Integration of dissipative PDEs: a case study

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Abstract. We develop a computer-assisted technique to construct and analyze orbits of dissipative evolution equations. As a case study, the methods are applied to the Kuramoto-Sivashinski equation. We prove the existence of a hyperbolic periodic orbit.

1. Introduction

In this paper we consider the problem of investigating the flow of dissipative parabolic equations via computer-assisted methods. The type of equations that we have in mind are of the form

$$\partial_t u + (i\partial_x)^m u + H(u, \partial_x u, \ldots, \partial_x^{m-1} u) = 0\,, \tag{1.1}$$

with $m > 0$ even, H real analytic, and $u = u(t,x)$ satisfying suitable boundary conditions on a bounded spatial domain.

Our goal is to obtain estimates for the time-t map and its derivative, for small times $t > 0$, and to combine these estimates with shadowing arguments, in order to control the long-time dynamics. For periodic orbits, this includes bounds on the eigenvalues of the linearized return map. Our method takes advantage of the fact that the solutions $u(t,x)$ of equation (1.1) are real analytic in x, when $t > 0$, due to the dissipation and the analyticity of H. This allows us to obtain accurate bounds in a relatively straightforward and general way.

As a case study, we consider the unidimensional Kuramoto-Sivashinski equation, which has been the focus of numerous analytical and numerical investigations [3–11]. Considering Dirichlet boundary conditions on $[0,\pi]$, this equation can be written in the form

$$\begin{aligned} \partial_t u + 4\partial_x^4 u + \alpha\bigl(\partial_x^2 u + 2u\partial_x u\bigr) &= 0\,, \\ u(t,0) = u(t,\pi) &= 0\,, \end{aligned} \qquad x \in [0,\pi]\,, \quad t \in \mathbb{R}\,. \tag{1.2}$$

It is a well known numerical result [2] that this system exhibits chaotic dynamics when $\alpha > 133$. Obtaining rigorous bounds for such large values of α is more difficult, since high frequency modes contribute more heavily to the dynamics. In order to test the robustness of our algorithm, and to make a first step towards the analysis of chaotic dynamics, we consider the value $\alpha = 150$, which is well above the chaotic threshold. Our main result on the Kuramoto-Sivashinski equation is the following.

Theorem 1.1. *Equation (1.2) with $\alpha = 150$ admits a hyperbolic periodic orbit of period $\tau = 0.00214688\ldots$. The Poincaré map associated with some transversal hyperplane is compact, and its eigenvalues lie in the disk $|\mu| < 0.69$, except for a simple eigenvalue $|\mu_1| > 4.8$.*

Somewhat similar results were obtained recently in [12], namely the existence of periodic orbits, for several values of α between 30 and 134. However, the methods used in [12] are purely topological and given no information on the stability of the orbit.

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The remaining part of the paper is structured as follows. In Section 2, we outline the general strategy and then describe the various steps in more detail. This part applies to dynamical systems in any separable Banach space, although the approach is motivated by the intended applications. A more specific functional setting is introduced in Section 3, where we discuss the integral equation used to control the time-t map. Section 4 describes our implementation of the various steps, and further details of our computer-assisted proof.

2. Strategy and techniques

Consider an evolution equation of the type

$$\partial_t u = Lu + G(u)\,, \qquad u(0) = \nu\,, \tag{2.1}$$

for a function $u = u(t)$, defined for $t \in [0, \infty)$, that takes values in some Banach space $\mathcal{X}$. Here, L and G are operators on $\mathcal{X}$, with L linear. In the applications that we have in mind, which include the Kuramoto-Sivashinski equation, $\mathcal{X}$ is a space of real-valued functions on a bounded domain in $\mathbb{R}^n$, satisfying suitable boundary conditions.

Assume that we can compute numerically an approximate periodic orbit $\bar{u}$, starting at some point $\bar{u}_0$ at time $t = 0$, and returning to the same point at a later time $t = \bar{\tau}$. After choosing a codimension-one hyperplane $S_0 \subset \mathcal{X}$ that intersects the curve $\bar{u}$ transversally at $\bar{u}_0$, we define the corresponding Poincaré return map $\Psi : S_0 \to S_0$, by setting $\Psi(u) = u(t)$, where $t = t(\bar{u}_0)$ is the first return time to the section S_0. Here, and in what follows, a Poincaré return map need only be defined locally. If the system is dissipative, then the derivative of Ψ is compact, and in particular, it has only finitely many eigenvalues of modulus larger than 1. We will refer to the corresponding eigenvectors as the "expanding directions". Our strategy for proving the existence of hyperbolic orbits involves several steps and technical tools. The basic tool is a computer-assisted technique for computing bounds on the time-t map Φ_t of the system, and of its derivative $D\Phi_t$. Then we choose a sequence of intermediate Poincaré sections S_j along the approximate orbit and compute rigorous estimates on the intermediate Poincaré maps $P_j : S_{j-1} \to S_j$ and their derivatives. By means of a shadowing technique, applied to the sequence of map P_j, we prove the existence of a true periodic orbit close to $\bar{u}$. This information is then used to estimate the derivatives of the intermediate Poincaré maps P_j. After verifying appropriate cone conditions for these derivatives, we obtain the desired bounds on the full Poincaré map.

2.1. Integration

Our first goal is to integrate equation (2.1) on a sequence of time intervals $[T_{n-1}, T_n]$, starting with $T_0 = 0$. Since the problem is autonomous, it suffices to consider the initial value problem on $[0, T]$. We turn (2.1) into an integral equation as usual, by rewriting the equation as

$$\partial_t \left[e^{-tL} u\right] = e^{-tL} G(u)\,, \qquad u(0) = \nu\,, \tag{2.2}$$

integrating both sides, and then multiplying by e^{tL},

$$u(t) = e^{tL}\nu + \int_0^t e^{(t-s)L} G(u(s))\,ds\,, \qquad 0 \le t \le T\,. \tag{2.3}$$

After defining a Banach space $\mathcal{X}$ of admissible initial conditions ν, our aim is to solve equation (2.3) by iteration, on a Banach space $\mathcal{X}_T$ of continuous $\mathcal{X}$-valued functions on the interval $J=[0,T]$. Setting $\Phi_t(\nu)=u(t)$ then defines the time-t map Φ_t on $\mathcal{X}$, for $t\in J$.

To be more precise, suppose that L has a compact inverse, and a sequence of eigenvectors $\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots$ whose span is dense in $\mathcal{X}$. Let $-\lambda_1,-\lambda_2,\ldots$ be the corresponding eigenvalues, and assume that $\lambda_k>0$ for sufficiently large k. Then $t\mapsto e^{tL}$ is a continuous semigroup on $\mathcal{X}$. Substituting the formal expansion $u(t)=\sum_k u_k(t)\boldsymbol{v}_k$ into equation (2.3) yields a system of (infinitely many) coupled ODEs

$$u_k(t)=e^{-\lambda_k t}\nu_k+\int_0^t e^{-\lambda_k(t-s)}g_k(s)\,ds\,,\qquad t\in J\,,\tag{2.4}$$

where $\{\nu_k\}$ and $\{g_k(t)\}$ are the expansion coefficients for the vectors ν and $g(t)=G(u(t))$, respectively. Our aim is to work with a finite truncation of this system of ODEs, say $k\le N$, and to control the truncation errors. The truncation is defined in terms of the spectral projection $\mathbb{P}_L$ onto the span of the first N eigenvectors of L.

The same approach will be used to analyze the derivative $D\Phi_t$ of the time-t map: The function $w=D\Phi_t(u)\omega$ is the solution of the initial value problem

$$\partial_t w=Lw+DG(u)w\,,\qquad w(0)=\omega\,,\tag{2.5}$$

which can be reduced to an integral equations similar to (2.3) and (2.4). Further details are provided in Section 3.

2.2. Intermediate Poincaré maps

A well known problem with computer-assisted integration is that the errors accumulate along the orbit, making the computation useless after a certain amount of time. In our approach to the Kuramoto-Sivashinski equation, this time is significantly shorter than what would be needed to estimate the Poincaré map Ψ directly. (And we do not expect the situation to be much better with other PDEs.) Adapting an approach that has been developed for finite dimensional dynamical systems [13], we proceed by smaller time steps as follows.

Consider a partition of the approximate period $[0,\bar\tau]$ into M subintervals $[t_j,t_{j+1}]$, where $t_j=j\bar\tau/M$ for $j=0,1,\ldots,M$. Then the points $\bar u_j=\bar u(t_j)$ define a discretization of the approximate orbit $\bar u$. For each j, we choose a nonzero functional $\eta_j:\mathcal{X}\to\mathbb{R}$ and define a Poincaré section $S_j=\bar u_j+X_j$, where X_j is the null space of η_j. After verifying that S_j is transversal to the flow, as defined below, a Poincaré map $P_j:S_{j-1}\to S_j$ is defined in the usual way as $x\mapsto\Phi_t(x)$, where $t=t(x)$ is the smallest positive number such that $\Phi_t(x)\in S_j$. The composition of all these "intermediate" Poincaré maps yields the full local Poincaré map Ψ.

In our implementation of this procedure, we choose for each j an ordered basis in $\mathbb{P}_L\mathcal{X}$. The second basis vector is chosen to point roughly in the direction of the flow at $\bar u_j$. Then we define $\eta_j(x)$ to be the second coordinate of $\mathbb{P}_L(x-\bar u_j)$.

We will use (verify) the following notion of transversality. Let $B\subset\mathcal{X}$ be a fixed set of initial conditions. Assume that $\eta:\mathcal{X}\to\mathbb{R}$ is continuous and nonzero.

Definition 2.1. *Consider a differentiable flow $(t,\nu)\mapsto u(t)$, defined on $[a,c]\times B$. Let $s\in\mathbb{R}$. We say that the section $S=\eta^{-1}(s)$ is transversal to the flow if, for all initial conditions $\nu\in B$, the interval bounded by $\eta(u(a))$ and $\eta(u(c))$ contains s, and the derivative of $t\mapsto\eta(u(t))$ is bounded away from zero on $[a,c]$, uniformly in ν.*

2.3. Shadowing

Our aim is to prove that there exists a true orbit that closely "shadows" the approximate orbit $\bar{u}$. In addition, we would like to show that the full Poincaré map has no eigenvalue on the unit circle, and to estimate the spectral gap between the least expanding and least contracting eigenvalue.

Our main tool for the first step is Theorem 2.1 below, which is an extension to the infinite dimensional setting of Theorem 4 in [14]. (See also [12] for an approach to the infinite dimensional case.)

Theorem 2.2. *Consider a Banach space $X=\mathbb{R}\oplus Z$, and let V be the closed unit ball in Z. Let F be a continuous and compact map from $[-1,1]\times V$ to $X=\mathbb{R}\oplus Z$. Writing $F(u,v)=(g(u,v),h(u,v))$, assume that $g(-1,v)\le -1$ and $g(1,v)\ge 1$ and $\|h(u,v)\|\le 1$, for all $u\in[-1,1]$ and $v\in V$. Then F has a fixed point in $[-1,1]\times V$.*

Proof. We may assume that the norm on X is given by $\|(u,v)\|=\max\{|u|,\|v\|\}$. By our assumption on F, the closure K of $F([1,1]\times V)$ is compact. Thus, given $\varepsilon>0$, there exist points $w_1,w_2,\ldots w_n\in K$, such that the balls $W_i=\{w\in X:\|w-w_i\|<\varepsilon\}$ cover the set K. For $w\in K$ define

$$\phi_\varepsilon(w)=\frac{\sum_{i=1}^n\varphi_i(w)w_i}{\sum_{i=1}^n\varphi_i(w)}\,,\qquad \varphi_i(w)=\begin{cases}\varepsilon-\|w-w_i\| & \text{if } w\in W_i\,;\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, $\phi_\varepsilon:K\to X$ is continuous, and $\|\phi_\varepsilon(w)-w\|\le\varepsilon$ for all $w\in K$. Assuming $\varepsilon<\frac{1}{2}$, define $\psi_\varepsilon(u,v)=((1+2\varepsilon)u,(1-2\varepsilon)v)$. Then $F_\varepsilon=\psi_\varepsilon\circ\phi_\varepsilon\circ F$ is continuous on $[-1,1]\times V$ and satisfies $\|F(x)-F_\varepsilon(x)\|\le c\varepsilon$, for some fixed constant $c>0$.

Denote by X_ε the subspace of X spanned by the vectors $\{w_0,w_1,\ldots,w_n\}$, where $w_0=(1,0)$. Notice that F_ε takes values in X_ε. The restriction of F_ε to X_ε satisfies the hypotheses of Theorem 4 in [14], and therefore it admits a fixed point in $[-1,1]\times V$. If x_m is such a fixed point, obtained with $\varepsilon\le\frac{1}{cm}$, then $\|F(x_m)-x_m\|=\|F(x_m)-F_\varepsilon(x_m)\|\le\frac{1}{m}$. Since F is compact, we can choose $m\mapsto x_m$ in such a way that $F(x_m)\to x$ for some x in $[-1,1]\times V$. Then

$$\|x-x_m\|\le\|x-F(x_m)\|+\|F(x_m)-x_m\|\to 0\,.$$

Thus, $x_m\to x$, and $F(x)=\lim_{m\to\infty}F(x_m)=x$ by the continuity of F. **QED**

In order to apply Theorem 2.2 to the sequence of Poincaré maps described in the previous subsection we use the following definitions, which are analogues of similar definitions in [13,14,15]. Assume that $\mathcal{X}=\mathcal{Y}\oplus\Theta\oplus\mathcal{Z}$, where $\mathcal{Y}$ and Θ are one-dimensional subspaces of $\mathcal{X}$. Denote by U and V the closed unit balls in $\mathcal{Y}$ and $\mathcal{Z}$, respectively.

Definition 2.3. *A section (of $\mathcal{X}$) is codimension-one affine subspace of $\mathcal{X}$. A box in a section S is the image of $U \times V$ under a bi-continuous affine map $\psi : \mathcal{Y} \times \mathcal{Z} \to S$.*

Definition 2.4. *Let $B_1 = \psi_1(U \times V)$ and $B_2 = \psi_2(U \times V)$ be boxes in two section S_1 and S_2, respectively. Given a map $f : B_1 \to S_2$, we say that B_1 f-covers B_2 if the map $F : U \times V \to \mathcal{Y} \times \mathcal{Z}$, defined by $F = \psi_2^{-1} \circ f \circ \psi_1$, satisfies the hypotheses of Theorem 2.2. For simplicity, we identified here $\mathcal{Y}$ with $\mathbb{R}$, and U with $[-1, 1]$.*

Roughly speaking, the set B_1 is stretched along B_2 in the direction $\psi_1(U)$ and compressed in the other directions by the map f. Clearly, this definition could be extended to a larger number of expanding directions.

Consider now the Poincaré maps $P_j : S_{j-1} \to S_j$ described in Subsection 2.2, and for each j, let B_j be a box in S_j. By periodicity, we identify $j = M$ with $j = 0$.

Corollary 2.5. *If for each j, the box B_{j-1} P_j-covers B_j, then the Poincaré map $\Psi : S_0 \to S_0$, defined by $\Psi = P_M \circ \ldots \circ P_2 \circ P_1$, has a fixed point in B_0.*

2.4. Derivatives of the Poincaré map

In our application of Corollary 2.5, we estimate the image of a box B_j under the flow via bounds on the derivative of the flow. The same bounds can also be used to prove hyperbolicity. To be more specific, let u be a periodic orbit, denote by u_j its intersection with the section S_j, and define $\dot{u}_j = Lu_j + G(u_j)$, for all j. A simple calculation shows that

$$DP_j(u_{j-1})w = D\Phi_{t(u_{j-1})}(u_{j-1})w - \frac{\eta_j(D\Phi_{t(u_{j-1})}(u_{j-1})w)}{\eta_j(\dot{u}_j)}\dot{u}_j \,. \tag{2.6}$$

Thus, what we need are accurate estimates on the velocities $\dot{u}_j$.

The low-frequency parts $\ell_j = \mathbb{P}_L \dot{u}_j$ are estimated explicitly in our construction of the orbit, since $G(u(s))$ appears in the integral equation (2.3), and $\ell_j = \mathbb{P}_L G(u_j)$. Consider now the high-frequency part $h_j = \mathbb{P}_H \dot{u}_j$. Here, and in what follows, $\mathbb{P}_H = \mathbb{I} - \mathbb{P}_L$. After proving that $\dot{u}_j \in \mathcal{X}$ for all j, we can use that $\dot{u}_j = D\Phi_{t(u_{j-1})}(u_{j-1})\dot{u}_{j-1}$. This immediately yields the following

Proposition 2.6. *Let $k_j = \mathbb{P}_H D\Phi_{t(u_{j-1})}(u_{j-1})\ell_{j-1}$ and $D_j = \mathbb{P}_H D\Phi_{t(u_{j-1})}(u_{j-1})\mathbb{P}_H$. Then*

$$\|h_j\| \le \|k_j\| + \|D_j\|\|h_{j-1}\|\,, \qquad j = 1, 2, \ldots, M.$$

In particular, if $\|k_j\| \le b$ and $\|D_j\| \le a < 1$ for all j, then $\|h_j\| \le (1 - a)^{-1}b$.

2.5. Hyperbolicity

In order to compute a lower bound on the modulus of the expanding eigenvalue of $D\Psi(u_0)$, and an upper bound on the moduli of the contracting eigenvalues, we replace the P_j-covering conditions by cone conditions on the derivatives DP_j. This corresponds roughly to replacing Theorem 2.2 by the following theorem.

Theorem 2.7. *Let $A \neq 0$ be a bounded linear operator on a real Banach space $X = Y \oplus Z$, with Y one-dimensional. Thus, if $y \in Y$ and $z \in Z$, we have a unique decomposition*

$$A(y+z) = y' + z' \,, \qquad y' \in Y,\ z' \in Z \,. \tag{2.7}$$

Assume now that A is compact, and that there exist positive real numbers $\beta < \alpha$, such that $\|z'\| \leq \beta \max\{\|y\|, \|z\|\}$, and such that $\|y'\| \geq \alpha\|y\|$ whenever $\|y\| \geq \|z\|$. Then A has a simple eigenvalue λ of modulus $|\lambda| \geq \alpha$, and no other eigenvalue of modulus $> \beta$.

Proof. Consider the double cone $\mathcal{C}$, defined by $\|y\| \geq \|z\|$. It is preserved by A. For any $x = y + z$ in $\mathcal{C}$ we have $\|y'\| \geq \alpha\|y\|$, and thus $\|A^n x\| \geq c\alpha^n$ for all $n \geq 0$, with $c > 0$. This guarantees e.g. that A has a spectral radius $\rho(A) \geq \alpha$.

Let λ be an eigenvalue of A, and let $\bar{x} = \bar{y} + \bar{z}$ be an eigenvector for this eigenvalue. If $|\lambda| < \alpha$, then $\bar{x}$ cannot belong to C, since $\|A^n \bar{x}\|$ grows slower than $c\alpha^n$. And if $\bar{x} \notin C$, then our condition $\|z'\| \leq \beta \max\{\|y\|, \|z\|\}$ implies that $|\lambda| \leq \beta$. Thus, if $|\lambda| > \beta$, then $|\lambda| \geq \alpha$ and $\bar{x} \in \mathcal{C}$.

Assume from now on that $|\lambda| \geq \alpha$. Then $\bar{x} \in \mathcal{C}$, and in particular, $\bar{y} \neq 0$. Replacing A by $-A$, if necessary, we may assume that $A\bar{y} \in c\bar{y} + Z$ with $c > 0$. Denote by $\mathcal{C}^{\pm}$ the two cones whose union is $\mathcal{C}$, with $\mathcal{C}^+$ being the one containing $\bar{y}$. Clearly, $\mathcal{C}^+$ is invariant under A. This implies e.g. that λ is real and positive.

Notice that $\mathcal{C}^+$ has a non-empty interior. Thus, by the Krein-Rutman theorem, $\rho(A)$ is a simple eigenvalue of A. Furthermore, A is strongly positive: $\mathcal{C}^+ \setminus \{0\}$ is mapped into the interior of $\mathcal{C}^+$, since $\|z'\| \leq \beta\alpha^{-1}\|y'\|$ whenever $y + z$ belongs to $\mathcal{C}^+$. This guarantees [9, Theorem 1.2] that $\rho(A)$ is the only eigenvalue of A that has an eigenvector in $\mathcal{C}^+$. Consequently, our eigenvalue λ is simple and equal to $\rho(A)$. This concludes the proof of Theorem 2.7. **QED**

Definition 2.8. *Let $X = \mathcal{Y} \oplus \mathcal{Z}$, and let $\alpha > \beta$ be positive real numbers. Given two sections $\nu_1 + X_1 = \psi_1(X)$ and $\nu_2 + X_2 = \psi_2(X)$ of $\mathcal{X}$, and a linear map $B : X_1 \to X_2$, we say that B satisfies the (α, β) cone condition, if $A = D\psi_2^{-1} B D\psi_1$ satisfies the hypotheses of Theorem 2.7.*

Consider again the Poincaré maps $P_j : S_{j-1} \to S_j$ described in Section 2.2. Denote by u_j the intersection of our periodic orbit with the Poincaré plane S_j.

Corollary 2.9. *If for each j, the derivative $DP_j(u_j)$ satisfies an (α_j, β_j) cone condition, then $D\Psi(u_0)$ has a simple eigenvalue $\mu_1 \geq \prod_j \alpha_j$ and no other spectrum outside the disk $|\mu| \leq \prod_j \beta_j$.*

As far as the proof of Theorem 1.1 is concerned, our task is now reduced to verifying the hypotheses of Corollary 2.5, and of Corollary 2.9, with $\beta_j < 1 < \alpha_j$ for all j.

3. Functional setting

3.1. The integral operator

It is convenient to rewrite the integral equation (2.3) as a fixed point problem $K_\nu(w) = w$ for the function $w(t) = u(t) - e^{tL}\nu$. The operator K_ν can be written as $K_\nu = \Lambda \circ \Gamma_\nu$, where

$$(\Lambda w)(t) = \int_0^t e^{(t-s)L} M w(s) ds\,, \quad \bigl(\Gamma_\nu(w)\bigr)(t) = M^{-1} G\bigl(w + e^{tL}\nu\bigr)(t)\,, \tag{3.1}$$

for $0 \leq t \leq T$. Here, M is some invertible linear operator on $\mathcal{X}$ that we are free to choose later. The problem with estimating K_ν on a computer is that the function $t \mapsto e^{tL}\nu$, and thus the integrand in Λ, can vary very rapidly near $t = 0$. Such functions have to be considered e.g. when estimating the derivative of the flow.

We deal with this problem by partitioning $J = [0, T]$ into n subintervals $J_i = [t_{i-1}, t_i]$, with the partition being much finer near $t_0 = 0$, than near $t_n = T$, and controlling our functions on each subinterval J_i separately. Define $\mathcal{X}_T$ to be space of all continuous functions $u : J \to \mathcal{X}$ with a finite norm

$$\|u\| = \max_i \sum_{k=1}^\infty \sup_{t \in J_i} \|u_k(t)\boldsymbol{v}_k\|\,, \qquad u(t) = \sum_{k=1}^\infty u_k(t)\boldsymbol{v}_k\,. \tag{3.2}$$

We note that this norm depends on the partition $\{J_i\}$ of J. But the norms associated with two different partitions are equivalent. Our only reason for subdividing J is to get more accurate estimates on the computer. For the results in this section, it suffices to consider just the trivial partition $\{J\}$.

We recall that the functions $\boldsymbol{v}_k$ are assumed to be eigenvectors of L, with eigenvalues $-\lambda_k$ that tend to $-\infty$ as $k \to \infty$. In order to avoid growing factors $e^{-\lambda_k t}$ in the equation (2.4), we assume from now on that $\lambda_k \geq 0$ for all k. This represents no loss of generality, since a positive part of L can always be incorporated into the nonlinear part G of the vector field $L + G$.

Lemma 3.1. *Let $0 < a \leq b < 1$, set $M = (\mathbb{I} - L)^a$, and assume that $\sum_{k>m} \lambda_k^{b-1} < \infty$, for some $m > 0$. Then Λ is compact on $\mathcal{X}_T$, and $\|\Lambda\| \leq c(T + T^{b-a})$, for some constant c that is independent of T.*

Proof. Let $w \in \mathcal{X}$ and $\widetilde{w} = \Lambda w$. Let $k \geq 1$. If $\lambda_k < 1$, then $|\widetilde{w}_k(t)| \leq 2t\|w_k\|$, where $\|w_k\| = \sup_{t\in J} |w_k(t)|$. If $\lambda_k \geq 1$ then

$$\begin{aligned} \bigl|\widetilde{w}_k(t)\bigr| &\leq \int_0^t e^{-\lambda_k(t-s)}(1+\lambda_k)^a |w_k(s)|\, ds \\ &\leq 2\lambda_k^{b-1} t^{b-a} \bigl[(\lambda_k t)^{a-b}\bigl(1 - e^{-\lambda_k t}\bigr)\bigr] \|w_k\|. \end{aligned} \tag{3.3}$$

The expression $[\cdots]$ in this equation is bounded, uniformly in k and t. Multiplying both sides of (3.3) by $\|\boldsymbol{v}_k\|$, and summing over k, yields $\|\widetilde{w}\| \leq c(T + T^{b-a})\|w\|$, with c independent of w and T.

Denote by $\mathbb{P}_k$ the canonical projection onto the one-dimensional subspace spanned by $\boldsymbol{v}_k$. Then $\Lambda\mathbb{P}_k$ is compact, by Arzelà-Ascoli. The sum $\sum_k \Lambda\mathbb{P}_k$ converges in norm to Λ, due to the factor λ_k^{b-1} in (3.3). Thus, Λ is compact. **QED**

What remains to be proved is that Γ_ν is continuous and bounded on some appropriate domain in $\mathcal{X}_T$. Then $K_\nu = \Lambda \circ \Gamma_\nu$ is compact. Now it suffices to find a closed and convex set C in this domain, such that $K_\nu(C) \subset C$. Then K_ν has a fixed point in C. On the computer, we determine such a set C by starting with a singleton $C_0 = \{\bar{w}\}$, where $\bar{w}$ is an approximate fixed point of K_ν. Using a suitable enlargement map $C \mapsto C'$ for sets, we compute for $n = 1, 2, \ldots$ an enclosure C_n for $K_\nu(C'_{n-1})$, until $C_n \subset C'_{n-1}$. The set $C = K(C_n)$ for the final value of n is a bound on the fixed point w of K_ν, in the sense that $C \ni w$. This in turn yields a bound on $u(t)$ for $t \in J$, and a bound (in a sense that will be made more precise later) on the time-t map $\nu \mapsto u(t)$.

How exactly these bounds are implemented depends on the specific equation. In the remaining part of this paper, we describe the details for the Kuramoto-Sivashinski equation.

3.2. The Kuramoto-Sivashinski equation

The one-dimensional Kuramoto-Sivashinski (KS) equation can be written as

$$\partial_t u = \mathbb{L}u - \alpha\partial_x(u^2)\,, \qquad \mathbb{L} = -4\partial_x^4 - \alpha\partial_x^2\,. \tag{3.4}$$

As mentioned earlier, we consider this equation for $\alpha = 150$. But for now, α can be any positive real number.

The boundary conditions considered are $u(t,0) = u(t,\pi) = 0$, so the eigenvectors of $\mathbb{L}$ are given by $\boldsymbol{v}_k(x) = \sin(kx)$, where k ranges over the set of all positive integers $\mathbb{N}$. The corresponding eigenvalues are $\alpha k^2 - 4k^4$. As indicated earlier, we extract a negative part L from $\mathbb{L}$. To this end, let $\kappa \geq \sqrt{\alpha}/2$, so that $\alpha k^2 - 4k^4 \leq 0$ for $k \geq \kappa$. Then define L and L' by the equations $L\boldsymbol{v}_k = -\lambda_k \boldsymbol{v}_k$ and $L'\boldsymbol{v}_k = -\lambda'_k \boldsymbol{v}_k$, respectively, where

$$\lambda_k = \begin{cases} 0, & \text{if } k \leq \kappa; \\ 4k^4 - \alpha k^2 & \text{if } k > \kappa; \end{cases} \qquad \lambda'_k = \begin{cases} 4k^4 - \alpha k^2 & \text{if } k \leq \kappa; \\ 0 & \text{if } k > \kappa; \end{cases} \tag{3.5}$$

Clearly, $\mathbb{L} = L + L'$, and L has no positive eigenvalues. The KS equation (3.4) can now be written in the standard form (2.1),

$$\partial_t u = Lu + G(u)\,, \qquad G(u) = L'u - \alpha\partial_x(u^2)\,. \tag{3.6}$$

At this point, we need to define some function spaces. Let ρ be a fixed positive real number. In addition to $\boldsymbol{v}_k(x) = \sin(kx)$ for integers $k > 0$, we also consider $\boldsymbol{v}_k(x) = \cos(kx)$ for $k \leq 0$. Given a real number $p > 0$, and a non-empty set of integers $\mathcal{K}$, define $\mathcal{A}_p^{\mathcal{K}}$ to be the completion of $\operatorname{Span}\bigl(\{\boldsymbol{v}_k\}_{k\in\mathcal{K}}\bigr)$ with respect to the norm

$$\|f\|_p = \sum_{k\in\mathcal{K}} |f_k|[k]^p e^{\rho|k|}\,, \qquad f = \sum_{k\in\mathcal{K}} f_k \boldsymbol{v}_k\,, \tag{3.7}$$

where $[k] = \max(|k|, 1)$. Notice that the functions in $\mathcal{A}_p^{\mathcal{K}}$ have analytic and bounded extensions to the strip $|\mathrm{Im}(x)| < \rho$. The space $\mathcal{X}$ referred to in earlier sections is $\mathcal{A}_p^{\mathbb{N}}$. For now, $p > 0$ is arbitrary; but later on, we will choose $p = 0$. A slightly different (but equivalent) norm on $\mathcal{X}$ is used in our definition of the boxes B_j, as will be described later.

The space of continuous curves $u : J \to \mathcal{A}_p^{\mathcal{K}}$ with a finite norm

$$\|u\|_p = \max_i \sum_{k\in\mathcal{K}} [k]^p e^{\rho|k|} \sup_{t\in J_i} |u_k(t)| , \qquad u(t) = \sum_{k\in\mathcal{K}} u_k(t)\boldsymbol{v}_k , \tag{3.8}$$

will be denoted by $\mathcal{C}(J, \mathcal{A}_p^{\mathcal{K}})$. In particular, $\mathcal{C}(J, \mathcal{X})$ agrees with the space $\mathcal{X}_T$ defined in Subsection 3.1. A useful property of the spaces $\mathcal{A}_p^{\mathbb{Z}}$ and $\mathcal{C}(J, \mathcal{A}_p^{\mathbb{Z}})$ is that they are both Banach algebras.

As described in the previous subsection, we solve (3.6) via the fixed point equation $K_\nu(w) = w$ for $w(t) = u(t) - e^{tL}\nu$, where $K_\nu = \Lambda \circ \Gamma_\nu$, with Γ_ν and Λ as defined by equation (3.1). Notice that this map K_ν is independent of the choice of the operator M.

Theorem 3.2. *K_ν is a compact map on $\mathcal{X}_T$, for each $\nu \in X$. If $B \subset \mathcal{X}$ is bounded and non-empty, and if C is a closed convex subset of $\mathcal{X}_T$, such that $K_\nu(C) \subset C$, for every $\nu \in B$, then K_ν has a unique fixed point $w \in C$, for every $\nu \in B$. If B is open, then the map $\nu \mapsto w$ is of class C^1 on B.*

Proof. Let $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Then the hypothesis of Lemma 3.1 is satisfied, implying that Λ is compact. Furthermore, since $\mathcal{C}(J, \mathcal{A}_p^{\mathbb{Z}})$ is a Banach algebra, the map $u \mapsto u^2$ is of class C^2 on all of $\mathcal{A}_p^{\mathbb{Z}}$. Clearly, the same holds for Γ_ν. Thus, K_ν is compact, and the existence of a fixed point follows from Schauder's fixed point theorem.

Let w_1 and w_2 be two such fixed points. Assume for contradiction that $w_1 \neq w_2$, and let $T_0 = \inf\{t \in [0, T] : w_1(t) \neq w_2(t)\}$. Replacing w_i by $t \mapsto w_i(T_0 + t) - e^{tL}w_i(T_0)$, if necessary, we can assume that $T_0 = 0$. In what follows, a curve or operator associated with a time interval $[0, s]$ will be given a subscript s.

Let $r > 0$. By Lemma 3.1, $K_{\nu,s}$ is a contraction on the ball $\|w_s\|_p < r$, if $s > 0$ is sufficiently small. By continuity, both $w_{1,s}$ and $w_{2,s}$ belong to this ball, if $s > 0$ has been chosen sufficiently small. As a result, $w_{1,s} = w_{2,s}$, contradicting the assumption that $w_1 \neq w_2$.

Let v be an eigenvector of $DK_\nu(w_1)$, for some eigenvalue λ, and define $T_0 = \inf\{t \in [0, T] : v(t) \neq 0\}$. After translating w_1 and v in time by T_0, if necessary, we can assume that $T_0 = 0$. Then v_s is an eigenvector of $DK_{\nu,s}(w_{1,s})$, with eigenvalue λ; and by taking $s \downarrow 0$, we see that $\lambda = 0$. In particular, $DK_\nu(w_1)$ has no eigenvalue 1. Thus, by the implicit function theorem, the map $\nu \mapsto w_1$ is of class C^1. **QED**

Consider now a section $S = \eta^{-1}(s)$, defined by some nonzero continuous functional $\eta : \mathcal{X} \to \mathbb{R}$, and some $s \in \mathbb{R}$. Let B be a non-empty bounded set in $\mathcal{X}$. In what follows, differentiability of a function on B means differentiability in an open neighborhood of B.

Theorem 3.3. *Let $0 < a < c \leq T$. The flow $(\nu, t) \mapsto u(t)$ is of class C^1, and compact, as a map from $[a, c] \times B$ to $\mathcal{A}_p^{\mathbb{N}}$, for any given $p > 0$. If the section S is transversal to the* *flow, then the corresponding Poincaré map $P : B \to S$ is well defined, of class C^1, and compact.*

Proof. We may assume that B is open. Let $0 < b < T$. Consider the map $Q : u \mapsto u(t)$ defined by the integral equation (2.3), where we set $\nu = u(0)$. By using that $\mathcal{C}(J, \mathcal{A}_p^{\mathbb{Z}})$ is a Banach algebra, the difference $Q(u) - Q(u')$ can be bounded by

$$\|u(t) - u'(t)\|_{p+3} \leq \bigl(A_1 + A_2\|u + u'\|_p\bigr)\|u - u'\|_p\,, \tag{3.9}$$

for all $t \in [b, T]$, where A_1 and A_2 are constants that are independent of u and u'. This, and the quadratic nature of Q, implies that Q defines a C^1 map from $\mathcal{C}(J, \mathcal{A}_p^{\mathbb{N}})$ to $\mathcal{A}_{p+3}^{\mathbb{N}}$.

Composing Q with the map $u(0) \mapsto u$ described in Theorem 3.2, it follows that Φ_t is a C^1 map from B to $\mathcal{A}_q^{\mathbb{N}}$, for $q = p + 3$. By composing several such time-t maps, this generalizes to any given $q > 0$. The same holds for the map $u(0) \mapsto \dot{u}(t)$ as well, since $L + G$ is a smooth map from $\mathcal{A}_{r+4}^{\mathbb{N}}$ to $\mathcal{A}_r^{\mathbb{N}}$, for any $r > 0$, This in turn implies that the flow $\Phi : (t, \nu) \mapsto \Phi_t(\nu)$ is of class C^1 on $[a, c] \times B$.

Assume now that S is transversal to this flow, in the sense of Definition 2.1. Then, by the implicit function theorem, the equation $\eta(\Phi_t(\nu)) = s$ has a unique solution $t = t(\nu)$, and this crossing time is a C^1 function on B. Composing $\nu \mapsto (t(\nu), \nu)$ with Φ yields the Poincaré map $P : B \to S$, and this map is C^1 by the chain rule. **QED**

Convention. From now on, we only consider $p = 0$ and drop the index p.

The remaining part of the paper is devoted to the proof of the following theorem, which, together with Corollary 2.5 and Corollary 2.9, implies Theorem 1.1.

Theorem 3.4. *For every integer j, modulo $M = 4294$, there exists a section $S_j \subset \mathcal{A}^{\mathbb{N}}$, a box $B_j \subset S_j$, a time interval $[a_j, c_j]$, and two positive real numbers $\beta_j < 1 < \alpha_j$, such that the following holds. For each j, the flow $\Phi : [a_j, c_j] \times B_{j-1} \to \mathcal{A}^{\mathbb{N}}$ is transversal to the section S_j, and the box B_{j-1} P_j-covers B_j. If $j \mapsto u_j \in B_j$ satisfies $P_j(u_{j-1}) = u_j$, for each j, then the derivative $DP_j(u_{j-1})$ satisfies the (α_j, β_j) cone condition, for each j. Furthermore, $\prod_j \alpha_j \geq 4.8$ and $\prod_j \beta_j \leq 0.69$.*

4. Implementation and further details

4.1. Integration

In order to obtain reasonably accurate error bounds for the integral operator K_ν, we decompose this operator into several parts and estimate each separately. In a first step, we write $K_\nu(w) = P(\nu, w) + Q(\nu, w)$, where P is linear and Q bilinear. Then $Q(\nu, w)$ is split into three terms $Q^{(n)}(\nu, w)$ that are homogeneous of degree $n = 0, 1, 2$ in w. After substituting the sine-series for ν and w, we end up with integrals like

$$\bigl(Q_m^{(1+)}(\nu, w)\bigr)(t) = -\alpha m \sum_{k+\ell=m} \nu_k \int_0^t e^{-\lambda_m(t-s)} e^{-\lambda_k s} w_\ell(s)\, ds\,. \tag{4.1}$$

To be more precise, this term is the first of two contributions to the m-th Fourier coefficient of $Q^{(1)}(\nu,w)$. The two contributions correspond to the two terms in the identity $\partial_x\bigl[2\sin(kx)\sin(\ell x)\bigr]=(k+\ell)\sin\bigl((k+\ell)x\bigr)-|k-\ell|\sin\bigl(|k-\ell|x\bigr)$.

The integral in (4.1) can be computed explicitly if w_ℓ is a polynomial. For other terms, we use the following estimate. Define $\|f\|_i=\sup_{t\in J_i}|f(t)|$ and

$$G_i(k,\ell)=\frac{k+\ell}{(\lambda_{k+\ell}-\lambda_k)+2/t_i}\,. \tag{4.2}$$

Proposition 4.1. *Let $\mathcal{K}$ and $\mathcal{L}$ be the supports of $k\mapsto\nu_k$ and $\ell\mapsto w_\ell$, respectively. Then*

$$\bigl\|Q^{(1+)}(\nu,w)\bigr\|_i\le 2\alpha\|\nu\|\|w\|\sup_{\substack{k\in\mathcal{K}\\ \ell\in\mathcal{L}}}G_i(k,\ell)e^{-\lambda_k t_{i-1}}\,. \tag{4.3}$$

The supremum in equation (4.3) can be determined in a finite computation, using the following monotonicity properties of G_i.

Proposition 4.2. *For $\ell>0$, the set of all $k>\kappa$ such that $G_i(k+1,\ell)<G_i(k,\ell)$ is of the form $\{k'_\ell,k'_\ell+1,\ldots\}$ with $\ell\mapsto k'_\ell$ non-increasing for $\ell>(\alpha/12)^{1/2}$. An analogous statement holds with the roles of k and ℓ exchanged. Furthermore, $G_i(k,\ell+1)\le G_i(k+1,\ell)$ for all k and ℓ.*

Similar bounds and monotonicity properties can be obtained for the remaining terms in the splitting of $K_\nu(\omega)$. For details we refer to the package `ContFuns.CE.Ops.KS` of computer programs [22]. The proofs of these propositions are elementary and thus will not be given here.

4.2. Choice of boxes

Our boxes B_j are centered at points $\bar u_j=\bar u(j\bar\tau/M)$ along an approximate periodic orbit $\bar u$, where $0\le j<M$. Consider now j fixed. Below we will describe a choice of basis $\Sigma_j=(\sigma_{j,1},\sigma_{j,2},\ldots,\sigma_{j,N})$ in $\mathbb{P}_L\mathcal{X}$. In particular, $\sigma_{j,2}$ is roughly the flow direction at $\bar u_j$. Using the unique representation $x=\sum_k c_k(x)\sigma_{j,k}+\mathbb{P}_H x$ for vectors $x\in\mathcal{X}$, we define $S_j=\{x\in\mathcal{X}:\eta_j(x)=\eta_j(\bar u_j)\}$, where $\eta_j(x)=c_2(x)$.

Definition 2.3 of a box requires a splitting $\mathcal{X}=\mathcal{Y}\oplus\Theta\oplus\mathcal{Z}$. We choose $\mathcal{Y}$ and Θ to be the one-dimensional spaces spanned by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively. Then $\mathcal{Z}$ is the closure of $\operatorname{Span}(\{\boldsymbol{v}_3,\boldsymbol{v}_4,\ldots\})$. In order to specify the box B_j, we choose (determine experimentally) positive real numbers $(r_{j,1},r_{j,3},\ldots,r_{j,N},\varepsilon_j)$, where $N=40$. An affine map $\psi_j:\mathcal{Y}\times\mathcal{Z}\to\mathcal{X}$ is now defined by setting $\psi_j(\boldsymbol{v}_k)=\bar u_j+r_{j,k}\sigma_{j,k}$ for $k=1,3,\ldots,N$, and $\psi_j(h)=\bar u_j+\varepsilon_j h$ for $h\in\mathbb{P}_H\mathcal{X}$. The box B_j is now defined as the image of the unit ball (for the norm below) in $\mathcal{Y}\times\mathcal{Z}$ under the map ψ_j. To be more precise, the norm (on $\mathcal{X}$) used here is

$$\|x\|'=\max\bigl\{|x_1|,|x_2|,\ldots,|x_N|,\|\mathbb{P}_H x\|\bigr\}\,,\qquad x=\sum_{k=1}^{\infty}x_k\boldsymbol{v}_k\,. \tag{4.4}$$

The first 8 vectors in the basis Σ_j are the first 8 approximate eigenvectors (in decreasing order of the the |eigenvalues|) for the derivative of the time-$\bar\tau$ map at the point $\bar u_j$.

Thus, $\sigma_{j,1}$ is approximately the expanding direction, and $\sigma_{j,1}$ approximates the flow direction, which corresponds to an eigenvalue 1. For $k = 9, \ldots, 20$ we choose $\sigma_{j,k} = (\mathbb{I} - P)\boldsymbol{v}_k$, where P is an approximation for the spectral projection for the largest 8 (in modulus) eigenvalues of the linearized time-$\bar{\tau}$ map at $\bar{u}_0$. For $k = 21, \ldots, 40$, we simply choose $\sigma_{j,k} = \boldsymbol{v}_k$. A precise description of all these choices can be found in the source code of our computer programs [22].

4.3. Mapping boxes by using convexity

In order to prove the covering relations described in Theorem 3.4, we need to compute the image of the box B_j under the intermediate Poincaré map P_j, for each j. Let now j be fixed. The first step in estimating P_j is to find an interval $[a, c]$ such that the section S_j is transversal, in the sense of Definition 2.1, to the flow with initial conditions in $B = B_{j-1}$ and times in $[a, c]$. Assume that we have found such an interval. Then, for every $u \in B$ there exists $t \in [a, c]$, such that $P_j(u) = \mathbb{E}\Phi_t(u)$. Here, $\mathbb{E}$ denotes some fixed affine projection from $\mathcal{X}$ onto S_j. Thus, in order to verify that B P_j-covers B_j, it suffices to verify that B $\mathbb{E}\Phi_t$-covers B_j, for every $t \in [a, c]$. In what follows, let $P_j' = \mathbb{E}\Phi_t$, for some fixed but arbitrary $t \in [a, c]$.

Computing $P_j'(B)$ directly, or even $P_j'(\partial B)$, is a prohibitive task. Fortunately, it suffices to compute the images of the "corners" of B. To be more precise, fix $0 < n < N$, set $I = \{1, 3, \ldots, n+1\}$, and denote by $\mathbb{P}$ the canonical projection onto the span of $\{\sigma_{j-1,k}\}_{k \in I}$. In order to simplify notation, assume that $\bar{u}_{j-1} = 0$. Write B as the sum of the n-dimensional rectangle $R = \mathbb{P}B$ and the ball $b = (\mathbb{I} - \mathbb{P})B$. The ball b can be regarded as the "center" of $B = b + R$. And by "corners" of B we mean the sets $b + w_i$, where $\{w_1, w_2, \ldots, w_m\}$ are the $m = 2^n$ corners of R.

In order to see why it suffices to compute the corners of B, consider the map $f(x) = P_j'(x) - P_j'(0)$, from $X = X_{j-1}$ to X_j, which satisfies $f(0) = 0$. The following arguments apply to more general situations.

Bound on f from a bound on Df. Suppose that, for every point $x \in X$ we have a bound $F(x)$ on $Df(B)x$. That is, $F(x)$ is a closed convex set containing $Df(w)x$, for every $w \in B$. If $t \mapsto w(t)$ is any continuous curve in B, then $\int_o^1 dtDf(w(t))x$ is a convex combination of elements from $F(x)$ and thus belongs to $F(x)$. In particular, taking $w(t) = tx$, we see that $f(x) \in F(x)$. Thus, F is a bound for f on B.

Convex combinations of corners. Now we use that every point $x \in B$ admits a (unique) representation $x = \xi + \sum_i s_i w_i$, where $\xi \in b$, and where $0 \le s_i \le 1$ are real numbers that add up to 1. Thus

$$f(x) = \int_0^1 dt\, Df(tx)x = \sum_i s_i \int_0^1 dt\, Df(tx)(\xi + w_i)\,, \tag{4.5}$$

and by convexity, this point belongs to $\sum_i s_i F(\xi + w_i)$. If $F(b + w_i)$ denotes a convex set containing $F(\xi + w_i)$, for every $\xi \in b$, then the bounds $\{F(b + w_i)\}_{i=1}^m$ are sufficient to estimate $f(x)$ for arbitrary $x \in B$.

4.4. The computer-assisted proof of Theorem 3.4

We continue our top-down approach to the original problem by reducing the proof of Theorem 3.4 further, to a point where it suffices to check a finite number of inequalities between (representable) numbers. In this section, we explain the problem-specific part of this process. The remaining task is simple enough that it can be automated and carried out by a computer. For the details of this last part we refer to the source code of our programs [22].

Besides explicit computations (basic operations for numbers or function), the proof of Theorem 3.4 involves the solution of implicit equations. In such cases, we first determine an approximate solution. Then the problem is reduced to verifying a-posteriori estimates that imply the existence of a true solution nearby. In particular, the sequence of boxes B_j represent a numerical approximation on the flow and its eigendirections. The covering relations are a-posteriori estimates that guarantee the existence of a true orbit, including the points $u_j \in B_j$ described in Theorem 3.4. Similarly at the level of local solution curves: The first step is to find an approximate fixed point w_0 of the map K_ν. A bound C on the fixed point w is then obtained by verifying set inclusions in $\mathcal{A}^{\mathbb{Z}}$, as described in Subsection 3.1. These inclusion relations in turn reduce to simple inequalities in $\mathbb{R}$.

Our computer programs are structured accordingly, into a "dynamical systems" level that deals with objects like boxes, a level where the main objects are functions in $\mathcal{C}(J,\mathcal{A}^{\mathbb{Z}})$, another level that deals with maps between the spaces $\mathcal{A}^{\mathbb{Z}}$, and a `Scalar` level. Every computation eventually ends up at the `Scalar` level; it can be carried out either in rigorous mode (`Scalar => Interval`), or in purely numeric mode (`Scalar => Numeric`) if the goal is to find an approximate solution. The other levels merely organize the proof.

Many of these steps require accurate bounds to succeed, and this has to be achieved with a finite (and reasonable) number of operations. This is made possible by the fact that the map K_ν is uniformly approximable by finite dimensional mappings. It allows for accurate bounds that involve only finitely many inequalities.

The general approach is quite standard by now. We start by associating to a space X a collection $\mathrm{std}(X)$ of subsets of X that are representable on the computer. We will refer to these sets as "standard sets" for X. A "bound" on an element $w \in X$ is then a set $W \in \mathrm{std}(X)$ containing w, while a bound on a map $f : X \to Y$ is a map $F : D_F \to \mathrm{std}(Y)$, with domain $D_F \subset \mathrm{std}(X)$, such that $f(w) \in F(W)$ whenever $w \in W \in D_F$. Notice that the composition of two bounds, if defined, is a bound on the corresponding composed map. This and other properties allow us to combine bounds on elementary maps into bounds on more complex maps like K_ν, and thus to mechanize the necessary estimates. Bounds are implemented as procedure or functions in our programs. Any procedure (or function) that uses a theorem first tries to verify that the hypotheses of the theorem are satisfied, for the given input. If the hypotheses cannot be verified, then the procedure aborts with an `Error` message. In that sense, every implemented bound F knows its domain: If a procedure (and thus any other procedure that may get invoked in the process) terminates without generating an `Error`, then the input is by definition in the domain of F.

The basic bounds used in the present proof have been developed already in [16], up to the level of bounds on basic operations (like sums, products, antiderivatives, norms, etc.) involving the spaces $\mathbb{R}$ and $\mathcal{X}^{\mathbb{Z}}$, as well as between products of these spaces. Thus, in order to avoid undue repetition, the reader is referred to [16,17] for a description of the bounds used at this level.

Here, we also have to choose and represent appropriate sets in the space $\mathcal{C}(J,\mathcal{X}^{\mathbb{Z}})$. For functions in $C(J,\mathbb{R})$, we use approximating polynomials of degrees up to $m=20$, and error estimates on $n=10$ subintervals J_i. Given $G=(G_0,\ldots,G_m)$ in $\mathrm{std}(\mathbb{R})^{m+1}$ and $H=(H_1,\ldots,H_n)$ in $\mathrm{std}(\mathbb{R}_+)^n$, define $C_{G,H}$ to be the set of all functions $f\in C(J,\mathbb{R})$ that admit a representation

$$f(t)=\sum_{k=0}^{m} g_k t^k + h(t)\,,\qquad h\in C(J,\mathbb{R})\,, \tag{4.6}$$

such that $g_j\in G_j$ for all j, and $\|h\|_i\le h_i\in H_i$ for all i. We now define $\mathrm{std}(C(J,\mathbb{R}))$ to be the collection of all such sets $C_{G,H}$. These standard sets correspond to the data type `ContFun` in our programs.

Consider now the space $\mathcal{C}(J,\mathcal{A}^{\mathbb{Z}})$. Functions in this space can be represented as Fourier series $u(t)=\sum_k u_k(t)\boldsymbol{v}_k$ with coefficients $u_k\in C(J,\mathbb{R})$. The data type for function $\nu=\sum_k \nu_k\boldsymbol{v}_k$ in $\mathcal{A}^{\mathbb{Z}}$, and the corresponding collection of subsets $\mathrm{std}(\mathcal{A}^{\mathbb{Z}})$, have already been defined in [17]. In fact, this type was derived from a generic data type `Fourier`, with coefficients in some unspecified Banach algebra, whose standard sets are represented by a generic type `FCoeff`. We can take advantage of this by instantiating `FCoeff` with `ContFun`, to define a derived type `TFourier`. The sets associated with data of type `TFourier` are our standard sets for the space $\mathcal{C}(J,\mathcal{A}^{\mathbb{Z}})$. We note that the (bound on the) norm inherited from this procedure corresponds to the trivial partition $\{J\}$ of J. The norm for other choices of the partition has to be implemented afterwards.

It is now straightforward to implement bounds on the basic operations involving the spaces $\mathcal{A}^{\mathbb{Z}}$ and $\mathcal{C}(J,\mathcal{A}^{\mathbb{Z}})$. This includes e.g. the evaluation map $u\mapsto u(t)$ from $\mathcal{C}(J,\mathcal{A}^{\mathbb{Z}})$ to $\mathcal{A}^{\mathbb{Z}}$. A bound on the map K_ρ is straightforward as well, albeit tedious, as Subsection 4.1 indicates. Our bound on K_ν yields a bound on the time-t map Φ_t, as described earlier. The map whose fixed point solves (2.5) is very similar to K_ν, and we can use the exact same estimates as for K_ν. This in turn yields a abound on the derivative $D\Phi_t$.

For the intermediate Poincaré maps P_j we use two different bounds. Let us first describe the "simple" version and its application. We start with a numerical guess for a time-interval $[a,c]$ that should contain the crossing times $t(\nu)$, for all initial conditions $\nu\in B_j$. Then we verify the transversality condition in Definition 2.1, with $B=B_{j-1}$ and $S=S_j$, using the convexity argument described in Subsection 4.3, for both $f=\Phi_a$ and $f=\Phi_c$. A bound on P_j is now given by a bound on $P_j'=\mathbb{E}\Phi_t$ that is valid for every $t\in[a,c]$. Here, $\mathbb{E}x$ is the vector obtained from $x\in\mathcal{X}$ by setting its second coordinate (in the basis Σ_j) equal to zero.

Consider now the problem of verifying that B_{j-1} P_j'-covers B_j. Let $f=\psi_j^{-1}\circ P_j'\circ\psi_{j-1}$, where ψ_{j-1} and ψ_j are the affine maps associated with the boxes B_{j-1} and B_j, respectively. That is, $B_j=\psi_j(B)$, where B denotes the unit ball in the space $X=\mathcal{Y}\oplus\mathcal{Z}$, equipped with the norm (4.4), where $x_2=0$. Let $I=\{1,3,\ldots,20\}$. Denote by $\mathbb{P}$ the canonical projection in X onto the span of $\{\boldsymbol{v}_k\}_{k\in I}$. Then every $x\in X$ has a unique representation $x=\sum_{k\in I}c_k(x)\boldsymbol{v}_k+\xi(x)$, where $\xi(x)=(\mathbb{I}-\mathbb{P})x$.

Using the notation from Subsection 4.3, the box B has $m = 2^{19}$ corners $b + w_i$, with $c_k(w_i) = \pm 1$ and $\xi(w_i) = 0$. We check that $c_1(f(b + w_i)) \geq \alpha_j > 1$ whenever $c_1(w_i) = 1$, and that $c_1(f(b + w_i)) \leq -\alpha_j < -1$ whenever $c_1(w_i) = -1$. In addition, we verify that $|c_k(f(b+w_i))| \leq \beta_j < 1$ for $k \geq 3$, and that $\|\xi(f(b+w_i))\| \leq \beta_j < 1$. This is done by using a bound F on Df, as described in Subsection 4.3. By convexity, the above inequalities imply that B_{j-1} P'_j-covers B_j. This holds for every $t \in [a, c]$, so B_{j-1} P_j-covers B_j.

Once these covering relations have been verified, the existence of the periodic orbit u follows from Corollary 2.5. This allows us to bound the derivatives $DP_j(u_{j-1})$ by using equation (2.6) and Proposition 2.6. Now we repeat the steps described above, but with F a bound on the derivative of $f = \psi_j^{-1} \circ P_j \circ \psi_{j-1}$ on B. By linearity, this is equivalent to verifying an (α_j, β_j) cone condition.

For a detailed and complete description of all these steps, we refer to the source code (in Ada95) and input data for our computer programs [22]. These programs were run successfully on several different types of machines, using public versions of the GNAT compiler [23]).

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