r^2} \, dr + \int_n^{n+2} \, dr \Big) \\ &\leq C_0, \end{aligned} \tag{4.7}$$

for some positive constant C_0, independent of n. Here S_m is the surface area of the unit sphere in $\mathbb{R}^m$. If the inequality (4.3) is true then it would follow from (4.3) and (4.7) that

$$\begin{aligned} C_0 &\geq \|\psi_n\|_{L^{m/(m-1)}(\mathbb{R}^m,\mathbb{C}^\ell)} \\ &\geq \Big(\int_{|\mathbf{x}|\leq n} |\psi(\mathbf{x})|^{m/(m-1)}\, d\mathbf{x} \Big)^{(m-1)/m} \\ &\geq \text{const.}\, (\log n)^{(m-1)/m}. \end{aligned} \tag{4.8}$$

which is a contradiction. Therefore the inequality (4.3) does not hold. Instead, what is true is the following inequality.

Theorem 4.1. *There exists a positive constant $C_{1,m}$ such that*

$$\|f\|_{L^{m/(m-1),\infty}(\mathbb{R}^m,\mathbb{C}^\ell)} \leq C_{1,m}\, \|(\boldsymbol{\gamma}\cdot\mathbf{p})f\|_{L^1(\mathbb{R}^m,\mathbb{C}^\ell)} \tag{4.9}$$

for all $f \in C_0^\infty(\mathbb{R}^m, \mathbb{C}^\ell)$.

Proof. Let $f \in C_0^\infty(\mathbb{R}^m, \mathbb{C}^\ell)$, and define $g = (\boldsymbol{\gamma}\cdot\mathbf{p})f$. Since $(\boldsymbol{\gamma}\cdot\mathbf{p})^2 = -\Delta I$, we have that $(-\Delta)f = (\boldsymbol{\gamma}\cdot\mathbf{p})g$. By Stein [8, p.118, (7)],

$$J_2(-\Delta)u = u, \quad u \in C_0^\infty(\mathbb{R}^m, \mathbb{C}), \tag{4.10}$$

where

$$J_2(u) = \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} I_2(u), \quad I_2(u)(\mathbf{x}) = \int_{\mathbb{R}^m} \frac{1}{|\mathbf{x}-\mathbf{y}|^{m-2}}\, u(\mathbf{y})\, d\mathbf{y}. \tag{4.11}$$

It follows that

$$\begin{aligned} f(\mathbf{x}) &= J_2(-\Delta)f(\mathbf{x}) \\ &= \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} \int_{\mathbb{R}^m} \frac{1}{|\mathbf{x}-\mathbf{y}|^{m-2}}\, (\boldsymbol{\gamma}\cdot\mathbf{p})g(\mathbf{y})\, d\mathbf{y}. \end{aligned} \tag{4.12}$$

On integration by parts, this yields

$$f(\mathbf{x}) = \frac{\Gamma((m-2)/2)}{4\,\pi^{m/2}} \int_{\mathbb{R}^m} \frac{i\,\boldsymbol{\gamma}\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^m}\, g(\mathbf{y})\, d\mathbf{y}. \tag{4.13}$$

Then it follows that

$$\begin{aligned} |f(\mathbf{x})| &\leq \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} \int_{\mathbb{R}^m} \frac{1}{|\mathbf{x}-\mathbf{y}|^{m-1}} |g(\mathbf{y})|\, d\mathbf{y} \\ &= \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} \frac{2\pi^{(m+2)/2}}{\Gamma((m-1)/2)}\, I_1(|g|)(\mathbf{x}). \end{aligned} \tag{4.14}$$

Here $I_1(|g|)$ is the m-dimensional Riesz potential of $|g|$; see [2, Section 3.5]. In view of [2, Remark 3.5.7(i)], we see that the Riesz potential I_1 is of weak type $(1,\, m/(m-1); m,\, \infty)$, in particular, of weak type $(1,\, m/(m-1))$ (cf. [2, Theorem 3.5.13], [8, Theorem 1, pp.119 - 120]), which means that there exists a positive constant C such that for all $u \in L^1(\mathbb{R}^m)$

$$\|I_1(u)\|_{L^{m/(m-1),\infty}(\mathbb{R}^m)} \le C\|u\|_{L^1(\mathbb{R}^m)}. \tag{4.15}$$

The inequality (4.9) follows. □

The m-dimensional Hardy inequality for L^1 is

$$\int_{\mathbb{R}^m} \frac{|u(\mathbf{x})|}{|\mathbf{x}|} d\mathbf{x} \le (m-1)^{-1} \int_{\mathbb{R}^m} |\mathbf{p}\ u(\mathbf{x})| d\mathbf{x}, \quad u \in C_0^\infty(\mathbb{R}^m). \tag{4.16}$$

It can be proved as in the 3-dimensional case that its natural analogue

$$\int_{\mathbb{R}^m} \frac{|f(\mathbf{x})|}{|\mathbf{x}|} d\mathbf{x} \le C \int_{\mathbb{R}^m} |(\boldsymbol{\gamma}\cdot\mathbf{p}) f(\mathbf{x})| d\mathbf{x}, \quad f \in C_0^\infty(\mathbb{R}^m, \mathbb{C}^\ell) \tag{4.17}$$

is not true.

Theorem 4.2. *For all* $f \in C_0^\infty(\mathbb{R}^m, \mathbb{C}^\ell)$

$$\|f/|\cdot|\|_{L^{1,\infty}(\mathbb{R}^m,\mathbb{C}^\ell)} \le C_{2,m}\|(\boldsymbol{\gamma}\cdot\mathbf{p})f\|_{L^1(\mathbb{R}^m,\mathbb{C}^\ell)}, \tag{4.18}$$

where

$$C_{2,m} \le C_{1,m} \frac{\pi^{1/2}\, m}{\Gamma\big((m+2)/2\big)^{1/m}(m-1)^{1-1/m}},$$

and $C_{1,m}$ *is the optimal constant in Theorem* 4.1.

Proof. It is easy to see that $\|1/|\cdot|\|_{m,\infty} = (\omega_m)^{1/m}$, where ω_m denotes the volume of the m-dimensional unit ball, and is given by

$$\omega_m = \frac{\pi^{m/2}}{\Gamma((m+2)/2)}.$$

On applying the weak Hölder inequality in the Appendix with $p = m/(m-1)$ and $q = m$, we get

$$\|f/|\cdot|\|_{1,\infty} \le \big((m-1)^{1/m} + (m-1)^{-(m-1)/m}\big)\, \omega_m^{1/m}\|f\|_{m/(m-1),\infty}. \tag{4.19}$$

The theorem follows on combining this inequality with (4.9). □

5. Appendix

The proofs of Theorems 2.2 and 4.2 are consequences of the following Hölder - type inequality in weak L^p spaces, which we have been unable to find in the literature.

Theorem 5.1 (**Weak Hölder inequality**). *Let $p > 1$, $q > 1$ and $p^{-1}+q^{-1} = 1$. If $f \in L^{p,\infty}(\mathbb{R}^d)$ and $g \in L^{q,\infty}(\mathbb{R}^d)$, then $fg \in L^{1,\infty}$ and*

$$\|fg\|_{1,\infty} \leq \Big((q/p)^{1/q} + (p/q)^{1/p}\Big)\|f\|_{p,\infty}\|g\|_{q,\infty}. \tag{5.1}$$

Proof. Let $\varepsilon > 0$ be arbitrary, and set

$$\begin{aligned}
A &= \{\mathbf{x} \in \mathbb{R}^d \,:\, \varepsilon|f(\mathbf{x})| > t^{1/p}\,\} \\
B &= \{\mathbf{x} \in \mathbb{R}^d \,:\, \frac{1}{\varepsilon}|g(\mathbf{x})| > t^{1/q}\,\} \\
E &= \{\mathbf{x} \in \mathbb{R}^d \,:\, |f(\mathbf{x})g(\mathbf{x})| > t\,\}.
\end{aligned}$$

Since

$$|f(\mathbf{x})g(\mathbf{x})| \leq p^{-1}\big(\varepsilon|f(\mathbf{x})|\big)^p + q^{-1}\big(\frac{1}{\varepsilon}|g(\mathbf{x})|\big)^q, \tag{5.2}$$

we have

$$E \subset A \cup B, \tag{5.3}$$

which implies that

$$t\mu(E) \leq t\mu\big(\{\mathbf{x} \,:\, \varepsilon|f(\mathbf{x})| > t^{1/p}\,\}\big) + t\mu\big(\{\mathbf{x} \,:\, \frac{1}{\varepsilon}|g(\mathbf{x})| > t^{1/q}\,\}\big). \tag{5.4}$$

With

$$s := \frac{t^{1/p}}{\varepsilon}, \quad r := \varepsilon t^{1/q}, \tag{5.5}$$

it follows from (5.4) that

$$\begin{aligned}
&t\mu\big(\{\mathbf{x} \,:\, |f(\mathbf{x})g(\mathbf{x})| > t\,\}\big) \\
&\quad \leq \varepsilon^p s^p \mu\big(\{\mathbf{x} \,:\, |f(\mathbf{x})| > s\,\}\big) + \varepsilon^{-q} r^q \mu\big(\{\mathbf{x} \,:\, |g(\mathbf{x})| > r\,\}\big) \\
&\quad \leq \varepsilon^p \|f\|_{p,\infty}^p + \varepsilon^{-q}\|g\|_{q,\infty}^q.
\end{aligned} \tag{5.6}$$

The minimum value of this is the expression on the right-hand side of (5.1), this being attained when $\varepsilon = (q\|g\|_{q,\infty}^q / p\|f\|_{p,\infty}^p)^{1/pq}$. □

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Note added in proof. We are grateful to the referee for the comment that the first inequality in (1.3) is not a direct consequence of Ichinose and Saitō [3, Theorem 1.3(ii)], but can be established as follows. Since $(\sigma \cdot \mathbf{p})^2 = -\Delta$, one has

$$-i\partial_j(\sigma \cdot \mathbf{p})^{-1} = \{-i\partial_j/\sqrt{-\Delta}\}\{(\sigma \cdot \mathbf{p})/\sqrt{-\Delta}\} = \sum_{k=1}^{3} \sigma_k R_j R_k$$

where $R_j = -i\partial_j/\sqrt{-\Delta}$, $j = 1, 2, 3$, are the Riesz transforms. Since R_j is a pseudo-differential operator with simbol $\xi_j/|\xi|$, it is bounded on L^p by the Calderón-Zygmund theorem (see [8]). The first inequality in (1.3) therefore follows.

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