Fast solitons on star graphs

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Abstract

We define the Schrödinger equation with focusing, cubic nonlinearity on one-vertex graphs. We prove global strong and weak well-posedness and conservation laws for some self-adjoint boundary conditions at the vertex, i.e. Kirchhoff and the so called δ and δ'. Moreover, in the same setting we study the collision of a fast solitary wave with the vertex and we show that it splits in reflected and transmitted components. The outgoing waves preserve a soliton character over a time which depends on the logarithm of the velocity of the ingoing solitary wave. Over the same timescale the reflection and transmission coefficients of the outgoing waves coincide with the corresponding coefficients of the linear problem. In the analysis of the problem we follow ideas borrowed from the seminal paper [14] about scattering of fast solitons by a delta interaction on the line, by Holmer, Marzuola and Zworski; the present paper represents a generalization of their work to graphs and, as a byproduct, it shows how to extend the analysis of soliton scattering by other point interactions on the line, interpreted as a degenerate graph.

1 Introduction

In the present paper we begin the study of the *nonlinear* wave propagation on graphs. While there exists an extensive literature on the behaviour of linear wave and Schrödinger equations on graphs ([19, 20, 18, 4, 5], and references therein) and a certain activity concerning the so called *discrete* nonlinear Schrödinger equation (DNLSE) in chains with edges and graphs inserted ("decorations", interpreted as defects in the chain), both from the physical or numerical point of view (see for example [17, 6]), as far as we known the subject of nonlinear Schrödinger evolution is new.

The study of nonlinear propagation in ramified structures could be of relevance in several branches of pure and applied science, from condensed matter physics to nonlinear fiber optics, hydrodynamics and fluid transport (a non traditional example is blood flow in veins and arterias) and finally neural networks (see for example the study of *reaction diffusion type* Fitz-Nagumo-Rall equations on networks in [9], and references therein).

Of course in all these examples there is a strong dependence on the modelization. The nonlinear Schrödinger equation with cubical nonlinearity is especially suitable to describe nonlinear electromagnetic pulse propagation in optical fibers and, under the name of Gross-Pitaevskii equation, the dynamics of Bose condensates. Better suited for other applications, for example the hydrodynamic flow, is the KdV equation or its relatives, which we do not treat here.

A word has to be spent about the model of the ramified structure. Real world is not strictly one dimensional, and an abstract graph, which is just a set of copies of $\mathbb{R}^+$ ("edges", or "branches") with functions living on it satisfying certain boundary conditions at 0 (see section 2 for the rigorous notion) lacks some of the geometric meaningful characteristics of a real network, such as thickness and curvature of the branch or orientation between edges. On the other hand these problems are far from being well understood also for the linear propagation (see for example [3, 7]), and so we content ourselves with posing and analysing the nonlinear problem in the idealized and simplified framework of an abstract graph.

We study in this paper the special case of a star graph with three edges. A generalization to a star graph with n edges would be straightforward, but here our interest is in clarifying the main behaviours and techniques involved in the analysis. A preliminary point and not a trivial issue is the definition of the dynamics. Let us recall that for a star graph $\mathcal{G}$, the linear Schrödinger dynamics is defined by giving a self-adjoint operator H on the product of n copies of $L^2(\mathbb{R}^+)$ (briefly $L^2(\mathcal{G})$), with a domain $D(H)$ in which appears a linear condition involving the values at 0 of the functions of the domain and of their derivatives. The admissible boundary conditions characterize the interaction at the vertex of the star graph. For the nonlinear problem, we establish the well-posedness of the dynamics in the case of a star graph with some distinguished boundary condition at the vertex, namely the free (or Kirchhoff), and the δ and δ' boundary conditions (see section 2 for the relevant definitions).

To be clear about the exact problem with we are faced to, the differential equation to be studied is of the form

$$i\frac{d}{dt}\Psi_t \ = \ H\Psi_t - |\Psi_t|^2\Psi_t \tag{1.1}$$

where the function Ψ is, for a three-edge graph, a column vector

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} ,$$

that lies in the domain $D(H)$ of the linear Hamiltonian H on the graph, so emboding the relevant boundary conditions. This is the abstract strong form of the equation, which, as it is apparent, is equivalent to a particular nonlinear coupled system of scalar equations. The coupling is not due to the nonlinearity, because of the definition

$$|\Psi|^2\Psi \equiv \begin{pmatrix} |\psi_1|^2\psi_1 \\ |\psi_2|^2\psi_2 \\ |\psi_3|^2\psi_3 \end{pmatrix} ,$$

but to the boundary condition at the vertex. For example, in the simple case of a Kirchhoff boundary condition, the coupling between various components of the generic domain element is given by

$$\Psi \in L^2(\mathcal{G}) \text{ s.t. } \psi_i \in H^2(\mathbb{R}^+),\ \psi_1(0) = \psi_2(0) = \psi_3(0),\ \psi_1'(0) + \psi_2'(0) + \psi_3'(0) = 0 \ .$$

The δ or δ' boundary conditions allow a coupling between the values of the function Ψ and values of their derivatives at the origin, but in principle the nature of the problem is unaltered.
In the present paper, for several reasons, we prefer to write the dynamics in weak form, which is the following

$$\Psi_t \ = \ e^{-iHt}\Psi_0 + i\int_0^t e^{-iH(t-s)}|\Psi_s|^2\Psi_s\, ds \ ; \tag{1.2}$$

here the Ψ is in the form domain $\mathcal{D}(\mathcal{E}_{lin})$, where $\mathcal{E}_{lin}$ is the quadratic form of the linear Hamiltonian H. We interpret, according to the use, the form domain as the finite energy space for the problem. An adaptation of the methods in [2] gives the local well-posedness of the equation (1.2) for every initial datum in $\mathcal{D}(\mathcal{E}_{lin})$. Moreover, charge and energy conservation laws hold true for such weak solutions, and as a consequence, the

NLS on graph admits global solutions. Similar results could be shown without essential additional difficulties in the case of a general boundary condition at the vertex. These generalizations will be treated elsewhere.

Apart from well-posedness, the main goal of this paper is to provide information on the interaction between a solitary wave and the boundary condition at the vertex. As it is well known, the NLS on the line admits a family of solitary non dispersive solutions, rigidly translating with a fixed velocity. The family of solitary solutions ("solitons") is given by the action of the Galilei group on the function

$$\phi(x) \;=\; \sqrt{2}\cosh^{-1} x \;,$$

or explicitly,

$$\phi_{x_0,v}(x,t) = e^{i\frac{v}{2}x}e^{-it\frac{v^2}{4}}e^{it}\phi(x - x_0 - vt)\,.$$

We show that, after the collision of a similar solitary wave with the vertex there exists a timescale during which the dynamics can be described as the scattering of three split solitary waves, one reflected on the same branch where the originary soliton was running asymptotically in the past, and two transmitted solitary waves on the other branches. On the same timescale, the amplitudes of the reflected and transmitted solitary waves are given by the scattering matrix of the linear dynamics on the graph. The soliton-like character persists over time intervals that depend on the velocity of the impinging original soliton: the faster is the original soliton, the (logarithmically in the velocity) longer is the survival time of the solitary wave behaviour on every branch of the graph. The non-trivial point is that the persistence time of solitary behaviour after collision with the vertex is much longer, for fast solitons, than the time over which it is reasonable to approximate the nonlinear dynamics with the linear one. The same timescale of the order $\ln v$ of persistence of solitary behaviour appears in the paper [1] where the collision of two solitary waves with an underlying smooth potential is studied, and in the paper by Holmer, Marzuola and Zworski [14] on the fast NLS-soliton scattering by a delta potential on the line, which is the main source of inspiration for the present paper.

We give now an outline of our result and its proof.
The initial datum is of the following form

$$\Psi_0(x) \;=\; \begin{pmatrix} \chi(x)e^{-i\frac{v}{2}x}\phi(x - x_0) \\ 0 \\ 0 \end{pmatrix}$$

where χ is a cut off function, that is $\chi \in C^{\infty}(\mathbb{R}^+)$, $\chi = 1$ in $(2, +\infty)$ and $\chi = 0$ in $(0, 1)$. Apart from a small tail term truncated by the cutoff function, the first component is the initial datum of a free (i.e. without external potentials) NLS which on the line yields to a solitary wave running with velocity v; the center x_0 of the initial soliton is chosen far from the vertex; the precise condition, given in term of the velocity, is $x_0 \geq v^{1-\delta}$ with $\delta \in (0, 1)$. We are interested in the evolution Ψ_t of this initial datum.

The analysis of the dynamics can be divided in several phases. The pre-interaction phase, where the evolved initial datum is far from the vertex, and the undisturbed NLS evolution dominates. At the end of this phase, the solution enters the vertex zone, and differs (in L^2 norm) from the evolved solitary wave for an exponentially small error in the velocity v. The second phase is the interaction phase, in which a substantial fraction of the mass of the initial soliton has reached the vertex, and the linear dynamics dominates due to the shortness of the interaction time, leaving the system at the end of this phase with three scattered waves, the amplitudes of which are given by the action of the scattering matrix of the associated linear graph on the incoming solitary wave. The size of the corresponding error is (again in L^2 norm) a suitable negative inverse power ($v^{-\frac{1}{2}\delta}$) of the velocity. In both the phases one and two, the main technical tool consists in an accurate use of Strichartz estimates to control the deviations between the unperturbed NLS flow, the global NLS flow and the linear flow on the graph in the relevant time intervals. Finally there is the post-interaction phase, where the free NLS dynamics dominates again; however, now the initial data are not exact solitary waves, but waves with solitonlike profiles and "wrong" amplitudes (due to the scattering process in the interaction phase).

The true evolution is confronted with a reference dynamics given by the superposition of the nonlinear evolution of the outgoing scattered profiles, and it turns out that the error is, in L^2 norm, of the order of an inverse power of velocity (which power depending on the size of the time interval of approximation). For a precise formulation one has to tackle the problem of representing the reference soliton dynamics to be confronted with the true dynamics. This problem arises because one would like to use crucial and known properties of NLS on the line (such as existence of infinite constants of motion), and various associated estimates, while on a star graph one has a NLS on halflines, jointly with boundary conditions. The problem occurs, of course, in each of the three phases in which the dynamics is decomposed. The details are in sections 4 and 5, and here to state the main result of the paper, we refer succinctly to the phase three only.

Our choice of reference dynamics is the following. We associate to every edge of the star graph a companion edge chosen between the other two, in such a way to have three fictitious lines; then we glue the soliton on every single edge with the right tail on the companion edge, respecting the free nonlinear dynamics. One of the main technical point in the analysis of the true dynamics is to have a control in the errors brought by this schematization. More precisely, let us define

$$\tilde{\Phi}_t^1(x_1,x_2,x_3) \equiv \begin{pmatrix} \tilde{r}e^{-i\frac{v^2}{4}t}e^{i\frac{v}{2}x_1}e^{it}\phi(x_1+x_0-vt) \\ \tilde{r}e^{-i\frac{v^2}{4}t}e^{-i\frac{v}{2}x_2}e^{it}\phi(x_2-x_0+vt) \\ 0 \end{pmatrix}$$

$$\tilde{\Phi}_t^2(x_1,x_2,x_3) \equiv \begin{pmatrix} 0 \\ \tilde{t}e^{-i\frac{v^2}{4}t}e^{i\frac{v}{2}x_2}e^{it}\phi(x_2+x_0-vt) \\ \tilde{t}e^{-i\frac{v^2}{4}t}e^{-i\frac{v}{2}x_3}e^{it}\phi(x_3-x_0+vt) \end{pmatrix}$$

$$\tilde{\Phi}_t^3(x_1,x_2,x_3) \equiv \begin{pmatrix} \tilde{t}e^{-i\frac{v^2}{4}t}e^{-i\frac{v}{2}x_1}e^{it}\phi(x_1-x_0+vt) \\ 0 \\ \tilde{t}e^{-i\frac{v^2}{4}t}e^{i\frac{v}{2}x_3}e^{it}\phi(x_3+x_0-vt) \end{pmatrix}$$

Each of these vectors represents a soliton on the fictitious line given by an edge and its companion, multiplied by the scattering coefficients of the linear dynamics considered, Kirchhoff, δ or δ' (and here unspecified). Up to small error, these functions represent outgoing waves at the end ($t = t_2$) of interaction phase, which is essentially a scattering process. Taking these as initial data for the free nonlinear dynamics on the pertinent fictitious line, we define their time evolution Φ_t^j as given by

$$\Phi_t^j = e^{-iH_j(t-t_2)}\tilde{\Phi}_{t_2}^j + i\int_{t_2}^t ds\, e^{-iH_j(t-s)}|\Phi_s^j|^2\Phi_s^j \qquad j=1,2,3$$

where the Hamiltonians H_j are the linear Hamiltonians with Kirchhoff boundary conditions in which the j-branch is excluded, in fact a free dynamics (see subsection 2.3 for the precise definition of H_j and remark 4.2 for a different characterization of the nonlinear evolution of Φ^j).

The $\{\Phi_t^j\}$ represent our reference dynamics (see complete definitions in section 4, where we adopt the slightly heavier notation $\Phi_{H,t}^j$, to take in account differences induced by the various vertex boundary conditions).

With these premises, the main result of the paper is the following.

Theorem 1.1. *There exist $\tau_* > 0$ and $T_* > 0$ such that for $0 < T < T_*$ one has*

$$\|\Psi_t - \Phi_t^1 - \Phi_t^2 - \Phi_t^3\|_{L_x^2(\mathcal{G})} \leqslant Cv^{-\frac{T_*-T}{\tau_*}}$$

for every time t in the interval $t_2 < t < t_2 + T\ln v$, where C is a positive constant independent of t and v.

In a different terminology (see the cited paper [14] for an analogous result) the previous approximation can be written as

$$\Psi_t = \Phi_t^1 + \Phi_t^2 + \Phi_t^3 + \mathcal{O}_{L_x^2}(v^{-\frac{T_*-T}{\tau_*}}) .$$

To get the previous estimate, Strichartz estimates do not suffice, and more direct properties coming from the integrable character of the cubical NLS are needed. In particular, thanks to the existence of infinite integrals of motion, in [14] a spatial localization property of the solution of NLS with smooth data it is proven, with a bound polinomial (and not exponential) in time. This gives a control on the tails of the difference between the solution and the reference modified solitary dynamics. An analogous method applies in our case, with the help of standard Strichartz estimates.

A further consequence of Theorem 1.1 is the fact that fast solitons have reflection and transmission coefficients which, up to negligible corrections, coincide with the corresponding coefficient of the linear graph. For example, a definition of the transmission coefficients along the branches $j = 2, 3$ could be given considering the ratio between the amount of mass on the $j-$edge and the total mass, in the limit $t \to \infty$.

$$\lim_{t\to\infty} \frac{\|\Psi_t^j\|}{\|\Psi_t\|} \equiv |t^j(v)| \ .$$

where Ψ^j denotes the restriction of the solution to the $j-$edge. In our case we do not have at our disposal the rigorous asymptotics for $t \to \infty$; nevertheless, we can obtain a weaker result. We have the results of Theorem 1.1, which gives, in the time interval $t_2 < t < t_2 + T \ln v$, the estimate

$$\frac{\|\Psi_t^j\|}{\|\Psi_t\|} = |\tilde{t}^j| + \mathcal{O}(v^{-\sigma})$$

for a certain $\sigma > 0$ and where $\tilde{t}^j$ is the scattering coefficient of the *linear* Hamiltonian which describes the vertex. So, in the limit of fast solitons, i.e. $v \to \infty$, one can assert that the ratio which defines the nonlinear scattering coefficient converges to the corresponding linear scattering coefficient. And analogously for the case of reflection coefficient $r(v)$, i.e. $j = 1$, one has (for $t > t_2$, as before)

$$\lim_{v\to\infty} \frac{\|\Psi_t^1\|}{\|\Psi_t\|} = |\tilde{r}| \ .$$

This is true for every coupling between the ones considered, i.e. Kirchhoff, δ and δ'.

We add some other remarks on the result and further analysis and generalizations.
To begin with, let us note that the real line with a point interaction at 0 can be interpreted as a degenerate graph with two edges. The cited paper [14] treat the special case of a δ interaction on the line, and our description shows how it could be possible to extend their results to other point interaction; the example here treated, apart δ and Kirchhoff conditions is (a version of) the δ' interaction, but in principle the analysis is applicable to other point interaction, also of more singular nature of the δ.

Concerning more general items, a sharper description of the post interaction phase can be achieved by an explicit characterization of the evolution of the modified solitary profiles, i.e. of the Φ_t^j . This last part is somewhat delicate, and intersects with contemporary intense work on asymptotics for solitons in integrable and quasi integrable PDE, so we limit ourselves to the following remarks. In the case analysed in [14], the asymptotic behaviour of nonlinear Schrödinger evolution of solitary waveforms with modified amplitudes is given, and making use of inverse scattering theory it is shown (appendix A of the cited paper) that the evolution is close to a soliton up to times of order $\ln v$ and an error the order of which is an inverse power of velocity v. Borrowing from these results, it is possible to get in our case too, but we omit details, the asymptotics of the Φ_t^j, i.e. of the free NLS evolution of the modified solitary profiles outgoing from the phase two. It turns out that these outgoing wavefunctions can be approximated, on the same logarithmic timescale of Theorem 1.1, as new solitons with the same waveform of the unperturbed dynamics, modified amplitudes and phases, plus a dispersive ("radiation") contribution. The meaning of this statement is that the L_x^∞ norm of the difference between the evolved modified solitary profiles Φ_t^j and such final outgoing solitons has the usual dispersive behaviour, $|t - t_2|^{-\frac{1}{2}}$. Let us note that at the end of the phase three, there are two types of errors: errors due to the approximation procedure in phase one and two ($\mathcal{O}_{L_x^2}$) ; and errors arising from neglecting dispersion in the reconstruction of the outgoing solitons ($\mathcal{O}_{L^\infty}$) .

An important question concerns the possibility of extending the timescale of validity of approximation by the solitary outgoing waves. As a quite generic remark, this possibility could be related to the asymptotic stability of the system, or of systems immediately related to it.
More concretely, in a different type of model (scattering of two solitons on the line) in the already cited paper [1], some considerations are given on obtaining longer timescales of quasiparticle approximation in dependence of the initial data and external potential, but it is unclear if similar considerations apply to the present case.
Another issue is the nonlinearity. The fundamental asymptotics proved in [14] and used in the present paper relies on the integrable nature of cubic NLS, and it is not immediate to extend these results to more general nonlinearities. One can conjecture that for nonlinearities close to integrable which admit solitary waves, the outgoing waves are close to solitons over suitable timescales. Let us mention, however, the recent results of G.Perelman on the asymptotics of colliding solitons for nonlinearity close to integrable or L^2 critical on the line ([22] and results announced in [21]).

A final, less difficult, problem is the extension of results of the present work to more general graphs. We believe that results similar to the ones of the present papers could be valid for more general boundary condition at the vertex of a star graphs, with the same proof, under the condition of absence of the eigenvalues of the linear Hamiltonian describing the graph. In the presence of eigenvalues, some Strichartz estimates weaken, and a more refined analysis is needed (see [11] for the analogous problem on the line with an attractive δ interaction).
The extension to graphs of less trivial topology is an open problem.

1.1 Setting and notations

We consider a graph $\mathcal{G}$ given by three infinite half lines attached to a common vertex. In order to study a quantum mechanical problem on $\mathcal{G}$, the natural Hilbert space is then $L^2(\mathcal{G}) \equiv L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+)$. We denote the elements of $L^2(\mathcal{G})$ by capital greek letters, while functions in $L^2(\mathbb{R}^+)$ are denoted by lowercase greek letters. It is convenient to represent functions in $L^2(\mathcal{G})$ as column vectors of functions in $L^2(\mathbb{R}^+)$, namely

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}.$$

The norm of L^2-functions on $\mathcal{G}$ is naturally defined by

$$\|\Psi\|_{L^2(\mathcal{G})} := \left(\sum_{j=1}^3 \|\psi_j\|^2_{L^2(\mathbb{R}^+)} \right)^{\frac{1}{2}}.$$

Analogously, given $1 \leqslant p \leqslant \infty$, we define the space $L^r(\mathcal{G})$ as the set of functions on the graph whose components are elements of the space $L^r(\mathbb{R}^+)$, and the norm is correspondingly defined by

$$\left\|\Psi\right\|_{L^r(\mathcal{G})} = \left(\sum_{j=1}^3 \|\psi_j\|^r_{L^r(\mathbb{R}^+)} \right)^{\frac{1}{r}}.$$

When a functional norm is referred to a function defined on the graph, we omit the symbol $\mathcal{G}$. Furthermore, from now on, when such a norm is L^2, we drop the subscript, and simply write $\|\cdot\|$. Accordingly, we denote by $(\cdot,\cdot)$ the scalar product in L^2.

As it is standard when dealing with Strichartz's estimates, we make use of spaces of functions that are measurable as functions of both time (on the interval $[T_1, T_2]$) and space (on the graph). We denote such spaces by $L^p_{[T_1,T_2]}L^r(\mathcal{G}))$, and indices $1 \leqslant r \leqslant \infty$, $1 \leqslant p \leqslant \infty$; we endow them with the norm

$$\left\|\Psi\right\|_{L^p_{[T_1,T_2]}L^r(\mathcal{G})} = \left(\int_{T_1}^{T_2} \left\|\Psi_s\right\|^p_{L^r} ds \right)^{1/p}.$$

The extension to the case $p=\infty$ or $r=\infty$ is straightforward.
Besides, we need to introduce the spaces

$$H^1(\mathcal{G}) \equiv H^1(\mathbb{R}^+)\oplus H^1(\mathbb{R}^+)\oplus H^1(\mathbb{R}^+), \qquad H^2(\mathcal{G}) \equiv H^2(\mathbb{R}^+)\oplus H^2(\mathbb{R}^+)\oplus H^2(\mathbb{R}^+),$$

equipped with the norms

$$\|\Psi\|^2_{H^1(\mathcal{G})} = \sum_{i=1}^3 \|\psi_i\|^2_{H^1(\mathbb{R}^+)}, \qquad \|\Psi\|^2_{H^2(\mathcal{G})} = \sum_{i=1}^3 \|\psi_i\|^2_{H^2(\mathbb{R}^+)}.$$

The product of functions is defined componentwise,

$$\Psi\Phi \equiv \begin{pmatrix} \psi_1\phi_1 \\ \psi_2\phi_2 \\ \psi_3\phi_3 \end{pmatrix}, \qquad \text{so that} \qquad |\Psi|^2\Psi \equiv \begin{pmatrix} |\psi_1|^2\psi_1 \\ |\psi_2|^2\psi_2 \\ |\psi_3|^2\psi_3 \end{pmatrix}.$$

We denote by $\mathbb{I}$ the 3×3 identity matrix, while $\mathbb{J}$ is the 3×3 matrix whose elements are all equal to one. When an element of $L^2(\mathcal{G})$ evolves in time, we use in notation a subscript t: for instance, Ψ_t. Sometimes we shall write $\Psi(t)$ in order to highlight the dependence on time, or whenever such a notation is more understandable.

2 Summary on linear dynamics on graphs

2.1 Hamiltonians and quadratic forms

Standard references about Schrödinger equation on graphs are [4, 5, 19, 20, 18], to which we refer for complete treatment. Here we only give the definitions needed to have a self-contained exposition.

We consider three Hamiltonian operators, denoted by H_F, H_δ^α, $H_{\delta'}^\beta$, and called, respectively, the Kirchhoff, the Dirac's delta, and the delta-prime Hamiltonian. The three of those operators act as

$$\Psi \longmapsto \begin{pmatrix} -\psi_1'' \\ -\psi_2'' \\ -\psi_3'' \end{pmatrix} \tag{2.1}$$

on some subspace of $H^2(\mathcal{G})$, to be defined by suitable boundary conditions at the vertex.

Here and in the following subsection we collect some well-known facts (see [18], [19],[20] [4]) on H_F, H_δ^α, and $H_{\delta'}^\beta$.

The Kirchhoff Hamiltonian H_F acts on the domain

$$\mathcal{D}(H_F) := \{\Psi \in H^2(\mathcal{G}) \text{ s.t. } \psi_1(0)=\psi_2(0)=\psi_3(0),\ \psi_1'(0)+\psi_2'(0)+\psi_3'(0)=0\}. \tag{2.2}$$

It is well known, see [18], that (2.2) and (2.1) define a self-adjoint Hamiltonian on $L^2(\mathcal{G})$. Boundary conditions in (2.2) are usually called Kirchhoff boundary conditions. We use the index F to remind that H_F reduces to the free Hamiltonian on the line for a degenerate graph composed of two half lines.
The quadratic form $\mathcal{E}_F$ associated to H_F is defined on the subspace

$$\mathcal{D}(\mathcal{E}_F) = \{\Psi \in H^1(\mathcal{G}) \text{ s.t. } \psi_1(0)=\psi_2(0)=\psi_3(0)\}$$

and reads

$$\mathcal{E}_F[\Psi] = \sum_{i=1}^3 \int_0^{+\infty} |\psi_i'(x)|^2\, dx\,.$$

The Dirac's delta Hamiltonian is defined on the domain

$$\mathcal{D}(H_\delta^\alpha) := \{\Psi \in H^2(\mathcal{G}) \text{ s.t. } \psi_1(0) = \psi_2(0) = \psi_3(0),\, \psi_1'(0) + \psi_2'(0) + \psi_3'(0) = \alpha\psi_1(0)\}$$

Again, H_δ^α is a self-adjoint opeator on $L^2(\mathcal{G})$ ([18]). It appears that H_δ^α generalizes the ordinary Dirac's delta interaction on the line.
The quadratic form $\mathcal{E}_\delta^\alpha$ associated to H_δ^α is defined on

$$\mathcal{D}(\mathcal{E}_\delta^\alpha) = \{\Psi \in H^1(\mathcal{G}) \text{ s.t. } \psi_1(0) = \psi_2(0) = \psi_3(0)\}$$

and is given by

$$\mathcal{E}_\delta^\alpha[\Psi] = \sum_{i=1}^{3} \int_0^{+\infty} |\psi_i'(x)|^2\, dx\, + \alpha|\psi_1(0)|^2.$$

The delta-prime Hamiltonian is defined on the domain

$$\mathcal{D}(H_{\delta'}^\beta) := \{\Psi \in H^2(\mathcal{G}) \text{ s.t. } \psi_1'(0) = \psi_2'(0) = \psi_3'(0),\, \psi_1(0) + \psi_2(0) + \psi_3(0) = \beta\psi_1'(0)\}$$

Again, $H_{\delta'}^\beta$ is a self-adjoint opeator on $L^2(\mathcal{G})$ ([18]).
The quadratic form $\mathcal{E}_{\delta'}^\beta$ associated to $H_{\delta'}^\beta$ is defined on $\mathcal{D}(\mathcal{E}_{\delta'}^\beta) = H^1(\mathcal{G})$ and is given by

$$\mathcal{E}_{\delta'}^\beta[\Psi] = \sum_{i=1}^{3} \int_0^{+\infty} |\psi_i'(x)|^2\, dx\, + \frac{1}{\beta}\left|\sum_{i=1}^{3} \psi_i(0)\right|^2.$$

Notice that $H_{\delta'}^\beta$ does not reduce to the standard δ' interaction on the line when it is restricted to a two edges graph. Here we are following the notation in [19], [20]. The present δ' vertex is called sometimes δ_s' graph, where s is for symmetric. A discussion of the correct extension of the usual δ' interaction is given in [12] and [5]. For completeness, we give the operator domain (the action is the same as the other cases). We use the denomination $\tilde{\delta}'$ to avoid confusion with the previously defined interaction.

$$\mathcal{D}(H_{\tilde{\delta}'}^{\tilde{\beta}}) := \{\Psi \in H^2(\mathcal{G}) \text{ s.t. } \psi_j(0) - \psi_k(0) = \frac{\tilde{\beta}}{n}(\psi_j'(0) - \psi_k'(0))\ , \quad j,k = 1,2,...n\}\ .$$

Throughout the paper we restrict to the case of repulsive delta and delta-prime (δ') interaction, i.e., $\alpha, \beta > 0$. It is easily proved, for example by inspection of the operator resolvents in the following subsection, that such a condition prevents the corresponding Hamiltonian operator from possessing bound states.

2.2 Resolvents, scattering, and propagators

First we consider the case of the Kirchhoff Hamiltonian. For any complex number k with $\text{Im } k > 0$ we denote by $R_F(k)$ the resolvent of H_F, namely $R_F(k) := (H_F - k^2)^{-1}$. The operator $R_F(k)$ is the integral operator on $L^2(\mathcal{G})$ whose kernel reads

$$R_F(k;x,y) = \frac{i}{2k}e^{ik|x-y|}\mathbb{I} + \frac{i}{2k}e^{ik(x+y)}\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

From the expression of the resolvent one immediately has the reflexion and transmission coefficient:

$$r(k) \;=\; -\frac{1}{3}, \qquad t(k) \;=\; \frac{2}{3}. \tag{2.3}$$

We define the function U_t as

$$U_t(x) \;:=\; \frac{e^{i\frac{x^2}{4t}}}{\sqrt{4\pi i t}}\,.$$

In the following we shall use the same symbol U_t to denote the operator in $L^2(\mathbb{R})$ defined by

$$\big[U_t\psi\big](x) := \int_{-\infty}^{\infty} U_t(x-y)\psi(y)dy\,.$$

Moreover, we define two integral operators $U_t^{\pm}$ acting on $L^2(\mathbb{R}^+)$

$$U_t^{\pm}\ :\ L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+),\qquad \big[U_t^{\pm}\psi\big](x) = \int_0^{+\infty} U_t(x\pm y)\psi(y)\,dy\,.$$

We stress that, according to our definitions, the operators U_t^- and U_t have the same integral kernel, but act on different Hilbert spaces.

With this notation the propagator $e^{-iH_F t}$ is given by

$$e^{-iH_F t} \;=\; U_t^-\mathbb{I} + U_t^+\frac{1}{3}\begin{pmatrix} -1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix} \;=\; (U_t^- - U_t^+)\mathbb{I} + \frac{2}{3}U_t^+\mathbb{J}. \tag{2.4}$$

Concerning the Hamiltonian H_δ^α, the related resolvent $R_\delta^\alpha(k) := (H_\delta^\alpha - k^2)^{-1}$ reads

$$R_\delta^\alpha(k;x,y) = \frac{i}{2k}e^{ik|x-y|}\mathbb{I} - \frac{i}{2k}\frac{e^{ik(x+y)}}{\alpha-3ik}\begin{pmatrix} \alpha-ik & 2ik & 2ik\\ 2ik & \alpha-ik & 2ik\\ 2ik & 2ik & \alpha-ik\end{pmatrix},$$

therefore the scattering coefficients read

$$r_{H_\delta^\alpha}(k) \;=\; -\frac{k+i\alpha}{3k+i\alpha},\qquad t_{H_\delta^\alpha}(k) \;=\; \frac{2k}{3k+i\alpha}\,. \tag{2.5}$$

We use the standard formula

$$U_{\delta,t}^\alpha(x,y) \;=\; \frac{1}{\pi i}\int_{-\infty}^{+\infty} e^{-ik^2t}\; R_\delta^\alpha(k;x,y)\,kdk,$$

where $U_{\delta,t}^\alpha(x,y)$ denotes the integral kernel that represents the unitary propagator of the operator $e^{-iH_\delta^\alpha t}$, and then, owing to the identity

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{ik(x+y)}}{a-ik}\,e^{-ik^2t}\,dk \;=\; \int_0^{+\infty} e^{-au}\,\frac{e^{i\frac{(x+y+u)^2}{4t}}}{\sqrt{4\pi it}}\,du,$$

we finally obtain

$$U_{\delta,t}^\alpha(x,y) \;=\; [U_t(x-y)-U_t(x+y)]\mathbb{I} + \frac{2}{3}\left[U_t(x+y) - \frac{\alpha}{3}\int_0^{+\infty} du\, e^{-\frac{\alpha}{3}u}U_t(x+y+u)\right]\mathbb{J}. \tag{2.6}$$

For the delta-prime interaction one can proceed analogously: first, the resolvent $R_{\delta'}^\beta(k) := (H_{\delta'}^\beta - k^2)^{-1}$ is known and coincides with the integral operator in $L^2(\mathcal{G})$ whose kernel reads

$$R_{\delta'}^\beta(k;x,y) = \frac{i}{2k}e^{ik|x-y|}\mathbb{I} - \frac{i}{2k}\frac{e^{ik(x+y)}}{3-i\beta k}\begin{pmatrix} -1+i\beta k & 2 & 2\\ 2 & -1+i\beta k & 2\\ 2 & 2 & -1+i\beta k\end{pmatrix}.$$

$$r_{H_{\delta'}^\beta}(k) \;=\; \frac{\beta k+i}{\beta k+3i},\qquad t_{H_{\delta'}^\beta}(k) \;=\; -\frac{2i}{\beta k+3i}. \tag{2.7}$$

Again, using (2.2) we get the integral kernel $U^{\beta}_{\delta',t}(x,y)$ of the propagator $e^{-iH^{\beta}_{\delta'}t}$, namely

$$U^{\beta}_{\delta',t}(x,y) \;=\; [U_t(x-y)+U_t(x+y)]\mathbb{I} - \frac{2}{\beta}\int_0^{+\infty} du\, e^{-\frac{3}{\beta}u}U_t(x+y+u)\mathbb{J}. \tag{2.8}$$

Throughout the paper we'll need some auxiliary dynamics to be compared with the dynamics described by (1.1), so, for later convenience, we introduce the two-edge Hamiltonians H_j and the corresponding two-edge propagators e^{-iH_jt}, $j=1,2,3$.
Let H_j be defined by:

$$\mathcal{D}(H_j) := \{\Psi \in H^2(\mathcal{G}) \text{ s.t. } \psi_j(0)=\psi_{j+1}(0)\,,\, \psi_j'(0)+\psi_{j+1}'(0)=0\,, \psi_k(0)=0\,,\, k\neq j,j+1\}$$

$$H_j\Psi := \begin{pmatrix} -\psi_1'' \\ -\psi_2'' \\ -\psi_3'' \end{pmatrix},$$

where in equation (2.2) it is understood that $j=\{1,2,3\}$ modulo 3.
The Hamiltonian H_j couples the edges j and $j+1$ with a Kirchhoff boundary condition and sets a Dirichlet boundary condition for the remaining edge, so that there is free propagation between the edges j and $j+1$ and no propagation between them and the edge $j+2$.
With a straightforward computation we have

$$e^{-iH_jt} \;=\; U_t^-\mathbb{I} + U_t^+\mathbb{T}_j\,, \tag{2.9}$$

where $\mathbb{T}_j$ are the matrices

$$\mathbb{T}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\;;\quad \mathbb{T}_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\;;\quad \mathbb{T}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

2.3 Strichartz's estimates

A key tool in our method is the extension of the standard Strichartz's estimates (see e.g. [10]) to the dynamics on $\mathcal{G}$ described by the propagators e^{-iH_Ft}, $e^{-iH^{\alpha}_{\delta}t}$, and $e^{-iH^{\beta}_{\delta'}t}$.
In this subsection we use the symbol H to denote any of the three Hamiltonians of interest.

As a preliminary step, we remark that from equations (2.4), (2.6), (2.8), the standard dispersive estimate immediately follows:

$$\|e^{-iHt}\Psi\|_{L^\infty} \leqslant \frac{c}{t^{1/2}}\|\Psi\|_{L^1}\,. \tag{2.10}$$

Proposition 2.1 (Strichartz Estimates for e^{-iHt})**.**
Let $\Psi_0\in L^2(\mathcal{G})$, $\Gamma\in L^q_{\mathbb{R}}L^k$, with $1\leqslant q,k\leqslant 2$, $\frac{2}{q}+\frac{1}{k}=\frac{5}{2}$, and define

$$\Psi(t) = e^{-iHt}\Psi_0 \qquad \Phi(t) = \int_0^t ds\, e^{-iH(t-s)}\Gamma(s)\,.$$

The following estimates hold true:

$$\|\Psi\|_{L^p_{\mathbb{R}}L^r} \leqslant c\|\Psi_0\| \tag{2.11}$$

$$\|\Gamma\|_{L^p_{\mathbb{R}}L^r} \leqslant c\,\|\Phi(\cdot)\|_{L^q_{\mathbb{R}}L^k} \tag{2.12}$$

for any pair of indices (r,p) satisfying

$$2\leqslant p,r\leqslant\infty\,,\; \frac{2}{p}+\frac{1}{r}=\frac{1}{2}. \tag{2.13}$$

The constants c in (2.11) and (2.12) are independent of T.

Proof. The proof is standard due to the dispersive estimate (2.10), see for instance [10] and [16]. $\square$

Remark 2.2. If $H = H_\delta^\alpha$ ($H_{\delta'}^\beta$), the constants appearing in (2.10), (2.11) and (2.12) are independent of α (β). Indeed, by the change of variable $u \to u\alpha$ ($u \to u/\beta$) the integral term in (2.6) ((2.8)) can be easily estimated independently of α (β), obtaining a dispersive estimate (2.10) independent of the parameters and therefore, by the standard Strichartz machinery, uniform inequalities (2.11) and (2.12).

3 Well-posedness and conservation laws

Here we treat the problem of the well-posedness, i.e., the existence and uniqueness of the solution to equation (1.2) in the energy domain of the system. Such a domain turns out to coincide with the form domain of the linear part of equation (1.1). Throughout this section, such a linear part is denoted by H, and, according to the particular case under consideration, it can be understood as the Hamiltonian operator H_F, H_δ^α, or $H_{\delta'}^\beta$. Correspondingly, we denote the associated energy domain simply by $\mathcal{D}(\mathcal{E})$. All of the following formulas can be specialized to the particular cases $\mathcal{D}(\mathcal{E}_F)$, $\mathcal{D}(\mathcal{E}_\delta^\alpha)$, or $\mathcal{D}(\mathcal{E}_{\delta'}^\beta)$.

Let us stress that throughout the paper we do not approximate the dynamics in H^1, but rather in L^2. Furthermore, local well-posedness in L^2 is ensured by Strichartz estimates (proposition 2.1), as is easily seen following the line exposed in [10], chapters 2 and 3. Nonetheless, we prefer to deal with functions in the energy domain, since they are physically more meaningful.

We follow the traditional line of proving, first of all, local well-posedness, and then extending it to all times by means of a priori estimates provided by the conservation laws.
For a more extended treatment of the analogous problem for a two-edge vertex (namely, the real line with a point interaction at the origin), see [2].

Let us preliminarily define a norm in the energy domain:

$$\|\Psi\|_{\mathcal{D}(\mathcal{E})}^2 := \|\Psi\|^2 + \|\Psi'\|^2 ,$$

where we used the notation

$$\Psi' := \begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix}.$$

Furthermore, we denote $\mathcal{D}(\mathcal{E})^\star$ the dual of $\mathcal{D}(\mathcal{E})$, i.e., the set of the continuous linear functionals on $\mathcal{D}(\mathcal{E})$. We denote the dual product of $\Gamma \in \mathcal{D}(\mathcal{E})^\star$ and $\Psi \in \mathcal{D}(\mathcal{E})$ by $\langle \Gamma, \Psi \rangle$. In such a bracket we'll often exchange the place of the factor in $\mathcal{D}(\mathcal{E})^\star$ with the place of the factor in $\mathcal{D}(\mathcal{E})$: indeed, the duality product follows the same algebraic rules of the standard scalar product.
As usual, one can extend the action of H to the space $\mathcal{D}(\mathcal{E})$, with values in $\mathcal{D}(\mathcal{E})^\star$, by

$$\langle H\Psi_1, \Psi_2 \rangle := (H^{\frac{1}{2}}\Psi_1, H^{\frac{1}{2}}\Psi_2).$$

Furthermore, for any $\Psi \in \mathcal{D}(\mathcal{E})$ the identity

$$\frac{d}{dt} e^{-iHt}\Psi = -iHe^{-iHt}\Psi \tag{3.1}$$

holds in $\mathcal{D}(\mathcal{E})^\star$ too. To prove it, one can first test the functional $\frac{d}{dt}e^{-iHt}\Psi$ on an element Ξ in the operator domain $\mathcal{D}(H)$, obtaining

$$\left\langle \frac{d}{dt} e^{-iHt}\Psi, \Xi \right\rangle = \lim_{h \to 0} \left(\Psi, \frac{e^{iH(t+h)}\Xi - e^{iHt}\Xi}{h} \right) = (\Psi, iHe^{iHt}\Xi) = \langle -iHe^{-iHt}\Psi, \Xi \rangle.$$

Then, the result can be extended to $\Xi \in \mathcal{D}(\mathcal{E})$ by a density argument.

Besides, by (3.1), the differential version (1.1) of the Schrödinger equation holds in $\mathcal{D}(\mathcal{E})^\star$.

In order to prove a well-posedness result we need to generalize standard one-dimensional Gagliardo-Nirenberg estimates to graphs, i.e.

$$\|\Psi\|_{L^p} \leqslant C\|\Psi'\|_{L^2}^{\frac{1}{2}-\frac{1}{p}}\|\Psi\|_{L^2}^{\frac{1}{2}+\frac{1}{p}}, \tag{3.2}$$

where the constant C depends on the index p only. The proof of (3.2) follows immediately from the analogous estimates for functions of the real line, considering that any function in $H^1(\mathbb{R}^+)$ can be extended to an even function in $H^1(\mathbb{R})$, and applying this reasoning to each component of Ψ.

Proposition 3.1 (Local well-posedness in $\mathcal{D}(\mathcal{E})$)**.** *For any $\Psi_0 \in \mathcal{D}(\mathcal{E})$, there exists $T > 0$ such that the equation* (1.2) *has a unique solution $\Psi \in C^0([0,T),\mathcal{D}(\mathcal{E})) \cap C^1([0,T),\mathcal{D}(\mathcal{E})^\star)$.*
Moreover, eq. (1.2) *has a maximal solution $\Psi^{\max}$ defined on the interval $[0,T^\star)$, and the following "blow-up alternative" holds: either $T^\star = \infty$ or*

$$\lim_{t\to T^\star}\|\Psi_t^{\max}\|_{\mathcal{D}(\mathcal{E})} = +\infty,$$

where we denoted by $\Psi_t^{\max}$ the function $\Psi^{\max}$ evaluated at time t.

Proof. We define the space $\mathcal{X} := L^\infty([0,T),\mathcal{D}(\mathcal{E}))$, endowed with the norm $\|\Psi\|_{\mathcal{X}} := \sup_{t\in[0,T)}\|\Psi_t\|_{\mathcal{D}(\mathcal{E})}$. Given $\Psi_0 \in \mathcal{D}(\mathcal{E})$, we define the map $G : \mathcal{X} \longrightarrow \mathcal{X}$ as

$$G\Phi := e^{-iH\cdot}\Psi_0 + i\int_0^{\cdot} e^{-iH(\cdot-s)}|\Phi_s|^2\Phi_s\,ds.$$

Notice that the nonlinearity preserves the space $\mathcal{D}(\mathcal{E})$. Indeed, since for any component ψ_i of Ψ, ψ_i' belongs to $L^2(\mathbb{R}^+)$, then $|\psi_i|^2\psi_i$ belongs to $L^2(\mathbb{R}^+)$ too, and so the energy space for the delta-prime case is preserved. Furthermore, the product preserves the continuity at zero required by the Kirchhoff and the delta case. By estimates (3.2) one obtains

$$\||\Phi_s|^2\Phi_s\|_{\mathcal{D}(\mathcal{E})} \leqslant C\|\Phi_s\|_{\mathcal{D}(\mathcal{E})}^3,$$

so

$$\|G\Phi\|_{\mathcal{X}} \leqslant \|\Psi_0\|_{\mathcal{D}(\mathcal{E})} + C\int_0^T \|\Phi_s\|_{\mathcal{D}(\mathcal{E})}^3\,ds \leqslant \|\Psi_0\|_{\mathcal{D}(\mathcal{E})} + CT\|\Phi\|_{\mathcal{X}}^3\,. \tag{3.3}$$

Analogously, given $\Phi, \Xi \in \mathcal{D}(\mathcal{E})$,

$$\|G\Phi - G\Xi\|_{\mathcal{X}} \leqslant CT\left(\|\Phi\|_{\mathcal{X}}^2 + \|\Xi\|_{\mathcal{X}}^2\right)\|\Phi - \Xi\|_{\mathcal{X}}\,. \tag{3.4}$$

We point out that the constant C appearing in (3.3) and (3.4) is independent of Ψ_0, Φ, and Ξ. Now let us restrict the map G to elements Φ such that $\|\Phi\|_{\mathcal{X}} \leqslant 2\|\Psi_0\|_{\mathcal{D}(\mathcal{E})}$. From (3.3) and (3.4), if T is chosen to be strictly less than $(8C\|\Psi_0\|_{\mathcal{D}(\mathcal{E})}^2)^{-1}$, then G is a contraction of the ball in $\mathcal{X}$ of radius $2\|\Psi_0\|_{\mathcal{D}(\mathcal{E})}$, and so, by the contraction lemma, there exists a unique solution to (1.2) in the time interval $[0,T)$. By a standard one-step boostrap argument one immediately has that the solution actually belongs to $C^0([0,T),\mathcal{D}(\mathcal{E}))$, and due to the validity of (1.1) in the space $\mathcal{D}(\mathcal{E})^\star$ we immediately have that the solution Ψ actually belongs to $C^0([0,T),\mathcal{D}(\mathcal{E}))) \cap C^1([0,T),\mathcal{D}(\mathcal{E})^\star)$.

The proof of the existence of a maximal solution is standard, while the blow-up alternative is a consequence of the fact that, whenever the $\mathcal{D}(\mathcal{E})$-norm of the solution is finite, it is possible to extend it for a further time by the same contraction argument. □

The next step consists in the proof of the conservation laws for our problem.

Proposition 3.2. *For any solution* $\Psi \in C^0([0,T),\mathcal{D}(\mathcal{E})) \cap C^1([0,T),\mathcal{D}(\mathcal{E})^\star)$ *to the problem* (1.2)*, the following conservation laws hold at any time* t*:*

$$\|\Psi_t\| \;=\; \|\Psi_0\|, \qquad \mathcal{E}(\Psi_t) \;=\; \mathcal{E}(\Psi_0),$$

where the symbol $\mathcal{E}$ *denotes the* energy functional

$$\mathcal{E}(\Psi_t) \;:=\; \frac{1}{2}\mathcal{E}_{lin}(\Psi_t) - \frac{1}{4}\|\Psi_t\|^4_{L^4}.$$

Here the functional $\mathcal{E}_{lin}$ *coincides with* $\mathcal{E}_F$, $\mathcal{E}^\alpha_\delta$ *or* $\mathcal{E}^\beta_{\delta'}$*, according to the case one considers.*

Proof. First, one can directly prove that $e^{iHt}\Psi_t$ is differentiable as a function of t with values in $\mathcal{D}(\mathcal{E})$, and

$$\frac{d}{dt}e^{iHt}\Psi_t \;=\; ie^{iHt}|\Psi_t|^2\Psi_t. \tag{3.5}$$

Then, according to proposition 3.1, given $\Xi \in \mathcal{D}(\mathcal{E})$ one has

$$\left\langle \frac{d}{dt}\Psi_t, \Xi \right\rangle \;:=\; \frac{d}{dt}(\Psi_t, \Xi) \;=\; \frac{d}{dt}(e^{iHt}\Psi_t, e^{iHt}\Xi) \;=\; (i|\Psi_t|^2\Psi_t, \Xi) + \langle -iH\Psi_t, \Xi, \rangle$$

where we used (3.5) and (3.1). Therefore, the conservation of the L^2-norm can be immediately obtained by

$$\frac{d}{dt}\|\Psi_t\|^2 \;=\; 2\,\mathrm{Re}\left\langle \Psi_t, \frac{d}{dt}\Psi_t \right\rangle \;=\; 2\,\mathrm{Im}\,\langle \Psi_t, H\Psi_t\rangle.$$

In order to prove the conservation of the energy, first we assume that $\langle \Psi_t, H\Psi_t\rangle$ is differentiable as a function of time and

$$\frac{d}{dt}(\Psi_t, H\Psi_t) \;=\; 2\,\mathrm{Re}\left\langle \frac{d}{dt}\Psi_t, H\Psi_t \right\rangle \;=\; 2\,\mathrm{Im}\,\langle |\Psi_t|^2\Psi_t, H\Psi_t\rangle, \tag{3.6}$$

where, in the last equality, we used (1.1) and the self-adjointness of H. Furthermore,

$$\frac{d}{dt}(\Psi_t, |\Psi_t|^2\Psi_t) \;=\; \frac{d}{dt}(\Psi_t^2, \Psi_t^2) \;=\; 4\,\mathrm{Im}\,\langle |\Psi_t|^2\Psi_t, H\Psi_t\rangle. \tag{3.7}$$

From (3.6) and (3.7) one then obtains

$$\frac{d}{dt}\mathcal{E}(\Psi_t) \;=\; \frac{1}{2}\frac{d}{dt}\langle \Psi_t, H\Psi_t\rangle - \frac{1}{4}\frac{d}{dt}(\Psi_t, |\Psi_t|^2\Psi_t)_{L^2} \;=\; 0$$

and the proposition is proved.

It remains to show (3.6). To this purpose, it suffices to notice that

$$\frac{1}{h}\left[\langle \Psi_{t+h}, H\Psi_{t+h}\rangle - \langle \Psi_t, H\Psi_t\rangle\right] \;=\; \left\langle \frac{\Psi_{t+h}-\Psi_t}{h}, H\Psi_{t+h} \right\rangle + \left\langle H\Psi_t, \frac{\Psi_{t+h}-\Psi_t}{h} \right\rangle$$

and then pass to the limit. □

Corollary 3.3. *The solutions are globally defined in time.*

Proof. By estimate (3.2) with $p = \infty$ and conservation of the L^2-norm, there exists a constant M, that depends on Ψ_0 only, such that

$$\mathcal{E}(\Psi_0) \;=\; \mathcal{E}(\Psi_t) \;\geq\; \frac{1}{2}\|\Psi_t'\|^2 - M\|\Psi_t'\|$$

Therefore one obtains a uniform (in t) bound on $\|\Psi_t'\|^2$. As a consequence, one has that no blow-up in finite time can occur, and therefore, by the blow-up alternative, the solution is global in time. □

4 Main Result

In this section we describe the asymptotic dynamics of a particular initial state, which resembles a soliton for the standard cubic NLS on the line.
We use the notation

$$\phi(x) \;=\; \sqrt{2}\cosh^{-1} x\,,$$

and define

$$\phi_{x_0,v}(x,t) = e^{i\frac{v}{2}x}e^{-it\frac{v^2}{4}}e^{it}\phi(x-x_0-vt)\,. \tag{4.1}$$

The function $\phi_{x_0,v}$ represents a soliton for the cubic NLS on the line centered at time $t=0$ in $x=x_0$ moving with speed v. Therefore, $\phi_{x_0,v}$ is the solution of the integral equation

$$\phi_{x_0,v}(x,t) = \left[U_t e^{i\frac{v}{2}\cdot}\phi(\cdot - x_0)\right](x) + i\int_0^t \left[U_{t-s}|\phi_{x_0,v}(\cdot,s)|^2\phi_{x_0,v}(\cdot,s)\right](x)ds\,. \tag{4.2}$$

We take as initial datum the following function

$$\Psi_0(x) \;=\; \begin{pmatrix} \chi(x)e^{-i\frac{v}{2}x}\phi(x-x_0) \\ 0 \\ 0 \end{pmatrix}$$

where χ is a cut off function, that is $\chi \in C^\infty(\mathbb{R}^+)$, $\chi = 1$ in $(2,+\infty)$ and $\chi = 0$ in $(0,1)$. For later use we define also $\chi_+ = \chi_{[0,+\infty)}$ and $\chi_- = \chi_{(-\infty,0]}$, where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$.
Furthermore, let x_0 and v be two positive constants and let the centre of the soliton be located at

$$x_0 \;\geqslant\; v^{1-\delta}$$

with $0<\delta<1$.

According to section 3, we use the symbol H to generically denote the linear part of the evolution, regardless of the fact that we are considering the Kirchhoff, delta, or delta-prime boundary conditions. When necessary, we will distinguish between the three of them.

Let $\Psi_{H,t}$ be the solution of the equation

$$\Psi_t \;=\; e^{-iHt}\Psi_0 + i\int_0^t ds\, e^{-iH(t-s)}|\Psi_s|^2\Psi_s\,. \tag{4.3}$$

Let us set $t_2 := x_0/v + v^{-\delta}$ and define the following functions:

$$\Phi^1_{H,t_2}(x_1,x_2,x_3) \equiv \begin{pmatrix} \tilde r_H e^{-i\frac{v^2}{4}t_2}e^{i\frac{v}{2}x_1}e^{it_2}\phi(x_1+x_0-vt_2) \\ \tilde r_H e^{-i\frac{v^2}{4}t_2}e^{-i\frac{v}{2}x_2}e^{it_2}\phi(x_2-x_0+vt_2) \\ 0 \end{pmatrix} \tag{4.4}$$

$$\Phi^2_{H,t_2}(x_1,x_2,x_3) \equiv \begin{pmatrix} 0 \\ \tilde t_H e^{-i\frac{v^2}{4}t_2}e^{i\frac{v}{2}x_2}e^{it_2}\phi(x_2+x_0-vt_2) \\ \tilde t_H e^{-i\frac{v^2}{4}t_2}e^{-i\frac{v}{2}x_3}e^{it_2}\phi(x_3-x_0+vt_2) \end{pmatrix} \tag{4.5}$$

$$\Phi^3_{H,t_2}(x_1,x_2,x_3) \equiv \begin{pmatrix} \tilde t_H e^{-i\frac{v^2}{4}t_2}e^{-i\frac{v}{2}x_1}e^{it_2}\phi(x_1-x_0+vt_2) \\ 0 \\ \tilde t_H e^{-i\frac{v^2}{4}t_2}e^{i\frac{v}{2}x_3}e^{it_2}\phi(x_3+x_0-vt_2) \end{pmatrix} \tag{4.6}$$

They represent solitons on the line multiplied by the scattering coefficients of the linear dynamics $\tilde{r}_H$ and $\tilde{t}_H$, that, in the particular regime we consider, are defined as follows:

$$\begin{aligned}
\tilde{r}_{H_F} &= -1/3, & \tilde{t}_{H_F} &= 2/3 \\
\tilde{r}_{H_\delta^\alpha} &= -\frac{1+2i\tilde{\alpha}}{3+2i\tilde{\alpha}}, & \tilde{t}_{H_\delta^\alpha} &= \frac{2}{3+2i\tilde{\alpha}} \\
\tilde{r}_{H_{\delta'}^\beta} &= \frac{\tilde{\beta}+2i}{\tilde{\beta}+6i}, & \tilde{t}_{H_{\delta'}^\beta} &= -\frac{4i}{\tilde{\beta}+6i}
\end{aligned} \tag{4.7}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the rescaled interaction parameters, namely

$$\tilde{\alpha} := \alpha/v, \qquad \tilde{\beta} := \beta v.$$

Remark 4.1. Notice that the coefficients $\tilde{r}_H$ and $\tilde{t}_H$ can be obtained by the scattering coefficients (2.5), (2.7), identifying k with $v/2$ and replacing α by $\tilde{\alpha}v$ and β by $\tilde{\beta}/v$. This is due to the fact that we implicitly considered a particle with mass equal to $1/2$, therefore the momentum k is linked to the speed v by $k = v/2$.

For any $t > t_2$ we define the vectors $\Phi^j_{H,t}$ as the evolution of Φ^j_{H,t_2} with the nonlinear flows generated by H_j, i.e., they are solutions of the equation

$$\Phi^j_{H,t} = e^{-iH_j(t-t_2)}\Phi^j_{H,t_2} + i\int_{t_2}^t ds\, e^{-iH_j(t-s)}|\Phi^j_{H,s}|^2\Phi^j_{H,s} \qquad j = 1,2,3\,. \tag{4.8}$$

Remark 4.2. The vectors $\Phi^j_{H,t}$ can be represented by

$$\Phi^1_{H,t}(x_1,x_2,x_3) = \begin{pmatrix} e^{-i\frac{v^2}{4}t_2}e^{it_2}\phi^{ref}_{t-t_2}(x_1) \\ e^{-i\frac{v^2}{4}t_2}e^{it_2}\phi^{ref}_{t-t_2}(-x_2) \\ 0 \end{pmatrix} \;;\quad \Phi^2_{H,t}(x_1,x_2,x_3) = \begin{pmatrix} 0 \\ e^{-i\frac{v^2}{4}t_2}e^{it_2}\phi^{tr}_{t-t_2}(x_2) \\ e^{-i\frac{v^2}{4}t_2}e^{it_2}\phi^{tr}_{t-t_2}(-x_3) \end{pmatrix} \tag{4.9}$$

$$\Phi^3_{H,t}(x_1,x_2,x_3) = \begin{pmatrix} e^{-i\frac{v^2}{4}t_2}e^{it_2}\phi^{tr}_{t-t_2}(-x_1) \\ 0 \\ e^{-i\frac{v^2}{4}t_2}e^{it_2}\phi^{tr}_{t-t_2}(x_3) \end{pmatrix} \tag{4.10}$$

where the functions ϕ^{ref}_t and ϕ^{tr}_t are the solutions to the following NLS on the line

$$\phi^{ref}_t(x) = \tilde{r}_H\int_{-\infty}^{\infty} U_t(x-y)e^{i\frac{v}{2}y}\phi(y-v^{1-\delta})dy + i\int_0^t ds\int_{-\infty}^{\infty} U_{t-s}(x-y)|\phi^{ref}_s(y)|^2\phi^{ref}_s(y)dy \tag{4.11}$$

$$\phi^{tr}_t(x) = \tilde{t}_H\int_{-\infty}^{\infty} U_t(x-y)e^{i\frac{v}{2}y}\phi(y-v^{1-\delta})dy + i\int_0^t ds\int_{-\infty}^{\infty} U_{t-s}(x-y)|\phi^{tr}_s(y)|^2\phi^{tr}_s(y)dy\,. \tag{4.12}$$

Our main result is summarized in the following theorem:

Theorem 4.3. *There exists $\tau_* > 0$ such that if we put $T_* = \tau_*\delta/2$ and we take $0 < T < T_*$ then for $t_2 < t < t_2 + T\ln v$ we have*

$$\|\Psi_{H,t} - \Phi^1_{H,t} - \Phi^2_{H,t} - \Phi^3_{H,t}\| \leqslant Cv^{-\frac{T_*-T}{\tau_*}}$$

where C is a positive constant independent of t and v.

The proof of the theorem will be broken into three steps or equivalently we break the time evolution of $\Psi_{H,t}$ into three phases.

5 Proof of theorem 4.3

In the proof we drop the subscript H. When convenient, we specify the particular Hamiltonian operator we refer to.

We introduce the soliton

$$\Phi_t(x_1, x_2, x_3) = \begin{pmatrix} e^{-i\frac{v^2}{4}t} e^{-i\frac{v}{2}x_1} e^{it} \phi(x_1 - x_0 + vt) \\ e^{-i\frac{v^2}{4}t} e^{i\frac{v}{2}x_2} e^{it} \phi(x_2 + x_0 - vt) \\ 0 \end{pmatrix}. \tag{5.1}$$

Then, the function Φ_t satisfies the equation

$$\Phi_t = e^{-iH_1 t}\Phi_0 + i\int_0^t ds\, e^{-iH_1(t-s)} |\Phi_s|^2 \Phi_s \,. \tag{5.2}$$

5.1 Phase 1

We call "phase 1" the dynamics in the time interval $(0, t_1)$ with $t_1 = \frac{x_0}{v} - v^{-\delta}$. In this interval we approximate the solution by the soliton (5.1). The content of this subsection is the estimate of the error due to such approximation, that is contained in proposition 5.3. Before proving it, we need to prove two lemmas.

Lemma 5.1. *Given $0 \leqslant t_a \leqslant t_b \leqslant t_1$, for the functions*

$$\begin{aligned} K_1(t,x) &:= \int_0^\infty U_{t-t_a}(x+y) e^{-i\frac{v}{2}y} \phi(y - x_0)\, dy + i\int_{t_a}^t ds \int_0^\infty U_{t-s}(x+y) e^{-i\frac{v}{2}y} e^{-is\frac{v^2}{4}} e^{is} \phi^3(y - x_0 + vs)\, dy \\ K_2(t,x) &:= \int_0^\infty U_{t-t_a}(x+y) e^{i\frac{v}{2}y} \phi(y + x_0)\, dy + i\int_{t_a}^t ds \int_0^\infty U_{t-s}(x+y) e^{i\frac{v}{2}y} e^{-is\frac{v^2}{4}} e^{is} \phi^3(y + x_0 - vs)\, dy \end{aligned} \tag{5.3}$$

the following estimate holds:

$$\|K_i\|_{X_{t_a,t_b}(\mathbb{R}^+)} \leqslant Ce^{-x_0 + vt_b}, \qquad i = 1, 2,$$

where $X_{t_a,t_b}(\mathbb{R}^+) := L^\infty_{[t_a,t_b]} L^2(\mathbb{R}^+) \cap L^6_{[t_a,t_b]} L^6(\mathbb{R}^+)$.

Proof. Let us start with K_1. Adding and subtracting a contribution on the negative y one can write

$$\begin{aligned} K_1(t,x) &= \int_{-\infty}^\infty U_{t-t_a}(x+y) e^{-i\frac{v}{2}y} \phi(y - x_0)\, dy + i\int_{t_a}^t ds \int_{-\infty}^\infty U_{t-s}(x+y) e^{-i\frac{v}{2}y} e^{-is\frac{v^2}{4}} e^{is} \phi^3(y - x_0 + vs)\, dy \\ &\quad - \int_{-\infty}^0 U_{t-t_a}(x+y) e^{-i\frac{v}{2}y} \phi(y - x_0)\, dy - i\int_{t_a}^t ds \int_{-\infty}^0 U_{t-s}(x+y) e^{-i\frac{v}{2}y} e^{-is\frac{v^2}{4}} e^{is} \phi^3(y - x_0 + vs)\, dy \\ &= e^{i\frac{v}{2}x} e^{-it\frac{v^2}{4}} e^{it} \phi(x + x_0 - vt) \\ &\quad - \int_{-\infty}^0 U_{t-t_a}(x+y) e^{-i\frac{v}{2}y} \phi(y - x_0)\, dy - i\int_{t_a}^t ds \int_{-\infty}^0 U_{t-s}(x+y) e^{-i\frac{v}{2}y} e^{-is\frac{v^2}{4}} e^{is} \phi^3(y - x_0 + vs)\, dy \end{aligned} \tag{5.4}$$

where we used the integral equation (4.2). By a straightforward computation, the $X_{t_a,t_b}(\mathbb{R}^+)$-norm of the first term can be estimated as $Ce^{-(x_0 - vt_b)}$. To evaluate the size of the second term, let us write it as follows:

$$\int_0^\infty U_{t-t_a}(x-y) e^{i\frac{v}{2}y} \phi(y + x_0) dy = [U^-_{t-t_a} e^{i\frac{v}{2}\cdot} \phi(\cdot + x_0)](x) = [U_{t-t_a}\chi_+ e^{i\frac{v}{2}\cdot} \phi(\cdot + x_0)](x), \quad x > 0. \tag{5.5}$$

Using the one-dimensional homogeneous Strichartz's estimates for U_{t-t_a}, namely, the analogous of (2.11) for functions of the half line, we can estimate the $X_{t_a,t_b}(\mathbb{R}^+)$-norm of this term as

$$\|U_{t-t_a}\chi_+ e^{i\frac{v}{2}\cdot}\phi(\cdot+x_0)\|_{X_{t_a,t_b}(\mathbb{R})} \leqslant C\|\chi_+ e^{i\frac{v}{2}\cdot}\phi(\cdot+x_0)\| \;\leqslant Ce^{-x_0},$$

where we used the notation $X_{t_a,t_b}(\mathbb{R}) := L^\infty_{[t_a,t_b]}L^2(\mathbb{R}) \cap L^6_{[t_a,t_b]}L^6(\mathbb{R})$.
The norm of the last term in the r.h.s. of equation (5.4) can be estimated in a similar way by

$$\left\| \int_{t_a}^{\cdot} \left[U_{\cdot-s}|\phi_{-x_0,v}(s)|^2\phi_{-x_0,v}(s)\right] ds \right\|_{X_{t_a,t_b}(\mathbb{R}^+)} \leqslant C\|\phi^3_{-x_0,v}\|_{L^1_{[t_a,t_b]}L^2(\mathbb{R}^+)} \leqslant C\frac{1}{v}e^{-x_0+vt_b}. \tag{5.6}$$

Therefore, from (5.4), (5.5), and (5.6) we get

$$\|K_1\|_{X_{t_a,t_b}(\mathbb{R}^+)} \;\leqslant\; Ce^{-x_0+vt_b}.$$

To estimate K_2, the first term in its definition (5.3) can be treated as in (5.5), while the second is estimated following the line of (5.6). □

Lemma 5.2. *Given $0 \leqslant t_a \leqslant t_b \leqslant t_1$, let a and b two positive numbers, with*

$$b \leqslant \frac{1}{8a^2+4a}.$$

Moreover, let y be a real, continuous function such that $0 \leqslant y(t_a) \leqslant a$, and

$$0 \leqslant y(t) \leqslant a + by^2(t) + by^3(t), \qquad \text{for any} \;\; t \in [t_a,t_b]. \tag{5.7}$$

Then,

$$\max_{t\in[t_a,t_b]} y(t) \;\leqslant\; 2a.$$

Proof. Consider the function $f_b(x) := bx^3 + bx^2 - x + a$. Denoted $\bar{b} := \frac{1}{8a^2+4a}$, one has $f_{\bar{b}}(2a) \;= 0$.
If $b \leqslant \bar{b}$, then $f_b(x) \leqslant f_{\bar{b}}(x)$ for any $x > 0$. Besides, notice that $f_b(0) = a > 0$, then there must be a point $\tilde{x} \in (0, 2a]$ s.t. $f_b(\tilde{x}) = 0$. Finally, since the function y is continuous, in order to satisfy the constraint (5.7) one must have

$$y(t) \leqslant \tilde{x} \leqslant 2a.$$

□

Proposition 5.3. *Let Ψ_t be the solution of the equation (4.3), and Φ_t be the solution of equation (5.2). There exists $C > 0$, independent of t and v, such that*

$$\|\Psi_t - \Phi_t\| \;\leqslant\; Ce^{-v^{1-\delta}} \tag{5.8}$$

for any $t \in [0, t_1]$.

Proof. Let us define $\Xi_t := \Psi_t - \Phi_t$, and fix $t_a \in [0, t_1]$. Then, from equations (4.3) and (5.2), we have

$$\begin{aligned}
\Xi_t &= e^{-iH(t-t_a)}\Xi_0 + (e^{-iH(t-t_a)} - e^{-iH_1(t-t_a)})\Phi_0 + i\int_{t_a}^t (e^{-iH(t-s)} - e^{-iH_1(t-s)})|\Phi_s|^2\Phi_s\, ds \\
&\quad + i\int_{t_a}^t e^{-iH(t-s)}\left[|\Xi_s|^2\Xi_s + |\Xi_s|^2\Phi_s + |\Phi_s|^2\Xi_s + 2\mathrm{Re}(\overline{\Xi_s}\Phi_s)\Xi_s + 2\mathrm{Re}(\overline{\Xi_s}\Phi_s)\Phi_s\right] \\
&= e^{-iH(t-t_a)}\Xi_0 + F(t_a,t) \\
&\quad + i\int_{t_a}^t e^{-iH(t-s)}\left[|\Xi_s|^2\Xi_s + |\Xi_s|^2\Phi_s + |\Phi_s|^2\Xi_s + 2\mathrm{Re}(\overline{\Xi_s}\Phi_s)\Xi_s + 2\mathrm{Re}(\overline{\Xi_s}\Phi_s)\Phi_s\right],
\end{aligned}$$

where we defined

$$F(t_a,t) \;:=\; (e^{-iH(t-t_a)}-e^{-iH_1(t-t_a)})\Phi_0 + i\int_{t_a}^{t}(e^{-iH(t-s)}-e^{-iH_1(t-s)})|\Phi_s|^2\Phi_s\,ds$$

Let us fix $t_b \in [t_a,t_1]$, and denote $X_{t_a,t_b} = L^\infty_{[t_a,t_b]}L^2 \cap L^6_{[t_a,t_b]}L^6$. Then

$$\begin{aligned}\|\Xi\|_{X_{t_a,t_b}} &\leqslant \|e^{-iH(\cdot-t_a)}\Xi_{t_a}\|_{X_{t_a,t_b}} + \|F(t_a,\cdot)\|_{X_{t_a,t_b}} \\ &+ \left\|\int_{t_a}^{\cdot} e^{-iH(\cdot-s)}\left[|\Xi_s|^2\Xi_s + |\Xi_s|^2\Phi_s + |\Phi_s|^2\Xi_s + 2\mathrm{Re}(\overline{\Xi_s}\Phi_s)\Xi_s + 2\mathrm{Re}(\overline{\Xi_s}\Phi_s)\Phi_s\right]\right\|_{X_{t_a,t_b}}.\end{aligned} \tag{5.9}$$

Using (2.11) the first term in the r.h.s can be estimated as

$$\|e^{-iH(\cdot-t_a)}\Xi_{t_a}\|_{X_{t_a,t_b}} \;\leqslant\; C\|\Xi_{t_a}\|.$$

We estimate the integral term in the r.h.s. of (5.9) also using Strichartz's estimates. Let us analyse in detail the cubic term. Since both pairs of indices $(\infty,2)$ and $(6,6)$ fulfil (2.13), in (2.12) we can choose $q=k=6/5$ and obtain

$$\left\|\int_{t_a}^{\cdot} e^{-iH(\cdot-s)}|\Xi_s|^2\Xi_s\,ds\right\|_{X_{t_a,t_b}} \leqslant C\||\Xi_\cdot|^2\Xi_\cdot\|_{L^{6/5}_{[t_a,t_b]}L^{6/5}}. \tag{5.10}$$

Moreover, by standard Hölder estimates,

$$\||\Xi_\cdot|^2\Xi_\cdot\|_{L^{6/5}_{[t_a,t_b]}L^{6/5}} \;\leqslant\; \left\|\|\Xi_\cdot\|^2_{L^6}\|\Xi_\cdot\|_{L^2}\right\|_{L^{6/5}_{[t_a,t_b]}} \;\leqslant\; \|\Xi\|^2_{L^6_{[t_a,t_b]}L^6}\|\Xi\|_{L^2_{[t_a,t_b]}L^2} \;\leqslant\; (t_b-t_a)^{\frac12}\|\Xi\|^2_{L^6_{[t_a,t_b]}L^6}\|\Xi\|_{L^\infty_{[t_a,t_b]}L^2}. \tag{5.11}$$

Then, by (5.10) and (5.11),

$$\left\|\int_{t_a}^{\cdot} e^{-iH(\cdot-s)}|\Xi_s|^2\Xi_s\,ds\right\|_{X_{t_a,t_b}} \leqslant C(t_b-t_a)^{1/2}\|\Xi\|^3_{X_{t_a,t_b}}. \tag{5.12}$$

Notice that the constant C can be chosen independently of t_a, t_b, and of the boundary condition at the vertex.
The other terms in the integral in the r.h.s. of (5.9) can be estimated analogously. One ends up with

$$\|\Xi\|_{X_{t_a,t_b}} \leqslant C\;\|\Xi_{t_a}\| + \|F(t_a,\cdot)\|_{X_{t_a,t_b}} + C(t_b-t_a)^{\frac12}\|\Xi\|^3_{X_{t_a,t_b}} + C(t_b-t_a)^{\frac23}\|\Xi\|^2_{X_{t_a,t_b}} + C(t_b-t_a)^{\frac56}\|\Xi\|_{X_{t_a,t_b}},$$

where the arising norms of Φ were absorbed in the constant C.
If t_b and t_a are sufficiently close, then $C(t_b-t_a)^{\frac56} < 1/2$. Furthermore, since the quantity t_b-t_a is upper bounded, one can estimate $(t_b-t_a)^{\frac23}$ by $C(t_b-t_a)^{\frac12}$. So

$$\|\Xi\|_{X_{t_a,t_b}} \leqslant \widetilde{C}\;\|\Xi_{t_a}\| + \widetilde{C}\;\|F(t_a,\cdot)\|_{X_{t_a,t_b}} + \widetilde{C}(t_b-t_a)^{\frac12}\left[\|\Xi\|^3_{X_{t_a,t_b}} + \|\Xi\|^2_{X_{t_a,t_b}}\right]. \tag{5.13}$$

Applying lemma 5.2 to the function $y(t) = \|\Xi\|_{X_{t_a,t}}$, which is continuous and monotone, one has that, if

$$t_b - t_a \;\leqslant\; \left(8\widetilde{C}^3\;(\;\|\Xi_{t_a}\| + \|F(t_a,\cdot)\|_{X_{t_a,t_b}})^2 + 4\widetilde{C}^2\;(\|\Xi_{t_a}\| + \|F(t_a,\cdot)\|_{X_{t_a,t_b}})\right)^{-2},$$

then $\|\Xi\|_{X_{t_a,t_b}} \leqslant\; 2\widetilde{C}\;\|\Xi_{t_a}\| + 2\widetilde{C}\;\|F(t_a,\cdot)\|_{X_{t_a,t_b}}$. From the immediate estimates

$$\|\Xi_{t_a}\| \;\leqslant\; 4, \qquad \|F(t_a,\cdot)\|_{X_{t_a,t_b}} \;\leqslant\; \|F(0,\cdot)\|_{X_{0,t_1}},$$

if one denotes

$$\tau \;:=\; \left(8\widetilde{C}^3\;(4 + \|F(t_a,\cdot)\|_{X_{0,t_1}})^2 + 4\widetilde{C}^2\;(4 + \|F(t_a,\cdot)\|_{X_{0,t_1}})\right)^{-2},$$

then for any $t \in [0, t_1)$ $\|\Xi\|_{X_{t,t+\tau}} \leqslant 2\widetilde{C}\ \|\Xi_t\| + 2\widetilde{C}\ \|F(t,\cdot)\|_{X_{t,t+\tau}}$.

We divide the interval $[0, t_1]$ in $N+1$ subintervals as follows

$$[0,t_1] \;=\; \left(\cup_{j=0}^{N-1}[j\tau,(j+1)\tau]\right) \cup [N\tau, t_1],$$

where

$$N := \left[\frac{x_0 - v^{1-\delta}}{v\tau}\right], \qquad [\cdot] = \text{ integer part.}$$

Making use of lemma 5.2, and noting that $\|\Xi_{(j+1)\tau}\| \leqslant \|\Xi\|_{X_{j\tau,(j+1)\tau}}$, one proves by induction that

$$\begin{aligned}
\|\Xi\|_{X_{j\tau,(j+1)\tau}} &\leqslant (2\widetilde{C})^{j+1}\|\Xi_0\| + \sum_{k=0}^{j}(2\widetilde{C})^{j+1-k}\|F(k\tau,\cdot)\|_{X_{k\tau,(k+1)\tau}}, \qquad j = 0,\ldots,N-1,\\
\|\Xi\|_{X_{N\tau,t_1}} &\leqslant (2\widetilde{C})^{N+1}\|\Xi_0\| + \sum_{k=0}^{N-1}(2\widetilde{C})^{N+1-k}\|F(k\tau,\cdot)\|_{X_{k\tau,(k+1)\tau}} + 2\widetilde{C}\ \|F(N\tau,\cdot)\|_{X_{N\tau,t_1}}
\end{aligned} \tag{5.14}$$

where the last inequality comes from the fact that $t_1 - N\tau \leqslant \tau$, so lemma 5.2 applies to this last step too. The norm of Ξ as a function of the whole time interval $[0, t_1]$ can be estimated by

$$\begin{aligned}
\|\Xi\|_{X_{0,t_1}} &\leqslant \sum_{j=0}^{N-1}\|\Xi\|_{X_{j\tau,(j+1)\tau}} + \|\Xi\|_{X_{N\tau,t_1}}\\
&\leqslant \sum_{j=0}^{N}(2\widetilde{C})^{j+1}\|\Xi_0\| + \sum_{j=0}^{N}\sum_{k=0}^{j}(2\widetilde{C})^{j+1-k}\|F(k\tau,\cdot)\|_{X_{k\tau,\min\{t_1,(k+1)\tau\}}}.
\end{aligned} \tag{5.15}$$

In order to prove the theorem using (5.15), we need more precise estimates for $\|\Xi_0\|$ and $\|F(j\tau,\cdot)\|_{X_{j\tau,(j+1)\tau}}$. First,

$$\|\Xi_0\|^2 \leqslant \int_0^2 \phi^2(x-x_0)\,dx + \int_0^\infty \phi^2(x+x_0) = 2(1-\tanh(x_0-2)) \leqslant Ce^{-2x_0}. \tag{5.16}$$

To estimate $\|F(t_a,\cdot)\|_{X_{t_a,t_b}}$ we specialize F to the three cases under analysis. From the explicit propagators (2.4), (2.6), (2.8),

$$\begin{aligned}
F_F(t_a,t) &= \frac{1}{3}U^+_{t-t_a}\begin{pmatrix} -1 & -1 & 2\\ -1 & -1 & 2\\ 2 & 2 & 2\end{pmatrix}\Phi_0 + \frac{i}{3}\int_{t_a}^{t}U^+_{t-s}\begin{pmatrix} -1 & -1 & 2\\ -1 & -1 & 2\\ 2 & 2 & 2\end{pmatrix}|\Phi_s|^2\Phi_s\,ds\\
F^\alpha_\delta(t_a,t) &= F_F(t_a,t) - \frac{2}{9}\alpha\int_0^{+\infty}du\,e^{-\frac{\alpha}{3}u}\left(U^+_{t-t_a}\,\mathbb{J}\,\Phi_0\right)(\cdot+u) - i\frac{2}{9}\alpha\int_{t_a}^{t}ds\int_0^{+\infty}du\,e^{-\frac{\alpha}{3}u}\left(U^+_{t-s}\,\mathbb{J}\,|\Phi_s|^2\Phi_s\right)(\cdot+u)\\
F^\beta_{\delta'}(t_a,t) &= U^+_{t-t_a}\begin{pmatrix} 1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}\Phi_0 + i\int_{t_a}^{t}U^+_{t-s}\begin{pmatrix} 1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}|\Phi_s|^2\Phi_s\,ds\\
&\quad - \frac{2}{\beta}\int_0^{+\infty}du\,e^{-\frac{3}{\beta}u}\left(U^+_{t-t_a}\,\mathbb{J}\,\Phi_0\right)(\cdot+u) - i\frac{2}{\beta}\int_{t_a}^{t}ds\int_0^{+\infty}du\,e^{-\frac{3}{\beta}u}\left(U^+_{t-s}\,\mathbb{J}\,|\Phi_s|^2\Phi_s\right)(\cdot+u)
\end{aligned}$$

It is immediately seen that

$$F_F(t_a,t,x) \;=\; \frac{1}{3}\begin{pmatrix} -K_1(x,t) - K_2(x,t)\\ -K_1(x,t) - K_2(x,t)\\ 2K_1(x,t) + 2K_2(x,t),\end{pmatrix},$$

where K_1 and K_2 were defined in (5.3). Lemma 5.1 yields

$$\|F_F(t_a,\cdot)\|_{X_{t_a,t_b}} \leqslant Ce^{-x_0+vt_b}. \tag{5.17}$$

Furthermore, since

$$F_\delta^\alpha(t_a,t,x) \;=\; F_F(t_a,t,x) - \frac{2}{9}\alpha\int_0^{+\infty} du\, e^{-\frac{\alpha}{3}u}\begin{pmatrix} K_1(x+u,t)+K_2(x+u,t)\\ K_1(x+u,t)+K_2(x+u,t)\\ K_1(x+u,t)+K_2(x+u,t)\end{pmatrix},$$

we conclude

$$\|F_\delta^\alpha(t_a,\cdot)\|_{X_{t_a,t_b}} \;\leqslant\; Ce^{-x_0+vt_b}. \tag{5.18}$$

Finally,

$$F_{\delta'}^\beta(t_a,t,x) \;=\; \begin{pmatrix} K_1(x,t)-K_2(x,t)\\ -K_1(x,t)+K_2(x,t)\\ 0\end{pmatrix} - \frac{2}{\beta}\int_0^{+\infty} du\, e^{-\frac{3}{\beta}u}\begin{pmatrix} K_1(x+u,t)+K_2(x+u,t)\\ K_1(x+u,t)+K_2(x+u,t)\\ K_1(x+u,t)+K_2(x+u,t)\end{pmatrix}$$

yields

$$\|F_{\delta'}^\beta(t_a,\cdot)\|_{X_{t_a,t_b}} \;\leqslant\; Ce^{-x_0+vt_b}. \tag{5.19}$$

Now we go back to estimate (5.15). Due to (5.16), (5.17), (5.18), and (5.19), and estimating any geometric sum by the double of its largest term, we get

$$\begin{aligned}
\|\Xi\|_{X_{0,t_1}} &\leqslant 2C\widetilde{C}e^{-x_0}\sum_{j=0}^{N}(2\widetilde{C})^j + 2C\widetilde{C}e^{-x_0+v\tau}\sum_{j=0}^{N-1}(2\widetilde{C})^j\sum_{k=0}^{j}\left(\frac{e^{v\tau}}{2\widetilde{C}}\right)^k \\
&\quad + Ce^{-x_0+v\tau}(2\widetilde{C})^{N+1}\sum_{k=0}^{N-1}\left(\frac{e^{v\tau}}{2\widetilde{C}}\right)^k + 2C\widetilde{C}e^{-x_0+vt_1} \\
&\leqslant 2C(2\widetilde{C})^{N+1}e^{-x_0} + 4C\widetilde{C}e^{-x_0+v\tau}\sum_{j=0}^{N-1}e^{jv\tau} + 8C\widetilde{C}^2e^{-x_0+Nv\tau} + 2C\widetilde{C}e^{-x_0+vt_1} \\
&\leqslant 2C(2\widetilde{C})^{N+1}e^{-x_0} + 8C\widetilde{C}e^{-x_0+Nv\tau} + 8C\widetilde{C}^2e^{-x_0+Nv\tau} + 2C\widetilde{C}e^{-x_0+vt_1}.
\end{aligned} \tag{5.20}$$

Concerning the first term in the r.h.s. of (5.20), we have

$$(2\widetilde{C})^N e^{-x_0} \;\leqslant\; \left(\frac{2\widetilde{C}^{\frac{1}{v\tau}}}{e}\right)^{x_0-v^{1-\delta}} e^{-v^{1-\delta}} \;\leqslant\; Ce^{-v^{1-\delta}}. \tag{5.21}$$

From (5.20) and (5.21) we get $\|\Xi\|_{X_{0,t_1}} \leqslant Ce^{-v^{1-\delta}}$, so (5.8) follows and the proof is concluded. □

5.2 Phase 2

We call "phase 2" the evolution of the system in the time interval (t_1,t_2) with $t_2=\frac{x_0}{v}+v^{-\delta}$. Let us define the vector

$$\Phi_t^S := \Phi_t^{S,in} + \Phi_t^{S,out}$$

with

$$\Phi_t^{S,in} := \begin{pmatrix}\phi_{x_0,-v}(t)\\ \\ 0\\ \\ 0\end{pmatrix}, \qquad \Phi_t^{S,out} := \begin{pmatrix}\tilde{r}\,\phi_{-x_0,v}(t)\\ \\ \tilde{t}\,\phi_{-x_0,v}(t)\\ \\ \tilde{t}\,\phi_{-x_0,v}(t)\end{pmatrix},$$

where the function $\phi_{x_0,v}$ was defined in equation (4.1) and the reflection and transmission coefficients, $\tilde{r}$ and $\tilde{t}$, must be chosen accordingly to the Hamiltonian H taken in the equation (4.3). The explicit expressions of $\tilde{r}$ and $\tilde{t}$ in all the cases $H=H_F$, H_δ^α, $H_{\delta'}^\beta$ can be read in formula (4.7).

Proposition 5.4. *Let $t \in (t_1, t_2)$ then there exists $v_0 > 0$ such that for all $v > v_0$*

$$\|\Psi_t - \Phi_t^S\| \leqslant C_1 v^{-\frac{\delta}{2}}\,, \tag{5.22}$$

moreover

$$\|\Psi_{t_2} - \Phi_{t_2}^{S,out}\| \leqslant C_2 v^{-\frac{\delta}{2}}\,, \tag{5.23}$$

where C_1 and C_2 are positive constants which do not depend on t and v.

Proof. From the definition of Ψ_t, see equation (4.3),

$$\Psi_t = \; e^{-iH(t-t_1)}\Psi_{t_1} + i\int_{t_1}^t ds\, e^{-iH(t-s)}|\Psi_s|^2\Psi_s\,.$$

We start with the trivial estimate

$$\begin{aligned}&\big\|\Psi_t - \Phi_t^S\big\| \\ &\leqslant \big\|\Psi_t - e^{-iH(t-t_1)}\Psi_{t_1}\big\| + \big\|e^{-iH(t-t_1)}\Psi_{t_1} - e^{-iH(t-t_1)}\Phi_{t_1}^{S,in}\big\| + \big\|e^{-iH(t-t_1)}\Phi_{t_1}^{S,in} - \Phi_t^S\big\|\end{aligned} \tag{5.24}$$

and estimate the r.h.s. term by term. The estimates involved in the analysis of the first term are similar to the ones used in the previous proposition then we omit the details. Similarly to what was done above we set $X_{t_1,t_2} = L^\infty_{[t_1,t_2]}L^2 \cap L^6_{[t_1,t_2]}L^6$, then by Strichartz (see equation (5.12) and proposition 2.1)

$$\big\|\Psi - e^{-iH(\cdot-t_1)}\Psi_{t_1}\big\|_{X_{t_1,t_2}} \leqslant \bigg\|\int_{t_1}^{\cdot} ds\, e^{-iH(\cdot-s)}|\Psi_s|^2\Psi_s\bigg\|_{X_{t_1,t_2}} \leqslant C(t_2-t_1)^{1/2}\big\|\Psi\big\|^3_{X_{t_1,t_2}}\,, \tag{5.25}$$

$$\big\|e^{-iH(\cdot-t_1)}\Psi_{t_1}\big\|_{X_{t_1,t_2}} \leqslant C\big\|\Psi_{t_1}\big\|\,,$$

which imply

$$\big\|\Psi\big\|_{X_{t_1,t_2}} \leqslant C\big\|\Psi_{t_1}\big\| + C(t_2-t_1)^{1/2}\big\|\Psi\big\|^3_{X_{t_1,t_2}}\,.$$

By lemma 5.2 one has that if $(t_2 - t_1) \leqslant \big[8C^3\|\Psi_{t_1}\|^2 + 4C^2\|\Psi_{t_1}\|\big]^{-2}$, then $\|\Psi\|_{X_{t_1,t_2}} \leqslant 2C\|\Psi_{t_1}\|$; using this estimate in the inequality (5.25) we get

$$\big\|\Psi_t - e^{-iH(t-t_1)}\Psi_{t_1}\big\| \leqslant \big\|\Psi - e^{-iH(\cdot-t_1)}\Psi_{t_1}\big\|_{X_{t_1,t_2}} \leqslant C(t_2-t_1)^{1/2}\big\|\Psi_{t_1}\big\|^3 \leqslant Cv^{-\delta/2} \tag{5.26}$$

where we used $t_2 - t_1 = 2v^{-\delta}$.

We proceed now with the estimate of the second term in the r.h.s. of inequality (5.24). Let us set

$$\Psi_{t_1} = \Psi_{t_1} - \Phi_{t_1} + \Phi_{t_1}^{S,in} + \Phi_{t_1}^{tail}\,.$$

Where the vector Φ_t was defined in equation (5.1), and we introduced the vector

$$\Phi_{t_1}^{tail} := \begin{pmatrix} 0 \\ \\ \phi_{-x_0,v}(t_1) \\ \\ 0 \end{pmatrix}\,.$$

We remark that $\Phi_{t_1}^{S,in} + \Phi_{t_1}^{tail} = \Phi_{t_1}$. The following trivial inequality holds true

$$\big\|e^{-iH(t-t_1)}\Psi_{t_1} - e^{-iH(t-t_1)}\Phi_{t_1}^{S,in}\big\| \leqslant \big\|\Psi_{t_1} - \Phi_{t_1}\big\| + \big\|\Phi_{t_1}^{tail}\big\| \leqslant Ce^{-v^{1-\delta}} \tag{5.27}$$

where in the last estimate we used proposition 5.3 and the fact that $\|\Phi_{t_1}^{tail}\| \leqslant 2e^{-v^{1-\delta}}$.

Let us consider now the last term in the r.h.s. of inequality (5.24). We are going to prove that for all $t \in (t_1, t_2)$ and for v big enough

$$\left\| e^{-iH(t-t_1)} \Phi^{S,in}_{t_1} - \Phi^S_t \right\| \leqslant C v^{-\delta} . \tag{5.28}$$

Let us introduce the functions

$$\phi^-_t(x) := \int_0^\infty U_{t-t_1}(x-y)\phi_{x_0,-v}(y,t_1)dy \tag{5.29}$$

and

$$\phi^+_t(x) := \int_0^\infty U_{t-t_1}(x+y)\phi_{x_0,-v}(y,t_1)dy . \tag{5.30}$$

First we prove a preliminary formula for the vector $e^{-iH(t-t_1)}\Phi^{S,in}_{t_1}$ (see equations (5.35) and (5.36) below). For any constant a, not dependent on v let us consider the term

$$\begin{aligned} &va \int_0^\infty e^{-uva}\big[U^+_{t-t_1}\phi_{x_0,-v}(t_1)\big](u+x)du \\ =&va\int_0^\infty du \int_0^\infty dy\, e^{-uva} U_{t-t_1}(u+x+y) e^{i\varphi(t_1)} e^{-i\frac{v}{2}y}\phi(y-x_0+vt_1) \end{aligned}$$

where we set $\varphi(t) := -t\frac{v^2}{4} + t$. By integrating by parts we obtain the equality

$$\begin{aligned} &va \int_0^\infty e^{-uva}\big[U^+_{t-t_1}\phi_{x_0,-v}(t_1)\big](u+x)du \\ =&2ia\int_0^\infty du \int_0^\infty dy\, e^{-uva} U_{t-t_1}(u+x+y) e^{i\varphi(t_1)} \left[\frac{d}{dy} e^{-i\frac{v}{2}y}\right]\phi(y-x_0+vt_1) \\ =&A_1(x,t)+A_2(x,t)+A_3(x,t) \end{aligned} \tag{5.31}$$

with

$$A_1(x,t) := -2ia\int_0^\infty du\, e^{-uva} U_{t-t_1}(u+x) e^{i\varphi(t_1)}\phi(-x_0+vt_1)$$

$$A_2(x,t) := -2ia\int_0^\infty du \int_0^\infty dy\, e^{-uva}\left[\frac{d}{dy}U_{t-t_1}(u+x+y)\right] e^{i\varphi(t_1)} e^{-i\frac{v}{2}y}\phi(y-x_0+vt_1)$$

$$A_3(x,t) := -2ia\int_0^\infty du \int_0^\infty dy\, e^{-uva} U_{t-t_1}(u+x+y) e^{i\varphi(t_1)} e^{-i\frac{v}{2}y}\left[\frac{d}{dy}\phi(y-x_0+vt_1)\right] .$$

We notice that

$$\left\|\int_0^\infty du\, e^{-uva}U_{t-t_1}(u+\cdot)\right\|_{L^2(\mathbb{R}^+)} \leqslant \left\|\int_0^\infty du\, e^{-uva}U_{t-t_1}(u+\cdot)\right\|_{L^2(\mathbb{R})} = \|\chi_+ e^{-va\cdot}\|_{L^2(\mathbb{R})} = \sqrt{\frac{1}{2va}}.$$

Then the following estimate for the term A_1 holds true

$$\|A_1(t)\|_{L^2(\mathbb{R}^+)} \leqslant 2a \left\|\int_0^\infty du\, e^{-uva}U_{t-t_1}(u+\cdot)\right\|_{L^2(\mathbb{R}^+)} |\phi(-x_0+vt_1)| \leqslant C\,\frac{e^{-v^{1-\delta}}}{v^{1/2}} . \tag{5.32}$$

The term A_3 is estimated by

$$\begin{aligned} \|A_3(t)\|_{L^2(\mathbb{R}^+)} \leqslant& 2a\int_0^\infty du\, e^{-uva} \left\| \big[U_{t-t_1}\chi_+ e^{-i\frac{v}{2}\cdot}\phi'(\cdot - x_0+vt_1)\big]\big(-(u+\cdot)\big)\right\|_{L^2(\mathbb{R}^+)} \\ \leqslant& 2\|\phi'\|_{L^2(\mathbb{R})} a \int_0^\infty du\, e^{-uva} \leqslant \frac{C}{v} , \end{aligned} \tag{5.33}$$

where we used the equality $[U_t^+ f](x) = [U_t \chi_+ f](-x)$.
We compute finally the term A_2. By integration by parts

$$\begin{aligned} A_2(x,t) =& -2ia \int_0^\infty du \int_0^\infty dy\, e^{-uva} \left[\frac{d}{du} U_{t-t_1}(u+x+y) \right] e^{i\varphi(t_1)} e^{-i\frac{v}{2}y} \phi(y - x_0 + vt_1) \\ =& 2ia\phi_t^+(x) - 2iva^2 \int_0^\infty e^{-uva} \left[U_{t-t_1}^+ \phi_{x_0,-v}(t_1) \right](u+x) du\,, \end{aligned}$$

where the function ϕ_t^+ was defined in equation (5.30). Using the last equality in the equation (5.31) we get

$$va \int_0^\infty e^{-uva} \left[U_{t-t_1}^+ \phi_{x_0,-v}(t_1) \right](u+x) du = \frac{2ia}{1+2ia} \phi_t^+(x) + \frac{A_1(x,t) + A_3(x,t)}{1+2ia}\,. \tag{5.34}$$

From the definition of $\Phi_t^{S,in}$ and using the last equality with $a = \tilde{\alpha}/3$ in the formula for the integral kernel of $e^{-iH_\delta^\alpha t}$, see equation (2.6), it follows that

$$e^{-iH_\delta^\alpha (t-t_1)} \Phi_{t_1}^{S,in} = e^{-iH_\delta^\alpha (t-t_1)} \begin{pmatrix} \phi_{x_0,-v}(t_1) \\ \\ 0 \\ \\ 0 \end{pmatrix} = \begin{pmatrix} \phi_t^- - \frac{1+2i\tilde{\alpha}}{3+2i\tilde{\alpha}} \phi_t^+ \\ \\ \frac{2}{3+2i\tilde{\alpha}} \phi_t^+ \\ \\ \frac{2}{3+2i\tilde{\alpha}} \phi_t^+ \end{pmatrix} - \frac{2}{3} \begin{pmatrix} A_{\tilde{\alpha}}(t) \\ \\ A_{\tilde{\alpha}}(t) \\ \\ A_{\tilde{\alpha}}(t) \end{pmatrix} \tag{5.35}$$

where the function ϕ_t^- was defined in equation (5.29) and we set $A_{\tilde{\alpha}}(x,t) := \left[(A_1(x,t) + A_3(x,t))/(1 + 2ia) \right]\big|_{a=\tilde{\alpha}/3}$. Similarly using equality (5.34) with $a = 3/\tilde{\beta}$ in the formula for the integral kernel of the propagator $e^{-iH_{\delta'}^\beta t}$, see equation (2.8), we get

$$e^{-iH_{\delta'}^\beta (t-t_1)} \Phi_{t_1}^{S,in} = \begin{pmatrix} \phi_t^- + \frac{\tilde{\beta}+2i}{\tilde{\beta}+6i} \phi_t^+ \\ \\ -\frac{4i}{\tilde{\beta}+6i} \phi_t^+ \\ \\ -\frac{4i}{\tilde{\beta}+6i} \phi_t^+ \end{pmatrix} - \frac{2}{3} \begin{pmatrix} A_{\tilde{\beta}}(t) \\ \\ A_{\tilde{\beta}}(t) \\ \\ A_{\tilde{\beta}}(t) \end{pmatrix} \tag{5.36}$$

where we introduced the notation $A_{\tilde{\beta}}(x,t) := \left[(A_1(x,t) + A_3(x,t))/(1 + 2ia) \right]\big|_{a=3/\tilde{\beta}}$. We notice that the analogous formula for $e^{-iH_F(t-t_1)} \Phi_{t_1}^{S,in}$ can be obtained from equation (5.35) by setting $\tilde{\alpha} = 0$ and $A_{\tilde{\alpha}} = 0$.

To get the estimate (5.28) we show that, at the cost of an error of the order $(t_2 - t_1)$, for $t \in (t_1, t_2)$, the functions $\phi_t^-(x)$ and $\phi_t^+(x)$ can be approximated with the solitons $\phi_{x_0,-v}(x,t)$ and $\phi_{-x_0,v}(x,t)$ respectively. We consider first the function ϕ_t^+, by adding and subtracting a suitable term to the r.h.s. of equation (5.30) we get

$$\begin{aligned} \phi_t^+(x) =& \int_{-\infty}^\infty U_{t-t_1}(x+y) \phi_{x_0,-v}(y,t_1) + i \int_{t_1}^t ds \int_{-\infty}^\infty U_{t-s}(x+y) |\phi_{x_0,-v}(y,s)|^2 \phi_{x_0,-v}(y,s) dy \\ & - \int_{-\infty}^0 U_{t-t_1}(x+y) \phi_{x_0,-v}(y,t_1) dy - i \int_{t_1}^t ds \int_{-\infty}^\infty U_{t-s}(x+y) |\phi_{x_0,-v}(y,s)|^2 \phi_{x_0,-v}(y,s)\, dy \\ =& \phi_{-x_0,v}(x,t) + I(x,t) + II(x,t)\,, \end{aligned} \tag{5.37}$$

where we used the fact that $\phi_{x_0,-v}(-x,t) = \phi_{-x_0,v}(x,t)$ and we set

$$I(x,t) := -\int_{-\infty}^0 U_{t-t_1}(x+y) \phi_{x_0,-v}(y,t_1) dy$$

and

$$II(x,t) := -i\int_{t_1}^{t} ds \int_{-\infty}^{\infty} U_{t-s}(x+y)|\phi_{x_0,-v}(y,s)|^2\phi_{x_0,-v}(y,s)\,dy\,.$$

For the term I we use the estimate

$$\|I\|_{L^2(\mathbb{R}^+)} \leqslant \|\phi(\cdot - x_0 + vt_1)\|_{L^2(\mathbb{R}^-)} \leqslant 2e^{-v^{1-\delta}}\,.$$

The term II is estimated by

$$\|II\|_{L^2(\mathbb{R}^+)} \leqslant (t-t_1)\|\phi^3\|_{L^2(\mathbb{R})} \leqslant Cv^{-\delta}\,.$$

Similarly, for the function $\phi^-_{t_2}$, we get

$$\phi^-_t(x) = \phi_{x_0,-v}(x,t) + III(x,t) + IV(x,t) \tag{5.38}$$

where we set

$$III(x,t) := -\int_{-\infty}^{0} U_{t-t_1}(x-y)\phi_{x_0,-v}(t_1,y)dy$$

and

$$IV(x,t) := -i\int_{t_1}^{t} ds \int_{-\infty}^{\infty} U_{t-s}(x-y)|\phi_{x_0,-v}(s,y)|^2\phi_{x_0,-v}(s,y)\,dy\,.$$

For $t \in (t_1,t_2)$ the estimates

$$\|III\|_{L^2(\mathbb{R}^+)} \leqslant 2e^{-v^{1-\delta}}\;; \quad \|IV\|_{L^2(\mathbb{R}^+)} \leqslant Cv^{-\delta}$$

are similar to the ones given above for the terms I and II. Then from equations (5.37) and (5.38) we have

$$\|\phi^+_t - \phi_{-x_0,v}(t)\|_{L^2(\mathbb{R}^+)} \leqslant C[e^{-v^{1-\delta}} + v^{-\delta}]\,, \quad \|\phi^-_t - \phi_{x_0,-v}(t)\|_{L^2(\mathbb{R}^+)} \leqslant C[e^{-v^{1-\delta}} + v^{-\delta}]$$

for all $t \in (t_1,t_2)$. By using the last estimates and the estimates (5.32) and (5.33) in equations (5.35) and (5.36) we get

$$\left\|e^{-iH(t-t_1)}\Phi^{S,in}_{t_1} - \Phi^S_t\right\| \leqslant C\left[e^{-v^{1-\delta}} + v^{-\delta} + \frac{e^{-v^{1-\delta}}}{v^{1/2}} + \frac{1}{v}\right]$$

which in turn implies that for v big enough the estimate (5.28) holds true.

Using estimates (5.26), (5.27) and (5.28) in the inequality (5.24) we get the estimate (5.22).

The estimate (5.23) is a consequence of estimate (5.22) and of the fact that $\|\Phi^S_{t_2} - \Phi^{S,out}_{t_2}\| = \|\Phi^{S,in}_{t_2}\| \leqslant Ce^{-v^{1-\delta}}$. □

Remark 5.5. *Notice that estimate* (5.22)*, although non strictly necessary for the proof of theorem 4.3, enforces the picture that in the phase 2 a scattering event is occurring. The true wavefunction can be approximated by the superposition of an incoming and an outgoing wavefunction. At the end of this phase, only the outgoing wavefunction is not negligibile.*

5.3 Phase 3

Let us put $t_3 = t_2 + T\ln v$. We call "phase 3" the evolution of the system in the time interval (t_2,t_3). The approximation of Ψ_t during this time interval is the content of theorem 4.3.
We recall the following proposition (see [14]).

Proposition 5.6. *Let ϕ^{tr}_t and ϕ^{ref}_t be defined as it was done in equations* (4.11) *and* (4.12) *above. Then $\forall k \in \mathbb{N}$ there exist two constants $c(k) > 0$ and $\sigma(k) > 0$ such that*

$$\|\phi^\gamma_t\|_{L^2(\mathbb{R}^-)} + \|\phi^\gamma_t\|_{L^\infty(\mathbb{R}^-)} \leqslant \frac{c(k)(\ln v)^{\sigma(k)}}{v^{k(1-\delta)}} \tag{5.39}$$

for $\gamma = \{ref, tr\}$, uniformly in $t \in [0, T\ln v]$.

Notice that also the norms $\|\phi_t^\gamma\|_{L^p(\mathbb{R}^-)}$, for $2 \leqslant p \leqslant \infty$, are estimated by the r.h.s. of (5.39). Now we can prove theorem 4.3

Proof of Theorem 4.3. The strategy of the proof closely follows proposition 5.3. We will just sketch the common part of the proof while proving in details the different estimates.

Let us define $\Xi_t := \Psi_t - \sum_{j=1}^3 \Phi_t^j$ where the vectors Φ_t^j were given in equation (4.8) and fix $t_a \in [t_2, t_3]$. From equations (4.3) and (4.8) it follows that the vector Ξ_t satisfies the following integral equation

$$\begin{aligned}
\Xi_t =& e^{-iH(t-t_a)}\Xi_{t_a} \\
&+ \sum_{j=1}^3 \left[\left(e^{-iH(t-t_a)} - e^{-iH_j(t-t_a)} \right) \Phi_{t_a}^j + i \int_{t_a}^t ds \left(e^{-iH(t-s)} - e^{-iH_j(t-s)} \right) |\Phi_s^j|^2 \Phi_s^j \right] \\
&+ i \int_{t_a}^t ds\, e^{-iH(t-s)} \sum_{j_1,j_2,j_3} (1 - \delta_{j_1 j_2}\delta_{j_2 j_3}) \overline{\Phi_s^{j_1}} \Phi_s^{j_2} \Phi_s^{j_3} \\
&+ i \int_{t_a}^t ds\, e^{-iH(t-s)} \left[\left| \Xi_s + \sum_{j=1}^3 \Phi_s^j \right|^2 \Xi_s + |\Xi_s|^2 \sum_{j=1}^3 \Phi_s^j + 2\,\mathrm{Re} \left[\overline{\Xi_s} \sum_{j=1}^3 \Phi_s^j \right] \sum_{j=1}^3 \Phi_s^j \right] \\
=& e^{-iH(t-t_a)}\Xi_{t_a} + G(t_a, t) \\
&+ i \int_{t_a}^t ds\, e^{-iH(t-s)} \left[\left| \Xi_s + \sum_{j=1}^3 \Phi_s^j \right|^2 \Xi_s + |\Xi_s|^2 \sum_{j=1}^3 \Phi_s^j + 2\,\mathrm{Re} \left[\overline{\Xi_s} \sum_{j=1}^3 \Phi_s^j \right] \sum_{j=1}^3 \Phi_s^j \right].
\end{aligned} \tag{5.40}$$

Let us fix $t_b \in [t_a, t_3]$ and let $X_{t_a,t_b} = L^\infty_{[t_a,t_b]} L^2 \cap L^6_{[t_a,t_b]} L^6$. Using the Strichartz estimates as it was done in proposition 5.3 (see equations (5.9) - (5.13)), it is straightforward to prove that

$$\|\Xi\|_{X_{t_a,t_b}} \leqslant \widetilde{C} \left[\|\Xi_{t_a}\| + \|G(t_a, \cdot)\|_{X_{t_a,t_b}} + (t_b - t_a)^{1/2} \left(\|\Xi\|^2_{X_{t_a,t_b}} + \|\Xi\|^3_{X_{t_a,t_b}} \right) \right]$$

where $\widetilde{C}$ depends only on the constants appearing in the Strichartz estimates. Using lemma 5.2 as it was done in the proof of proposition 5.3 it follows that there exists $\tau > 0$ such that, for any $t \in [t_2, t_3)$ one has $\|\Xi\|_{X_{t,t+\tau}} \leqslant 2\widetilde{C}\, \|\Xi_t\| + 2\widetilde{C}\, \|G(t, \cdot)\|_{X_{t,t+\tau}}$.
We divide the interval $[t_2, t_3]$ in $N + 1$ subintervals: $[t_2 + j\tau, t_2 + (j + 1)\tau)$, with $j = 0, ..., N - 1$; and $[t_2 + N\tau, t_3)$. Where N is the integer part of $(t_3 - t_2)/\tau$.
Then proceeding by induction as we did in the proof of proposition 5.3, see equations (5.14) and (5.15), we get the inequality

$$\|\Xi\|_{X_{t_2,t_3}} \leqslant \sum_{j=0}^N (2\widetilde{C})^{j+1} \|\Xi_{t_2}\| + \sum_{j=0}^N \sum_{k=0}^j (2\widetilde{C})^{j+1-k} \|G(t_2 + k\tau, \cdot)\|_{X_{t_2+k\tau, \min\{t_3, t_2+(k+1)\tau\}}}. \tag{5.41}$$

Now we estimate the initial datum $\|\Xi_{t_2}\|$ and the source term $\|G(t_a, \cdot)\|_{X_{t_a,t_b}}$ with $t_2 \leqslant t_a \leqslant t_b \leqslant t_3$ and $t_b - t_a \leqslant \tau$.
By proposition 5.4 (estimate (5.33)) and using the definitions (4.4) - (4.6) one has

$$\Xi_{t_2} = \Psi_{t_2} - \sum_{j=1}^3 \Phi_{t_2}^j = \Psi_{t_2} - \Phi_{t_2}^{S,out} - \begin{pmatrix} \tilde{t}\, \phi_{x_0,-v}(t_2) \\ \tilde{r}\, \phi_{x_0,-v}(t_2) \\ \tilde{t}\, \phi_{x_0,-v}(t_2) \end{pmatrix},$$

with $\|\Psi_{t_2} - \Phi_{t_2}^{S,out}\| \leqslant C v^{-\delta/2}$. Since

$$\int_0^\infty |\phi_{x_0,-v}(x, t_2)|^2 dx = \int_0^\infty |\phi(x - x_0 + vt_2)|^2 dx = 2 \int_{v^{1-\delta}}^\infty \mathrm{sech}(x)^2 dx = \frac{4e^{-2v^{1-\delta}}}{1 + e^{-2v^{1-\delta}}} \leqslant 4e^{-2v^{1-\delta}}$$

we have

$$\|\Xi_{t_2}\| \leqslant C\big(v^{-\frac{\delta}{2}} + e^{-2v^{1-\delta}}\big) \leqslant Cv^{-\frac{\delta}{2}} . \tag{5.42}$$

Let us now consider the source term $G(t_a,t)$. We use the estimate $\|G(t_a,\cdot)\|_{X_{t_a,t_b}} \leqslant \|G(t_2,\cdot)\|_{X_{t_2,t_3}}$. To simplify the notation we set $G(t)\equiv G(t_2,t)$ and

$$G_1(t) := \sum_{j=1}^{3}\bigg[\Big(e^{-iH(t-t_2)} - e^{-iH_j(t-t_2)}\Big)\Phi^j_{t_2} + i\int_{t_2}^{t} ds\,\Big(e^{-iH(t-s)} - e^{-iH_j(t-s)}\Big)|\Phi^j_s|^2\Phi^j_s\bigg]$$

$$G_2(t) := i\int_{t_2}^{t} ds\, e^{-iH(t-s)}\sum_{j_1,j_2,j_3}(1-\delta_{j_1j_2}\delta_{j_2j_3})\overline{\Phi^{j_1}_s}\Phi^{j_2}_s\Phi^{j_3}_s .$$

By the definition of $G(t_a,t)$, see equation (5.40) it follows that

$$G(t) = G_1(t) + G_2(t)\,;$$

we estimate $G_1(t)$ and $G_2(t)$ separately.

We proceed first with the estimate of the term G_1. From equations (2.4), (2.6), (2.8) and (2.9) one can see that for any (column) vector $F=(F_1,F_2,F_3)\in L^2$

$$\big[e^{-iHt} - e^{-iH_jt}\big]F = \mathbb{M}_j\begin{pmatrix} U^+_tF_1 \\ U^+_tF_2 \\ U^+_tF_3\end{pmatrix} - \frac{2}{3}\mathbb{J}\begin{pmatrix} va\int_0^\infty e^{-uva}\big[U^+_tF_1\big](u+\cdot)du \\ va\int_0^\infty e^{-uva}\big[U^+_tF_2\big](u+\cdot)du \\ va\int_0^\infty e^{-uva}\big[U^+_tF_3\big](u+\cdot)du\end{pmatrix} \qquad j=1,2,3\,,$$

where the constant a and the matrices $\mathbb{M}_j$ must be chosen accordingly to the Hamiltonian H:
for $H=H^\alpha_\delta$,

$$a = \frac{\tilde\alpha}{3}\,, \qquad \mathbb{M}_j = -\mathbb{I} + \frac{2}{3}\mathbb{J} - \mathbb{T}_j\,;$$

for $H = H^\beta_{\delta'}$,

$$a = \frac{3}{\tilde\beta}\,, \qquad \mathbb{M}_j = \mathbb{I} - \mathbb{T}_j\,;$$

and the formula for $H=H_F$ can be obtained by setting $\tilde\alpha = 0$ in the formula for $H = H^\alpha_\delta$.
Then, denoting by $\big(G_1(x,t)\big)_l$, $l=1,2,3$, the l-th component of the vector G_1 one has

$$\big(G_1(x,t)\big)_l = \big(\widetilde G_1(x,t)\big)_l + \big(\widehat G_1(x,t)\big)_l\,,$$

with

$$\big(\widetilde G_1(x,t)\big)_l := \sum_{j,k=1}^{3}(\mathbb{M}_j)_{lk}\bigg[\big[U^+_{t-t_2}\big(\Phi^j_{t_2}\big)_k\big](x) + i\int_{t_2}^t ds\big[U^+_{t-s}\big|\big(\Phi^j_s\big)_k\big|^2\big(\Phi^j_s\big)_k\big](x)\bigg]$$

$$\big(\widehat G_1(x,t)\big)_l := -\frac{2va}{3}\sum_{j,k=1}^{3}\int_0^\infty e^{-vua}\bigg[\big[U^+_{t-t_2}\big(\Phi^j_{t_2}\big)_k\big](u+x) + i\int_{t_2}^t ds\big[U^+_{t-s}\big|\big(\Phi^j_s\big)_k\big|^2\big(\Phi^j_s\big)_k\big](u+x)\bigg]du\,.$$

From the definition of the vectors Φ^j_t, see equations (4.9) - (4.10), we see that for each $l=1,2,3$ the function $\big(\widetilde G_1(x,t)\big)_l$ is a linear combination of four functions, $f^{\gamma,+}_t$ and $f^{\gamma,-}_t$, with γ being equal to ref and tr, given by

$$f^{\gamma,+}_t(x) := e^{-i\frac{v^2}{4}t_2}e^{it_2}\bigg[\int_0^\infty U_{t-t_2}(x+y)\phi^\gamma_0(y)dy + i\int_{t_2}^t ds\int_0^\infty U_{t-s}(x+y)|\phi^\gamma_{s-t_2}(y)|^2\phi^\gamma_{s-t_2}(y)dy\bigg] \tag{5.43}$$

and

$$f_t^{\gamma,-}(x) := e^{-i\frac{v^2}{4}t_2}e^{it_2}\left[\int_0^\infty U_{t-t_2}(x+y)\phi_0^\gamma(-y)dy + i\int_{t_2}^t ds\int_0^\infty U_{t-s}(x+y)|\phi_{s-t_2}^\gamma(-y)|^2\phi_{s-t_2}^\gamma(-y)dy\right]$$

where the functions ϕ_t^{ref} and ϕ_t^{ref} were defined in equations (4.11) and (4.12) respectively.

Similarly one can see that for each $l = 1,2,3$ the function $\big(\widehat{G}_1(x,t)\big)_l$ is a linear combination of

$$va\int_0^\infty e^{-vua}f_t^{ref,+}(u+x)du\ , \quad va\int_0^\infty e^{-vua}f_t^{tr,+}(u+x)du\ ,$$

$$va\int_0^\infty e^{-vua}f_t^{ref,-}(u+x)du\ , \quad va\int_0^\infty e^{-vua}f_t^{tr,-}(u+x)du\ .$$

First we study the function $f_t^{\gamma,+}$. We notice that, adding and subtracting a suitable term in equation (5.43) and using the definitions (4.11) and (4.12), $f_t^{\gamma,+}(x)$ can be written as

$$f_t^{\gamma,+}(x) = I(x,t) + II(x,t) + III(x,t)\,,$$

with

$$I(x,t) := e^{-i\frac{v^2}{4}t_2}e^{it_2}\phi_{t-t_2}^\gamma(-x)$$

$$II(x,t) := -e^{-i\frac{v^2}{4}t_2}e^{it_2}\int_0^\infty U_{t-t_2}(x-y)\phi_0^\alpha(-y)dy$$

$$III(x,t) := -ie^{-i\frac{v^2}{4}t_2}e^{it_2}\int_{t_2}^t ds\int_0^\infty U_{t-s}(x-y)|\phi_{s-t_2}^\gamma(-y)|^2\phi_{s-t_2}^\gamma(-y)dy\,.$$

Similarly to what was done above, we set $X_{t_2,t_3}(\mathbb{R}^\pm) = L^\infty_{[t_2,t_3]}L^2(\mathbb{R}^\pm)\cap L^6_{[t_2,t_3]}L^6(\mathbb{R}^\pm)$. By proposition 5.6 , we have

$$\|I\|_{X_{t_2,t_3}(\mathbb{R}^+)} = \|\phi^\gamma_{\cdot-t_2}\|_{X_{t_2,t_3}(\mathbb{R}^-)} \leqslant \frac{c'(k)(\ln v)^{\sigma'(k)}}{v^{k(1-\delta)}} \tag{5.44}$$

where $c'(k)$ and $\sigma'(k)$ are different constants from the one appearing in proposition 5.6. For our purposes we do not need to compute them.

Using the one dimensional Strichartz estimates for U_t, we have

$$\|II\|_{X_{t_2,t_3}(\mathbb{R}^+)} \leqslant C\|\chi_-\phi_0^\gamma\|_{L^2(\mathbb{R})} \leqslant Ce^{-2v^{1-\delta}}\,. \tag{5.45}$$

Finally, the term III can be estimated using the inhomogeneous Strichartz estimate and proposition 5.6.

$$\|III\|_{X_{t_2,t_3}(\mathbb{R}^+)} \leqslant C\|(\chi_-\phi^\gamma_{\cdot-t_2})^3\|_{L^1_{[t_2,t_3]}L^2(\mathbb{R})} \leqslant C\left[\frac{c(k)(\ln v)^{\sigma(k)'}}{v^{k(1-\delta)}}\right]^3. \tag{5.46}$$

Collecting the estimates (5.44), (5.45) and (5.46), it follows that

$$\|f_\cdot^{\gamma,+}\|_{X_{t_2,t_3}(\mathbb{R}^+)} \leqslant C\left[\frac{c(k)(\ln v)^{\sigma(k)'}}{v^{k(1-\delta)}}\right].$$

The estimate of $f_t^{\gamma,-}$ is similar and we omit it. We have proved that for some $c'(k)$ and $\sigma'(k)$ possibly bigger than $c(k)$ and $\sigma(k)$ we have:

$$\|\widetilde{G}_1\|_{X_{t_2,t_3}} \leqslant C\left[\frac{c'(k)(\ln v)^{\sigma'(k)}}{v^{k(1-\delta)}}\right]$$

where $\widetilde{G}_1$ is the vector in L^2 with components $\big(\widetilde{G}_1(x,t)\big)_l$, $l = 1,2,3$.

The estimate of $\widehat{G}_1(t) = \big((\widehat{G}_1(t))_1, (\widehat{G}_1(t))_2, (\widehat{G}_1(t))_3\big)$ is a trivial consequence of the fact that

$$\left\| va \int_0^\infty e^{-vua} f_\cdot^{\gamma,\pm}(u+\,\cdot\,)du \right\|_{X_{t_2,t_3}(\mathbb{R}^+)} \leqslant \|f_\cdot^{\gamma,\pm}\|_{X_{t_2,t_3}(\mathbb{R}^+)} \leqslant C\left[\frac{c(k)(\ln v)^{\sigma(k)'}}{v^{k(1-\delta)}}\right],$$

from which it follows that

$$\|\widehat{G}_1\|_{X_{t_2,t_3}} \leqslant C\left[\frac{c'(k)(\ln v)^{\sigma'(k)}}{v^{k(1-\delta)}}\right];$$

and

$$\|G_1\|_{X_{t_2,t_3}} \leqslant C\left[\frac{c'(k)(\ln v)^{\sigma'(k)}}{v^{k(1-\delta)}}\right]. \tag{5.47}$$

We analyse now the term G_2. Due to the presence of $(1-\delta_{j_1j_2}\delta_{j_2j_3})$ the components of the vector

$$(1-\delta_{j_1j_2}\delta_{j_2j_3})\overline{\Phi_s^{j_1}}\Phi_s^{j_2}\Phi_s^{j_3}$$

contains only terms (up to a phase) like

$$\phi^{\gamma_1}_{t-t_2}(x)\phi^{\gamma_2}_{t-t_2}(x)\phi^{\gamma_3}_{t-t_2}(-x) \quad \text{or} \quad \phi^{\gamma_1}_{t-t_2}(x)\phi^{\gamma_2}_{t-t_2}(-x)\phi^{\gamma_3}_{t-t_2}(-x)$$

where γ_1, γ_2 and γ_3 can be ref or tr. This can be easily seen by using equations (4.9) - (4.10). By Strichartz, it is sufficient to estimate the $L^1_{[t_2,t_3]}L^2(\mathbb{R}^+)$ norm of these terms. Then using Hölder inequality we have, for istance

$$\|\phi^{\gamma_1}_{t-t_2}\phi^{\gamma_2}_{t-t_2}\phi^{\gamma_3}_{t-t_2}(-\cdot)\|_{L^2(\mathbb{R}^+)} \leqslant \|\phi^{\gamma_1}_{t-t_2}\|_{L^4(\mathbb{R}^+)}\|\phi^{\gamma_2}_{t-t_2}\|_{L^4(\mathbb{R}^+)}\|\phi^{\gamma_3}_{t-t_2}\|_{L^\infty(\mathbb{R}^-)} \leqslant C\frac{c(k)(\ln v)^{\sigma(k)}}{v^{k(1-\delta)}}.$$

The second kind of terms can be estimated in the same way and we obtain

$$\|G_2\|_{X_{t_2,t_3}} \leqslant C\left[\frac{c'(k)(\ln v)^{\sigma'(k)}}{v^{k(1-\delta)}}\right],$$

which, together with the estimate (5.47), gives

$$\|G\|_{X_{t_2,t_3}} \leqslant C\left[\frac{c'(k)(\ln v)^{\sigma'(k)}}{v^{k(1-\delta)}}\right]. \tag{5.48}$$

Fix k such that $k(1-\delta) > 2$ then for v sufficiently large (5.48) implies

$$\|G\|_{X_{t_2,t_3}} \leqslant \frac{1}{v}. \tag{5.49}$$

Using estimates (5.42) and (5.49) in the inequality (5.41) we get

$$\begin{aligned}\|\Xi\|_{X_{t_2,t_3}} &\leqslant Cv^{-\delta/2}\sum_{j=0}^{N}(2\widetilde{C})^{j+1} + v^{-1}\sum_{j=0}^{N}\sum_{k=0}^{j}(2\widetilde{C})^{j+1-k}\\ &\leqslant 2Cv^{-\delta/2}(2\widetilde{C})^{N+1} + v^{-1}\sum_{j=0}^{N}\left[(2\widetilde{C})^{j+1} + (2\widetilde{C})^{j} + \ldots + (2\widetilde{C})^2 + (2\widetilde{C})\right]\\ &\leqslant 2Cv^{-\delta/2}(2\widetilde{C})^{N+1} + 2v^{-1}N(2\widetilde{C})^{N+1} \leqslant \widehat{C}Nv^{-\delta/2}(2\widetilde{C})^N.\end{aligned}$$

Since N is the integer part of $(t_3-t_2)/\tau = \frac{T}{\tau}\ln v$

$$Nv^{-\delta/2}(2\widetilde{C})^N = \exp\left[\ln N - \frac{\delta}{2}\ln v + N\ln(2\widetilde{C})\right] \leqslant \exp\left[\Big(-\frac{\delta}{2} + \frac{T}{\tau}(\ln(2\widetilde{C})+1)\Big)\ln v\right].$$

We can finally set $\tau_* \equiv \tau/(\ln(2\widetilde{C})+1)$ and $T_* = \delta\tau_*/2$ and obtain

$$\|\Xi\|_{L^\infty_{[t_2,t_3]}L^2} \leqslant \|\Xi\|_{X_{t_2,t_3}} \leqslant \widehat{C}v^{-\frac{T_*-T}{\tau_*}}.$$

which concludes the proof of theorem (4.3). $\square$

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