ERASURE ENTROPIES AND GIBBS MEASURES

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ABSTRACT. Recently Verdù and Weissman introduced *erasure entropies*, which are meant to measure the information carried by one or more symbols given all of the remaining symbols in the realization of a random process or field. A natural relation to *Gibbs measures* has also been observed. In this short note we study this relation further, review a few earlier contributions from statistical mechanics, and provide the formula for the erasure entropy of a Gibbs measure in terms of the corresponding potential. For some 2-dimensional Ising models, for which Verdú and Weissman suggested a numerical procedure, we show how to obtain an exact formula for the erasure entropy.

1. INTRODUCTION

1.1. **Erasure entropies.** Recently, Verdù and Weisman [21, 22], motivated by questions from information theory, introduced and studied erasure entropies. In one dimension the description is as follows. Let $\mathscr{X} = \{X_n\}_{n\in\mathbb{Z}}$ be a stationary process, taking values in a finite alphabet A. The erasure entropy of a collection of random variables $\{X_1, \dots, X_n\}$ is defined as

$$\mathsf{H}^-(X_1, \dots, X_n) = \sum_{i=1}^{n} \mathsf{H}(X_i|X_{\setminus i}), \tag{1}$$

where $X_{\setminus i} = \{X_j :\ j = 1, \dots, n,\ j \neq i\}$, and $\mathsf{H}(X_i|X_{\setminus i}) = \mathsf{H}(X_1, \dots, X_n) - \mathsf{H}(X_{\setminus i})$ is the conditional entropy.

Definition 1 ([21])**.** ***The erasure entropy rate*** *of the process* $\mathscr{X} = \{X_n\}_{n\in\mathbb{Z}}$ *is defined as*

$$\mathsf{h}^-(\mathscr{X}) = \lim_{n\to\infty} \frac{1}{n} \mathsf{H}^-(X_1, \dots, X_n). \tag{2}$$

The erasure entropy measures the information required to recover an erased symbol, knowing all the other symbols.

Verdù and Weissman established the following properties of erasure entropies.

Theorem 2 ([21])**.** *The limit in* (2) *exists. Moreover,*

$$\mathsf{h}^-(\mathscr{X}) = \lim_{n\to\infty} \mathsf{H}(X_0|X_{-n}, \dots, X_{-1}, X_1, \dots, X_n), \tag{3}$$

and

$$\mathsf{h}^-(\mathscr{X}) \leq \mathsf{h}(\mathscr{X}) = \lim_{n\to\infty} \mathsf{H}(X_0|X_{-1}, \dots, X_{-n}). \tag{4}$$

If $\mathscr{X} = \{X_n\}_{n\in\mathbb{Z}}$ is a k-step Markov chain then

$$\mathsf{h}(\mathscr{X}) = \frac{\mathsf{h}^-(\mathscr{X}) + \mathsf{H}(X_1,\ldots,X_k|X_{-1},\ldots,X_{-k})}{k+1}. \tag{5}$$

The interest in objects like the erasure entropy (1) and the corresponding entropy rate (2) arose in information theory in connection to coding problems for discrete memoryless erasure channels (DME).

Suppose that $\mathscr{X} = \{X_i\}$ is a discrete stationary process with values in a finite alphabet $\mathscr{A} = \{1,\ldots,N\}$, and consider the process $\mathscr{Z} = \{Z_i\}$ with values in alphabet $\mathscr{A}\cup\{*\}$, defined for all i by

$$Z_i = \begin{cases} X_i, & \text{if } E_i = 0,\\ *, & \text{if } E_i = 1,\end{cases}$$

where $\{E_i\}$ is a sequence of Bernoulli random variables with

$$\mathbb{P}(E_i = 1) = p.$$

The process $\{E_i\}$ is also assumed to be independent of $\mathscr{X}$. The resulting process $\mathscr{Z}$ is the response of the DME with parameter p over $\mathscr{X}$. The information rate at which the observer of $\mathscr{Z}$ has to be supplied in order to reconstruct the erased symbols almost surely is given by

$$\mathsf{h}(\mathscr{X}|\mathscr{Z}) = \lim_{n\to\infty}\frac{1}{n}\mathsf{H}(X_1,\ldots,X_n|Z_1,\ldots,Z_n)$$

and, generically, is very difficult to evaluate. In [22], it was shown that

$$\mathsf{h}(\mathscr{X}|\mathscr{Z}) \ge p\,\mathsf{h}^-(\mathscr{X}) \quad \forall p\in[0,1),$$

and

$$\mathsf{h}(\mathscr{X}|\mathscr{Z}) = p\,\mathsf{h}^-(\mathscr{X}) + \mathsf{o}(p) \quad \text{as } p\to 0.$$

2. Bilaterally deterministic processes: source of counterexamples

A stationary process $\mathscr{X} = \{X_n\}_{n\in\mathbb{Z}}$ is called ***deterministic*** if X_0 is measurable with respect to the σ-algebra $\mathcal{B}(X_k, k<0)$. By stationarity this means that for every $n\in\mathbb{N}$

$$\mathcal{B}(X_k, k\le -n) = \mathcal{B}(\ldots,X_{-1},X_0,X_1,\ldots),$$

and hence X_0 can be predicted with zero error probability from its past $(\ldots,X_{-2},X_{-1})$.

Analogously, a stationary process $\mathscr{X} = \{X_n\}_{n\in\mathbb{Z}}$ is called ***bilaterally deterministic*** if for every $n\in\mathbb{N}$,

$$\mathcal{B}(X_k, |k|\le n) = \mathcal{B}(\ldots,X_{-1},X_0,X_1,\ldots),$$

which is equivalent to requiring that given the past $\{\ldots,X_{-n-1},X_{-n}\}$ and the future $\{X_n,X_{n+1},\ldots\}$, the value X_0 can be reconstructed with zero error probability. Therefore, any bilaterally deterministic process has erasure entropy rate 0.

Gurevič [6] constructed a first example of a non-deterministic, but bilaterally deterministic, stochastic process; for a strongly mixing example see [4]. Any deterministic process is clearly bilaterally deterministic. The converse is false in quite a strong sense, as the following somewhat counter-intuitive result obtained by Ornstein and Weiss [14] shows.

Theorem 3. *Given any ergodic finite-state stationary stochastic process, there is an isomorphic ergodic finite-state stationary stochastic process which is bilaterally deterministic.*

Isomorphism here is understood in measure-theoretical sense. More specifically, in [14] a finite *generating* partition $\beta = \{B_1, \dots, B_m\}$ of $A^{\mathbb{Z}}$ is built such that the corresponding factor process $\mathscr{Y} = \{Y_n\}$ given by

$$Y_n = j \quad \iff \quad \sigma^n(\{X_k\}_{k\in\mathbb{Z}}) \in B_j \quad \text{for all } n,$$

where $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is the left shift, is bilaterally deterministic. The fact that the partition β is generating means that given $\{Y_n\}$ one can reconstruct $\{X_n\}$ almost surely.

Theorem 3 implies that the erasure entropy rate $\mathsf{h}^-(\mathscr{X})$ is not a measure-theoretic invariant of the process $\mathscr{X} = \{X_n\}$, while the entropy rate $\mathsf{h}(\mathscr{X})$ is well known to be such an invariant.

3. Erasure entropies of Gibbs states

A natural relation between erasure entropies and thermodynamic (Gibbs) states was observed in the original paper [21]. In particular, the application to Gibbs sampling of Markov random fields, a well-known problem in stochastic image analysis, served as a motivation for considering this question also on higher-dimensional lattices, see in particular their section **4F**, in which they provide an approximate expression for the erasure entropy of a two-dimensional nearest-neighbour Ising model. Let us elaborate on this issue a bit further.

3.1. Gibbs measures.

Gibbs measures for a given interaction Φ, according to the definition introduced by Dobrushin, Lanford and Ruelle, are defined as measures whose conditional probabilities of finite-volume configurations X_Λ, given external configurations X_{Λ^c}, are of the the Gibbsian form

$$\mathbb{P}(X_\Lambda | X_{\Lambda^c}) = \frac{1}{Z_{X_{\Lambda^c}}} \exp\Big(-H_\Lambda^\Phi(X)\Big) =: \gamma_\Lambda^\Phi(X_\Lambda | X_{\Lambda^c})$$

where the Hamiltonian H_Λ^Φ contains all interaction terms having a non-empty intersection with Λ:

$$H_\Lambda^\Phi(X) = \sum_{A\cap\Lambda\neq\varnothing} \Phi_A(X),$$

where the normalization constant $Z_{X_{\Lambda^C}}$ is the appropriate partition function.
This should hold for all volumes $\Lambda \subset \mathbb{Z}^d$, and all choices of the configurations X_Λ and X_{Λ^c}. See e.g. [8, 11].

3.2. **Erasure entropy rates of random fields.** For a finite volume $\Lambda \subseteq \mathbb{Z}^d$ and any $\mathbf{k} \in \mathbb{Z}^d$, we describe the shifted volume $\mathbf{k}+\Lambda = \{\mathbf{k}+\mathbf{i} | \mathbf{i} \in \Lambda\}$.

Definition 4. *Let $\{X_n\}_{n\in\mathbb{Z}^d}$ be a translation-invariant random field taking values in a finite alphabet A, and let $\mathbb{P}$ be the corresponding probability measure on $A^{\mathbb{Z}^d}$. The erasure entropy rate of $\mathbb{P}$ for the finite volume $\Lambda \subset \mathbb{Z}^d$ is defined as*

$$\mathsf{h}^-_\Lambda(\mathbb{P}) = \lim_{V\nearrow\mathbb{Z}^d} \frac{1}{|V|} \sum_{\mathbf{k}\in V} \mathsf{H}\big(X_{(\mathbf{k}+\Lambda)\cap V} | X_{V\setminus(\mathbf{k}+\Lambda)}\big) \tag{6}$$

The interpretation of the erasure entropy for a finite volume remains unchanged: $\mathsf{h}^-_\Lambda(\mathbb{P})$ is the number of bits required to reconstruct the configuration on Λ given the configuration outside of Λ. In fact, it is also convenient to consider the erasure entropy normalized by the volume $\frac{1}{|\Lambda|}\mathsf{h}^-_\Lambda(\mathbb{P})$.

Theorem 5. *The limit (6) exists, and*

$$\mathsf{h}^-_\Lambda(\mathbb{P}) = \lim_{V\nearrow\mathbb{Z}^d} \mathsf{H}(X_\Lambda | X_{V\setminus\Lambda}) = -\int \log \mathbb{P}(X_\Lambda | X_{\Lambda^c}) \mathbb{P}(dX). \tag{7}$$

For Gibbs measures one can say more.

Theorem 6. *If $\mathbb{P}$ is a translation invariant Gibbs measure on $A^{\mathbb{Z}^d}$ for interaction Φ, then the erasure entropy rate can be expressed in terms of the Gibbs specification γ^Φ_Λ as follows:*

$$\mathsf{h}^-_\Lambda(\mathbb{P}) = -\int \log \gamma^\Phi_\Lambda(X_\Lambda | X_{\Lambda^c})\, \mathbb{P}(dX). \tag{8}$$

Hence, $\mathsf{h}^-_\Lambda(\cdot)$ is an affine functional on the set of Gibbs measures consistent with specification γ^Φ. Moreover,

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \mathsf{h}^-_\Lambda(\mathbb{P}) = \mathsf{h}(\mathbb{P}). \tag{9}$$

Proof. Proofs of such results are rather standard and rely on application of (strong) subadditivity, martingale convergence and ergodic theorems, [7, 19, 21]. Expression (8) is an immediate consequence of the Gibbs property of $\mathbb{P}$. To show (9) we note that

$$\begin{aligned} &-\frac{1}{|\Lambda|}\int \log \gamma^\Phi_\Lambda(X_\Lambda | X_{\Lambda^c})\, \mathbb{P}(dX) \\ &\qquad = \int \frac{1}{|\Lambda|} H^\Phi_\Lambda(X) \mathbb{P}(dX) + \int \frac{1}{|\Lambda|} \log Z(X_{\Lambda^c}) \mathbb{P}(dX) \to \mathsf{h}(\mathbb{P}) \end{aligned}$$

by [6, Corollary 15.35]. □

Let us conclude this section with a number of remarks on the relation between erasure entropies and the theory of Gibbs states of Equilibrium Statistical Mechanics.

(A) By (7), the erasure entropy rate can also be seen as the difference in entropies of the system with and without a finite number of spins inside an interval or a volume. As such, erasure entropy is an example of an *inner entropy* (in some literature called local or conditional entropy) introduced earlier in Statistical Mechanics, see [10, 16] or [19, 20].

Such conditional entropies have been used before to establish "local" or "inner" variational principles [5, 16, 17, 19, 20]. From (9) it is known that for Gibbs measures the limit conditional entropy density and the standard entropy density coincide [7, 20]. This property is obviously not true in general, due to the Gurevič and Ornstein-Weiss examples, see e.g. [19, 20].

(B) One extra remark to be made is that Gibbs measures can be characterised, not only by a local variational principle for general volumes, but even by a single-site variational principle, using only the (Verdù-Weissman) single-site erasure entropy. This was proven in a short paper by J. Fernando Perez and R.H. Schonmann [9]. More specifically, suppose $\Phi = \{\Phi_\Lambda\}$ is an interaction and $\mathbb{P}$ is an arbitrary probability measure (state). The *free-energy content of a region* Λ *in a state* $\mathbb{P}$ is defined by

$$F_\Lambda(\mathbb{P}) = \int H_\Lambda d\mathbb{P} + \mathsf{h}^-_\Lambda(\mathbb{P}).$$

Here we do not assume $\mathbb{P}$ to be translation invariant, and hence the erasure entropy $\mathsf{h}^-_\Lambda(\mathbb{P})$ is understood as the limit

$$\mathsf{h}^-_\Lambda(\mathbb{P}) = \lim_{V \nearrow \mathbb{Z}^d} \mathsf{H}(X_\Lambda | X_{V\setminus\Lambda}).$$

Following Sewell [17], a state $\mathbb{P}$ is called ***locally thermodynamically stable*** if for every finite volume Λ one has

$$F_\Lambda(\tilde{\mathbb{P}}) \geq F_\Lambda(\mathbb{P})$$

for all measures $\tilde{\mathbb{P}}$ such that $\tilde{\mathbb{P}}_{\Lambda^c} = \mathbb{P}_{\Lambda^c}$. In other words, the locally thermodynamically stable state minimizes the free-energy content with respect to variations of the state inside Λ, for all Λ.

It turns out that for a fairly large class of finite-norm interactions, from the fact that $\mathbb{P}$ minimizes F_Λ, one can deduce that $\mathbb{P}$ satisfies the Dobrushin-Lanford-Ruelle (DLR) conditions on Λ, see [9, Theorem 2]. Moreover, it also known that if a translation-invariant state $\mathbb{P}$ satisfies DLR condition on $\Lambda = \{0\}$, then $\mathbb{P}$ is a Gibbs state. (Or, more generally, if the DLR conditions hold for all single sites they hold in general).

Hence, we can formulate a variational principle for Gibbs states in terms of the erasure entropy $\mathsf{h}^-_0(\mathbb{P})$.

Although [9] formulate their result for finite-range interactions, it is easy to see that their arguments generalise beyond that, due to the analysis in [5, 16] of the LTS condition and its equivalence to other equilibrium conditions.

(C) It follows from the bilaterally deterministic examples discussed in Section 2 that neither the erasure entropy, nor the Gibbs property are measure-theoretical invariants, c.f. [13].

4. Computing erasure entropies for Ising models in two dimensions

In [22] a question arose about computation of a 1-site (i.e., $\Lambda = \{0\}$) erasure entropy rate for the standard nearest-neighbour Ising model on the two-dimensional integer lattice. In fact, in contrast to what was suggested there, this erasure entropy rate can be computed explicitly.

Because of the Markov property of the Gibbs measures of the 2-dimensional Ising model (immediate from the nearest-neighbour character of the interaction), it is sufficient to consider nearest-neighbour environments of the spin at the center. Let us denote the 5 spins we will need to consider $\sigma_C, \sigma_E, \sigma_N, \sigma_W, \sigma_S$ (for Center, East, North, West, and South).

	σ_N	
σ_W	σ_C	σ_E
	σ_S	

Figure 1. Ising spins

We first remark that the probabilities of the 16 configurations of the E., N., W., and S. spins, in the spin-flip-invariant infinite-volume Gibbs measure $\mathbb{P} = \frac{1}{2}(\mathbb{P}^+ + \mathbb{P}^-)$, fall into 4 classes.

	$+$			$+$			$+$			$+$	
$+$	$\cdot$	$+$	$+$	$\cdot$	$-$	$-$	$\cdot$	$-$	$+$	$\cdot$	$-$
	$+$			$+$			$+$			$-$	

Figure 2. Four boundary types (upto rotations and spin-flips)

Denote by P_i, $i = 1, \ldots, 4$ the probability of the corresponding boundary configuration. The appropriate combination of these 4 probabilities adds up to 1:

$$2P_1 + 8P_2 + 2P_3 + 4P_4 = 1. \tag{10}$$

By (8), the erasure entropy $\mathsf{h}^-_{\{0\}}(\mathbb{P})$ is given by:

$$\begin{aligned}\mathsf{h}^-_{\{0\}}(\mathbb{P}) =& 2P_1 \cdot h\Big(\frac{e^{4J}}{e^{4J} + e^{-4J}}\Big) + 8P_2 \cdot h\Big(\frac{e^{2J}}{e^{2J} + e^{-2J}}\Big)\\ &+ (2P_3 + 4P_4) \cdot h\Big(\frac{1}{2}\Big),\end{aligned}$$

where $h(p) = -p\log p - (1-p)\log(1-p)$ for $p \in [0,1]$, and J is the nearest-neighbour coupling constant. Note that in [22] the third and the fourth boundary types are not distinguished (since they contribute equally to the entropy). However, to actually compute the erasure entropy $\mathsf{h}^-_{\{0\}}(\mathbb{P})$ we must distinguish these types.

By Pfaffian methods (albeit in somewhat implicit form, see [12]), one can compute three relevant correlation functions, namely the 4-point function and the two pair correlation functions one can build with these four spins, that is $\mathbb{E}(\sigma_E\sigma_W), \mathbb{E}(\sigma_E\sigma_N)$, and

$\mathbb{E}(\sigma_E\sigma_N\sigma_W\sigma_S)$. Finally, we derive remaining equations for probabilities P_i, by noting that

$$\begin{aligned} \mathbb{E}(\sigma_E\sigma_N\sigma_W\sigma_S) &= 2P_1 - 8P_2 + 2P_3 + 4P_4, && (11)\\ \mathbb{E}(\sigma_E\sigma_W) &= 2P_1 + 2P_3 - 4P_4, && (12)\\ \mathbb{E}(\sigma_E\sigma_N) &= 2P_1 - 2P_3. && (13) \end{aligned}$$

As an immediate consequence, one gets

$$P_1 = \frac{1 + \mathbb{E}(\sigma_E\sigma_N\sigma_W\sigma_S) + 2\mathbb{E}(\sigma_E\sigma_W) + 4\mathbb{E}(\sigma_E\sigma_N)}{16},$$
$$P_2 = \frac{1 - \mathbb{E}(\sigma_E\sigma_N\sigma_W\sigma_S)}{16},$$

which is sufficient to express the erasure entropy in terms of correlation functions.

In conclusion, we note that for obvious symmetry reasons

$$\mathsf{h}^-_{\{0\}}(\mathbb{P}^-) = \mathsf{h}^-_{\{0\}}(\mathbb{P}^+),$$

and, by the affine property, the erasure entropies of both phases coincide with $\mathsf{h}^-_{\{0\}}(\mathbb{P})$ obtained above. For the same reasons, the erasure entropy of any Gibbs measure of the Ising model coincides with $\mathsf{h}^-_{\{0\}}(\mathbb{P})$.

Definitions of erasure entropies extend readily to more general lattices. For the correspoding nearest-neighbour Ising models the Pfaffian method of computing correlations will apply to some other two-dimensional planar lattices, see again [1,3,12] and specifically [2]. In case of the hexagonal lattice, we propose another approach, which is less general, in the sense that everything we need can be derived from just knowing the free energy density, but already leads to an explicit analytic expression of the erasure entropy.

Consider the two-dimensional hexagonal (honeycomb) lattice. One now needs to consider only three neighbours – denoted σ_I, σ_{II}, and σ_{III}, of the central spin σ_C. There are obviously only 2 boundary types.

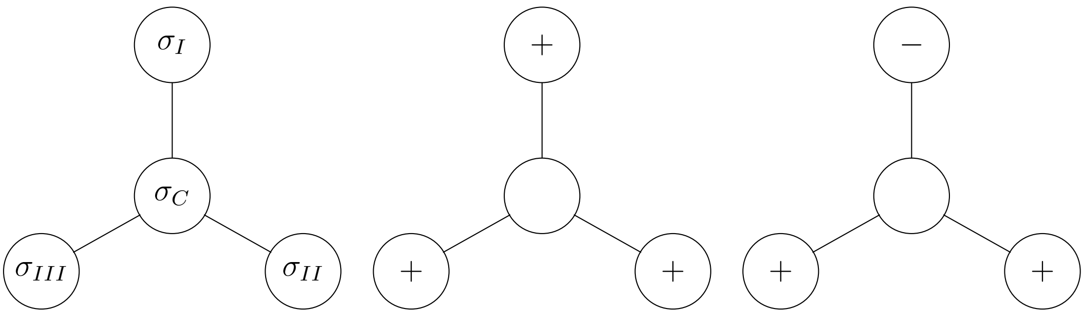

FIGURE 3. Ising spins on hexagonal lattice and two boundary types.

Consider again the zero-field situation and the symmetric Gibbs measure $\mathbb{P}$. Let P_i, $i = 1, 2$, denote the probability of the corresponding boundary type. Thus we have

$$2P_1 + 6P_2 = 1. \qquad (14)$$

For each quadruple $(\sigma_I, \sigma_{II}, \sigma_{III}, \sigma_C) \in \{-1,1\}^4$, one has

$$\begin{aligned}\mathbb{P}(\sigma_I, \sigma_{II}, \sigma_{III}, \sigma_C) &= \mathbb{P}(\sigma_C|\sigma_I, \sigma_{II}, \sigma_{III})\mathbb{P}(\sigma_I, \sigma_{II}, \sigma_{III})\\ &= \frac{e^{J\sigma_C(\sigma_I+\sigma_{II}+\sigma_{III})}}{2\cosh\left(J(\sigma_I+\sigma_{II}+\sigma_{III})\right)}P_i,\end{aligned}$$

where the boundary type i is determined by $(\sigma_I, \sigma_{II}, \sigma_{III})$.

Consider now the correlation function $\mathbb{E}(\sigma_I\sigma_C)$,

$$\mathbb{E}(\sigma_I\sigma_C) = \sum_{(\sigma_I,\sigma_{II},\sigma_{III},\sigma_C)\in\{-1,1\}^4} \sigma_I\sigma_C\mathbb{P}(\sigma_I, \sigma_{II}, \sigma_{III}, \sigma_C),$$

and after a straightforward manipulation we conclude that

$$\mathbb{E}(\sigma_I\sigma_C) = P_1\tanh(3J) + P_2\tanh(J).$$

Hence,

$$P_1 = \frac{6\mathbb{E}(\sigma_I\sigma_C) - \tanh(J)}{6\tanh(3J) - 2\tanh(J)}, \quad P_2 = \frac{\tanh(3J) - 2\mathbb{E}(\sigma_I\sigma_C)}{6\tanh(3J) - 2\tanh(J)}, \tag{15}$$

and the erasure entropy $\mathsf{h}_0^-(\mathbb{P})$

$$\mathsf{h}_0^-(\mathbb{P}) = 2P_1\cdot h\Big(\frac{e^{3J}}{e^{3J}+e^{-3J}}\Big) + 6P_2\cdot h\Big(\frac{e^{J}}{e^{J}+e^{-J}}\Big) \tag{16}$$

can be expressed in terms of the correlation function $\mathbb{E}(\sigma_I\sigma_C)$.

In order to obtain the expression for $\mathbb{E}(\sigma_I\sigma_C)$ we recall that the zero-field Ising model on the hexagonal lattice is an exactly solvable model, and the pressure function at inverse temperature β is given by

$$\begin{aligned}p(\beta) = \frac{3}{4}\log 2 + \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log\Big(&\cosh^3(2\beta J) + 1 - \sinh^2(2\beta J)\times\\ &\Big[\cos(\theta_1-\theta_2) + \cos(\theta_1) + \cos(\theta_2)\Big]\Big)d\theta_1 d\theta_2,\end{aligned} \tag{17}$$

see e.g. [18, eq. (A.6). p.322]. The pressure function $p(\beta)$ is an everywhere differentiable function of β; the phase transition in the Ising model on hexagonal lattice is of second order.

Finally, as is well known [11, 15], Gibbs states are tangent functionals to the pressure, and hence, if the interaction $\Phi = \{\Phi_\Lambda\}$ is such that the pressure function $p(\beta) = P(\beta\Phi)$ is differentiable as a function of β at $\beta = \beta_0$, then

$$p'(\beta_0) = \int\Big(-\sum_{\Lambda\ni 0}\frac{1}{|\Lambda|}\Phi_\Lambda(\sigma)\Big)\mathbb{P}(d\sigma), \tag{18}$$

where $\mathbb{P}$ is a Gibbs state for the Hamiltonian $\beta H = \beta\sum_\Lambda \Phi_\Lambda$. In case of the Ising model on the hexagonal lattice, we thus conclude that

$$p'(1) = -\frac{3J}{2}\,\mathbb{E}(\sigma_I\sigma_C). \tag{19}$$

Combining (19), (17), (15), and (16), we arrive to an analytic expression for $\mathsf{h}_0^-(\mathbb{P})$. Similarly to the case of rectangular lattice, the erasure entropy is independent of the phase.

Final remarks A possible generalisation rests on the fact that the coupling constants in the three different directions are allowed to be different, without affecting the solubility of the model. In that case the corresponding correlation functions need to be separately computed.

Also, the next nearest-neighbour correlation functions $\mathbb{E}(\sigma_I \sigma_{II})$ etc, can be computed as the *nearest neighbour* correlation functions of an Ising model on a triangular lattice, related to the hexagonal model by a star-triangle transformation, see again [18]. This offers a slightly different, though essentially equivalent, route to computing exact expressions for the Verdù-Weissman erasure entropy

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