Lifshitz tails for continuous Laplacian in the site percolation case.

W. Kirsch [1] and H. Najar [2]

Abstract

In this paper we study Lifshitz tails for continuous Laplacian in a continuous site percolation situation. By this we mean that we delete a random set Γ_ω from $\mathbb{R}^d$ and consider the Dirichlet or Neumann Laplacian on $D = \mathbb{R}^d \setminus \Gamma_\omega$. We prove that the integrated density of states exhibits Lifshitz behavior at the bottom of the spectrum when we consider Dirichlet boundary conditions, while when we consider Neumann boundary conditions , it is bounded from below by a van Hove behavior. The Lifshitz tails are proven independently of the percolation probability, whereas for the van Hove case we need some assumption on the volume of the sets taken out as well as on the percolation probability.

2000 Mathematics Subject Classification:15A52, 35P05, 37A30, 47F05.
Keywords and phrases:spectral theory, random operators, integrated density of states, Lifshitz tails, percolation, random graphs.

[1]Fakultät für Mathematik und Informatik FernUniversität Hagen D-58084 Hagen, Germany. e-mail: werner.kirsch@fernuni-hagen.de

[2]Département de Mathématiques Physiques I.S.M.A.I. Kairouan, Abd Assed Ibn Elfourat Kairouan 3100. Tunisie.

1 Introduction

The present study deals with the behavior of the integrated density of states for Laplacians in a site percolation setting on the continuum.

We start by defining the main object of our study, the integrated density of states (IDS). The IDS is a concept of fundamental importance in condensed matter physics. Indeed, it is considered to measure the number of energy levels per unit volume, below a given energy. For P, a finite-dimensional orthogonal projection, we denote by $tr(P)$, the dimension of its range. Let $P_{(-\infty,E]}$ be the spectral projection of a random Schrödinger operator H_ω and Λ_L be a cube of side length L around the origin. The restriction of $P_{(-\infty,E]}$ to the cube Λ_L has $tr(\chi_{\Lambda_L} P_{(-\infty,E]})$ as a dimension range. Here for a set A we denote by χ_A, the characteristic function of the set A. We denote by $|E|$, the Lebesgue measure of E. We consider

$$N(E) = \lim_{L\to\infty} \frac{1}{\mid \Lambda_L \mid} tr(\chi_{\Lambda_L} P(-\infty, E]). \tag{1.1}$$

It is called the **integrated density of states** of H_ω (IDS). See [10] for alternative definition of the IDS.
The question we are interested in here concerns the behavior of N at the bottom of the spectrum of H_ω.
In 1964, Lifshitz [15] argued that, for a Schrödinger operator of the form $H_\omega = -\Delta + V_\omega$, there exists $c_1, c_2 > 0$ such that $N(E)$ satisfies the asymptotic:

$$N(E) \simeq c_1 \exp(-c_2(E - E_0)^{-\frac{d}{2}}), \quad \text{as } E \searrow E_0. \tag{1.2}$$

Here E_0 is the bottom of the spectrum of H_ω. The behavior (1.2) is known as **Lifshitz tails**. In the last thirty years, there has been vast literature, both physical and mathematical, concerning Lifshitz tails and related phenomena. We do not try to give an exhaustive account of this literature. The work of Kirsch and Metzger [11] gives a survey of such results and basic references on this subject. Below, we give results on the IDS behavior in the context of percolation.

There is a long history of works which consider Hamiltonians on percolation graphs. These graphs are obtained by removing sites (site percolation) or bonds (bond percolation) from a graph, for example $\mathbb{Z}^d$, in a random way. For example, one might remove sites in the graph $\mathbb{Z}^d$ independently of each other with a probability p which is the same for each site, in other words there is a site at a given $i \in \mathbb{Z}^d$ with probability $q = 1 - p$. The percolation graph (consisting of the sites *not* removed) has various connected components, the so called clusters. It is known that there is a critical value p_c which depends on the dimension d, such that for $q < p_c$ there are only finite clusters, while for $q > p_c$ (the percolation regime) there is also an infinite cluster. For $d \geq 2$ the critical value satisfies $0 < p_c < 1$. The above results are discussed and proved in [5], they were mainly obtained by Hammersley in the late fifties. We remark that the fact, that there is *only* one infinite cluster, was proven thirty years later by Aizenman, Kesten and Newmann, see [5].

Quantum Hamiltonian with a percolation structure were first studied by de Gennes et al in [3, 4], where the Hamiltonian of a binary alloy was considered. It is proved that the spectrum of these percolation Hamiltonian is pure point if the fraction q (of conducting sites) is less than the critical value p_c. For the percolation regime it is argued in [3] that the spectrum contains a continuous part.

In [2] the authors considered the site percolation model on the lattice $\mathbb{Z}^d$, for $d \geq 2$. They investigated the density of states for the tight-binding Hamiltonian projected on the infinite cluster. It is shown that, almost surely, the IDS is discontinuous on a set of energies which is dense in the band. See also [20] for this type of questions.

There is also a collection of papers concerning the asymptotic behavior of the density of states for Laplacians on percolation graphs (see e. g . [1],[6], [7],[12],[16] and [17]). For example, the papers [12] and [17] investigate the asymptotic behavior of the integrated density of states $N_D(E)$ and $N_N(E)$ near the bottom E_0 of the spectrum of the bond percolation Laplacian on $\mathbb{Z}^d$ with Dirichlet repectively Neumann boundary conditions. These authors prove that $N_D(E)$ has Lifshitz tails with Lifshitz exponent $\frac{d}{2}$ for any $0 < q < 1$. The Neumann Laplacian on the percolation graph has an infinitely degenerate eigenvalue at the bottom $E_0 = 0$ of its spectrum and its density of states is discontinuous there. So, it is reasonable to consider $\tilde{N}(E) = N_N(E) - N_N(0)$. The results from [12] and [17] show that for $q < p_c$ the quantity $\tilde{N}(E)$ has Lifshitz behavior with Lifshitz exponent $\frac{1}{2}$, independent of the dimension. For $q > p_c$ $\tilde{N}(E)$ has a van Hove behavior near $E_0 = 0$, i.e. $\tilde{N}(E) \simeq CE^{\frac{d}{2}}$.

In this paper, we consider quantum percolation problems on $\mathbb{R}^d$. In one particular model we remove a set $S_i = S + i$ near the point $i \in \mathbb{Z}^d$ with probability p. More precisely, let $\{\xi_i\}_{i\in\mathbb{Z}^d}$ be a collection of independent $\{0,1\}$-valued random variable with $P(\xi = 1) = p$. We set $D = \mathbb{R}^d \setminus \bigcup_{\xi_i=1} S_i$ and denote by H_D^N and H_D^D the Laplacian restricted to D with Neumann and Dirichlet boundary conditions respectively. As the set D is random, the operators H_D^N and H_D^D are random operators.

We note that the set D may be a connected set for all ω if the set S is small, e.g. if $S \subset \Lambda_a = \{x \in \mathbb{R}^d \mid -a \le x_i \le a \text{ for } i = 1 \dots d\}$ with $a < 1$. On the other hand, if S is big and p is small, the set D is a union of bounded connected components.

We consider the behavior of the integrated density of states for these two classes of operators. We will obtain Lifshitz tails for the Dirichlet case under mild assumptions. For the Neumann case we prove that $N(E) \ge CE^{\frac{d}{2}}$ if the set S is not too big. The result on the Neumann case is based on a deterministic result (Theorem 4.1) which we believe is of some interest on its own. We apply it to other percolation models as well, for example to a Poisson percolation model.

2 Model and results

Let us first define the model we are studying.

2.1 The Model

Consider the probability space $\Omega = \{0,1\}^{\mathbb{Z}^d}$, which is endowed with the usual product sigma-algebra, generated by finite dimensional cylinder sets, and equipped with a product probability measure $\mathbb{P}$. Elementary events in Ω are sequences $\omega \equiv (\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$, we assume their entries to be independently and identically distributed according to a Bernoulli law. i.e

$$\mathbb{P}\{\omega_0 = 1\} = p. \tag{2.3}$$

Here p is the *site* probability, a parameter in $]0,1[$.
Let $\mathbb{Z}^d$ be the lattice, considered as a subset of $\mathbb{R}^d$. We set Λ_L the hypercube defined by $\Lambda_0 = [-\frac{L}{2}, \frac{L}{2}[^d$. Let $\mathcal{S}$ be a bounded subset of $\mathbb{R}$.
We set

$$\Gamma_\omega = \bigcup_{\gamma\in\mathbb{Z}^d} \omega_\gamma \Big(\gamma + \mathcal{S}\Big) = \bigcup_{\omega_\gamma = 0} \Big(\gamma + \mathcal{S}\Big)$$

where the notation $0\,\mathcal{S} = \emptyset$ and $1\,\mathcal{S} = \mathcal{S}$ was used.

Let $D^\omega = \mathbb{R}^d \backslash \Gamma_\omega$, be the domain obtained from $\mathbb{R}^d$ by taking off the random set Γ_ω. We denote by H^D_ω the restriction of the Laplacian $-\Delta$ to D^ω, with Dirichlet boundary condition and by H^N_ω the restriction of $-\Delta$ to D_ω with Neumann boundary condition. H^D_ω and H^N_ω are self-adjoint linear operators on $L^2(\mathbb{R}^d)$, we call them **Dirichlet** respectively **Neumann Laplacian for site-percolation in** $\mathbb{R}^d$. We denote by N_D, respectively by N_N the IDS of H^D_ω respectively of H^N_ω.

Remark 2.1. *As in [2], we can write H^D_ω formally, in the Anderson form with some random potential V_ω i.e*

$$H^D_\omega = -\Delta + V_\omega,$$

$$V_\omega = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma f(x-\gamma). \quad \textit{with} \quad f = \infty, \textit{on} \quad \mathcal{S} \textit{ and } \quad 0 \textit{ elsewhere}\ .$$

Let us consider the map $\mathcal{P}$, from Ω, to the set of the self-adjoint operator on $L^2(\mathbb{R}^d)$, such that to ω associate $\mathcal{P}(\omega) = H_\omega = H^\bullet_\omega = -\Delta^\bullet_{\lceil \mathbb{R}^d\backslash\Gamma_\omega}, \bullet \in$

$\{N, D\}$. $\mathcal{P}$ is measurable. Sometimes we suppress the superscript D and respectively N and write only H_ω. In this cases the statement remains true for both H_ω^D and H_ω^N.
Let $\mathcal{T}_i$ be the unitary translation operator on $L^2(\mathbb{Z}^d)$, i.e

$$\mathcal{T}_i\psi(x) = \psi(x-i), \quad \forall \psi \in L^2(\mathbb{R}^d) \text{ and } \ x \in \mathbb{R}^d.$$

As the probability measure $\mathbb{P}$ is ergodic with respect to the group of translation $(\mathcal{T}_i)_{i\in\mathbb{Z}^d}$, acting as $\mathcal{T}_i(\omega) = (\omega_{\gamma+i})_{\gamma\in\mathbb{Z}^d}$, we get

$$\mathcal{T}_i^{-1} H_\omega \mathcal{T}_i = H_{(\mathcal{T}_i\omega)}, \quad \forall i \in \mathbb{Z}^d, \ \omega \in \Omega. \tag{2.4}$$

By this, we deduce that H_ω is a measurable family of self-adjoint and ergodic operators. According to [8, 9, 18] we know that there exists $\Sigma, \Sigma_{pp}, \Sigma_{ac}$ and Σ_{sc} closed and non-random sets of $\mathbb{R}$ such that Σ is the spectrum of H_ω with probability one and such that if σ_{pp} (respectively σ_{ac} and σ_{sc}) denote the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of H_ω, then $\Sigma_{pp} = \sigma_{pp}, \Sigma_{ac} = \sigma_{ac}$ and $\Sigma_{sc} = \sigma_{sc}$ with probability one.
The following Lemma gives the precise location of the spectrum.

Lemma 2.2. *The spectrum Σ, of H_ω is $[0, +\infty[$ with probability one.*

Proof: First let us notice that for any $\omega \in \Omega$, we have

$$H_\omega \geq 0. \tag{2.5}$$

This gives that

$$\Sigma \subset [0, +\infty[.$$

So one needs to show the opposite inclusion, i.e

$$[0, +\infty[\subset \Sigma \text{ for } \mathbb{P} - \text{almost every } \omega \in \Omega. \tag{2.6}$$

For this, let $\widetilde{\Omega}$, be the following events

$$\widetilde{\Omega} = \Big\{\omega \in \Omega : \quad \text{for any } k \in \mathbb{N}, \qquad \text{There exists } \Lambda_k^\omega \subset \mathbb{R}^d, \text{such that } D^\omega_{\Lambda_k^{(\omega)}} = \mathbb{R}^d_{\Lambda_k^{(\omega)}}\Big\}. \tag{2.7}$$

Here $A_{\Lambda_k^{(\omega)}}$ is the set of points which are both in A and Λ_k^ω. In (2.7) we asked that all sites inside of $\Lambda_k^{(\omega)}$ to be present. Let $E \in [0, +\infty[= \Sigma(-\Delta)$ be arbitrarily fixed. Using Weyl criterion, we know that there exists a Weyl sequence $(\varphi_{E,n})_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$, for $-\Delta$. Thus $\|\varphi_{E,n}\| = 1$, for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|(\Delta + E \cdot \mathbb{I})\varphi_{E,n}\| = 0 \tag{2.8}$$

Notice that for any $i \in \mathbb{Z}^d$, $(\mathcal{T}_i\varphi_{E,n})_{n\in\mathbb{N}}$ is also a Weyl sequence. Without loss of generality, we assume that the sequence $(\varphi_{E,n})_{n\in\mathbb{N}}$ is compactly supported. So for any $\omega \in \widetilde{\Omega}$, there exists a Weyl sequences $(\varphi^\omega_{E,n})_{n\in\mathbb{N}}$ for $(-\Delta)$ on $\mathbb{R}^d$ with the property that all the supports are contained inside the cubes of (2.7). So for every $\omega \in \widetilde{\Omega}$ and any $n \in \mathbb{N}$ there exists an integer k_n^ω and a cube $\Lambda^{(\omega)}_{k_n^\omega}$ and $\varphi^\omega_{E,n}$ as in (2.7) such that $\mathrm{supp}(\varphi^\omega_{E,n})$ is contained in $\Lambda^{(\omega)}_{k_n^\omega}$. That is,

$$\min\{| x - y |: x \in \mathrm{supp}\varphi^\omega_{E,n};\ y \in \mathbb{R}^d\backslash\Lambda^{(\omega)}_{k_n^\omega}\} > 0.$$

So, for any $n \in \mathbb{N}$ and $\omega \in \widetilde{\Omega}$, we get

$$\|(H_\omega - E\mathbb{I})\varphi^\omega_{E,n})\| = \|(\Delta + E \cdot \mathbb{I})\varphi^\omega_{E,n}\|. \tag{2.9}$$

Hence, $(\varphi^\omega_{E,n})_{n\in\mathbb{N}}$ is also a Weyl sequence for H_ω. So we get (2.6) for any $\omega \in \widetilde{\Omega}$. Now it suffices to check that $\mathbb{P}(\widetilde{\Omega}) = 1$. For this let λ be an integer bigger than 2. $(\Lambda_{k,\lambda})_{\lambda\in\mathbb{N}} \subset \mathbb{R}^d$ be a sequence of disjoint cubes in $\mathbb{R}^d$ i.e $\Lambda_{k,\lambda_1} \cap \Lambda_{k,\lambda_2} = \emptyset$ whenever $\lambda_1 \neq \lambda_2$. We set $\Omega_{k,\lambda} = \{\omega \in \Omega : D^\omega_{\Lambda_{k,\lambda}} = \mathbb{R}^d_{\Lambda_{k,\lambda}}\}$. So $(\Omega_{k,\lambda})_{\lambda\in\mathbb{N}}$, is a sequence of two by two statistically independent sets, with non-zero probability and independent of $\lambda \in \mathbb{N}$. So, Using the Borel-Cantelli lemma, we get that $\mathbb{P}(\Omega_k) = 1$ for any $k \in \mathbb{N} + 2$, where

$$\mathbb{P}(\Omega_k) = \limsup_{\lambda\to+\infty} \Omega_{k,\lambda}.$$

The proof of Lemma 2.2 is ended by noting that

$$\cap_{k\in\mathbb{N}+2}\Omega_k \subset \widetilde{\Omega}.$$

$\square$

2.2 Main results

Our study is on the bottom of the almost sure spectrum of H_ω. We recall that $|\mathcal{S}|$ denotes the volume of $\mathcal{S}$

Theorem 2.3. *For $p \in]0,1[$, we have*

$$\lim_{E \to 0^+} \frac{\log|\log N_D(E)|}{\log E} = -\frac{d}{2}. \tag{2.10}$$

Remark 2.4. *We notice that the set D_ω may have unbounded connected component depending on the shape of the set $\mathcal{S}$ and on the probability p. Indeed the result of the above theorem is independent on the existence or nonexistence of such unbounded clusters.*

Theorem 2.5. *If $|\mathcal{S}| \cdot p^{\frac{1}{d}} < \dfrac{1}{2^d}$ is a bounded set in $\mathbb{R}^d$, we get*

$$\lim_{E \to 0^+} \frac{\log N_N(E)}{\log(E)} \leq \frac{d}{2}. \tag{2.11}$$

Remark 2.6.

- *In fact we prove a slightly better result then (2.11). Indeed we prove that*

$$N(E) \geq C \cdot E^{\frac{d}{2}}, \tag{2.12}$$

with C a constant which can be computed from our proof below.

- *In the generality of Theorem 2.5, we can't bound $N(E)$ from above, as we discuss at the end of the paper.*

- *The result given in Theorem 2.5 is deduced from a more general result of Theorem 4.1 which gives a deterministic lower bound of the numbers of eigenvalues for Neumann operators.*

3 The proof of Theorem 2.3:

In this section, we prove Theorem 2.3.

3.1 Preliminary

We start by recalling the following result and giving some properties of the IDS.

Proposition 3.1. *Let $\varphi \in C_0^\infty(\mathbb{R}^d)$, then*

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} tr(\varphi(H_\omega)\chi_{\Lambda_L}) = \mathbb{E}\Big(tr(\chi_{\Lambda_1}\varphi(H_\omega)\chi_{\Lambda_1})\Big), \tag{3.13}$$

for $\mathbb{P}$-almost all ω. Here $\mathbb{E}$, is the expectation with respect to the probability measure $\mathbb{P}$.

Proof: First we write $\chi_{\Lambda_L} = \sum_{i\in\Lambda_L\cap\mathbb{Z}^d} \chi_{\Lambda_1(i)}$, Here $\Lambda_1(i)$, is the cube of center i end side length 1. We set $\zeta_i = tr(\varphi(H_\omega)\chi_{\Lambda_1(i)})$. So ζ_i is an ergodic sequence (with respect to $\mathbb{Z}^d$) of random variables. So

$$\frac{1}{|\Lambda_L|} tr(\varphi(H_\omega)\chi_{\Lambda_L}) = \frac{1}{|\Lambda_L|} \sum_{i\in\Lambda_L\cap\mathbb{Z}^d} \zeta_i. \tag{3.14}$$

By the Birkhoff's ergodic theorem, the sum in (3.14) converges to its expectation value. This ends the proof of (3.13). $\square$

Now, we notice that both sides of (3.13), are positive linear functionals on the bounded, continuous functions. So, they define positives measures μ_L and μ respectively, i.e

$$\int_{\mathbb{R}} \varphi(\lambda) d\mu_L(\lambda) = \frac{1}{|\Lambda_L|} tr(\varphi(H_\omega)\chi_{\Lambda_L})$$

and

$$\int_{\mathbb{R}} \varphi(\lambda) d\mu(\lambda) = \mathbb{E}\Big(tr(\chi_{\Lambda_1}\varphi(H_\omega)\chi_{\Lambda_1})\Big).$$

For those two measures we have the following result proved in [20],

Theorem 3.2. *For any $\varphi \in C_0^\infty(\mathbb{R})$ and for almost all $\omega \in \Omega$ we have*

$$\lim_{k\to\infty} \langle \varphi, d\mu_L \rangle = \langle \varphi, d\mu \rangle.$$

Remark 3.3. *We call the non-random probability measure μ the density of states measure. It satisfies the following fundamental properties*

$$N(E) = \mu((-\infty, E]),$$

$$\Sigma(H_\omega) = supp(\mu).$$

Let $H^D_{\Lambda_L}(\omega)$ be the operator $H^D_\omega(\omega)$ restricted to Λ_L with Dirichlet boundary condition also on $\partial\Lambda_L$, while $H^N_{\Lambda_L}(\omega)$ when we consider Neumann boundary condition on $\partial\Lambda_L$. As H_ω is an elliptic operator, $H^\bullet_{\Lambda_L}(\omega)$, $\bullet \in \{D, N\}$, has a compact resolvent . So $H^\bullet_{\Lambda_L}(\omega)$ has a discrete spectrum with possibility of accumulation at the infinity [19]. Let us denote the sequences of eigenvalues by

$$E^\bullet_1(\Lambda_L) \leq E^\bullet_2(\Lambda_L) \leq \cdots \leq E^\bullet_n(\Lambda_L) \leq \cdots \tag{3.15}$$

For $E \in \mathbb{R}$ we denote by $N(H^\bullet_{\Lambda_L}(\omega), E)$ the number of eigenvalues of $H^\bullet_{\Lambda_L}(\omega)$ less or equal to E, of course counted with their multiplicities.
To prove Theorem 2.3, we prove a lower and an upper bounds on $N(E)$. The upper and lower bounds are proven separately and based on the following result (Theorem 5.25 p. 110 of [18]).

$$\frac{1}{\mid \Lambda_L \mid}\mathbb{E}\{N(H^D_{\Lambda_L}(\omega), E)\} \leq N(E) \leq \frac{1}{\mid \Lambda_L \mid}\mathbb{E}\{N(H^N_{\Lambda_L}(\omega), E)\}. \tag{3.16}$$

Inequalities in (3.16) are based on Akcoglu-Krengel subergodic theorem. Indeed

$$H^N_{\Lambda_1}(\omega) \oplus H^N_{\Lambda_2} \leq H^N_{\Lambda_1 \cup \Lambda_2}(\omega) \tag{3.17}$$

and

$$H^D_{\Lambda_1 \cup \Lambda_2}(\omega) \leq H^D_{\Lambda_1}(\omega) \oplus H^D_{\Lambda_2}(\omega), \tag{3.18}$$

hold on $L^2(\Lambda_1 \cup \Lambda_2)$, for all bounded cubes $\Lambda_1, \Lambda_2 \subset \mathbb{R}^d$ whenever $\Lambda_1 \cap \Lambda_2 = \emptyset$ [19].

3.2 The Dirichlet case:

3.2.1 The upper bound

The upper bound is proved by a comparison procedure. Indeed, let,

$$\widetilde{V}_\omega = \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma \chi_{\mathcal{S}}(x - \gamma),$$

and

$$\widetilde{H}_\omega = -\Delta + \widetilde{V}_\omega$$

For any $\Lambda_L \subset \mathbb{R}^d$, we set

$$\mathcal{Q}_\omega^{\Lambda_L}(\varphi, \psi) = \langle \varphi, H_0 \psi \rangle, \quad \varphi, \psi \in H_0^1(\Lambda_L \backslash \Gamma_\omega) = \mathcal{D}_{\Lambda_L}, \tag{3.19}$$

and

$$\widetilde{\mathcal{Q}}_\omega^{\Lambda_L}(\varphi, \psi) = \langle \varphi, \widetilde{H}_\omega \psi \rangle, \quad \varphi, \psi \in H_0^1(\Lambda_L) = \widetilde{\mathcal{D}}_{\Lambda_L}. \tag{3.20}$$

From [19], we recall the following result,

Lemma 3.4. *For any $L, k \in \mathbb{N}^*$, we have*

$$\sup_{\varphi_1, \cdots, \varphi_{k-1} \in \widetilde{\mathcal{D}}_{\Lambda_L}} \inf_{\psi \in [\varphi_1, \cdots, \varphi_{k-1}]^\perp \cap \widetilde{\mathcal{D}}_{\Lambda_L}, \|\psi\| = 1} \widetilde{\mathcal{Q}}_\omega^{\Lambda_L}(\psi, \psi) \leq \sup_{\psi_1, \cdots, \psi_{k-1} \in \mathcal{D}_{\Lambda_L}} \inf_{\varphi \in [\psi_1, \cdots, \psi_{k-1}]^\perp \cap \mathcal{D}_{\Lambda_L}, \|\varphi\| = 1} \mathcal{Q}_\omega^{\Lambda_L}(\varphi, \varphi). \tag{3.21}$$

From Lemma 3.4, one deduces that for any $n \in \mathbb{N}^*$, we have

$$E_n(\widetilde{H}_\omega(\Lambda_L)) \leq E_n(H_\omega(\Lambda_L)). \tag{3.22}$$

Thus, we get that for any $E \in \mathbb{R}$,

$$N(H_{\Lambda_L}(\omega), E) \leq N(\widetilde{H}_{\Lambda_L}(\omega), E) \tag{3.23}$$

We notice that for $\widetilde{H}_\omega$ it is already known that it exhibits Lifshitz tails, by the result of Kirsch and Simon [13]. This ends the proof of the upper bound. $\square$

3.2.2 The lower bound

Let H_1^D be the operator H_ω^D with $\omega_\gamma = 1$ for any $\gamma \in \mathbb{Z}^d$. We recall that for any $\Lambda_L \subset \mathbb{R}^d$, we have

$$H_\omega \le H_{\Lambda_L}^D(\omega) \le H_{1,\Lambda_L}^D. \tag{3.24}$$

So by the min-max argument we get that

$$E_1^D(\Lambda_L) \le E_1^D(H_{1,\Lambda_L}). \tag{3.25}$$

Using equation (3.16) one gets,

$$\begin{aligned} N(E) &\ge \frac{1}{L^d} \cdot \mathbb{P}\{E_1^D(\Lambda_L) \le E\} \\ &\ge \frac{1}{L^d} \cdot \mathbb{P}\{E_1^D(\Lambda_L) \le E \text{ and } \forall \gamma \in \Lambda_L \cap \mathbb{Z}^d,\ \omega_\gamma = 0\} & (3.26) \\ &\ge \frac{1}{L^d} \cdot \mathbb{P}\{E_1^D(H_{0,\Lambda_L}) \le E \text{ and } \forall \gamma \in \Lambda_L \cap \mathbb{Z}^d,\ \omega_\gamma = 0\}. & (3.27) \end{aligned}$$

If L is such that

$$E_1^D(H_{0,\Lambda_L}) \le E, \tag{3.28}$$

then

$$(3.27) \ge \frac{1}{L^d} \cdot \mathbb{P}\{\omega_0 = 0\}^{|\Lambda_L|} = (1-p)^{L^d}. \tag{3.29}$$

We remark that (3.28) is satisfied for $L \approx E^{-\frac{1}{2}}$. This ends the proof of the lower bound.

4 The proof of Theorem 2.5

4.1 The deterministic results

The aim of this section is to prove the following theorem

Theorem 4.1. *Let Ω be a subset of $\mathbb{R}^d$. Let H^N be the self adjoint operator defined as the Laplacian operator, $-\Delta$ restricted to $L^2(\mathbb{R}^d \backslash \Omega)$ with Neumann boundary conditions on $\partial\Omega$. Assume there exists a sequence of cubes Λ_L such that for L big enough we have $\frac{2^d}{L^d} \mid \Omega_L \mid < 1$ ($\Omega_L = \Lambda_L \cap \Omega$). Then, then there is a constant C, such that for $E > 0$ small enough the IDS N of H^N satisfies*

$$\lim_{E \to 0^+} \frac{\log N_N(E)}{\log E} \leq \frac{d}{2}. \tag{4.30}$$

Proof: For the proof of we start by recalling the following Lemma from [14]:

Lemma 4.2. *Let $n \in \mathbb{N}$ and $\varphi_1, \cdots, \varphi_n \in \mathcal{H}$ be in the domain $\mathcal{D}$ of a self-adjoint operator A, which acts on a separable Hilbert space $\mathcal{H}$. Suppose there are constants $\alpha_1 \leq \cdots \leq \alpha_n \leq \alpha$ such that*

$$|\langle \varphi_i, \varphi_j \rangle - \delta_{i,j}| \leq \varepsilon_1 \quad \text{and} \ |\langle \varphi_i, A\varphi_j \rangle - \alpha_j \delta_{i,j}| \leq \varepsilon_2 \tag{4.31}$$

for all $i, j = 1, \cdots, n$. If $\varepsilon_1 < 1$, then

$$N(A, \frac{\alpha + \varepsilon_2}{1 - \varepsilon_1}) \geq n. \tag{4.32}$$

Before giving the proof of Lemma 4.2 let us use it to end the proof of Theorem 4.1. Let us denote by $H^{ND}_{\Lambda_L}$, the operator H^N restricted to Λ_L, with Dirichlet boundary condition on $\partial\Lambda_L$. $H^{ND}_{\Lambda_L}$ is the self adjoint operator associated with the following quadratic form

$$\mathcal{H}^{ND}_L(u, v) = \int_{\Lambda_L \backslash \Omega_L} \nabla u(x) \cdot \overline{\nabla u(x)} dx; \tag{4.33}$$

with domain

$$\mathcal{D}(\mathcal{H}^{ND}_L) = \{f \in H^1(\Lambda_L \backslash \Omega_L);\ f \lceil \partial\Lambda_L = 0\}.$$

$\mathcal{H}^{ND}_L$ is a densely-defined quadratic form. We denote by $H^{ND}_{\Lambda_L}$ the self-adjoint operator associated to $\mathcal{H}^{ND}_L$.

For $n \in \mathbb{N}^d$, we set

$$\alpha_n(L) = (\frac{\pi}{L})^2 \sum_{i=1}^d n_i^2,$$

and

$$\Phi_{n,L}(x) = (\frac{2}{L})^{d/2} \prod_{i=1}^{d} \varphi_{n_i}(\frac{x_i}{L}), \tag{4.34}$$

where

$$\varphi_k = \cos(k\pi x); k = 1, 3, 5, \cdots, .$$

We note that the family $(\Phi_{n,L})_{n\in\mathbb{N}^d;}$ lies in the domain $\mathcal{D}(\mathcal{H}^{ND})$ and that

$$|\Phi_{n,L}(x)|^2 \ \leq \ \frac{2^d}{L^d} \ .$$

We have

$$\mathcal{H}_L^{ND}(\Phi_{n,L}, \Phi_{n,L}) = \int_{\Lambda_L} |\nabla\Phi_{n,L}(x)|^2 dx - \int_{\Omega_L} |\nabla\Phi_{n,L}(x)|^2 dx. \tag{4.35}$$

As $(\Phi_{n,L})_{n\in\mathbb{N}^d}$ are the eigenfunctions of the Laplacian restricted to the box Λ_L with the appropriate boundary condition [19], we get the following properties,

- for any $n, m \in \mathbb{N}^d$ such that $n \neq m$, we have

$$\int_{\Lambda_L} \Phi_{n,L}(x) \cdot \overline{\Phi_{m,L}(x)} dx = \int_{\Lambda_L} \nabla\Phi_{n,L}(x) \cdot \overline{\nabla\Phi_{m,L}(x)} dx = 0, \tag{4.36}$$

- for any $n \in \mathbb{N}^d$,

$$\int_{\Lambda_L} |\nabla\Phi_{n,L}(x)|^2 dx = \alpha_n(L). \tag{4.37}$$

Now using equation (4.35) and taking into account (4.36) and (4.37) we get that,

$$|\mathcal{H}_L^{ND}(\Phi_{n,L}, \Phi_{n,L}) - \alpha_n(L)| \leq \alpha_n(L) \cdot (\frac{2}{L})^d |\Omega_L| \tag{4.38}$$

and

$$|\mathcal{H}_L^{ND}(\Phi_{n,L}, \Phi_{m,L})| \leq (\alpha_n(L))^{\frac{1}{2}} (\alpha_m(L))^{\frac{1}{2}} \cdot (\frac{2}{L})^d \cdot |\Omega_L|. \tag{4.39}$$

Applying Lemma 4.2, with

$$\alpha = E > 0, \ \varepsilon_1 = (\frac{2}{L})^d \cdot \mid \Omega_L \mid < 1, \ \varepsilon_2 = \alpha \cdot (\frac{2}{L})^d \cdot \mid \Omega_L \mid,$$

we get that there exist constants $C_1, C_2 > 0$ such that

$$N(H^{ND}_{\Lambda_L}, C_1 \cdot E) \geq C_2 \cdot E^{\frac{d}{2}} \cdot L^d. \tag{4.40}$$

The right hand side of equation (4.40) is due to the fact that for $-\Delta$, there exists $C_2 > 0$ such that [19]

$$\sharp\{n; \alpha_n(L) \leq E\} = C_2 \cdot E^{d/2} L^d. \tag{4.41}$$

The proof of Theorem 4.1 is ended by taking into account (4.40) and the following equation [9]

$$\frac{1}{L^d} \cdot N(H^{ND}_{\Lambda_L}, E) \leq N(E). \tag{4.42}$$

$\square$

The proof of Lemma 4.2: We start by recalling the following expression of the counting function given in terms of a supremum of the dimension of all linear subspace $\mathcal{E}$ in the domain of A which we denote by $\mathcal{D}$.

$$N(A, E) = \sup_{\mathcal{E} \subset \mathcal{D}} \{dim\mathcal{E} | \langle \varphi, A\varphi \rangle \leq E \langle \varphi, \varphi \rangle, \text{for all } \varphi \in \mathcal{E}\}. \tag{4.43}$$

When $\varepsilon_1 < 1$, $\varphi_1, \cdots, \varphi_n$ span a subspace $\mathcal{E}_n$ of dimension n. For any $\Phi \in \mathcal{E}_n$ there exist (non-unique) coefficients $c_1, \cdots, c_n \in \mathbb{C}$ such that

$$\Phi = \sum_{i=1}^{n} c_i \varphi_i.$$

Using (4.31), one gets

$$\langle \Phi, \Phi \rangle \geq \sum_{i=1}^{n} |c_i|^2 - \sum_{i,j}^{n} |c_i| \cdot |c_j| \cdot |\langle \varphi_i, \varphi_j \rangle - \delta_{i,j}| \geq (1 - \varepsilon_1) \sum_{i=1}^{n} |c_i|^2, \tag{4.44}$$

and

$$\langle \Phi, A\Phi \rangle \leq \sum_{i=1}^{n} \alpha_i |c_i|^2 + \sum_{i,j=1}^{n} |c_i| \cdot |c_k| \cdot |\langle \varphi_i, A\varphi_j, \rangle - \alpha_i \delta_{i,j}| \leq (\alpha + \varepsilon_2) \sum_{i=1}^{n} |c_i|^2. \tag{4.45}$$

The proof of Lemma 4.2 is ended by setting $E = \dfrac{\alpha + \varepsilon_2}{1 - \varepsilon_1}$.

4.2 Examples

Let us consider some particular cases.

4.2.1 The periodic case.

Consider

$$\Omega = \bigcup_{\gamma \in \mathbb{Z}^d} \Lambda_\beta(\gamma), \beta < 1,$$

here $\Lambda_\beta(\gamma)$ is the cube of center γ and side length β. In this particular case we get $|\Omega_L| = L^d \beta^d$, and the assumption of Theorem 4.1 is satisfied for $\beta < \frac{1}{2}$.

Remark 4.3. *The periodic model includes random displacement models.*

4.2.2 The Anderson case.

Consider now the case of the model described in subsection 2.1, which is the object of our studies.

$$\Omega = \bigcup_{\gamma \in \mathbb{Z}^d} \omega_\gamma (\mathcal{S} + \gamma).$$

In this situation we get:

For P-almost all ω and any $\alpha > \mid \mathcal{S} \mid \cdot p$ and all L large enough we have

$$\frac{1}{L^d} |\Omega_L| \leq \alpha \tag{4.46}$$

So the assumption of Theorem 4.1 is satisfied for $\mid \mathcal{S} \mid \cdot p < \frac{1}{2^d}$.

To get (4.46), we use the fact that $\mathcal{S}$ is bounded, so there exists L_0 such that $\mathcal{S} \subset \Lambda_{L_0}$, and we get

$$\Omega_L = \cup_{\gamma \in \mathbb{Z}^d} \omega_\gamma (\mathcal{S} + \gamma) \cap \Lambda_L \subset \cup_{\gamma \in \Lambda_{L_0 + L}} \omega_\gamma (\mathcal{S} + \gamma).$$

So

$$\begin{aligned} \frac{1}{L^d} \cdot |\Omega_L| &\leq \frac{1}{L^d} \sum_{\gamma \in \Lambda_{L_0+L}} \omega_\gamma \cdot |\mathcal{S}| && (4.47) \\ &\leq \frac{(L_0 + L)^d}{L^d} \cdot |\mathcal{S}| \cdot \Big(\frac{1}{(L_0 + L)^d} \sum_{\gamma \in \Lambda_{L_0+L}} \omega_\gamma\Big). && (4.48) \end{aligned}$$

Now by taking L big enough we get the result as the term between parenthisis in eqution (4.48), converges to $\mathbb{E}(\omega_0) = p$ by the strong law of large numbers.

4.2.3 The Poisson case.

Let us starts by giving the model. Let $m(dx)$ be a Poisson random measure on $\mathbb{R}^d$. This means that if $(A_1, \cdots, A_n)$ are pairwise disjoint Borel sets in $\mathbb{R}^d$, the random variables $m(A_1), \cdots m(A_n)$ are independent, and when A is a bounded Borel set, for $k = 0, 1, \cdots$, the random variable has the distribution

$$\mathbb{P}\{m(A) = k\} = e^{-c|A|}\frac{(c|A|)^k}{k!}.$$

Here c is the the concentration defined by $\mathbb{E}(m(A)) = c \cdot |A|$. If $(\xi_i)_i$ are the atoms of the Poisson measure then m can be written as

$$m(dx) = \sum_i \delta_\xi(x).$$

In the above notation we have

$$\Gamma_\omega = \Omega = \bigcup_i (S + \xi_i).$$

So we get that

$$\Omega_L \subset \bigcup_{\xi_i \in \Lambda_{L_0+L}} (\mathcal{S} + \xi_i).$$

In the same way as in the Anderson Model we get that

$$\begin{aligned} \frac{1}{L^d} \cdot |\Omega_L| &\leq \frac{1}{L^d} \cdot \cap_{\xi_i \in \Lambda_{L_0+L}} |(\mathcal{S} + \xi_i)| \\ &\leq \frac{(L_0 + L)^d}{L^d} \cdot |\mathcal{S}| \cdot (\frac{1}{(L_0 + L)^d} \cdot m(\Lambda_{L_0+L}). \end{aligned} \tag{4.49}$$

As

$$\frac{1}{|\Lambda_k|} m(\Lambda_k) \longrightarrow c \quad \text{if} \quad k \longrightarrow +\infty. \tag{4.50}$$

we get that the assumption of Theorem 4.1 is satisfied for

$$c \cdot |\mathcal{S}| < \frac{1}{2^d}.$$

Acknowledgements. *W. K. would like to thank the I.S.M.A.I. Kairouan for the warm hospitality extended to him there.*

H. N. gratefully acknowledges the financial support of the Institute für Mathematik and SFB/TR 12 of the Deutsche Forschungsgemeinschaft. He also thanks the Ruhr-Universität Bochum and the FernUniversität Hagen for the warm hospitality extended to him.

References

[1] M.T Barlow: *Random walks on supercritical percolation clusters.* Ann. Prob. **32** p. 1061-1101, (1995).

[2] J. T. Chayes, L. Chayes, J. R. Franz, J. P. Sethna and S.A. Trugman: *On the density of states for the quantum percolation problem.* J. Phy. A: Math. Gen. **19** p. 1173-1177, (1986).

[3] P-G. de Gennes, P. Lafore and J. Millot: *Amas accidentels dans les solutions solides désordonées* . Jour. of Phy. and Chem. of Solids. **11** (1-2) p. 105-110, (1959).

[4] P-G. de Gennes, P. Lafore and J. Millot: *Sur un phenomène de propagations dans un milieu désordonné.* Jour. of Phy. Rad. **20**, p. 624 (1959).

[5] G. Grimmett: *Percolation* 2nd Springer, Berlin (1999).

[6] D. Heicklen and C. Hoffman *Return probabilities of a simple random walk on percolation clusters.* Elec. Jour. Probab. **10** p. 250-302, (2005).

[7] O. Khorunzhiy, W. Kirsch and P. Müller: *Lifshitz tails for spectra of Erdös-Rényi random graphs.* Annals of Applied Probability **16**, p. 295-309, (2006).

[8] W. Kirsch and F. Martinelli: *On the Spectrum Of Schrödinger operators with a Random Potential.* Commun. Math. Phys. **85** p. 329-350, (1982).

[9] W. Kirsch: *Random Schrödinger operators.* A course. In: Schrödinger operators, Sønderborg 1988, Lecture Notes in Phys. **(345)**, p. 264-370, Springer, Berlin (1989).

[10] W. Kirsch and F. Martinelli: *On the density of states of Schrödinger operators with random potential.* Jour. Phy A: Math Gen p. 2139-2156 (1982).

[11] W. Kirsch and B. Metzger: *Integrated Density of States for Random Schrödinger Operators.* Proc. Sympos. Pure Math. 76 Amer. Math. Soc., Providence, RI, p. 649-696 , (2007).

[12] W. Kirsch and P.Müller: *Spectral properties of the Laplacian on bond-percolation graphs.* Math. Z. **252**, p. 899-916, (2006).

[13] Kirsch, W.; Simon, B. *Lifshitz tails for periodic plus random potentials.* J. Statist. Phys.
bf42, no. 5-6, p. 799-808, (1986).

[14] W. Kirsch and S. Warzel: *Anderson localization and Lifshitz tails for random surface potentials.* Jour. Func. Anal. **230** p. 222-250, (2006).

[15] I. Lifshitz: *Structure of the energy spectrum of impurity bands in disordered solid solutions.* Soviet Phys. JETP **17**, p. 1159-1170, (1963).

[16] P. Mathieu E. Remey: *Isopermitery and heat kernel decay on percolation clusters.* Ann. Probab. **32** p. 100-128, (2004).

[17] P. Müller and P. Stollmann: *Spectral asymptotics of the Laplacian on supercritical bond-percolation graphs.* Ann. Probab. **32** p. 100-128, (2004).

[18] L. Pastur and A. Figotin: *Spectra of Random and Almost-Periodic Operators.* Springer-Verlag Heidelberg 1992 .

[19] M. Reed and B. Simon: *Methods of Modern Mathematical vol IV: Analysis of Operators.* New York: Academic Press 1978.

[20] I. Veselić: *Spectral Analysis of percolation Hamiltonians.* Math. Ann. **331** (4) p. 841-865, (2005).