REFLECTION GROUPS AND COMPOSITE MAPPINGS IN CLASS $\mathcal{C}^r$

GÉRARD P. BARBANÇON

ABSTRACT. Let W be a finite reflection group acting orthogonally on $\mathbb{R}^n$, $P = (p_1, \ldots, p_n)$ be a set of basic polynomial invariants and h be the highest degree of the p_is which is the largest Coxeter number of the irreducible components of W. We first study the subspace of functions of the invariants that are invariant functions of class C^{hr}. This subspace, say $(P^*)^{-1}(\mathcal{C}^{hr}(\mathbb{R}^n)^W)$ is a space of "compositors" in $\mathcal{E}^r(P(\mathbb{R}^n))$. Then we show that the algebra of composite mappings $P^*(\mathcal{E}^r(P(\mathbb{R}^n)))$ is a Fréchet subspace of the space of r-regular jets of order hr. The algebra homomorphism P^* identifies this Fréchet space with the space of functions of class $\mathcal{C}^r$ of the polynomial invariants. This study needs the Whitney 1-regularity property of $P(\mathbb{R}^n)$. By lack of a reference, we give in appendix a proof of this regularity property by completing a result given in [7] with a lemma that was not proved for all Coxeter groups.

1. Introduction

Let W be a finite reflection group acting orthogonally on $\mathbb{R}^n$. The algebra of W-invariant polynomials is generated by n algebraically independent W-invariant homogeneous polynomials and the degrees of these basic invariants are uniquely determined [13], [5]. Let $p_1, \ldots, p_n$ be a set of basic invariants, we define the 'Chevalley' mapping

$$P : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad P(x) = \big(p_1(x), \ldots, p_n(x)\big).$$

Glaeser's theorem [9] shows that W-invariant functions of class $\mathcal{C}^\infty$, may be expressed as functions of class $\mathcal{C}^\infty$ of the basic invariants, and actually that the subalgebra $P^*(\mathcal{C}^\infty(\mathbb{R}^n))$ of composite mappings of the form $F \circ P$ with F of class $\mathcal{C}^\infty$, is closed in $\mathcal{C}^\infty(\mathbb{R}^n)$. In finite class of differentiability, the situation is not this simple. Let h be the highest degree of the coordinate polynomials in P, equal to the greatest Coxeter number of the irreducible components of W, in [4] it is shown that if a function $f \in \mathcal{C}^{hr}(\mathbb{R}^n)$ is invariant by W, there exists an F of class $\mathcal{C}^r$, such that $f = F \circ P$. A general counterexample shows that this result is the best possible. However if F is of class $\mathcal{C}^r$, in general $f = F \circ P$ is of class $\mathcal{C}^r$ and not of class $\mathcal{C}^{hr}$. So the following questions naturally arise: what is the space of functions of the invariants that are invariant functions of class $\mathcal{C}^{hr}$ on $\mathbb{R}^n$, and what is the space of W-invariant functions that are functions of class $\mathcal{C}^r$ of the polynomial invariants?

We first complement the result given in [4] by showing that the functions F of the invariants such that $f = F \circ P$ is of class $\mathcal{C}^{hr}$, belong to a subspace of rough compositors in $\mathcal{E}^r(P(\mathbb{R}^n))$, isomorphic to $\mathcal{C}^{hr}(\mathbb{R}^n)^W$.

Key words and phrases. Finite Coxeter groups, r-regular jets of order m, algebra of composite mappings, Whitney regularity property.

Then, we study the algebra of composite mappings $P^*(\mathcal{C}^r(\mathbb{R}^n))$. The functions in this algebra induce jets in a Fréchet space isomorphic to the space of functions of class $\mathcal{C}^r$ of the invariants.

This study uses the Whitney 1-regularity of the image set $P(\mathbb{R}^n)$.

Definition 1.1. [16] [14] A compact set $K \subset \mathbb{R}^n$, connected by rectifiable arcs, is *Whitney 1-regular* if the geodesic distance in K is equivalent to the Euclidean distance: there exists a constant $k_K > 0$ such that for all $(x, x') \in K^2$, there is a rectifiable arc from x to x' in K with length $\ell(x, x') \leq k_K |x - x'|$.

A closed set is Whitney 1-regular if it is a union of Whitney 1-regular compact sets.

The 1-regularity of $P(\mathbb{R}^n)$ was conjectured in [4] and already in [7]. By lack of a better reference we state and prove this property in the appendix, using the results of [7] and some results given in [11] for the symmetric group that apply for any Coxeter group, *mutatis mutandis*.

2. The Chevalley mapping

A detailed study may be found in [5] or [8].

When W is reducible, it is a direct product of its irreducible components, say $W = W^0 \times W^1 \times \ldots \times W^s$ and we may write $\mathbb{R}^n$ as an orthogonal direct sum $\mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_s}$. The first component W^0 is the identity on the W-invariant subspace $\mathbb{R}^{n_0}$, and for $i = 1, \ldots, s$, W^i is an irreducible finite Coxeter group acting on $\mathbb{R}^{n_i}$. We will choose coordinates that fit with this orthogonal direct sum. If $w = w_0 w_1 \ldots w_s \in W$ with $w_i \in W^i$, $0 \leq i \leq s$ we have $w(x) = w(x^0, x^1, \ldots, x^s) = (x^0, w_1(x^1), \ldots, w_s(x^s))$ for all $x \in \mathbb{R}^n$. The direct product of the identity P^0 on $\mathbb{R}^{n_0}$ and Chevalley mappings P^i associated with W^i acting on $\mathbb{R}^{n_i}$, $1 \leq i \leq s$, is a Chevalley map associated with the action of W on $\mathbb{R}^n$.

For an irreducible W (or for an irreducible component) we will assume as we may that the degrees of the coordinate polynomials $p_1, \ldots, p_n$ are in increasing order: $2 = k_1 \leq \ldots \leq k_n = h$, where h is the Coxeter number of W. In the reducible case, if there is an invariant subspace $\mathbb{R}^{n_0}$, the corresponding p_is say $P^0_j, 1 \leq j \leq n_0$ are of degree $k_0 = 1$. Then for each $i = 1 \ldots s$ the P^i_j are of degree k^i_j, for $j = 1$ to n_i. The k^i_1 are equal to 2, and $k^i_{n_i}$ is the Coxeter number of W^i. We denote with h is the maximal Coxeter number (or highest degree of the coordinate polynomials) of the irreducible components.

Let $\mathcal{R}$ be the set of reflections different from identity in W. The number of these reflections is $\mathcal{R}^{\#} = d = \sum_{i=1}^{n}(k_i - 1)$. For each $\tau \in \mathcal{R}$, let λ_τ be a linear form the kernel of which is the hyperplane $H_\tau = \{x \in \mathbf{R}^n | \tau(x) = x\}$.

The jacobian matrix J_P is block diagonal. Let $I_{n_0}, J_{P^1}, \ldots, J_{P^{n_s}}$ be the diagonal blocks. The $n_0 \times n_0$ minor in the upper left corner is 1 but the determinants $|J_{P^1}|, \ldots, |J_{P^s}|$, all vanish on $\mathbb{R}^{n_0}$. The jacobian determinant of P is $|J_P| = c \prod_{\tau \in \mathcal{R}} \lambda_\tau$ for some constant $c \neq 0$. The critical set is the union of the H_τ when τ runs through $\mathcal{R}$. The invariant subspace is the intersection $\bigcap_{\tau \in \mathcal{R}} H_\tau$ of the reflecting hyperplanes.

A Weyl Chamber C is a connected component of the regular set. The other connected components are obtained by the action of W and the regular set is $\bigcup_{w \in W} w(C)$. There is a stratification of $\mathbf{R}^n$ by the regular set, the reflecting hyperplanes H_τ and their intersections. The mapping P is neither injective nor surjective but it induces an analytic diffeomorphism of C onto the interior of $P(\mathbf{R}^n)$

and an homeomorphism that carries the stratification from the fundamental domain $\overline{C}$ onto $P(\mathbf{R}^n)$.

Finally, let us recall that the Chevalley mapping is proper and separates the orbits.

3. Whitney Functions and r-regular jets of order m.

For Whitney functions of class $\mathcal{C}^r$ one can see [14], the notations of which will be used freely.

A jet of order $m \in \mathbf{N}$, on a locally closed set $E \subset \mathbf{R}^n$ is a collection $A = (a_k)_{\substack{k\in\mathbf{N^n}\\|k|\le m}}$ of real valued functions a_k continuous on E. At each point $x \in E$ the jet A determines a polynomial $A_x(X)$, and we sometimes speak of continuous polynomial fields instead of jets. As a function, A_x acts upon vectors $x' - x$ tangent to $\mathbf{R}^n$ at x and we write somewhat inconsistently:

$$A_x : x' \mapsto A_x(x') = \sum_k \frac{1}{k!} a_k(x)\ (x' - x)^k.$$

The space $J^m(E)$ of jets of order m on E is naturally provided with the Fréchet topology induced by the family of semi-norms: $|A|_m^{K_s} = \sup_{\substack{x\in K_s\\|k|\le m}} |a_k(x)|$, where K_s runs through a countable exhaustive collection of compact sets of E.
By formal derivation of A of order $q \in \mathbb{N}^n$, $|q| \le m$, we get jets of the form $(a_{q+k})_{|k|\le m-|q|}$ inducing polynomials

$$(D^qA)_x(x') = a_q(x) + \sum_{\substack{k>q\\|k|\le m}} \frac{1}{(k-q)!} a_k(x)\ (x' - x)^{k-q}.$$

For $0 \le |q| \le r \le m$, we put:

$$(R_xA)^q(x') = (D^qA)_{x'}(x') - (D^qA)_x(x').$$

Definition 3.1. Let A be a jet of order m on E. For $r \le m$, A is r-regular on E, if and only if for all compact set K in E, for $(x, x') \in K^2$, and for all $q \in \mathbf{N}^n$ with $|q| \le r$, it satisfies the Whitney conditions.

$$(\mathcal{W}_q^r) \quad (R_xA)^q(x') = o(|x' - x|^{r-|q|}), \text{ when} |x - x'| \to 0.$$

Remark 3.2. If $m > r$ there is no need to consider the truncated field A^r instead of A in the conditions $(\mathcal{W}_q^r)$. Actually $(R_xA^r)^q(x')$ and $(R_xA)^q(x')$ differ by a sum of terms $[a_k(x)/(k-q)!]\,(x' - x)^{k-q}$, with a_k uniformly continuous on K and $|k| - |q| > r - |q|$.

The space of r-regular jets of order m on E, is naturally provided with the Fréchet topology defined by the family of semi-norms:

$$\|A\|_{r,m}^{K_s} = |A|_m^{K_s} + \sup_{\substack{(x,x')\in K_s^2\\x\ne x',\ |k|\le r}} \Big(\frac{|(R_xA)^k(x')|}{|x - x'|^{r-|k|}} \Big).$$

when $K_s, s \in \mathbb{N}$ runs through an exhaustive family of compact sets in E. This Fréchet space is denoted by $\mathcal{E}^{r,m}(E)$.

If $r = m$, $\mathcal{E}^r(E)$ is the space of Whitney fields of order r or Whitney functions of class $\mathcal{C}^r$ on E. When E is locally closed, we have:

Theorem 3.3. Whitney extension theorem [15]. *The restriction mapping of the space $\mathcal{E}^r(\mathbf{R}^n)$ of functions of class $\mathcal{C}^r$ on $\mathbf{R}^n$ to the space $\mathcal{E}^r(E)$ of Whitney fields of order r on E, is surjective. There is a linear section, continuous when the spaces are provided with their natural Fréchet topologies.*

In general the semi-norms $\|.\|_r^{K_s}$ and $|.|_r^{K_s}$ are not equivalent on $\mathcal{E}^r(E)$. They are when E is open. In this case $\mathcal{E}^r(E)$ is the space of fields of Taylor polynomials of functions in $\mathcal{C}^r(E)$ and the two spaces $\mathcal{E}^r(E)$ and $\mathcal{C}^r(E)$ may be identified.

When $A \in \mathcal{E}^r(E)$ and $|q| \leq r$, the formal derivation D^qA and the regular derivative $\partial^{|q|}A/\partial x^q$ define the same polynomial

$$(D^qA)_x(x') = \left(\frac{\partial^{|q|}A}{\partial x^q}\right)_x (x')$$

and they can be identified.

If E is open and if for all q with $|q| \leq m$, a_q is a continuous partial derivative, then A is m-regular. So, when E is open, if $A \in \mathcal{E}^{r,m}(E)$ but $A \notin \mathcal{E}^m(E)$, for $|q| > r$ the a_q may or may not be interpreted as partial derivatives. The identification of D^qA and $\partial^{|q|}A/\partial x^q$ may not be possible and for $|\ell| < r$, $\partial^{|\ell|}A/\partial x^\ell$ is in $\mathcal{E}^{r-\ell}(E)$, but in general it is not in $\mathcal{E}^{r-\ell,m-\ell}(E)$.

Whitney showed that on a closed set the 1-regularity brings the equivalence of the semi-norms:

Proposition 3.4. [16], [14] *If the compact set K is 1-regular, the norms $\|.\|_r^K$ and $|.|_r^K$ are equivalent on $\mathcal{E}^r(K)$.*

Let E be the 1-regular closure of its interior and A be in $\mathcal{E}^{r,m}(E)$. The proposition implies that as in the case of an open set, if for all q with $|q| \leq m$, a_q is a partial derivative and is continuous, A is m-regular. Accordingly, considering $\mathcal{E}^{r,m}(E)$ makes sense if either the $a_q, |q| > r$ are not necessarily partial derivatives of order $|q|$, or are not necessarily continuous (*e.g.* they are continuous on the interior of E but not on its border.)

Conversely, assuming that the compact K is connected by rectifiable arcs (or is a finite union of sets connected by rectifiable arcs), Glaeser has proved :

Proposition 3.5. [8], [14] *If the norms $\|.\|_1^K$ and $|.|_1^K$ are equivalent on $\mathcal{E}^1(K)$, then K is 1-regular.*

In appendix A we will show that:

Theorem 3.6. *The image of the Chevalley mapping $P(\mathbb{R}^n)$ is Whitney 1-regular.*

As a consequence we have:

Corollary 3.7. *On $\mathcal{E}^r(P(\mathbb{R}^n))$ the families of semi-norms $\|.\|_r^{K_s}$ and $|.|_r^{K_s}$ are equivalent.*

Besides a geometric property of $P(\mathbb{R}^n)$ which is of interest by itself, Whitney 1-regularity plays an important part when studying the algebras of composite mappings invariant by reflection groups in finite class of differentiability.

4. Algebra of W-invariant functions of class $\mathcal{C}^{hr}$

There exists a linear and continuous mapping (see [4] [1]):

$$L : \mathcal{C}^{hr}(\mathbb{R}^n)^W \longrightarrow, \quad L(f) = F, \quad f = F \circ P.$$

Clearly, L is injective and surjective on its image.

On the interior of $P(\mathbb{R}^n)$, F is of class $\mathcal{C}^{hr}$. Let S be a stratum of $\overline{C}$, intersection of p reflecting hyperplanes. On $P(S)$ the loss of differentiability is governed by W_S, the isotropy subgroup of S, and F is of class $[hr/h_S]$, where h_S is the highest degree of the W_S-invariants. Moreover we have:

Proposition 4.1. *$L(\mathcal{C}^{hr}(\mathbb{R}^n)^W$ is a space of rough "compositors". More specifically if $F \circ P = f$, the partial derivatives of F of order $\beta = (\beta_1, \dots, \beta_n)$ with $\sum_{i=1}^n \beta_i k_i \leq hr$ are continuous on $P(\mathbb{R}^n)$.*

When $\sum_{i=1}^n \beta_i k_i > hr$, the $\partial^{|\beta|}F/\partial P^\beta$ do not exist on some strata of $P(\mathbb{R}^n)$ but there are continuous weights $\varepsilon_\beta \geq 0$, such that $\varepsilon_\beta(x)(\partial^{|\beta|}F/\partial P^\beta) \circ P(x)$ tends to zero when $x \in C$ tends to a stratum S of $\overline{C}$ where $(\partial^{|\beta|}F/\partial P^\beta) \circ P$ does not exist. If we put $\varepsilon_\beta(x)(\partial^{|\beta|}F/\partial P^\beta) \circ P(x) = 0$ at these points, the $\varepsilon_\beta(\partial^{|\beta|}F/\partial P^\beta) \circ P$ are continuous on $\overline{C}$.

The proof of the first part of the statement was almost complete in [4]. Let us briefly recall that it is sufficient to study the problem for each W^i acting on $\mathbb{R}^{n_i}$ so that we may assume W to be irreducible. We are to solve the system:

$$\left(\frac{\partial f}{\partial x}\right) = \left(\left(\frac{\partial p_i}{\partial x_j}\right)_{\substack{1\leq i\leq n\\1\leq j\leq n}}\right)\left(\frac{\partial F}{\partial p} \circ P\right). \tag{4.1}$$

For the derivation with respect to p_j we get:

$$\left\{ c\,(\prod_{\tau\in\mathcal{R}} \lambda_\tau)\ \frac{\partial F}{\partial p_j} \circ P = \sum_{i=1}^n (-1)^{i+j} M_{i,j} \frac{\partial f}{\partial x_i},\ j = 1 \dots, n \right. \tag{4.2}$$

There is a loss of differentiability of $1 + d - s_j = k_j$ units, where s_j is the degree of $M_{i,j}$. This loss of differentiability is clear at the origin. In the neighborhood of a stratum S in $\overline{C}$, we write $P = Q \circ V$ with Q invertible and V a Chevalley mapping for the isotropy subgroup W_S of S. We have $M_{i,j} = \sum_{k=1}^n V_{k,j} Q_{i,k}$. The $V_{k,j}$ and accordingly the $M_{i,j}$ are $(s_p^j - 1)$-flat on S, where s_p^j is the degree of $V_{k,j}$. Since p is the number of reflections in W_S, $1 - s_p^j + p$ is the degree of the j^{th} polynomial of V which is less than or equal to k_j. The remaining part of the proof is exactly as in [4] where the details are given for the case $j = n$. The continuity on $P(\mathbb{R}^n)$ of the partial derivatives of F of order $\beta = (\beta_1, \dots, \beta_n)$ with $\sum_{i=1}^n \beta_i k_i \leq hr$ follows at once.

The study of the adherence of the ideal generated by $|J_P| = c \prod_{\tau\in\mathcal{R}} \lambda_\tau$ in $\mathcal{C}^k(\mathbb{R}^n)$ and in spaces of multipliers shows that even after compensation by the $M_{i,j}$, the functions $(\partial F/\partial p_j) \circ P$ and more generally $(\partial^{|\beta|}F/\partial P^\beta) \circ P$ are multipliers (see [10] and [3]) and this brings the second part of the statement.

When $\sum_{i=1}^n \beta_i k_i \leq hr$, we take $\varepsilon_\beta = 1$. When $\sum_{i=1}^n \beta_i k_i > hr$, the ε_β are determined by products of $\sum_{i=1}^n \beta_i k_i - hr$ factors $|\lambda_\tau|$. Using the 1-regular separation

---

[1] Even though the proof would be basically the same, the 1-regularity makes it simpler since the r-continuity on $P(\mathbb{R}^n)$ of the field F brings its r-regularity and there is no need for an extension to $P^{-1}(\mathbb{R}^n)$.

of the linear strata of $\overline{C}$, it would be possible to express these ε_β as powers of distances to the strata. The complete determination of the weights ε_β is tedious. We will not do it here.

When computing $\partial^{|\ell|}(F \circ P)/\partial x^\ell$, $|\ell| \leq hr$, by the Faa di Bruno formula, the derivative $(\partial^{|\beta|} F/\partial P^\beta) \circ P$ is multiplied by the product $\prod_{j=1}^n (D^1 p_j)^{\mu_1^j} \dots (D^q p_j)^{\mu_q^j}$, with $\mu_1^j + \dots + \mu_q^j = \beta_j$, and $\sum_j (1\mu_1^j + \dots + q\mu_q^j) = |\ell| \leq hr$.

Since the p_j are homogeneous of degree k_j, up to a multiplicative numerical constant $(\partial^{|\beta|} F/\partial P^\beta) \circ P$ is multiplied by $|x|^{\sum_j \beta_j k_j - |\ell|} \leq |x|^{\sum_j \beta_j k_j - hr}$ and the product tends to 0 with x.

When x tends to a stratum S, we write as above $P = Q \circ V$. Each $D^s(Q \circ V)$ in the Faa di Bruno formula, may in turn be expanded by the same formula, and the $D^t V$ thus obtained provide the power of the distance $d(x, S)$ we need.

It is possible to determine the ε_β from these considerations. Actually a good applicant that satisfies all the requirements is: $\varepsilon_\beta = \prod_{j=1}^n (D^1 p_j)^{\mu_1^j} \dots (D^q p_j)^{\mu_q^j}$, with $\mu_1^j + \dots + \mu_q^j = \beta_j$ for $j = 1, \dots, n$ and $\sum_j (1\mu_1^j + \dots + q\mu_q^j) = hr$.

If we define the semi-norms:

$$\|F\|^{hr}_{P(K_s)} = \max_{\substack{|\beta| \leq hr \\ x \in K_s}} |\frac{\partial^{|\beta|} F}{\partial P^\beta} \circ P(x) \varepsilon_\beta(x)|,$$

the Faa di Bruno formula shows directly that $\|f\|^{hr}_{K_s} \leq C_{K_s} \|F\|^{hr}_{P(K_s)}$, and through an induction it also shows that actually the norms are equivalent.

L is a Fréchet isomorphism of $\mathcal{C}^{hr}(\mathbb{R}^n)^W$ onto the subspace of compositors in $\mathcal{E}^r(P(\mathbb{R}^n))$ on which P^* is a section for L.

5. Algebra of composite mappings $P^*(\mathcal{E}^r(P(\mathbb{R}^n)))$

We consider the homomorphism induced by P:

$$P^* : \mathcal{E}^r(P(\mathbb{R}^n)) \longrightarrow \mathcal{E}^r(\mathbb{R}^n), \quad P^*(F) = F \circ P.$$

For any $(a, x) \in \mathbb{R}^n \times \mathbb{R}^n$, by the Taylor formula for F between $P(a)$ and $P(x)$, we have:

(5.1)

$$F[P(x)] = F[P(a)] + \sum_{1 \leq |\beta| \leq r} \frac{1}{\beta!} D^\beta F[P(a)] \ (P(x) - P(a))^\beta + o(|P(x) - P(a)|^r),$$

either by using an extension of F to $\mathcal{E}^r(\mathbb{R}^n)$ given by theorem 3.3 [2] or by using a Taylor integral remainder along an integrable path satisfying the inequality given by the 1-regularity property [16].

Expanding $P(x) - P(a)$ by the polynomial Taylor formula we get a polynomial in $x - a$ of degree h. Hence for $f = F \circ P$:

$$f(x) = f(a) + \sum_{1 \leq |\alpha| \leq hr} \frac{1}{\alpha!} f_\alpha(a) \ (x - a)^\alpha + o(|P(x) - P(a)|^r). \tag{5.2}$$

On a compact $K \subset \mathbb{R}^n$ containing $[a, x]$, there exists a constant C_K such that $|P(x) - P(a)|^r \leq C_K |x - a|^r$, and $f \in \mathcal{E}^{r,hr}(\mathbb{R}^n)$.

When $|\alpha| \leq r$ the f_α are partial derivatives, but when $r < |\alpha| \leq hr$, the f_α are obtained by the composition process with $|\beta| \leq r$ and are not partial derivatives in

[2]We might also consider $\mathcal{C}^r(\mathbb{R}^n)$ as the source of P^*, instead of $\mathcal{E}^r(P(\mathbb{R}^n))$.

general. Nevertheless, they allow the estimate of $f_x(x)-f_a(x)=o(|P(x)-P(a)|^r)$ instead of $f_x(x)-f_a(x)=o(|x-a|^r)$ as we would have with the truncated f^r.

The derivatives of f of order $|\alpha|\leq r$ are in $\mathcal{E}^{r-|\alpha|,h(r-|\alpha|)}(\mathbb{R}^n)$ by reference to the derivatives of F of the same order. For an example:

$$\frac{\partial f}{\partial x_i}(x)=\sum_{j=1}^{n}\frac{\partial p_j}{\partial x_i}(a)[\frac{\partial F}{\partial p_j}\circ P(a)+\sum_{1\leq|\beta|\leq r-1}\frac{1}{\beta!}\frac{\partial^{|\beta|+1}}{\partial P^\beta\partial p_j}F[P(a)]\ (P(x)-P(a))^\beta]$$
$$+o(|P(x)-P(a)|^{r-1}).$$

Definition 5.1. We say that $f\in J^{hr}(\mathbb{R}^n)$ pointwise belongs to $P^*(J^r(P(\mathbb{R}^n)))$, if for each x there exists an $F\in J^r(P(\mathbb{R}^n))$ such that $f_x=(F\circ P)_x$.

This means that for each $x\in\mathbb{R}^n$, the polynomial $f_x(X)$ is W-invariant. Of course it is necessary that f be W-invariant but it is not sufficient. Let $g\in J^r(\mathbb{R}^n)^W$, from [4] we know that there is an $F\in J^r(P(\mathbb{R}^n))$ such that $g=(F\circ P)^r$. Here we consider $f=F\circ P$ before truncation and it is of degree hr.

Lemma 5.2. *If $f\in J^{hr}(\mathbb{R}^n)$ pointwise belongs to $P^*(J^r(P(\mathbb{R}^n)))$, and we write $f=F\circ P$, then for $0\leq|\beta|\leq r$, $F_\beta\circ P(x)$ is a linear combination of some $f_\alpha(x)$, $0\leq|\alpha|\leq hr$. In particular the F_β are continuous, $F\in J^r(P(\mathbb{R}^n)))$ and $f\in P^*(J^r(P(\mathbb{R}^n)))$.*

Clearly $f_0(x)=F_0(P(x))$.
By induction, let us first assume $r=1$. We identify

$$f_x(x')=f_0(x)+\sum_{1\leq|\alpha|\leq h}\frac{1}{\alpha!}f_\alpha(x)(x'_1-x_1)^{\alpha_1}...(x'_\ell-x_\ell)^{\alpha_l},\ \text{with}$$

$$f_x(x')=F_0(P(x))+\sum_{1}^{l}F_i\circ P(x)(\sum_{|\alpha|=1}^{k_i}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}p_i}{\partial x^\alpha}(x)(x'-x)^\alpha)$$

where F_i stands for F_β, $\beta_i=1,\beta_j=0$ if $i\neq j$.
For $|\alpha|=k_i$, $\alpha_1\geq\alpha_2\geq...\geq\alpha_n$, we get:

$$f_\alpha=F_i\circ P\frac{\partial^{k_i}p_i}{\partial x^\alpha}+\sum_{s>i}F_s\circ P\frac{\partial^{k_i}p_s}{\partial x^\alpha}. \tag{5.3}$$

In particular for α with $|\alpha|=h$, $\alpha_1\geq\alpha_2\geq...\geq\alpha_n$, we have:

$$f_\alpha=F_n\circ P\frac{\partial^h p_n}{\partial x^\alpha},$$

and since p_n is of degree h, there is a $\partial^h p_n/\partial x^\alpha$ which is not 0. Hence the result for F_n.
Solving the equations (5.3) in succession gives the result for the $F_i\circ P$, $i=1,...,n$.

For more explicit computations, observe that if W is reducible, it would be sufficient to study each irreducible component in each subspace $\mathbb{R}^{n_i}$ and gather the results at the end. For an irreducible component, we may use the polynomial invariants given in [12]. Disregarding D_n for a while, for all the other groups the k_i are distinct and there is an invariant set of real linear forms $\{L_1,\dots,L_v\}$ such that their symmetric functions $\sum_{i=1}^{v}L_i^k$ are W-invariant, and we may take $p_i(X)=\sum_{j=1}^{v}[L_j(X)]^{k_i}$ with k_i as determined in [6]. At least one of the $L_j(X)$ contains a monomial in X_1, bringing in $p_i(X)$ a monomial in $X_1^{k_i}$ that cannot be canceled since the k_i are all even, with 2 exceptions: A_n and $I_2(p)$. For $I_2(p)$ we may choose $p_1(X) = X_1^2 + X_2^2$ and $p_2(X) = \sum_{j=1}^p (X_1 \cos 2j\theta + X_2 \sin 2j\theta)^p$ in which the coefficient of X_1^p is $\sum_{j=1}^p (\cos 2j\theta)^p \neq 0$. For A_n we may take $L_i(X) = X_i, i = 1, \ldots, n+1$ and there is no possible cancelation either. Finally, for D_n if we choose as basic invariant polynomials $p_j(x) = \sum_{i=1}^n x_i^{2j}$, $j = 1, \ldots, n-1$ and $q(x) = x_1 x_2 \ldots x_n$[3], we may use the above method when $1 \leq j \leq n-1$, and consider $\partial^n q/\partial x_1 \ldots \partial x_n = 1$ to get the continuity of $(\partial F/\partial q) \circ P$.

Anyway $\partial^{k_n} p_n/\partial x_1^{k_n} = k_n! c_n$ for some coefficient $c_n \neq 0$, while for $j < n$, $\partial^{k_n} p_j/\partial x_1^{k_n} = 0$, since the greatest exponent of x_1 in $p_j(x)$ is $k_j < k_n$.
The identification shows that $c_n F_n \circ P(x) = (1/k_n!) f_{k_n,0,\ldots,0}(x)$ with $c_n \neq 0$.
Assuming that when $s > i$, the $F_s \circ P$ are linear combinations of f_α, $|\alpha| \leq h$, since $p_i(x)$ contains a monomial in $x_1^{k_i}$, we have $\partial^{k_i} p_i/\partial x_1^{k_i} = k_i! c_i$ for some coefficient $c_i \neq 0$, while as above for $j < i$, $\partial^{k_i} p_j/\partial x_1^{k_i} = 0$. The identification now gives: $(1/k_i!) f_{k_i,0,\ldots,0} = c_i F_i \circ P + \sum_{s>i} F_s \circ P \; (1/k_i!) \; \partial^{k_i} p_s/\partial x_1^{k_i}$.
By using the induction assumption we get the result for $F_i \circ P$, and by decreasing induction for all the $F_j \circ P$, $j = 1, \ldots, n$.

Let us assume that the lemma is true when $r \leq k$. When $|\beta| = k$, $F_\beta \circ P$ is a linear combination of some f_αs with $|\alpha| \leq hk$. By using the basis step of the induction for the function $g(x) = F_\beta \circ P(x) = G \circ P(x)$, we get that the $G_i \circ P$ are linear combinations of $g_\alpha, |\alpha| \leq h$ and using the induction assumption for g we get that for $|\gamma| = k+1$, $F_\gamma \circ P$ is a linear combination of f_αs with $|\alpha| \leq h(k+1)$. This achieves the induction and the proof of the first part of the lemma.

Since $f \in J^{hr}(\mathbb{R}^n)$, the $f_\alpha, |\alpha| \leq hr$ are continuous. Hence the $F_\beta \circ P$ are also continuous for $|\beta| \leq r$ and since P is proper, the F_β themselves are continuous, so that $F \in J^r(P(\mathbb{R}^n))$. $\square$

From the lemma, we get that there exists a constant C_K depending only on K and W, such that $|F|_r^{P(K)} \leq C_K |f|_{hr}^K$. Then, by using the 1-regularity there is a constant C_K^1 such that $\|F\|_r^{P(K)} \leq C_K^1 |f|_{hr}^K$.

Theorem 5.3. *$P^*(\mathcal{E}^r(P(\mathbb{R}^n))$ is closed in $\mathcal{E}^{r,rh}(\mathbb{R}^n)$. P^* is an isomorphism of Fréchet spaces from $\mathcal{E}^r(P(\mathbb{R}^n)$ onto its image.*

The algebra homomorphism: $P^* : \mathcal{E}^r(P(\mathbb{R}^n) \longrightarrow \mathcal{E}^{r,rh}(\mathbb{R}^n)$, is injective and surjective onto its image. The inverse $(P^*)^{-1}$ exists and is continuous by the above inequality.

Let $(f_k)_{k\in N}$ be a Cauchy sequence in $P^*(\mathcal{E}^r(P(\mathbb{R}^n))$. This sequence converges in $\mathcal{E}^{r,rh}(\mathbb{R}^n)$. For each k there exists an $F_k \in \mathcal{E}^r(P(\mathbb{R}^n))$ such that $f_k = F_k \circ P$, and since $\|F\|_{P(K)}^r \leq C_K^1 |f|_K^{hr}$, the sequence $(F_k)_{k\in N}$ is a Cauchy sequence for the topology defined by the semi-norms $\|F\|_{P(K)}^r$. Hence $(F_k)_{k\in N}$ converges to an $F \in \mathcal{E}^r(P(\mathbb{R}^n)$. If we take the limit in $f_k = F_k \circ P$ we see that $f = F \circ P$ is also in $P^*(\mathcal{E}^r(P(\mathbb{R}^n))$. So $P^*(\mathcal{E}^r(P(\mathbb{R}^n))$ is closed in the Fréchet space $\mathcal{E}^{r,rh}(\mathbb{R}^n)$ and then a Fréchet space itself. Then the Banach theorem shows that P^* is a topological isomorphism onto its image.

Remark 5.4. From (5.1), (5.2), we see that a 'truncated' Faa di Bruno formula gives $|f|_{hr}^K \leq C_K^2 |F|_r^{P(K)}$ for some constant C_K^2 that depends on the compact K, *a*

[3]With this choice, the degree of p_n is n. The highest degree $h = 2n-2$ is the degree of p_{n-1}.

fortiori $|f|_{hr}^K \leq C_K^2 \|F\|_r^{P(K)}$. This shows the continuity of the linear mapping P^* directly and without using the Whitney 1-regularity.

Proposition 5.5. *$P^*(\mathcal{E}^r(P(\mathbb{R}^n))$ is closed in $\mathcal{E}^{r,rh}(\mathbb{R}^n)$ if and only if $P(\mathbb{R}^n)$ is Whitney 1-regular.*

By theorem 5.3, the 1-regularity is sufficient for $P^*(\mathcal{E}^r(P(\mathbb{R}^n))$ to be closed. Conversely, let us assume that $P^*(\mathcal{E}^r(P(\mathbb{R}^n))$ is closed and consider a sequence $(F_n)_{n\in N}$ in $\mathcal{E}^1(P(\mathbb{R}^n)$, which is Cauchy for the topology induced by the semi-norms $|.|_1^{P(K)}$. The sequence $(f_n = F_n \circ P)_{n\in N}$ in $P^*(\mathcal{E}^1(P(\mathbb{R}^n)) \subset \mathcal{E}^{1,h}(\mathbb{R}^n)$ which is also Cauchy by Faa di Bruno's formula, converges to an f in $P^*(\mathcal{E}^1(P(\mathbb{R}^n))$ which is closed by assumption. As a consequence, the limit F of $(F_n)_{n\in N}$ with $f = F\circ P$ is in $\mathcal{E}^1(P(\mathbb{R}^n)$. Thus $\mathcal{E}^1(P(\mathbb{R}^n))$ is complete for the topology induced by the semi-norms $|.|_1^{P(K)}$ and the Banach isomorphism theorem shows that this topology is equivalent to the topology induced by the semi-norms $\|.\|_1^{P(K)}$. Glaeser's proposition 3.5 then shows that $P(\mathbb{R}^n)$ is Whitney 1-regular.

Remark 5.6. Finally, one might wish to prove the 1-regularity of $P(\mathbb{R}^n)$ by using Glaeser's proposition 3.5. By Banach theorem, we just have to prove that $\mathcal{E}^1(P(\mathbb{R}^n))$ is complete for the topology induced by the semi-norms $|.|_1^{P(K)}$. So we consider a sequence $(F_n)_{n\in N} \subset \mathcal{E}^1(P(\mathbb{R}^n)$ which is Cauchy for the topology induced by these semi-norms. The sequence $(f_n = F_n \circ P)_{n\in N}$ is Cauchy in $\mathcal{E}^{1,h}(\mathbb{R}^n)$ since as we already noticed $|f|_h^K \leq C_K^2 |F|_1^{P(K)}$, hence (f_n) converges to an $f \in \mathcal{E}^{1,h}(\mathbb{R}^n)$, which is of the form $F \circ P$ with $F = \lim_n F_n$.

The function f induces a formally holomorphic jet still denoted by $f \in \mathcal{H}^{1,h}(\mathbb{R}^n)$ (see [14] and [4]). If there was an extension $\tilde{f}$ to $\mathcal{H}^{1,h}(P^{-1}(\mathbb{R}^n))$ of the form $\tilde{f} = \tilde{F} \circ P$, $\tilde{F}$ identified with F on $P(\mathbb{R}^n)$, would be of class $\mathcal{C}^1$ on the regular image of P in $\mathbb{R}^n$, and 1-continuous everywhere by 5.2. Since the critical image is contained in the null set of the discriminant polynomial, $\tilde{F}$ would be of class $\mathcal{C}^1$ everywhere in $\mathbb{R}^n$, and in particular $F \in \mathcal{E}^1(P(\mathbb{R}^n))$. Unfortunately, no such extension is available. As mentioned above, Whitney's extension does not take into account the part of the jet f beyond the degree r. It is possible to provide a linear and continuous version of Łojasiewicz extension operator (see [14]) that would give an $\tilde{f} \in \mathcal{H}^{1,h}(P^{-1}(\mathbb{R}^n))$, but it is not an algebra homomorphism and $\tilde{f}$ would not be of the form $\tilde{F} \circ P$ with an $\tilde{F}$ of degree 1 and lemma 5.2 does not apply.

Appendix A. Whitney 1-Regularity of $P(R^n)$.

The Whitney 1-regularity of the image $P(\mathbb{R}^n)$ of the Chevalley mapping associated with any finite reflection group is a result which is of interest by itself.

In [7], theorem A.1 below was stated but lemma A.2 was not proved for all Coxeter groups.

Since the 1-regularity is not altered by diffeomorphism, it does not depend on the choice of coordinates. It does not depend on the choice of the set of basic invariants either, since a change of basic invariants is an invertible polynomial map on the target.

When W is reducible we choose coordinates such that the Chevalley map P is the product of the Chevalley maps P^is associated with the irreducible components W^is of W acting on the subspaces $\mathbb{R}^{n_i}$ of $\mathbb{R}^n$. If for each i, $P^i(\mathbb{R}^{n_i})$ is 1-regular, so is $P(\mathbb{R}^n)$. As a consequence it is sufficient to prove the regularity when W is irreducible and from now on, we will assume W to be a finite Coxeter group.

A.1. **Strata of $P(\mathbb{R}^n)$ and minors of the jacobian.** Let C be a Weyl Chamber. The walls of $\bar{C}$ are contained in n hyperplanes $(H_\omega)_{\omega\in\Omega}$, where Ω is a subset of cardinal n in $\mathcal{R}$.

A stratum S of dimension k in $\bar{C}$ is the intersection of $(n-k)$ of the H_ω. The λ_τ that are linear combinations of the λ_ω vanishing on S will also vanish there, so that $p \geq n-k$ hyperplanes $H_\tau, \tau \in \mathcal{R}$ will intersect along S.

The points in S have the same isotropy subgroup W_S, generated by the reflections in the p hyperplanes H_τ containing S. In a neighborhood of S, since P is W_S invariant we can write $P = Q \circ V$, where Q is invertible and V is a Chevalley mapping for W_S.
We write $W_S = W^0 \times W^1 \times \cdots \times W^\ell$ where W^0 is the identity on S and the W^is are the irreducible components of W_S. If we choose coordinates fitted to the orthogonal direct sum $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n_1} \oplus \cdots \oplus \mathbb{R}^{n_\ell}$, and we have $V = \mathrm{Id}_k \times V_1 \times \cdots \times V_\ell$.
The equation of an H_τ containing S depends of the $x_{k+1}, x_{k+2}, \ldots, x_n$ but perhaps not all of them. There is a partition among the x_is, $k+1 \leq i \leq n$ and a corresponding one among the λ_τ. Let $r < s$, x_{k+r} and x_{k+s} are equivalent if there is a sequence $(x_{k+\ell}), k+r < k+\ell < k+s$ and forms λ_τ such that any two consecutive $x_{k+\ell}$ appear in the equation of a form λ_τ. Two forms λ_τ are equivalent if two equivalent x_{k+r} appear in their equations. In each class, the H_τ are the reflecting hyperplanes of an irreducible component of W_S.
The jacobian matrix J_V is block diagonal. Let $I_k, J_{V_1}, \ldots, J_{V_\ell}$ be the diagonal blocks, the determinants $|J_{V_1}|, \ldots, |J_{V_\ell}|$, all vanish on S. The $k \times k$ minor $|I_k|$ in the upper left corner is 1 but all the bordering $(k+1) \times (k+1)$ minors vanish on S.

When restricted to a stratum S, $P_{|S} = Q \circ V_{|S}$ is an analytic isomorphism on its image and onto each of its projections on the spaces $\mathbb{R}^\ell$, $k \leq \ell \leq n-1$. The closure $\overline{S}$ of S, is a fundamental domain for a subgroup of W generated by reflections in hyperplanes that do not contain S, and Q is a polynomial mapping, invariant by this subgroup. On $\overline{S}$, $P = Q$ is an homeomorphism onto $P(\overline{S})$ and each of its projections on the spaces $\mathbb{R}^\ell$.

A.2. **The varieties $P_k^{-1}[P_k(x)]$.** We set $P_k = (p_1, p_2, \ldots, p_k) : \mathbb{R}^n \to \mathbb{R}^k$, and analogously $Q_k = (q_1, q_2, \ldots, q_k)$. We denote with Π_k the image $P_k(\mathbb{R}^n)$.

Let $m^k \in \Pi_k$, then

$$P_k^{-1}(m^k) = \{x \in \mathbb{R}^n; p_1(x) = m_1, \ldots, p_k(x) = m_k\}$$

is a compact algebraic variety of co-dimension k.

Theorem A.1. [7] *For almost all m^k, the intersection of a fundamental domain $\bar{C}$ and $P_k^{-1}(m^k)$ is either empty or contractible.*

By Sard's lemma, for almost all m^k, the variety $P_k^{-1}(m^k)$ is a non singular manifold. Of course if m^k is on the border of Π_k, the variety is singular, but even if m^k belongs to the interior of Π_k, it may be the image of both regular and singular points.

Lemma A.2. *Let $P_k^{-1}(m^k)$ be a non singular manifold. On this manifold, p_{k+1} is a Morse function with no critical point on the strata of positive dimension in $P_k^{-1}(m^k) \cap \bar{C}$. Its critical points are the zero dimensional strata of the intersection.*

Proof. On the non critical manifold $P_k^{-1}(m^k)$, there is a $k \times k$ minor of the matrix $\partial(p_1, \ldots, p_k)/\partial(x_1, \cdots, x_n)$ that does not vanish. We may assume that it is the principal minor $\partial(p_1, \ldots, p_k)/\partial(x_1, \ldots, x_k)$ which is not 0.

The critical points of p_{k+1} on $P_k^{-1}(m^k)$ are the points where:

$$\frac{\partial p_{k+1}}{\partial x_i} - \lambda_k \frac{\partial p_k}{\partial x_i} - \ldots - \lambda_1 \frac{\partial p_1}{\partial x_i} = 0, \quad \text{for } i = 1, \ldots, n.$$

For this system to have solutions in λ_is, $1 \leq i \leq k$, the $(k+1) \times (k+1)$ bordering minors $\partial(p_1, \ldots, p_k, p_{k+1})/\partial(x_1, \ldots, x_k, x_{k+s})$, $s = 1, \ldots, n-k$ must all be zero. Therefore the jacobian and all minors of order $\geq (k+1) \times (k+1)$ are 0 and the rank of P is k at such points. These points belong to strata S of dimension k in $\bar{C}$. The critical points are the points of intersection of $P_k^{-1}(m^k)$ and strata S; they are points of 0-dimensional strata in $P_k^{-1}(m^k) \cap \bar{C}$.

If $P_k^{-1}(m^k)$ is not singular, strata of lower dimension do not intersect it since on such a stratum all $k \times k$ minors vanish. On strata of dimension $> k$, one of the $(k+1) \times (k+1)$ minors does not vanish and there may not be critical points.

Let us show that the critical points of p_{k+1} on $P^{-1}(m^k)$ are non degenerate. At a critical point

$$\frac{\partial p_{k+1}}{\partial x_i} - \lambda_k \frac{\partial p_k}{\partial x_i} - \ldots - \lambda_1 \frac{\partial p_1}{\partial x_i} = \sum_j \Big(\frac{\partial q_{k+1}}{\partial v_j} - \lambda_k \frac{\partial q_k}{\partial v_j} - \ldots - \lambda_1 \frac{\partial q_1}{\partial v_j} \Big) \Big(\frac{\partial v_j}{\partial x_i} \Big) = 0.$$

When $1 \leq i \leq k$, $\partial v_j/\partial x_i = 1$ if $i = j$, and $= 0$ otherwise. The only terms remaining in the sum are the $(\partial q_{k+1}/\partial v_i - \lambda_k \partial q_k/\partial v_i - \ldots - \lambda_1 \partial q_1/\partial v_i)$ and these are 0.
When $k+1 \leq i \leq n$, $\partial v_j/\partial x_i = 0$ either because v_j does not depend on x_i or is an homogeneous polynomial of degree ≥ 2 that vanishes with its derivatives on S, but the $(\partial q_{k+1}/\partial v_i - \lambda_k \partial q_k/\partial v_i - \ldots - \lambda_1 \partial q_1/\partial v_i)$ are not all 0, since Q is invertible.

Actually the vectors $(\partial P/\partial x_1, \ldots, \partial P/\partial x_k) = (\partial Q/\partial x_1, \ldots, \partial Q/\partial x_k)$ define the tangent plane to $P(S)$ of dimension k. The $\partial P/\partial x_s, k+1 \leq s \leq n$ are linear combinations of them, hence the vanishing of the bordering minors. The $\partial Q/\partial x_s$, $k+1 \leq s \leq n$, however, span the complement of the tangent space, and they are linearly independent. As a consequence the minors of order $\geq k+1$ in the jacobian of Q do not vanish, and for $i \geq k+1$, $\partial q_{k+1}/\partial v_i - \lambda_k \partial q_k/\partial v_i - \ldots - \lambda_1 \partial q_1/\partial v_i \neq 0$.

In restriction to the kernel of the first differential, which is the orthogonal of S, in the quadratic differential

$$\begin{aligned} &\frac{\partial^2 p_{k+1}}{\partial x_i \partial x_j} - \lambda_k \frac{\partial^2 p_k}{\partial x_i \partial x_j} - \ldots - \lambda_1 \frac{\partial^2 p_1}{\partial x_i \partial x_j} \\ = &\sum_{r,s \geq k+1} \Big(\frac{\partial^2 q_{k+1}}{\partial v_r \partial v_s} - \lambda_k \frac{\partial^2 q_k}{\partial v_r \partial v_s} - \ldots - \lambda_1 \frac{\partial^2 q_1}{\partial v_r \partial v_s}\Big)\Big(\frac{\partial v_r}{\partial x_i}\Big)\Big(\frac{\partial v_s}{\partial x_j}\Big) \\ &+ \sum_{r \geq k+1} \Big(\frac{\partial q_{k+1}}{\partial v_r} - \lambda_k \frac{\partial q_k}{\partial v_r} - \ldots - \lambda_1 \frac{\partial q_1}{\partial v_r}\Big)\Big(\frac{\partial^2 v_r}{\partial x_i \partial x_j}\Big) \end{aligned}$$

the mixed derivatives are all 0 either because v_j does not depend on x_i or x_j or is an homogeneous polynomial of degree > 2 or a sum of squares of x_is. For the same reasons many terms in the pure derivatives also vanish, but some do not. In each $V_s : \mathbb{R}^{n_s} \to \mathbb{R}^{n_s}$, let v_s^1 be the first W^s-invariant which is the sum of the squares of the x_i in $\mathbb{R}^{n_s}$. By the above remark, $\partial q_{k+1}/\partial v_s^1 - \lambda_k \partial q_k/\partial v_s^1 - \ldots - \lambda_1 \partial q_1/\partial v_s^1$ does not vanish, and for each $x_i \in \mathbb{R}^{n_s}$ of which v_s^1 actually depends, we have $\partial^2 v_s^1/\partial x_i^2 = 2$, so that:

$$\frac{\partial^2 p_{k+1}}{\partial x_i^2} - \lambda_k \frac{\partial^2 p_k}{\partial x_i^2} - \ldots - \lambda_1 \frac{\partial^2 p_1}{\partial x_i^2} = 2(\frac{\partial q_{k+1}}{\partial v_s^1} - \lambda_k \frac{\partial q_k}{\partial v_s^1} - \ldots - \lambda_1 \frac{\partial q_1}{\partial v_s^1}) \neq 0$$

Accordingly $p_{k+1} = q_{k+1} \circ V$ is a Morse function on $P_k^{-1}(m^k)$.
Observe that for each irreducible component, the quadratic differential is definite with the sign of $\partial q_{k+1}/\partial v_s^1 - \lambda_k \partial q_k/\partial v_s^1 - \ldots - \lambda_1 \partial q_1/\partial v_s^1$, but the sign may be different for different irreducible components.
By the equivariant Morse lemma [2], in the neighborhood of a critical point at the intersection of S and $P_k^{-1}(m^k)$, p_{k+1} is W_S-locally equivalent to a W_S-invariant quadratic form which is the direct sum of definite quadratic forms in each $\mathbb{R}^{n_i}$. $\square$

Lemma A.3. [7] *The reconstruction of the topology of a level set of a function on $\mathbb{R}_+^a \oplus \mathbb{R}_+^b$ in the neighborhood of the critical point $0 \oplus 0$ with the quadratic differential $Q_+ \oplus Q_-$ is trivial if $a, b > 0$ and consists of the birth (death) of a simplex otherwise.*

Theorem A.1 may now be proved by induction on k, see [7].

In particular, for almost all $m^k \in \Pi_k$, $P_k^{-1}(m^k) \cap \bar{C}$ is connected.

Corollary A.4. *Every variety $P_k^{-1}(m^k) \cap \bar{C}$, $k = 1, \ldots, n$, is connected or empty.*

The corollary may be derived from A.1, exactly as in [11]. Basically if m^k is not in the regular image, we have to consider two cases. First if for some j, $1 < j < k$, $m^j = (m_1, m_2, \ldots, m_j)$ belongs to the projection of a stratum of dimension less than j but m^k itself does not belong to the projection of any stratum of dimension less than k, then $P_k^{-1}(m^k)$ is connected since almost all the $P_k^{-1}(n^k)$ are, for n^k close enough to m^k. Then there is the case when m^k is on the projection of a stratum of dimension less than k. For a given m^{k-1} this may happen but for a finite number of m^k. Since $P_1^{-1}(m^1)$ is connected, this case is taken care of by an induction, using the following:

Lemma A.5. [11] *Let C be a connected compact set in $\mathbb{R}^n$, and f be a real valued function continuous on C. If all but a finite number of the level sets of f in C are connected, then they are all connected.*

As a consequence, $P(P_k^{-1}(m^k))$ is connected or empty and in particular, we have:

Corollary A.6. *The fibres of the projections: $\Pi_{k+1} \to \Pi_k$ are connected: they are points or intervals.*

A.3. **End of the proof.** Let us consider the lift-up $(p_1, \ldots, p_k) \mapsto p_{k+1}$ of the projection $P_k(S) \subset \Pi_k$ of a stratum of dimension k. The derivatives are obtained by solving the system:

$$\frac{\partial p_{k+1}}{\partial x_i} = \sum_{j=1}^{k} \frac{\partial p_{k+1}}{\partial p_j} \frac{\partial p_j}{\partial x_i}, \quad i = 1, \ldots, k$$

by Cramer's method. Considering the subgroup of W generated by the reflections in the hyperplanes that do not contain S, for $j = 1, \ldots, k$ the $\partial p_{k+1}/\partial p_j$, are

quotients of two polynomials anti-invariant by this subgroup. So, they are rational fractions the numerator and the denominator of which are invariant homogeneous polynomials that do not vanish but at the origin. Since the degree of the numerator is greater than the degree of the denominator the rational fractions have continuous extensions on $\bar{S}$.
If we restrict ourselves to some compact subset K of $P(\mathbb{R}^n)$, determined for instance by $p_1 \leq a,\ a > 0$, the x_is and as a consequence the $\partial p_{k+1}/\partial p_j,\ j = 1, \ldots, k$ are bounded on $\overline{S} \cap K$.
P is a homeomorphism of $\bar{S}$ onto its image $P(\bar{S})$, and so is P_k (on any compact it is continuous and one to one). Hence p_{k+1} which is continuous with respect to the variables $(x_1, \ldots, x_k)$ on $\bar{S}$, is also continuous in the variables $(p_1, \ldots, p_k)$ on $P_k(\bar{S})$ and moreover by the previous paragraph it is Lipschitz.

The border of Π_{k+1} is contained in the images $P_{k+1}(\bar{S})$ of closed strata of dimension k. These images are graphs of functions p_{k+1} on $P_k(\bar{S})$. By A.6, the graph of one of the p_{k+1}, say $p_{k+1}^{\max}$, is above the others and another, say the graph of $p_{k+1}^{\min}$, is below the others. In Π_k, the images of the closure of strata of dimension k will intersect along the images of strata of lesser dimension. Above such points of intersection the mapping $p_{k+1}^{\max}$ (resp. $p_{k+1}^{\min}$) may and will change but globally $p_{k+1}^{\max}$ (resp. $p_{k+1}^{\min}$) will still be continuous and Lipschitz since the functions above the two strata are glued by their common value above the stratum of lesser dimension along which they intersect.

Now, it would be easy to get for any two points A and B in a compact $K \subset P(\mathbb{R}^n)$ a continuous arc $AB \subset K$ of length $\ell(AB) \leq k_K|AB|$, following the method in [11] and already in [3]. This justifies the statement in [1]: *the prism between graphs of Lipschitz functions over a Whitney 1-regular domain is Whitney 1-regular.*

Π_1 is Whitney 1-regular. By induction, assuming Π_k to be Whitney 1-regular, the prism Π_{k+1} over it and between the graphs of the Lipschitz functions $p_{k+1}^{\min}$ and $p_{k+1}^{\max}$ will also be Whitney 1-regular. Hence all the Π_k and in particular $\Pi_n = P(\mathbb{R}^n)$ are Whitney 1-regular and we can state:

Theorem A.7. *The image of the Chevalley mapping $P(\mathbb{R}^n)$ is Whitney 1-regular.*

References

1. V.I. Arnold, Hyperbolic polynomials and Vandermonde mappings, Funct. Anal. Appl. **20** (2) (1986) 125-128.
2. V.I. Arnold, Wave fronts evolution and equivariant Morse lemma, Comm.Pure Appl. Math.**29** (1976), 557-582.
3. G. Barbançon, À propos du théorème de Newton pour les fonctions de classe $\mathcal{C}^m$, Ann. Fac. Sci. P.Penh (1969) 7-32.
4. G. Barbançon, Chevalley Theorem in class $\mathcal{C}^r$, Proc. Roy. Edinburgh **139A** (2009) 743-758.
5. C. Chevalley, Invariants of finite groups generated by reflections. *Amer. J. Math.* **77** (1955), 778-782.
6. H.S.M. Coxeter, The product of the generators of a finite group generated by reflections. *Duke Math. J.* **18** (1951) 765-782
7. A.B. Givental', Moments of random variables and the equivariant Morse lemma, Russian Math. Surveys, **42** (2) (1987) 275-276.
8. G. Glaeser, Étude de quelques algèbres tayloriennes. *J. Analyse Math.* **6** (1958), 1-124.
9. G. Glaeser, Fonctions composées différentiables. *Ann. of Math.*, **77** (1963), 193-209.
10. G.Glaeser, Multiplicateurs rugueux des fonctions différentiables et synthèse spectrale*Ann. Sci. École Norm. Sup.* (3) **79** (1962), 243-261.
11. V.P. Kostov, On the geometric properties of Vandermonde's mapping and on the problem of moments, Proc. Roy. Soc. Edinburgh, **112A** (1989), 203-211.

12. M.L.Mehta. Basic set of invariant polynomials for finite reflexion groups. *Commun.Alg.* **16**(1988), 1083-1098.
13. G.C.Shephard and J.A.Todd, Finite unitary reflection groups. *Canad.J.Math.* **6** (1954), 274-304.
14. J.C. Tougeron, Idéaux de fonctions différentiables. *Ergeb. Math. Grenzgeb.* **71** Springer (1972).
15. H. Whitney, Analytic extensions of differentiable functions defined in closed sets. *Trans. Amer. Math. Soc.* **36** (1934), 63-89.
16. H. Whitney, Functions differentiable on the boundaries of regions. *Ann. of Math.* **35** (1934), 482-485.

University of Texas at Austin
E-mail address: gbarbanson@yahoo.com