On the New Dimer λ_d x-Expansion, Triangular and Hexagonal Lattices too

Paul Federbush
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1043
(pfed@umich.edu)

April 9, 2010

Abstract

In recent work the author presented a formal expansion for λ_d associated to the dimer problem on a d-dimensional rectangular lattice. Expressed in terms of d it yielded a presumed asymptotic expansion for λ_d in inverse powers of d. We also considered an expansion in powers of x, a formal variable ultimately set equal to 1. We believe this series has better asymptotic properties than the expansion in inverse powers of d. We discuss this, and apply the same method to the two-dimensional triangular and hexagonal lattices. Viewed as a test of the x-expansion the results on those two lattices are satisfactory, if not thoroughly convincing.

The formalism we will be working within is presented in [1], a much clearer exposition than in the earlier reports on which [1] is based. The main result there is the formal asymptotic expansion for λ_d (called by some the entropy per site)

$$\lambda_d \sim \frac{1}{2}\ln(2d) - \frac{1}{2} + \frac{1}{8}\frac{1}{d} + \frac{5}{96}\frac{1}{d^2} + \frac{5}{64}\frac{1}{d^3}\cdots \tag{1}$$

Here d is the dimension of the rectangular lattice on which one is studying the dimer problem. However we prefer now to define d, instead, as one half the number of edges entering each vertex.

The key quantities determining the expansion are the $\bar{J}_n$, $n = 1, 2, \ldots$, certain cluster expansion kernels. For the rectangular lattices the $\bar{J}_n$, $n = 1, \ldots, 6$, are given in (31) - (36) of [1]. We now present the result of computer computations for some of these $\bar{J}_n$ for the triangular and hexagonal lattices. For the triangular lattice one has

$$\bar{J}_1 = 0 \tag{2}$$

$$\bar{J}_2 = \frac{1}{24} \tag{3}$$

$$\bar{J}_3 = 0 \tag{4}$$

$$\bar{J}_4 = -\frac{31}{1728} \tag{5}$$

$$\bar{J}_5 = -\frac{13}{6480}. \tag{6}$$

And for the hexagonal lattice

$$\bar{J}_1 = 0 \tag{7}$$
$$\bar{J}_2 = \frac{1}{12} \tag{8}$$
$$\bar{J}_3 = \frac{1}{27} \tag{9}$$
$$\bar{J}_4 = -\frac{7}{216} \tag{10}$$
$$\bar{J}_5 = -\frac{23}{405} \tag{11}$$
$$\bar{J}_6 = \frac{23787}{8748}. \tag{12}$$

The formal expression for λ_d, in terms of the $\bar{J}_n$ is given as

$$\lambda_d = \frac{1}{2}\ln(2d) - \frac{1}{2} + \frac{1}{N}\ln Z^* \tag{13}$$

where Z^* is detailed in (28) - (30) of [1]. If one expresses Z^* in terms of d, and expands in powers of ${}^1\!/_d$ one gets (1).

Alternatively we introduce a parameter x, replacing $\bar{J}_n$ by $x^{n-1}\bar{J}_n$. We then express $\frac{1}{N}\ln Z^*$ as a function of x and the $\bar{J}_n$ and expand in powers of x. This leads to a formal expression for λ_d as

$$\begin{aligned}\lambda =& \frac{1}{2}\ln(2d) - \frac{1}{2} + \left(\bar{J}_2\right)x + \left(\bar{J}_3 + 4\bar{J}_2^2\right)x^2 + \left(\frac{112}{3}\bar{J}_2^3 + 12\bar{J}_2\bar{J}_3 + \bar{J}_4\right)x^3 \\ &+ \left(\bar{J}_5 + 16\bar{J}_2\bar{J}_4 + 192\bar{J}_2^2\bar{J}_3 + 9\bar{J}_3^2 + 480\bar{J}_2^4\right)x^4 \\ &+ \left(\frac{36608}{5}\bar{J}_2^5 + \bar{J}_6 + 20\bar{J}_2\bar{J}_5 + 24\bar{J}_3\bar{J}_4 + 288\bar{J}_4\bar{J}_2^2 + 324\bar{J}_3^2\bar{J}_2 + 3520\bar{J}_2^3\bar{J}_3\right)x^5 \\ &+ \cdots \end{aligned} \tag{14}$$

with x set equal to 1 herein. We use λ instead of λ_d since we may apply the same expression for other lattices as the triangular and hexagonal. This is the x-expansion for λ.

If we substitute into equation (14) the values of the $\bar{J}_n$ given in (31) - (36) of [1] we obtain another expression for λ_d. Keeping only the terms in it with inverse powers of d less than or equal to 3 we recover (1). The expression obtained from (1) and that from (14) are each as far as one can go with the information in the $\bar{J}_n$ for $n \leq 6$.

We let B_r be obtained from (14) by keeping powers of x through x^r (and again then setting $x = 1$). Thus the expression on the right side of equation (14), before the dots, is B_5, and

$$B_0 = \frac{1}{2}\ln(2d) - \frac{1}{2}. \tag{15}$$

We assemble a table of B values for the rectangular lattices of dimensions 1, 2, and 3, columns 1, 2, and 3 below; and for the hexagonal and triangular lattices in columns 4 and 5 respectively.

	1	2	3	4	5
B_0	-.1534	.1931	.3959	.0493	.3959
B_1	-.0284	.2556	.4375	.1326	.4375
B_2	.1174	.2921	.4538	.1975	.4445
B_3	.2684	.2993	.4524	.2237	.4293
B_4	.4419	.2906	.4468	.2086	.4167
B_5	.7096	.2814	.4445	2.88	–

The values of λ for each of these are known as follows:

$$\lambda_1 = 0 \tag{16}$$
$$\lambda_2 = .29156\ldots \tag{17}$$
$$.440075 \leq \lambda_3 \leq .457547 \tag{18}$$
$$\lambda_{\text{hex}} = .1691\ldots \tag{19}$$
$$\lambda_{\text{tri}} = .4286\ldots \tag{20}$$

Equation (16) is an easy deduction. References for (17) and (18) are in [1], for (19) see [2], and for (20) see [3]. It is too bad we do not have many more known lattices to test our ideas against. Would that λ_3 and λ_4 were known say.

Once we hoped $\lim_{n\to\infty} B_n = \lambda$, now we believe instead that the x-expansion behaves as an asymptotic series. The prototype asymptotic series is a sum of terms that get smaller and smaller to some point and then quickly grow. In favorable situations a "good" sum for the series is obtained by keeping only the terms till they start growing (we will actually keep the smallest term times $1/2$, and throw out the succeeding terms). Thus we take as value of the sum $\frac{1}{2}\left(B_k + B_{k+1}\right)$ where k is such that $|B_{k+1} - B_k|$ is minimum in value. We take the size of the smallest term, $|B_{k+1} - B_k|$, as an estimate of the error of our approximation. Applying this procedure to obtain a value of λ from the x-expansion, we get in each of these five cases:

$$\lambda_1 \cong .03 \pm .13 \tag{21}$$
$$\lambda_2 \cong .296 \pm .007 \tag{22}$$
$$\lambda_3 \cong .453 \pm .001 \tag{23}$$
$$\lambda_{\text{hex}} \cong .216 \pm .015 \tag{24}$$
$$\lambda_{\text{tri}} \cong .441 \pm .007 \tag{25}$$

We believe, for λ_d, the results of this procedure will be increasingly strikingly accurate as $d \to \infty$, and even for $d = 2$ and 3, as here, are impressive. We take (23) as a good prediction for λ_3. For λ_{hex} and λ_{tri} the listed error estimates in (24) and (25) must be tripled and doubled respectively. We note eq. (1) gives for λ_1, λ_2, λ_3 the values .102, .278, .446. Both eq. (1) and the x-expansion procedure are pleasingly accurate at λ_3.

The expansions for λ are for dimension going to infinity. We should be pleased they yield interesting results even in two dimensions. Seeking a better way to extract λ from the sequence of $\bar{J}_n$ may be like searching for the pot of gold at the end of the rainbow.

References

[1] Paul Federbush, Phys. Lett. A **374** (2009) 131.

[2] J F Nagle, Phys. Rev. Lett. **34** (1975) 1150
G H Wannier, Phys. Rev. **79** (1950) 357
F Y Wu, Phys. Rev. **168** (1968) 539
P W Kasteleyn, J. Math. Phys. **4** (1963) 287
Veit Elser, J. Phys. A: Math. Gen. **17** (1984) 1509.

[3] J F Nagle, Phys. Rev. **152** (1966) 190
D S Gaunt, Phys. Rev. **179** (1969) 174
S Samuel, J. Math. Phys. **21** (1980) 2806
F Mila, Phys. Rev. Lett. **81** (1998) 2356
W F Wolff and J Zittartz, Z. Phys. B**49** (1982) 139
P Fendley, R Moessner, S L Sondhi, Phys. Rev. B **66** (2002) 214513.