An ultrametric state space with a dense discrete overlap distribution: Paperfolding sequences

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Abstract. We compute the Parisi overlap distribution for paperfolding sequences. It turns out to be discrete, and to live on the dyadic rationals. Hence it is a pure point measure whose support is the full interval $[-1, +1]$. The space of paperfolding sequences has an ultrametric structure. Our example provides an illustration of some properties which were suggested to occur for pure states in spin glass models.

1 Introduction

In [11] the study of the Parisi overlap distribution for various classes of non-periodically ordered sequences was undertaken.
The considered sequences were members of $\{-, +\}^{\mathbb{Z}}$ and their orbit closure forms typically a uniquely ergodic system, with a unique shift-invariant measure μ. This measure then can be a ground state for a translation-invariant interaction which one can construct [3, 28, 29], and the individual sequences will be the pure (extremal) ground states.
Their overlap distribution gives the behaviour under the product measure of μ with itself, describing a two-replica system, written as $\mu \times \mu'$ of the overlap between two randomly (from this product measure) chosen, bi-infinite sequences σ and σ':

$$q_{\sigma\sigma'} = lim \frac{1}{N} \sum_{i=1,\dots N} \sigma_i \sigma'_i. \tag{1}$$

Note that the overlap $q_{\sigma\sigma'}$ between two sequences is directly related to their Hamming distance $d_H(\sigma,\sigma') = \frac{1-q_{\sigma\sigma'}}{2}$.

If we take two random sequences, each chosen according to the same shift-invariant measure, with probability one the above limit will exist, as follows from the ergodic theorem. The product measure of two ergodic measures, however, although shift-invariant, in general will not be ergodic, thus one may obtain different values for the overlap with positive probability.

A simple example illustrating this is the symmetric measure which give equal weights to the two alternating sequences:

$$\mu_{alt}(\sigma) = \frac{1}{2}(\delta(+-...,\sigma) + \delta(-+...,\sigma)) \tag{2}$$

This measure is ergodic under the shift transformation, but the product measure of μ_{alt} with itself is not. For the overlaps it holds that with probability $\frac{1}{2}$ the same sequence is chosen (overlap one) and with probability $\frac{1}{2}$ two different sequences are chosen (overlap minus one). Thus in this simple example

$$p(q) = \frac{1}{2}(\delta(q,1) + \delta(q,-1)) \tag{3}$$

In more general situations the overlap can take any value in the interval $[-1,+1]$, and the induced measure, which is the overlap distribution, describable by a probability distribution function on this interval, can be discrete (that is, it is a pure point measure) or nondiscrete, and thus have continuous components.

In particular, it was found in [11] that whenever the atomic ("diffraction") spectrum of the unique translation-invariant measure on the sequences does not contain a pure point component, the overlap distribution is trivial; this applies in particular to all weakly mixing systems (in which case their product measures are ergodic), as well as to the (Prouhet-(Thue-))Morse system.

On the other hand for the Fibonacci sequences the overlap distribution was found to contain a(n absolutely) continuous part. In fact, in explicit form it is [11]

$$p(q)dq = (2\gamma - 1)\delta(q, 1 - 4(1-\gamma)) + \frac{1}{2}1_{[1-4(1-\gamma),1]}(q)dq \tag{4}$$

where $\gamma = \frac{2}{1+\sqrt{5}}$.

We notice here that the same argument which was used in [11] applies to more general Sturmian sequences, which thus also have a continuous part in their overlap distribution. The reason is that these are also rotation sequences, see e.g. [17]. They also have recently been studied as ground states of some explicit interaction functions on lattice systems in [1]).

Moreover, it was found that the Toeplitz (or period-doubling) sequences give rise to an overlap distribution which is concentrated on a countable number of values, with 1 as its only limit point. Also, the set of these Toeplitz sequences has a tree (ultrametric) structure, and the overlap distribution could be obtained via this structure.

Here we show that the paperfolding sequences (see [5, 9] and references mentioned there), which also display such an ultrametric structure, give rise to an overlap distribution living on a *dense* countable number of points. As such the paperfolding system, which, in contrast to the Toeplitz system, is symmetric under the spin-flip (plus-minus) symmetry, is even closer to what is expected in Parisi's replica symmetry breaking theory of the Sherrington-Kirkpatrick model [18, 27]. In Parisi's theory for the SK model, this countable overlap distribution is supposed to show up for a fixed disorder, while averaging over the disorder distribution is supposed to result in a continuous piece in the overlap distribution. Often the countable overlap distribution is interpreted as being related to or implying a countable number of pure states; however there seems not to be much justification for this [8].

Conceptually, our example illustrates a number of points:

Newman and Stein [21, 22, 23, 24, 25, 20] have shown that self-averaging arguments, based on the use of the ergodic theorem, apply to many finite-dimensional. disordered (spin-glass) models (although not to the equivalent-neighbour Sherrington-Kirkpatrick model). In particular they find that for systems with a spatial structure, in contrast to the Parisi theory's predictions for the Sherrington-Kirkpatrick model [18, 27], overlap quantities cannot depend on the realization of the disorder parameter, that is, they are "self-averaging". For example, for Edwards-Anderson-type spin-glass models there is no difference between a typical (Parisi SK-prediction: supported on countably many values, dense) and an averaged (Parisi SK-prediction: continuous) overlap distribution.

As mentioned before, for many nonperiodic systems, the individual (nonperiodic) sequences are ground states for some translation invariant deterministic interaction. Thus they are often used as models for non-periodic (quasi-)crystals, [3, 28, 29, 13, 19, 30, 31, 33]. It therefore becomes natural to consider various questions naturally arising in statistical mechanics about such systems, whether about their spectral properties [4, 12, 15, 16] or, as was done in [11, 10] and also here, about their overlap distributions. Typically, there are uncountably many such ground states, and as mentioned before, they can have nontrivial overlap distributions; somewhat surprisingly even without the presence of any disorder.

It turns out that:

- There is no general connection between the number of pure states -uncountable- and the number of possible overlap values –which can be one, countably or uncountably many–.
- Triviality, non-triviality, and ultrametricity of the Parisi overlap distribution all can be realized without any role for the disorder.
- A dense discrete overlap distribution, with a hierarchical structure, can be realised (by the paperfolding system).

2 Main Result

For background on the paperfolding system we refer to [5, 9, 2], and references mentioned there.

A paperfolding sequence can be constructed as follows:
In step 1, we choose k_1 to be a number from the set $\{1, 2, 3, 4\}$.

Now we fill all sites of the form $k_1 + 4\mathbb{Z}$ with pluses, and all sites of the form $k_1 + 2 + 4\mathbb{Z}$ with minuses.
Thus we have the period-4 structure covering half the sites:
..... $+ . - . + . - . + . - . + . - .$ ...
In step 2, we again choose k_2 to be a number from the set $\{1, 2, 3, 4\}$.
Now we occupy all sites of the form $k_1+1+2k_2+8\mathbb{Z}$ with pluses and all sites $k_1+5+2k_2+8\mathbb{Z}$ with minuses.

We repeat this step and in the mth step we again occupy one quarter of the remaining empty sites with pluses, periodically with period 2×2^m, and the quarter of the sites at distance 2^m from the pluses we occupy with minuses, periodically with the same period. Thus in each step one half of the remaining sites are filled.

After making a choice for k_m, for all $m \in \mathbb{N}$, all sites will be filled, and we have a paperfolding sequence. Notice that in every step the uncountable set of sequences is divided into four times as many subsets as in the previous step, which generates the hierarchical (ultrametric) structure of the uncountable set of sequences.

With probability $\frac{1}{2}$ we find the overlap value $q = 0$, which happens if in step 1 one "odd" and one "even" sequence was chosen, that is, $k_1 - k_1'$ is odd. With probability $\frac{1}{8}$ one finds overlap values q equal to plus or minus $\frac{1}{2}$ respectively, which happens if in the first step either the same ($k_1 = k_1'$), or the opposite ($|k_1 - k_1'| = 2$) choice was made, and in step 2 a different even-odd choice for k_2 and k_2'. We repeat this argument, and we find that with probability $\frac{1}{2^{2n+1}}$ one finds overlap values $\frac{(2p+1)}{2^n}$, with $p = -2^{n-1}, \ldots\ldots, 2^{n-1} - 1$ [14]. Thus in the end, the overlap distribution becomes concentrated on the dyadic rationals.
Summarizing we have proven:

Theorem 1 *The overlap distribution of paperfolding sequences is given by:*

$$p(q) = \sum_{n=0}^{\infty} {\sum_{m}}' \frac{1}{2} \left(\frac{1}{4}\right)^n \delta\left(q, \frac{m}{2^n}\right) \tag{5}$$

where the integers n, m must satisfy the following conditions:

- *if $n = 0$ then $m = 0$,*
- *if $n > 0$ then m is odd and $|m| < 2^n$.*

Comments:

Properties which are common between the period-doubling and the paperfolding system are:

1) Limit periodicity. The sequences are a countable union of periodic subsequences of pluses and minuses. This implies a countable number of possible values for the overlap.

2) There is a tree (hierarchical, ultrametric) structure in the set of sequences– the pure states–. As the overlap between two sequences depends on the level of the tree at which a different even-odd choice is made, the overlap distribution reflects this.

3) Both paperfolding and period-doubling sequences form model sets, obtained by a cut and project scheme, with an internal space of 2-adic numbers. [6, 5, 32]. This agrees with the existence of a connection between Parisi's replica-symmetry-breaking ideas and p-adic numbers as was suggested in [26].

However, in contrast to the period-doubling case, the set of paperfolding sequences is spin-flip symmetric, and the set on which the overlap distribution is concentrated now lies dense in the interval $[-1, +1]$.

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