SINGULAR MIURA TYPE INITIAL PROFILES FOR THE KDV EQUATION

SERGEI GRUDSKY AND ALEXEI RYBKIN

Abstract. We show that the KdV flow evolves any real singular initial profile q of the form $q = r' + r^2$, where $r \in L^2_{loc}$, $r|_{\mathbb{R}_+} = 0$ into a meromorphic function with no real poles.

1. Introduction

This note is closely related to the recent paper [10] by Kappeler et al and [14] by one of the authors.

More specifically, we are concerned with well-posedness (WP) of the initial value problem (IVP) for the Korteweg-De Vries (KdV) equation ($x \in \mathbb{R}, t \geq 0$)

$$\begin{cases} \partial_t u - 6u\partial_x u + \partial_x^3 u = 0 \\ u(x,0) = q(x) \end{cases} \tag{1.1}$$

with certain low regularity non-decaying initial profiles q.

The problem of WP of (1.1) was raised back in the late 60's at about the same time as the inverse scattering formalism for (1.1) was discovered and has drawn an enormous attention. We are not in a position to go over the extensive literature on the subject and refer to the book [16] by Tao where further literature is given.

The problem, of course, gets more difficult once we impose less regularity on the initial data in (1.1). Delta function type q's in (1.1) were rigorously treated by Kappeler in [8]. In the present century, a large amount of effort has been put into WP in the Sobolev space H^{-s} with negative index[1]. The sharpest result is $s = 3/4$ and due to Guo [7] which, in turn, sharpens the result by Colliander et al [3]. The space $H^{-3/4}$ includes such singular functions as δ, $1/x$, etc. However, the harmonic analytical methods employed in above papers break down on $s = -1$. On the other hand, the Schrödinger operator

$$L_q = -\partial_x^2 + q(x)$$

in the Lax pair associated with (1.1) is well-defined for $q \in H^{-1}$ (see e.g. [15]) suggesting that the global WP could be pushed to H^{-1}. It is exactly how Kappeler-Topalov [9] were able to extend WP to $H^{-1}(\mathbb{T})$ for periodic q's. It is natural to conjecture that the global WP for (1.1) also holds and could be achieved by a suitable extension of the inverse scattering transform (IST) method for L_q with

Date: August, 2011.

1991 *Mathematics Subject Classification.* 34B20,37K15, 47B35.

Key words and phrases. KdV equation, Titchmarsh-Weyl m-function, Hankel operators, Miura transformation.

Based on research supported in part by the NSF under grant DMS 1009673.

[1] H^{-s} is the space of distributions subject to $(1 + |x|)^{-s} \widehat{f}(x) \in L^2$.

$q \in H^{-1}$. An important step in this direction was done by Kappeler et al [10] where it was shown that (1.1) is globally well-posed in a certain sense if $q = r' + r^2$ with some $r \in L^2$. The transform

$$B(r) = r' + r^2 \quad , \quad r \in L^2_{loc}$$

is called Miura. Of course, $B(L^2)$ doesn't exhaust H^{-1} as H^{-1} consists of all functions $f = r' + p$ with some $r, p \in L^2$. It is easy to see that L_q is a positive operator for any $q = r' + r^2$ (we call such q a Miura potential) meaning that (1.1) admits so-called 'dispersive' solutions, i.e. solutions which disperse in time and do not have a soliton component. However, since $H^{-1} \supset H^{-3/4}$, singularity of such solutions is pushed all the way to[2] $s = -1$.

We note that all functions in $H^{-s}(\mathbb{R})$ exhibit certain decay at $\pm\infty$. On the other hand, there has been a significant interest in non-decaying solutions to (1.1) (other than periodic). The case of the so-called steplike initial profiles (i.e. when $q(x) \to 0$ sufficiently fast as $x \to +\infty$ $(-\infty)$ and $q(x)$ doesn't decay at $-\infty$ $(+\infty)$). is of physical interest and has attracted much attention since the early 70s. We refer to the recent paper [5] by Egorova-Grunert-Teschl for a comprehensive account of the (rigorous) literature on steplike initial profiles with specified behavior at infinity (e.g. q's tending to a constant, periodic function, etc.). In the recent preprint of one of the authors [14] (see also [13]), the case of q's rapidly decaying at $+\infty$ and sufficiently arbitrary at $-\infty$ is studied in great detail. Initial steplike profiles in these papers are at least locally integrable (i.e. regular).

The current note is concerned with treating Miura steplike initial data q in (1.1). Namely, we consider $q = r' + r^2$ for $r \in L^2_{loc}$ identically (for simplicity) vanishing on $(0, \infty)$. Even though q has very low regularity and is essentially arbitrary on $(-\infty, 0)$ the fact that q is zero on $(0, \infty)$ leads to an extremely strong smoothing effect. Dispersion instantaneously turns such initial profiles $q(x)$ into a function $u(x, t)$ meromorphic in x on the whole complex plane for any $t > 0$. The WP of the problem (1.1) can therefore be understood in a classical sense and moreover it comes with an explicit formula

$$u(x, t) = -2\partial_x^2 \log \det (1 + \mathbb{H}_{x,t}),$$

where $\mathbb{H}_{x,t}$ is the Hankel operator with symbol

$$\varphi_{x,t}(\lambda) = \frac{i\lambda - m(\lambda^2)}{i\lambda + m(\lambda^2)} e^{2i\lambda(4\lambda^2 t + x)}, \quad \lambda \in \mathbb{R}\,,\ x \in \mathbb{R}\,,\ t > 0,$$

where m is the Titchmarsh-Weyl m-function associated with L_q on $(-\infty, 0)$ with a Dirichlet boundary condition at 0.

The WP of our problem, among others, means that $u(x, t)$ has no real poles for any $t > 0$. I.e. no positon solution may occur in our situation.

Our approach is based on a suitable adaptation of the IST and analysis of Hankel operators with oscillatory symbols. To keep our note as short as possible, we will omit some technical issues and come back to them elsewhere in a more suitable setting.

The paper is organized as follows. In Section 2 we review Hankel operators and prove a new result related to a Hankel operator with a cubic oscillatory symbol. In Section 3 we discuss the Titchmarsh-Weyl m-function and reflection coefficient in

[2] As indicated in [15], L_q with $q \in H^{-s}$ for $s > 1$ is ill-defined.

the context of singular points. In the last Section 4 we state and prove our main result.

2. Hankel operators

Hankel operators naturally appear in linear algebra, operator theory, complex analysis, mathematical physics, and many other areas. In our note they play a crucial role. However, their formal definitions vary. In the context of integral operators, a Hankel operator is usually defined as an integral operator on $L^2_+ := L^2(\mathbb{R}_+)$ whose kernel depends on the sum of the arguments. I.e.

$$(\mathbb{H}f)(x) = \int_0^\infty H(x+y)f(y)dy \quad , \quad x \ge 0 \, , \; f \in L^2_+ \, , \tag{2.1}$$

with some function H.

In many situations, including ours, H is not a function but rather a distribution. It is convenient then to accept a regularized version of (2.1).

Let[3]

$$(\mathcal{F}f)(\lambda) = \frac{1}{\sqrt{2\pi}} \int e^{i\lambda x} f(x)dx$$

be the Fourier transform and χ the Heaviside function of $\mathbb{R}_+ := (0, \infty)$.

Definition 2.1. *Given $\varphi \in L^\infty$, we call the operator $\mathbb{H}_\varphi$ on L^2_+ defined for any $f \in L^2_+$ by*

$$\mathbb{H}_\varphi f = \chi \mathcal{F} \varphi \mathcal{F} f \tag{2.2}$$

the Hankel operator on L^2_+ with symbol φ.

It follows from a straightforward computation that (2.1) and (2.2) agree if $\varphi \in L^2 \cap L^\infty$ and $H = \mathcal{F}\varphi$. However if φ is merely L^∞ then $\mathcal{F}\varphi$ is not a function but a (tempered) distribution. The operator $\mathbb{H}$ given by (2.1) is no longer well-defined. But the one given by (2.2) is.

The Hankel operator $\mathbb{H}_\varphi$ is clearly bounded from (2.2). One immediately has

$$\|\mathbb{H}_\varphi\| \le \|\varphi\|_\infty \, . \tag{2.3}$$

Membership of $\mathbb{H}_\varphi$ in narrower Schatten-Von Neumann ideals is, however, a much more subtle issue which was completely resolved by Peller in about 1980 (see e.g. [12]).

We will be particularly concerned with the invertibility of $1 + \mathbb{H}_\varphi$. The first fact is trivial.

Lemma 2.2. *Let φ be such that $|\varphi(\lambda)| \le 1$ a.e. $\lambda \in \mathbb{R}$ and $|\varphi(\lambda)| < 1$ a.e. on a set S of positive Lebesgue measure. Then -1 is not an eignevalue of $\mathbb{H}_\varphi$.*

Proof. Assume -1 is an eigenvalue of $\mathbb{H}_\varphi$ and $f \ne 0$ is the corresponding normalized eigenvector (i.e. $\|f\|_{L^2_+} = 1$).

It follows from

$$f + \mathbb{H}_\varphi f = 0$$

that

$$1 + \int \varphi(\lambda) \widehat{f}(\lambda) \widehat{\overline{f}}(\lambda) d\lambda = 0$$

[3] For brevity we set $\int := \int_{-\infty}^{\infty}$

and hence

$$1+\operatorname{Re}\int \varphi(\lambda)\widehat{f}(\lambda)\overline{\widehat{f}}(\lambda)d\lambda = 0. \tag{2.4}$$

But

$$\begin{aligned}\operatorname{Re}\int \varphi(\lambda)\widehat{f}(\lambda)d\lambda &\leq \int \left|\varphi(\lambda)\widehat{f}(\lambda)\right|\cdot\left|\overline{\widehat{f}}(\lambda)\right|d\lambda \\ &\leq \left\|\varphi\widehat{f}\right\|_{L^2}\cdot\left\|\overline{\widehat{f}}\right\|_{L^2} = \left\|\varphi\widehat{f}\right\|_{L^2} \\ &\leq \int_S |\varphi(\lambda)|^2\cdot\left|\widehat{f}(\lambda)\right|^2 d\lambda \\ &< \int_S \left|\widehat{f}(\lambda)\right|^2 \leq 1. \end{aligned}\tag{2.5}$$

Comparing (2.4) and (2.5) leads to a contradiction. $\square$

The proof of Lemma 2.2 is no longer valid if $|\varphi(\lambda)| = 1$ for a.e. real λ.
However in our setting symbols φ have a very specific structure

$$\varphi(\lambda) = e^{i\lambda(\lambda^2+a)}I(\lambda) \tag{2.6}$$

where a is a real number and I is an inner function of the upper half plane (i.e.[4] $I \in H^\infty_+$ and $|I(\lambda)| = 1$ a.e. $\lambda \in \mathbb{R}$).

Lemma 2.3. *Let φ be given by* (2.6). *Then*

(1) $\mathbb{H}_\varphi$ *is a compact operator,*
(2) $1+\mathbb{H}_\varphi$ *is invertible.*

Proof. Our argument is based upon the factorization (see [1], [4] Section 5.10, [6])

$$e^{i\lambda(\lambda^2+a)} = B(\lambda)U(\lambda) \quad , \quad \lambda\in\mathbb{R}, \tag{2.7}$$

where $B(\lambda)$ is a Blaschke product with infinitely many zeros accumulating at infinity and U is a unimodular function from $C(\overline{\mathbb{R}})$, the class of continuous on $\mathbb{R}$ functions f subject to

$$\lim_{\lambda\to-\infty} f(\lambda) = \lim_{\lambda\to\infty} f(\lambda) \neq \pm\infty.$$

Since a product of an inner function and a $C(\overline{\mathbb{R}})$-function is in the algebra $H^\infty_+ + C(\overline{\mathbb{R}})$, by the Hartman theorem [11] $\mathbb{H}_\varphi$ is compact and (1) is proven.

Consider the Hankel operator (2.2) in the Fourier representation. Denoting $P_\pm$ the Riesz projection in L^2 onto $H^2_\pm$, we have

$$\begin{aligned}\mathcal{F}\mathbb{H}_\varphi\mathcal{F}^{-1} &= \mathcal{F}\chi\mathcal{F}\varphi\mathcal{F}\mathcal{F}^{-1} \\ &= P_+\mathcal{F}\mathcal{F}\varphi = P_+\mathcal{F}^2\varphi \\ &= P_+J\varphi = JP_-\varphi\end{aligned}$$

where $Jf(x) = f(-x)$. Thus, the operator

$$JP_-\varphi : H^2_+ \to H^2_+ \tag{2.8}$$

is unitarily equivalent to $\mathbb{H}_\varphi$. Let $\mathbb{T}_\varphi$ be the Toeplitz operator on H^2_+. I.e.

$$\mathbb{T}_\varphi f = P_+\varphi f, \quad f\in H^2_+.$$

[4] $H^p_\pm$ $(0 < p \leq \infty)$ are standard Hardy spaces of the upper (lower) half planes $\mathbb{C}^\pm$.

Note ([2] Ch. 2) that (2.7) implies left-invertibility of the operator $\mathbb{T}_\varphi$ and, by the Devinatz-Widom theorem ([2] p. 59), there exists a function $f \in H^\infty_+$, such that

$$\|\varphi - f\|_{L^\infty} < 1.$$

Thus, it immediately follows from the representation (2.8) that

$$\mathbb{H}_\varphi = \mathbb{H}_{\varphi - f}$$

and hence (2.3) and (2.7)

$$\|H_\varphi\| = \|\mathbb{H}_{\varphi-f}\| \leq \|\varphi - f\|_{L^\infty} < 1.$$

This proves (2) and the lemma is proven. $\square$

3. The Titchmarsh-Weyl m-function and the reflection coefficient

Denote $H^{-1}_{loc} := H^{-1}_{loc}(\mathbb{R})$ the local H^{-1} space (i.e. the set of all functions $\widetilde{\chi}_S f$, where $f \in H^{-1}$ and $\widetilde{\chi}_S$ is a smoothened characteristic function of a compact set $\mathbb{R}$). It is well-known that any $q \in H^{-1}_{loc}(\mathbb{R})$ can be represented as $q = Q'$ with some $Q \in L^2_{loc}$ and we rewrite

$$-y'' + qy = zy$$

as

$$-(y' - Qy)' - Qy' = zy \tag{3.1}$$

(the regularized Schrödinger equation).

Following the approach of [15] we introduce

$$\begin{cases} L_q := -\partial_x(\partial_x - Q) - Q\partial_x \\ \partial_x Q = q \end{cases} \tag{3.2}$$

the Schrödinger operator with a (singular) potential $q \in H^{-1}_{loc}(\mathbb{R})$.

As proven in [15], the operator (3.2) is well-defined. One can also extend the classical Titchmarsh-Weyl theory to L_q. In particular, the Weyl limit point/circle classification can be easily extended to singular q's. We plan to provide the details elsewhere and only mention here that regular derivatives ∂_x in classical Titchmarsh-Weyl theory should, where appropriate, be replaced by "quasi" derivative $\partial_x - Q$. Note that this doesn't change the Wronskian as

$$\det\begin{pmatrix} y_1 & y_2 \\ y_1' - Qy_1 & y_2' - Qy_2 \end{pmatrix} = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

if y_1, y_2 are a.c.[5]

Let's now define the (Dirichlet) Titchmarsh-Weyl m-function corresponding to $\mathbb{R}_-$. Assuming that $q = Q'$, with some $Q \in L^2_{loc}(\mathbb{R})$ and L_q is limit point case at $-\infty$ and $Q|_{\mathbb{R}_+} = 0$.

Denoting $\psi(x, z)$ the Weyl solution (i.e. $\psi \in L^2(\mathbb{R}_-)$ for any $z \in \mathbb{C}^+$ of (3.1) with $Q \in L^2_{loc}$ and $Q|_{\mathbb{R}_+} = 0$) we define the Titchmarsh-Weyl m-function as

$$m(z) = -\frac{\partial_x \psi(+0, z)}{\psi(+0, z)}. \tag{3.3}$$

Note that $\partial_x \psi(x, z)$ is not a.c. for $x \geq 0$ (whereas $\partial_x \psi - Q\psi$ is), but $\partial_x \psi(x, z) = \partial_x \psi(x, z) - Q(x)\psi(x, z)$ for $x > 0$ and $\partial_x \psi(+0, z)$ are well-defined. As its regular counterpart, the Titchmarsh-Weyl m-function has the following properties:

[5] a.c. abbreviates absolutely continuous.

Properties of m.

(1) m is analytic and Herglotz. I.e. $m : \mathbb{C}^+ \to \mathbb{C}^+$.

(2) Let Q_n be a sequence of smooth L^2_{loc} functions such that $\|Q - Q_n\|_{L^2_{loc}} \to 0$, $n \to \infty$, and $q = Q'$ is limit point case at $-\infty$. Then $m_n \to m$ uniformly on compact subsets of $\mathbb{C}^+$.

Define now the reflection coefficient R from the right incident of a singular potential $q \in H^{-1}_{loc}(\mathbb{R})$ such that $q|_{\mathbb{R}_+} = 0$.

Pick up a point $x_0 > 0$ and consider a solution to $L_q y = \lambda^2 y$ which is proportional to the Weyl solution on $(-\infty, x_0)$ and is equal to $e^{-i\lambda x} + re^{i\lambda x}$ on (x_0, ∞). From the continuity of this solution and its derivative at x_0 one has

$$r(\lambda, x_0) = e^{-2i\lambda x_0} \frac{i\lambda - \frac{\psi'(x_0,\lambda^2)}{\psi(x_0,\lambda^2)}}{i\lambda + \frac{\psi'(x_0,\lambda^2)}{\psi(x_0,\lambda^2)}}.$$

We define the right reflection coefficient by

$$R(\lambda) = \lim_{x_0 \to 0^+} r(\lambda, x_0) = \frac{i\lambda - m(\lambda^2)}{i\lambda + m(\lambda^2)}. \tag{3.4}$$

Example 3.1. *Let $q(x) = c\delta(x)$. The Weyl solution corresponding to $-\infty$ can be explicitly computed by ($C \neq 0$)*

$$\psi(x, \lambda^2) = C \begin{cases} e^{-i\lambda x} & , \quad x < 0 \\ \frac{1}{2i\lambda}\left(ce^{i\lambda x} + (2i\lambda - c)e^{-i\lambda x}\right) & , \quad x > 0 \end{cases}$$

and hence by (3.3) *and* (3.4)

$$m(\lambda^2) = i\lambda - c,$$
$$R(\lambda) = \frac{c}{2i\lambda - c}.$$

4. Main result

In the last section we state and prove our main result. As customary, given self-adjoint operator A we write $A \geq 0$ if A is positive.

Theorem 4.1. *Let q in* (1.1) *supported on $\mathbb{R}_-$ be in H^{-1} and such that the Schrödinger operator $L_q \geq 0$. Then there is a (unique) classical solution to* (1.1) *given by*

$$u(x, t) = -2\partial_x^2 \log \det (1 + \mathbb{H}_{x,t}) \tag{4.1}$$

where $\mathbb{H}_{x,t}$ is the trace class Hankel operator on $L^2(\mathbb{R}_+)$ with the symbol

$$\varphi_{x,t}(\lambda) = \frac{i\lambda - m(\lambda^2)}{i\lambda + m(\lambda^2)} e^{2i\lambda x + 8i\lambda^3 t}$$

where m is the (Dirichlet) Titchmarsh-Weyl m-function of L_q on $L^2(\mathbb{R}_-)$.

The solution $u(x, t)$ is meromorphic in $\mathbb{C}^+$ for any $t > 0$ except (double) poles none of which are real.

Proof. It is proven in [10] that

$$L_q \geq 0 \quad \Rightarrow \quad q \in B\left(L^2_{loc}\right) \subset H^{-1}_{loc}$$

where $B(r) = r' + r^2$ is the Miura map.

Since compactly supported smooth functions are dense in H^{-1}_{loc}, we can approximate our q by a sequence $\tilde q = \tilde r' + \tilde r^2$ where $\tilde r$'s are smooth and compactly supported.

For each $\tilde q$ there exists the (classical) right reflection coefficient $\widetilde R$. The (classical) Marchenko operator $\widetilde{\mathbb{H}}_{x,t}$ has no discrete component (since $L_{\tilde q} \geq 0$) and hence it takes the form

$$\left(\widetilde{\mathbb{H}}_{x,t} f\right)(\cdot) = \int_0^\infty \widetilde H_{x,t}(\cdot + y) f(y) dy \tag{4.2}$$

where

$$\widetilde H_{x,t}(\cdot) = \frac{1}{2\pi} \int e^{2i\lambda x + 8i\lambda^3 t} e^{i\lambda(\cdot)} \widetilde R(\lambda) d\lambda. \tag{4.3}$$

The reflection coefficient $\widetilde R$ can be computed by

$$\widetilde R(\lambda) = \frac{i\lambda - \widetilde m(\lambda^2)}{i\lambda + \widetilde m(\lambda^2)}$$

where $\tilde m$ is the Titchmarsh-Weyl m-function of $L^0_{\tilde q}$, the Dirichlet $-\partial_x^2 + \tilde q(x)$ on $\mathbb{R}_-$. Since the function $\widetilde R(\lambda)$ is analytic in $\mathbb{C}^+$ and $\widetilde R(\lambda) = O(1/\lambda)$, $\lambda \to \pm\infty$, and $\left|\widetilde R(\lambda)\right| \leq 1$, $\lambda \in \mathbb{C}^+$, one can obviously deform the contour of integration in (4.3) and (4.3) reads

$$\widetilde H_{x,t}(\cdot) = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = h} e^{2i\lambda x + 8i\lambda^3 t} e^{i\lambda(\cdot)} \widetilde R(\lambda) d\lambda. \tag{4.4}$$

for any $h > 0$. Since the integrand in (4.4) is clearly integrable along the line $\operatorname{Im}\lambda = h$, the operator $\widetilde{\mathbb{H}}_{x,t}$ is trace class (see [14]) and the function

$$\tilde u(x,t) = -2\partial_x^2 \log\det\left(1 + \widetilde{\mathbb{H}}_{x,t}\right) \tag{4.5}$$

is well-defined and solves (1.1) with initial data $\tilde q$.

We now pass to the limit in (4.5) as $\tilde r \to r$ in L^2_{loc}. By property (2) of the Titchmarsh-Weyl m-function,

$$\widetilde R(\lambda) = \frac{i\lambda - \widetilde m(\lambda^2)}{i\lambda + \widetilde m(\lambda^2)} \quad \longrightarrow \quad R(\lambda) = \frac{i\lambda - m(\lambda^2)}{i\lambda + m(\lambda^2)}$$

on each compact set in $\mathbb{C}^+$. The oscillatory factor $e^{2i\lambda x + 8i\lambda^3 t}$ exhibits a superexponential decay on $\operatorname{Im}\lambda = h > 0$. This means that (see [14] for)

$$\widetilde{\mathbb{H}}_{x,t} \longrightarrow \mathbb{H}_{x,t}$$

for any $x \in \mathbb{R}$, $t > 0$ in trace class norm and hence

$$\det\left(1 + \widetilde{\mathbb{H}}_{x,t}\right) \longrightarrow \det\left(1 + \mathbb{H}_{x,t}\right).$$

Note that $\widetilde H_{x,t}$ and

$$H_{x,t}(\cdot) = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = h} e^{2i\lambda x + 8i\lambda^3 t} e^{i\lambda(\cdot)} R(\lambda) d\lambda$$

are clearly entire with respect to x, $\forall\, t > 0$. It is quite easy to see that $\widetilde{\mathbb{H}}_{x,t}, \mathbb{H}_{x,t}$ are operator-valued functions entire with respect to x, $\forall\, t > 0$. This means that the functions

$$\tilde u(x,t) = -2\partial_x^2 \log\det\left(1 + \widetilde{\mathbb{H}}_{x,t}\right)$$

are meromorphic in x on the whole complex plane for any $t>0$ and converge to the meromorphic function

$$u(x,t)=-2\partial_x^2\log\det\left(1+\mathbb{H}_{x,t}\right)$$

as $\tilde{r}\to r$ in L^2_{loc}.

It remains to show that $\det(1+\mathbb{H}_{x,t})$ doesn't vanish on the real line for any $t>0$. Since $\mathbb{H}_{x,t}$ is trace class, this amounts to showing that -1 is not an eigenvalue of $\mathbb{H}_{x,t}$ for all $x\in\mathbb{R}$, $t>0$. We have two cases: L_q has some a.c. spectrum, L_q has no a.c. spectrum. The first case immediately follows from Lemma 2.2.

The second case is a bit more involved. If the a.c. spectrum of L_q is empty then the Titchmarsh-Weyl m-function is real a.e. on the real line and hence the reflection coefficient $|R(\lambda)|\leq 1$ in $\mathbb{C}^+$ and $|R(\lambda)|=1$ a.e. on $\mathbb{R}$. I.e. R is an inner function of the upper half plane. Lemma 2.3 then applies. □

Remark 4.2. *Theorem 4.1 implies very strong WP of the KdV equation with eventually any steplike Miura initial data supported on $(-\infty,0)$. Each such solution $u(x,t)$ is smooth and hence solves the KdV equation in the classical sense. It also has a continuity property in the sense that if $\{q_n\}$ is a sequence of smooth H^{-1}_{loc} functions convergent in H^{-1}_{loc} to q then the sequence of the corresponding solutions $\{u_n(x,t)\}$ converges in H^{-1}_{loc} to $u(x,t)$. This, in turn, implies uniqueness. The initial condition is satisfied in the sense that*

$$\left\|u(\cdot,t)-q\right\|_{H^{-1}_{loc}}\ \to\ 0\quad,\quad t\to 0.$$

Remark 4.3. *It is unlikely that, under our conditions, $\mathbb{H}_{x,t}$ in (4.1) is trace class for any x if $t=0$. We conjecture however that if Q is uniformly in L^2_{loc}, i.e. $\sup_{x\leq 0}\int_{x-1}^{x}|Q|^2<\infty$, then $\mathbb{H}_{x,0}$ is also trace class for any real x.*

Remark 4.4. *We assumed $q|_{\mathbb{R}_+}=0$ for simplicity and it can be replaced with a suitable decay condition but the consideration becomes much more involved due to serious technical circumstances. We plan to return to it elsewhere.*

References

[1] Bötcher, A.; Grudsky S.; Spitkovsky I. *Toeplitz operators with frequency modulated semi-almost periodic symbols.* J. Fourier Anal. and Appl. 7 (2001), no. 5, 523–35.

[2] Bötcher, A.; Silbermann B. *Analysis of Toeplitz operators.* Springer-Verlag, Berlin, 2002. 665 pp.

[3] Colliander, J.; Keel, M.; Staffilani, G.; Takaoka, H.; Tao, T. *Sharp global well-posedness for KdV and modified KdV on R and T.* J. Amer. Math. Soc. 16 (2003), no. 3, 705–49. MR1969209 (2004c:35352)

[4] Dybin, V.; Grudsky S. *Introduction to the theory of Toeplitz operators with infinite index.* Birkhäuser Verlag, Basel, 2002. xii+299 pp.

[5] Egorova, I.; Grunert K.; Teschl G. *On the Cauchy problem for the Korteweg-de Vries equation with steplike finite-gap initial data I. Schwarz-type perturbations.* Nonlinearity 22 (2009), 1431–57.

[6] Grudsky, S.M. *Toeplitz operators and the modelling of oscillating discontinuities with the help of Blaschke products.* Operator theory: Advances and Applications, v. 121, Birkhäuser Verlag, Basel, 2001, pp. 162-193.

[7] Guo, Zihua *Global Well-posedness of Korteweg-de Vries equation in $H^{-3/4}(\mathbb{R})$.* J. Math. Pures Appl. (9) 91 (2009), no. 6, 583–97.

[8] Kappeler, T. *Solutions to the Korteweg-de Vries equation with irregular initial data.* Comm. Partial Diff. Eq. 11 (1986), 927–45.

[9] Kappeler, T.; Topalov, P. *Global wellposedness of KdV in $H^{-1}(\mathbb{T},\mathbb{R})$.* Duke Math. J. Volume 135, Number 2 (2006), 327–60

[10] Kappeler, T.; Perry, P.; Shubin, M.; Topalov, P. *The Miura map on the line*. Int. Math. Res. Not. (2005), no. 50, 3091–133. MR2189502 (2006k:37191)

[11] Nikolski, N.K. *Operators, functions, and systems: An easy reading. Volume 1: Hardy, Hankel and Toeplitz.* Mathematical Surveys and Monographs, vol. 92, Amer. Math. Soc., Providence, 2002. 461 pp.

[12] Peller, Vladimir V. *Hankel operators and their applications.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2003. xvi+784 pp. ISBN: 0-387-95548-8.

[13] Rybkin, Alexei *Meromorphic solutions to the KdV equation with non-decaying initial data supported on a left half line.* Nonlinearity 23 (2010), no. 5, 1143–67.

[14] Rybkin, A. *The Hirota τ-function and well-posedness of the KdV equation with an arbitrary step like initial profile decaying on the right half line*, preprint (2010).

[15] Savchuk, A.M.; Shkalikov, A.A. *Sturm-Liouville operators with distribution potentials.* Trans. Moscow Math. Soc. 64 (2003), 143–92.

[16] Tao, Terence *Nonlinear dispersive equations*. Local and global analysis. CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp. ISBN: 0-8218-4143-2

Sergei M. Grudsky, Departamento de Matematicas, CINVESTAV del I.P.N. Aportado Postal 14-740, 07000 Mexico, D.F., Mexico

E-mail address: grudsky@math.cinvestav.mx

Alexei Rybkin, Department of Mathematics and Statistics, University of Alaska Fairbanks, PO Box 756660, Fairbanks, AK 99775

E-mail address: arybkin@alaska.edu