Systems of differential operators in anisotropic Roumieu classes

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Abstract

The aim of this work is to show that he ellipticity is a sufficient condition for the elliptic iterate property to hold for systems of linear differential operators in the anisotropic Roumieu classes

1 Introduction

In this work we show that the ellipticity is a sufficient condition for the elliptic iterate property to hold for systems of linear differential operators in the anisotropic classes $\{M_p\}$ of functions (also called Roumieu classes). The obtained result is an extension of the results of Nelson [6], Kotake-Narasimhan [4], Bolley-Camus [1] and Zanghirati [7]. Also we give here an answer to a problem asked by Lions-Magenes [5].

Let Ω be an open subset of $\mathbb{R}^n$, $P_j(x, D)$, $j = 1, .., N$, henceforth denoted $(P_j)_{j=1}^N$, linear differential operators of order m,

$$P_j(x, D) = \sum_{|\alpha| \leq m} a_{j\alpha}(x) D^\alpha .$$

We define the principal symbol of the operator $P_j(x, D)$ by

$$P_{jm}(x, \xi) = \sum_{|\alpha| = m} a_{j\alpha}(x) \xi^\alpha .$$

Let (M_p) be a sequence of positive real numbers satisfying the following conditions

$$M_p^2 \leq M_{p+1} M_{p-1}, \ \forall p \in \mathbb{N}^*; \tag{1}$$

$$\exists A > 0, \exists H > 0, \exists c > 0 : c C_p^j M_{p-j} M_j \leq M_p \leq A H^p M_{p-j} M_j, \ \forall p \in \mathbb{N}, j \leq p; \tag{2}$$

$$\forall m \geq 2, \exists d > 0, \forall p \in \mathbb{N}, \ 0 \leq q \leq m : (M_{pm})^{m-q} (M_{pm+m})^q \leq d (M_{pm+q})^m \tag{3}$$

Definition 1 *We call vector of class $\{M_p\}$ of the system $(P_j)_{j=1}^N$ every function $u \in C^\infty(\Omega)$ such that*

$$\forall K \text{ compact of } \Omega, \ \exists C > 0, \ \forall l \in \mathbb{N} : \|P_{i_1} ... P_{i_l} u\|_{L^2(K)} \leq C^{lm+1} M_{lm} , \tag{4}$$

where $1 \leq i_j \leq N,\ 1 \leq j \leq l.$

The space of these functions is denoted $R_M\left(\Omega, (P_j)_{j=1}^N\right).$

Definition 2 *Let* $M^1, .., M^n$ *be sequences satisfying* $(1)-(3)$. *An anisotropic Roumieu space in* Ω *denoted* $R_{M^1,...,M^n}(\Omega)$, *is the space of functions* $u \in C^\infty(\Omega)$ *such that*

$$\forall K \text{ compact de } \Omega,\ \exists C > 0 : \|D^\alpha u\|_{L^2(K)} \leq C^{|\alpha|+1} M^1_{\alpha_1} ... M^n_{\alpha_n} \ . \tag{5}$$

Definition 3 *the system* $(P_j)_{j=1}^N$ *is said elliptic in* Ω *if for every* $x_0 \in \Omega$, *we have*

$$\sum_{j=1}^N |P_{jm}(x_0, \xi)| \neq 0,\ \forall \xi \in \mathbb{R}^n \backslash \{0\} \ .$$

The elliptic iterate property for the system $(P_j)_{j=1}^N$ in the anisotropic Roumieu classes means the inclusion

$$R_M\left(\Omega, (P_j)_{j=1}^N\right) \subset R_{M^1,...,M^n}(\Omega),$$

more precisely, we will prove the following theorem.

Theorem 4 *Let* $M^1, .., M^n,\ M$ *be sequences satisfying* $(1)-(3)$ *and*

$$\frac{(i_1 + ... + i_n)!}{i_1!...i_n!} M^1_{i_1} ... M^n_{i_n} \leq M_{i_1+...+i_n} \leq B^{i_1+...+i_n} M^1_{i_1} ... M^n_{i_n}, \tag{6}$$

for some $B > 0$ *and for all* $i_j \in \mathbb{N},\ j = 1, ..., n.$
Let $(P_j)_{j=1}^N$ *be a system of differential operators with coefficients in* $R_{M^1,...,M^n}(\Omega)$, *then*

$$(P_j)_{j=1}^N \text{ is elliptic in } \Omega \Rightarrow R_M\left(\Omega, (P_j)_{j=1}^N\right) \subset R_{M^1,...,M^n}(\Omega) \ .$$

2 Preliminary lemmas

Let $x_0 \in \Omega$, and let $0 < R < 1$, for every $u \in C^\infty(\Omega),\ k \in \mathbb{Z}_+,\ \rho \leq R$, we denote

$$|u|_{k,\rho} = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|D^\alpha u\|_{L^2(B_\rho)} \ ,$$

where B_ρ designates the ball of center x_0 and radius ρ.

The following lemma is the lemma 3.3 of [4].

Lemma 5 *Let* $m \geq 2$, *then* $\exists C_m,\ \forall \varepsilon > 0,\ \forall u \in C^\infty(\Omega),$

$$|u|_{pm+q,\rho} \leq \varepsilon |u|_{pm+m,\rho} + C_m \varepsilon^{-\frac{q}{m-q}} |u|_{pm,\rho},\ \forall p, q \in \mathbb{Z}_+,\ q < m \ . \tag{7}$$

From the ellipticity of the system $(P_j)_{j=1}^N$, we have the following result (see proposition 2.2 of [2]).

Lemma 6 *Let $(P_j)_{j=1}^N$ be an elliptic system with coefficients in $C^\infty(\Omega)$, so there exists a small neighbourhood ω of x_0 and a constant $C>0$ such that*
$\forall u \in C^\infty(\omega), \forall \rho > 0, \forall \delta \in]0,1[$ *with* $\overline{B}_{\rho+\delta} \subset \omega, \forall p \in \mathbb{N}$,

$$|u|_{(p+1)m,\rho} \leq C\left(\sum_{j=1}^N |P_j u|_{pm,\rho+\delta} + \delta^{-m}|u|_{pm,\rho+\delta} + \sum_{j=1}^N \sum_{|\alpha|=pm} \frac{(pm)!}{\alpha!}|[P_j, D^\alpha]u|_{0,\rho+\delta}\right). \tag{8}$$

We give a convenient form to this lemma for systems with coefficients in $R_{M^1,\ldots,M^n}(\Omega)$.

Lemma 7 *Let $(P_j)_{j=1}^N$ be an elliptic system with coefficients in $R_{M^1,\ldots,M^n}(\Omega)$, ω as in lemma 6, $\varepsilon \in]0,1[$, so there exists $C_1 > 0$ and $C_2(\varepsilon) > 0$ such that*
$\forall u \in C^\infty(\omega), \forall \rho > 0, \forall \delta \in]0,1[$ *with* $\overline{B}_{\rho+\delta} \subset \omega, \forall p \in \mathbb{Z}_+^*$,

$$\begin{aligned} |u|_{(p+1)m,\rho} \quad &\leq \quad C_1\left(\sum_{j=1}^N |P_j u|_{pm,\rho+\delta} + (\varepsilon\delta)^{-m}|u|_{pm,\rho+\delta} + \varepsilon|u|_{pm+m,\rho+\delta}\right) \\ &\quad + \sum_{h=0}^p \frac{M_{pm+m}}{M_{hm}} C_2(\varepsilon)^{(p+1-h)m}|u|_{hm,\rho+\delta}, \end{aligned} \tag{9}$$

and

$$|u|_{m,\rho} \leq C_1\left(\sum_{j=1}^N |P_j u|_{0,\rho+\delta} + (\varepsilon\delta)^{-m}|u|_{0,\rho+\delta} + \varepsilon|u|_{m,\rho+\delta}\right) \tag{10}$$

Proof. The estimate (10) follows from (8) for $p=0$. For (9) we have to estimate the third term of the right-hand side of (8). By Leibniz formula , we have

$$|[a_{j\beta}, D^\alpha]u|_{0,\rho+\delta} \leq \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \left|D^\gamma u D^{\alpha-\gamma} a_{j\beta}\right|_{0,\rho+\delta}. \tag{11}$$

As the coefficients $a_{j\beta}$ are in $R_{M^1,\ldots,M^n}(\Omega)$, then

$$\exists C_1 > 0, \forall |\beta| \leq m,\ \forall j = 1,..,N,\ \forall \gamma \in \mathbb{Z}_+^n : \sup_{B_R} |D^\gamma a_{j\beta}| \leq C_1^{|\gamma|+1} \prod_{i=1}^n M_{\gamma_i}^i.$$

Substituting in (11), and taking account of (6) and (2) we obtain

$$\begin{aligned} \frac{|\alpha|!}{\alpha!}|[a_{j\beta}, D^\alpha]u|_{0,\rho+\delta} \quad &\leq \quad \sum_{\gamma<\alpha} \frac{|\alpha|!}{(\alpha-\gamma)!\gamma!} C_1^{|\alpha-\gamma|+1} \prod_{i=1}^n M_{\alpha_i-\gamma_i}^i |D^\gamma u|_{0,\rho+\delta} \\ &\leq \quad \sum_{\gamma<\alpha} \frac{|\alpha|!}{|\alpha-\gamma|!\gamma!} C_1^{|\alpha-\gamma|+1} M_{|\alpha-\gamma|} |D^\gamma u|_{0,\rho+\delta} \\ &\leq \quad \sum_{\gamma<\alpha} C_1^{|\alpha-\gamma|+1} \frac{1}{c} \frac{|\gamma|!}{\gamma!} \frac{M_{|\alpha|}}{M_{|\gamma|}} |D^\gamma u|_{0,\rho+\delta}, \end{aligned}$$

hence for every $p \in \mathbb{Z}_+^*$, we have

$$\sum_{|\alpha|=pm} \frac{(pm)!}{\alpha!} \left|[a_{j\beta}, D^\alpha]\, u\right|_{0,\rho+\delta} \leq \sum_{h=0}^{pm-1} C_2^{pm-h+1} \frac{M_{pm}}{M_h} |u|_{h,\rho+\delta} \ . \tag{12}$$

Now

$$[P_j, D^\alpha]u = \sum_{|\beta|\leq m} \left(a_{j\beta}(x)\, D^{\beta+\alpha}u - D^\alpha \left(a_{j\beta}(x)\, D^\beta u\right)\right) = \sum_{|\beta|\leq m} [a_{j\beta}, D^\alpha] D^\beta u \ ,$$

so from (12), we get

$$\begin{aligned} \sum_{|\alpha|=pm} \frac{(pm)!}{\alpha!} |[P_j, D^\alpha]u|_{0,\rho+\delta} &\leq \sum_{|\beta|\leq m} \sum_{|\alpha|=pm} \frac{(pm)!}{\alpha!} \left|[a_{j\beta}, D^\alpha] D^\beta u\right|_{0,\rho+\delta} \\ &\leq \sum_{k=0}^{m} \sum_{|\beta|=k} \sum_{h=0}^{pm-1} C_2^{pm-h+1} \frac{M_{pm}}{M_h} \left|D^\beta u\right|_{h,\rho+\delta} \\ &\leq \sum_{k=0}^{m} \sum_{h=k}^{pm+k-1} C_2^{pm+k-h+1} \frac{M_{pm}}{M_{h-k}} |u|_{h,\rho+\delta} \end{aligned} \tag{13}$$

In the other hand, it easy to check that the condition (1) gives

$$\forall h \leq p,\ \forall i \in \mathbb{Z}_+ : \frac{M_p}{M_h} \leq \frac{M_{p+i}}{M_{h+i}} \ . \tag{14}$$

In particular

$$\frac{M_{pm}}{M_{h-k}} \leq \frac{M_{pm+m}}{M_{h-k+m}} \leq \frac{M_{pm+m}}{M_h} \ ,$$

which gives with (13)

$$\begin{aligned} \sum_{|\alpha|=pm} \frac{(pm)!}{\alpha!} |[P_j, D^\alpha]u|_{0,\rho+\delta} &\leq (m+1) \sum_{h=0}^{pm+m-1} C_2^{pm+m-h+1} \frac{M_{pm+m}}{M_h} |u|_{h,\rho+\delta} \\ &\leq \sum_{l=0}^{p} \sum_{q=0}^{m-1} C_3^{pm+m-lm-q+1} \frac{M_{pm+m}}{M_{lm+q}} |u|_{lm+q,\rho+\delta} \end{aligned} \tag{15}$$

From lemma 5, we have for every $\varepsilon' \in [0,1]$,

$$|u|_{lm+q,\rho+\delta} \leq \varepsilon' |u|_{lm+m,\rho+\delta} + C_m \varepsilon'^{-\frac{q}{m-q}} |u|_{lm,\rho+\delta} \ .$$

Set $\varepsilon' = \varepsilon \dfrac{M_{lm+q}}{M_{lm+m}} C_3^{-m+q}$, so from (3), we obtain

$$\begin{aligned} C_3^{m-q} \frac{M_{pm+m}}{M_{lm+q}} |u|_{lm+q,\rho+\delta} &\leq \varepsilon \frac{M_{pm+m}}{M_{lm+m}} |u|_{lm+m,\rho+\delta} \\ &\quad + C_m \varepsilon^{-\frac{q}{m-q}} C_3^m \frac{M_{pm+m}}{M_{lm+q}} \left(\frac{M_{lm+m}}{M_{lm+q}}\right)^{\frac{q}{m-q}} |u|_{lm,\rho+\delta} \\ &\leq \varepsilon \frac{M_{pm+m}}{M_{lm+m}} |u|_{lm+m,\rho+\delta} + C_m \varepsilon^{-m} C_3^m d' \frac{M_{pm+m}}{M_{lm}} |u|_{lm,\rho+\delta} \ . \end{aligned}$$

With (15) we obtain

$$\begin{aligned}\sum_{|\alpha|=pm}\frac{(pm)!}{\alpha!}\left|[P_j,D^\alpha]u\right|_{0,\rho+\delta} &\leq m\varepsilon\sum_{l=0}^{p}C_3^{pm-lm+1}\frac{M_{pm+m}}{M_{lm+m}}\left|u\right|_{lm+m,\rho+\delta}\\ &\quad +mC_m\varepsilon^{-m}d'\sum_{l=0}^{p}C_3^{pm+m-lm+1}\frac{M_{pm+m}}{M_{lm}}\left|u\right|_{lm,\rho+\delta}\\ &\leq m\varepsilon C_3\left|u\right|_{pm+m,\rho+\delta}\\ &\quad +m\left(\varepsilon+C_m\varepsilon^{-m}d'\right)\sum_{l=0}^{p}C_3^{pm+m-lm+1}\frac{M_{pm+m}}{M_{lm}}\left|u\right|_{lm,\rho+\delta}\ .\end{aligned}$$

The last estimate and (8) give the requested estimate (9). ■

3 The theorem

For every $\lambda>0$, define

$$\sigma_p(u,\lambda)=\frac{1}{\lambda^p M_{pm}}\sup_{R/2\leq\rho<R}(R-\rho)^{pm}\left|u\right|_{pm,\rho}\ .$$

Lemma 8 *Let $(P_j)_{j=1}^N$ an elliptic system with coefficients in $R_{M^1,\ldots,M^n}(\Omega)$, ω as in lemma 6, so*
$\exists\lambda_0>0$ (*depending only of $(P_j)_{j=1}^N$ and R*), $\forall\lambda\geq\lambda_0,\ \forall u\in C^\infty(\omega),\ \forall p\in\mathbb{Z}_+,$

$$\sigma_{p+1}(u,\lambda)\leq\frac{M_{pm}}{M_{pm+m}}\sum_{j=1}^{N}\sigma_p(P_ju,\lambda)+\sum_{h=0}^{p}\sigma_h(u,\lambda)\ . \tag{16}$$

Proof. First we prove this lemma for $p\geq1$. For every $\rho\in[R/2,R[$ we take $\delta=\frac{R-\rho}{p+1}$, then we have $\rho+\delta\in[R/2,R[$ and $\left(\frac{R-\rho}{R-\rho-\delta}\right)^{pm}<e^m$.

Multiplying both sides of (9) by $\left(\lambda^{p+1}M_{pm+m}\right)^{-1}(R-\rho)^{pm+m}$ and taking the sup over $\rho\in[R/2,R[$, we obtain

$$\sigma_{p+1}(u,\lambda)\leq\sup_{\rho\in[R/2,R[}C_1\left(I_1+\varepsilon^{-m}I_2+\varepsilon I_3+I_4\right)\ , \tag{17}$$

where

$$\begin{aligned}
I_1 &= \frac{(R-\rho)^{pm+m}}{\lambda^{p+1}M_{pm+m}}\sum_{j=1}^{N}\left|P_ju\right|_{pm,\rho+\delta}\leq\sum_{j=1}^{N}\frac{e^m}{\lambda}\frac{M_{pm}}{M_{pm+m}}\sigma_p\left(P_ju,\lambda\right)\\
I_2 &= \frac{(R-\rho)^{pm+m}}{\lambda^{p+1}M_{pm+m}}\delta^{-m}\left|u\right|_{pm,\rho+\delta}\leq\frac{e^m}{\lambda}\frac{(p+1)^m M_{pm}}{M_{pm+m}}\sigma_p\left(u,\lambda\right)\\
I_3 &= \frac{(R-\rho)^{pm+m}}{\lambda^{p+1}M_{pm+m}}\left|u\right|_{pm+m,\rho+\delta}\ \leq(2e)^m\sigma_{p+1}\left(u,\lambda\right)\\
I_4 &= \frac{(R-\rho)^{pm+m}}{\lambda^{p+1}M_{pm+m}}\sum_{h=0}^{p}\frac{M_{pm+m}}{M_{hm}}C_2\left(\varepsilon\right)^{(p+1-h)m}\left|u\right|_{hm,\rho+\delta}\\
&\leq \frac{e^mC_2\left(\varepsilon\right)^m}{\lambda}\sum_{h=0}^{p}\left(\frac{C_2\left(\varepsilon\right)^m}{\lambda}\right)^{p-h}\sigma_h\left(u,\lambda\right)\ .
\end{aligned}$$

Set $\varepsilon=\left[2C_1\left(2e\right)^m\right]^{-1}$ in (17), so we get

$$\begin{aligned}
\sigma_{p+1}\left(u,\lambda\right) \quad &\leq \quad \frac{C_3}{\lambda}\frac{M_{pm}}{M_{pm+m}}\sum_{j=1}^{N}\sigma_p\left(P_ju,\lambda\right)+\frac{(p+1)^m M_{pm}}{M_{pm+m}}\frac{C_4}{\lambda}\sigma_p\left(u,\lambda\right)\\
&\quad +\frac{C_5}{\lambda}\sum_{h=0}^{p}\left(\frac{C_5}{\lambda}\right)^{p-h}\sigma_h\left(u,\lambda\right)\ . \qquad (18)
\end{aligned}$$

In the other hand we have from the condition (2),

$$C_{pm+m}^{m}M_mM_{pm}\leq\frac{1}{c}M_{pm+m}\ ,$$

which gives

$$\frac{M_{pm}}{M_{pm+m}}\left(pm+1\right)\cdot\cdot\left(pm+m\right)\leq\frac{1}{c}\frac{m!}{M_m}\ ,$$

and consequently

$$\frac{M_{pm}}{M_{pm+m}}\left(pm+1\right)^m\leq\frac{1}{c}\frac{m!}{M_m}\ .$$

This estimate and (18) give

$$\sigma_{p+1}\left(u,\lambda\right)\leq\frac{C_3}{\lambda}\frac{M_{pm}}{M_{pm+m}}\sum_{j=1}^{N}\sigma_p\left(P_ju,\lambda\right)+\frac{C_4'}{\lambda}\sigma_p\left(u,\lambda\right)+\frac{C_5}{\lambda}\sum_{h=0}^{p}\left(\frac{C_5}{\lambda}\right)^{p-h}\sigma_h\left(u,\lambda\right)\ .$$

If we take $\lambda_0=C_3+C_4'+C_5$, we obtain then the estimate (16).

For $p=0$, we take $\delta=\frac{R-\rho}{2}$ for every $\rho\in[R/2,R[$ and we use the estimate (10). By the same procedure we obtain the same result. ■

Lemma 9 *Let $(P_j)_{j=1}^N$ an elliptic system with coefficients in $R_{M^1,\dots,M^n}(\Omega)$, ω, λ_0 as in lemma 8. If $u \in C^\infty(\omega)$ and $\lambda \geq \lambda_0$, then for every $p \in Z_+$, we have*

$$\sigma_{p+1}(u,\lambda) \leq 2^{p+1}\sigma_0(u,\lambda)+\sum_{h=1}^{p+1} 2^{p+1-h}\binom{p+1}{h}\frac{1}{M_{hm}}\sum_{\substack{1\leq j\leq h \\ 1\leq i_j\leq N}} \sigma_0\left(P_{i_1}\cdot\cdot P_{i_j}\cdot\cdot P_{i_h}u,\lambda\right) . \tag{19}$$

Proof. For $p=0$, (19) is the estimate (16). Suppose that the estimate (19) is true for the order p and let us show that it remains true at the order $p+1$. From (16) and the recurrence hypothesis, we have

$$\begin{aligned}
\sigma_{p+2}(u,\lambda) &\leq \frac{M_{pm+m}}{M_{pm+2m}}\sum_{j=1}^N \sigma_{p+1}(P_j u,\lambda)+\sum_{h=0}^{p+1}\sigma_h(u,\lambda) \\
&\leq \frac{M_{pm+m}}{M_{pm+2m}}\sum_{j=1}^N 2^{p+1}\sigma_0(P_j u,\lambda) \\
&\quad +\frac{M_{pm+m}}{M_{pm+2m}}\sum_{j=1}^N\sum_{h=1}^{p+1} 2^{p+1-h}\binom{p+1}{h}\frac{1}{M_{hm}}\sum_{\substack{1\leq k\leq h \\ 1\leq i_k\leq N}}\sigma_0(P_{i_1}\cdots P_{i_h}P_j u,\lambda) \\
&\quad +\sigma_0(u,\lambda)+\sum_{h=0}^{p} 2^{h+1}\sigma_0(u,\lambda) \\
&\quad +\sum_{h=0}^{p}\sum_{l=1}^{h+1} 2^{h+1-l}\binom{h+1}{l}\frac{1}{M_{lm}}\sum_{\substack{1\leq k\leq l \\ 1\leq i_k\leq N}}\sigma_0(P_{i_1}\cdots P_{i_l}u,\lambda) \\
&\leq \frac{M_{pm+m}}{M_{pm+2m}}\sum_{h=1}^{p+2} 2^{p+2-h}\binom{p+1}{h-1}\frac{1}{M_{(h-1)m}}\sum_{\substack{1\leq k\leq l \\ 1\leq i_k\leq N}}\sigma_0(P_{i_1}\cdots P_{i_h}u,\lambda) \\
&\quad +2^{p+2}\sigma_0(u,\lambda)+\sum_{l=1}^{p+1} 2^{p+2-l}\binom{p+1}{l}\frac{1}{M_{lm}}\sum_{\substack{1\leq k\leq l \\ 1\leq i_k\leq N}}\sigma_0(P_{i_1}\cdots P_{i_l}u,\lambda) .
\end{aligned}$$

But from (14), we have

$$\frac{M_{pm+m}}{M_{pm+2m}}\frac{1}{M_{(h-1)m}} \leq \frac{1}{M_{hm}} ,$$

hence

$$\begin{aligned}
\sigma_{p+2}(u,\lambda) &\leq \sum_{h=1}^{p+2} 2^{p+2-h}\binom{p+1}{h-1}\frac{1}{M_{hm}}\sum_{\substack{1\leq k\leq h\\ 1\leq i_k\leq N}}\sigma_0\left(P_{i_1}\cdots P_{i_h}u,\lambda\right)\\
&\quad +2^{p+2}\sigma_0(u,\lambda)+\sum_{h=1}^{p+1}2^{p+2-h}\binom{p+1}{h}\frac{1}{M_{hm}}\sum_{\substack{1\leq k\leq h\\ 1\leq i_k\leq N}}\sigma_0\left(P_{i_1}\cdots P_{i_h}u,\lambda\right)\\
&\leq 2^{p+2}\sigma_0(u,\lambda)+\sum_{h=1}^{p+2}2^{p+2-h}\binom{p+2}{h}\frac{1}{M_{hm}}\sum_{\substack{1\leq j\leq h\\ 1\leq i_j\leq N}}\sigma_0\left(P_{i_1}\cdots P_{i_h}u,\lambda\right)\ .
\end{aligned}$$

■

Now we give the main result of this work.

Theorem 10 *Let $M^1,..,M^n$ and M be sequences satisfying $(1)-(3)$ such that*

$$\frac{(i_1+...+i_n)!}{i_1!...i_n!}M^1_{i_1}...M^n_{i_n}\leq M_{i_1+...+i_n}\leq B^{i_1+...+i_n}M^1_{i_1}...M^n_{i_n}, \tag{20}$$

for some $B>0$ and for all $i_j\in\mathbb{N},\ j=1,...,n,$ and let $(P_j)_{j=1}^N$ be an elliptic system with coefficients in $R_{M^1,...,M^n}(\Omega)$, so

$$(P_j)_{j=1}^N \text{ is elliptic in } \Omega\Rightarrow R_M\left(\Omega,(P_j)_{j=1}^N\right)\subset R_{M^1,...,M^n}(\Omega)\ .$$

Proof. It suffices to show the estimate (5) in the neighborhood of every point of Ω. Let $x_0\in\Omega$, so there exists a compact neighborhood ω of x_0 for which the above lemmas hold. Suppose that $u\in R_M\left(\Omega,(P_j)_{j=1}^N\right)$, then there exists $C>0$ such that

$$\|P_{i_0}\cdot\cdot P_{i_h}u\|_{L^2(\omega)}\leq C^{h+1}M_{hm}\ . \tag{21}$$

Let $R\in]0,1[$ such that $B_R\subset\omega$, so from (21), we have

$$\sigma_0\left(P_{i_0}\cdot\cdot P_{i_h}u,\lambda_0\right)\leq C^{h+1}M_{hm}\ ,$$

consequently from (19), we obtain

$$\begin{aligned}
\sigma_{p+1}(u,\lambda_0) &\leq 2^{p+1}C+\sum_{h=1}^{p+1}2^{p+1-h}\binom{p+1}{h}N^hC^{h+1}\\
&\leq C(2+NC)^{p+1}\ ,
\end{aligned}$$

which gives

$$\begin{aligned}
|u|_{(p+1)m,R/2} &\leq \lambda_0^{p+1}M_{pm+m}(2+NC)^{p+1}\left(\frac{2}{R}\right)^{pm}\\
&\leq C_1^{(p+1)m}M_{pm+m}\ .
\end{aligned}\tag{22}$$

Now, let $\alpha \in \mathbb{Z}_+^n$, so there exists positive integers p, q such that $|\alpha| = pm+q$, $q \leq m-1$. Set $\varepsilon = C_1^{q-m} \frac{M_{pm+q}}{M_{pm+m}}$, from (3) and (7), we have

$$\begin{aligned} |u|_{pm+q,R/2} &\leq C_1^{q-m} \frac{M_{pm+q}}{M_{pm+m}} |u|_{pm+m,R/2} + C_m C_1^q \left(\frac{M_{pm+m}}{M_{pm+q}} \right)^{\frac{q}{m-q}} |u|_{pm,R/2} \\ &\leq C_1^{q-m} \frac{M_{pm+q}}{M_{pm+m}} |u|_{pm+m,R/2} + C_1^q C_m d' \frac{M_{pm+q}}{M_{pm}} |u|_{pm,R/2} \ . \end{aligned}$$

Taking account of (20), we obtain from (22) and the last estimate

$$\begin{aligned} \|D^\alpha u\|_{L^2(B_{R/2})} &\leq |u|_{pm+q,R/2} \\ &\leq C_1^{pm+q} M_{pm+q} + C_1^{pm+q} C_m d' M_{pm+q} \\ &\leq C_2^{|\alpha|+1} M_{|\alpha|} \\ &\leq C_2^{|\alpha|+1} B^{|\alpha|} M_{\alpha_1}^1 ... M_{\alpha_n}^n, \end{aligned}$$

which shows that $u \in R_{M^1,...,M^n}(B_{R/2})$. ■

As consequences of Theorem 10 we obtain a result of Roumieu -Hypoellipticity of elliptic systems and the principal result of [7] in the scalar case where (M_p) is the Gevrey sequence $(p!)^s$.

Corollary 11 *Under the assumptions of Theorem 10, the following assertions are equivalent.*

(i) $u \in \mathcal{D}'(\Omega)$ *and* $P_j u \in R_{M^1,...,M^n}(\Omega)$, $\forall j = 1, ..., N$.

(ii) $u \in R_{M^1,...,M^n}(\Omega)$.

Corollary 12 *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and* $s \geq 1$, *and let* P *be a linear differential operator with cofficients in* $G^{s,q}(\Omega)$, *then*

$$P \textit{ is } q-\textit{quasi-elliptic in } \Omega \Longrightarrow G^s(\Omega, P) \subset G^{s,q}(\Omega)$$

References

[1] Bolley P.,Camus J., Powers and Gevrey regularity for a system of differential operators, Czechoslovak Math. J., Praha, 29 (104), (1979), 649-661.

[2] Bouzar C., Chaïli R., Régularité des vecteurs de Beurling de systèmes elliptiques, Maghreb Math. Rev., Vol. 9, no. 1 & 2, (2000), 43-53.

[3] John O., Sulla regolarità delle soluzioni delle equazioni lineari ellittiche nelle classi di Beurling, Boll. U.M.I., 4, 2, (1969), 183-195.

[4] Kotake T., Narasimhan M.S., Regularity theorems for fractional powers of a linear elliptic operator, Bull. Soc. Math. France, 90, (1962), 449-471.

[5] Lions J. L., Magenes E., Problèmes aux limites non homogenes et appliquations, Vol 3, Dunod, Paris, 1970.

[6] Nelson E., Analytic vectors, Ann. of Math., Vol. 70, no. 3, (1959), 572-615.

[7] Zanghirati L., Iterati di operatori quasi-ellittici e classi di Gevrey, Bollettino U.M.I., Vol. 5, no. 18-B, (1981), 411-428.