SPECTRAL PROPERTIES OF A LIMIT-PERIODIC SCHRÖDINGER OPERATOR IN DIMENSION TWO.

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Abstract. We study Schrödinger operator $H = -\Delta + V(x)$ in dimension two, $V(x)$ being a limit-periodic potential. We prove that the spectrum of H contains a semiaxis and there is a family of generalized eigenfunctions at every point of this semiaxis with the following properties. First, the eigenfunctions are close to plane waves $e^{i\langle \vec{k}, \vec{x}\rangle}$ at the high energy region. Second, the isoenergetic curves in the space of momenta $\vec{k}$ corresponding to these eigenfunctions have a form of slightly distorted circles with holes (Cantor type structure). Third, the spectrum corresponding to the eigenfunctions (the semiaxis) is absolutely continuous.

1. Introduction.

We study the operator

$$H = -\Delta + V(x) \tag{1.1}$$

in two dimensions, $V(x)$ being a limit-periodic potential:

$$V(x) = \sum_{r=1}^{\infty} V_r(x), \tag{1.2}$$

where $\{V_r\}_{r=1}^{\infty}$ is a family of periodic potentials with doubling periods and decreasing L_∞-norms. Namely, V_r has orthogonal periods $2^{r-1}\vec{d_1},\ 2^{r-1}\vec{d_2}$ and $\|V_r\|_\infty < \hat{C}exp(-2^{\eta r})$ for some $\eta > \eta_0 > 0$. Without loss of generality we assume that $\hat{C} = 1$, $\vec{d_1} = (d_1, 0)$, $\vec{d_2} = (0, d_2)$ and $\int_{Q_r} V_r(x)dx = 0$, Q_r being the elementary cell of periods corresponding to $V_r(x)$. We assume that all $V_r(x)$ are trigonometric polynomials with the lengths growing at most linearly with period. Namely, there exists a positive number $R_0 < \infty$, such that each potential admits Fourier representation:

$$V_r(x) = \sum_{q \in \mathbb{Z}^2 \setminus \{0\},\ 2^{-r+1}|q| < R_0} v_{r,q} \exp i\langle 2^{-r+1}\tilde{q}, x\rangle, \quad \tilde{q} = 2\pi\left(\frac{q_1}{d_1}, \frac{q_2}{d_2}\right),$$

$\langle \cdot, \cdot \rangle$ being the canonical dot product in $\mathbb{R}^2$.

The one-dimensional analog of (1.1), (1.2) is already thoroughly investigated. It is proven in [1]–[7] that the spectrum of the operator $H_1 u = -u'' + Vu$ is generically a Cantor type set. It has a positive Lebesgue measure [1, 6]. The spectrum is absolutely continuous [1, 2], [5]-[9]. Generalized eigenfunctions can be represented

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in the form of $e^{ikx}u(x)$, $u(x)$ being limit-periodic [5, 6, 7]. The case of a complex-valued potential is studied in [10]. Integrated density of states is investigated in [11]-[14]. Spectral properties of Schrödinger operators in $l^2(\mathbb{Z})$ with limit-periodic potentials are recently investigated in [15]. It regards such potentials as generated by continuous sampling along the orbits of a minimal translation of a Cantor group. It is shown that the spectrum is a Cantor set of positive Lebesgue measure and purely absolutely continuous for a dense set of sampling functions, and it is a Cantor set of zero Lebesgue measure and purely singular continuous for a dense G_δ set of sampling functions. Properties of eigenfunctions of discrete multidimensional limit-periodic Schrödinger operator are studied in [16]. As to the continuum multidimensional case, it is proven in [14] that the integrated density of states for (1.1) is the limit of densities of states for periodic operators.

We concentrate here on properties of the spectrum and eigenfunctions of (1.1), (1.2) in the high energy region. We prove the following results for the two-dimensional case.

(1) The spectrum of the operator (1.1), (1.2) contains a semiaxis. A proof of an analogous result by different means can be found in the paper [17]. In [17], the authors consider the operator $H=(-\Delta)^l+V$, $8l>d+3$, $d\neq 1(\mathrm{mod}4)$, d being the dimension of the space. This obviously includes our case $l=1$, $d=2$. However, there is an additional rather strong restriction on the potential $V(x)$ in [17], which we don't have here: in [17] all the lattices of periods Q_r of periodic potentials V_r need to contain a nonzero vector γ in common, i.e., V(x) is periodic in a direction γ.

(2) There are generalized eigenfunctions $\Psi_\infty(\vec{k},\vec{x})$, corresponding to the semiaxis, which are close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_\infty$ of $\mathbb{R}^2$, there is a solution $\Psi_\infty(\vec{k},\vec{x})$ of the equation $H\Psi_\infty=\lambda_\infty\Psi_\infty$ which can be described by the formula:

$$\Psi_\infty(\vec{k},\vec{x})=e^{i\langle\vec{k},\vec{x}\rangle}\left(1+u_\infty(\vec{k},\vec{x})\right), \tag{1.3}$$

$$\|u_\infty\|=_{|\vec{k}|\to\infty}O(|\vec{k}|^{-\gamma_1}),\quad \gamma_1>0, \tag{1.4}$$

where $u_\infty(\vec{k},\vec{x})$ is a limit-periodic function:

$$u_\infty(\vec{k},\vec{x})=\sum_{r=1}^{\infty}u_r(\vec{k},\vec{x}), \tag{1.5}$$

$u_r(\vec{k},\vec{x})$ being periodic with periods $2^{r-1}\vec{d_1}$, $2^{r-1}\vec{d_2}$. The eigenvalue $\lambda_\infty(\vec{k})$ corresponding to $\Psi_\infty(\vec{k},\vec{x})$ is close to $|\vec{k}|^2$:

$$\lambda_\infty(\vec{k})=_{|\vec{k}|\to\infty}|\vec{k}|^2+O(|\vec{k}|^{-\gamma_2}),\quad \gamma_2>0. \tag{1.6}$$

The "non-resonant" set $\mathcal{G}_\infty$ of the vectors $\vec{k}$, for which (1.3) – (1.6) hold, is an extensive Cantor type set: $\mathcal{G}_\infty=\cap_{n=1}^{\infty}\mathcal{G}_n$, where $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in $\mathbb{R}^2$. Each $\mathcal{G}_n$ has a finite number of holes in each bounded region. More and more holes appears when n increases, however holes added at each step are of smaller and smaller size. The set $\mathcal{G}_\infty$ satisfies the estimate:

$$\frac{|(\mathcal{G}_\infty \cap \mathbf{B_R})|}{|\mathbf{B_R}|} =_{R\to\infty} 1 + O(R^{-\gamma_3}), \quad \gamma_3 > 0, \tag{1.7}$$

where $\mathbf{B_R}$ is the disk of radius R centered at the origin, $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^2$.

(3) The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\vec{k})$,

$$\mathcal{D}_\infty(\lambda) = \left\{\vec{k} \in \mathcal{G}_\infty : \lambda_\infty(\vec{k}) = \lambda\right\},$$

is proven to be a slightly distorted circle with the infinite number of holes. It can be described by the formula:

$$\mathcal{D}_\infty(\lambda) = \{\vec{k} : \vec{k} = \varkappa_\infty(\lambda, \vec{\nu})\vec{\nu},\ \vec{\nu} \in \mathcal{B}_\infty(\lambda)\}, \tag{1.8}$$

where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_\infty(\lambda)$ can be interpreted as the set of possible directions of propagation for the almost plane waves (1.3). The set $\mathcal{B}_\infty(\lambda)$ has a Cantor type structure and an asymptotically full measure on S_1 as $\lambda \to \infty$:

$$L\left(\mathcal{B}_\infty(\lambda)\right) =_{\lambda\to\infty} 2\pi + O\left(\lambda^{-\gamma_3/2}\right), \tag{1.9}$$

here and below $L(\cdot)$ is the length of a curve. The value $\varkappa_\infty(\lambda, \vec{\nu})$ in (1.8) is the "radius" of $\mathcal{D}_\infty(\lambda)$ in a direction $\vec{\nu}$. The function $\varkappa_\infty(\lambda, \vec{\nu}) - \lambda^{1/2}$ describes the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect circle of the radius $\lambda^{1/2}$. It is proven that the deviation is asymptotically small:

$$\varkappa_\infty(\lambda, \vec{\nu}) =_{\lambda\to\infty} \lambda^{1/2} + O\left(\lambda^{-\gamma_4}\right), \quad \gamma_4 > 0. \tag{1.10}$$

(4) Absolute continuity of the branch of the spectrum (the semiaxis) corresponding to $\Psi_\infty(\vec{k}, \vec{x})$ is proven.

To prove the results listed above we develop a modification of the Kolmogorov-Arnold-Moser (KAM) method. This paper is inspired by [18, 19, 20], where the method is used for periodic problems. In [18] KAM method is applied to classical Hamiltonian systems. In [19, 20] the technique developed in [18] is applied to semi-classical approximation for multidimensional periodic Schrödinger operators at high energies.

We consider a sequence of operators

$$H_0 = -\Delta, \qquad H^{(n)} = H_0 + \sum_{r=1}^{M_n} V_r, \quad n \geq 1,\ M_n \to \infty \text{ as } n \to \infty.$$

Obviously, $\|H - H^{(n)}\| \to 0$ as $n \to \infty$, where $\|\cdot\|$ is the norm in the class of bounded operators. Clearly,

$$H^{(n)} = H^{(n-1)} + W_n, \quad W_n = \sum_{r=M_{n-1}+1}^{M_n} V_r. \tag{1.11}$$

We consider each operator $H^{(n)}$, $n \geq 1$, as a perturbation of the previous operator $H^{(n-1)}$. Every operator $H^{(n)}$ is periodic, however the periods go to infinity as $n \to$

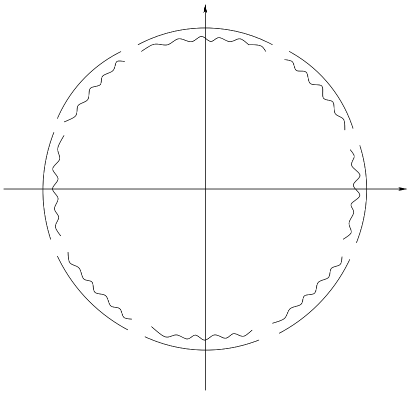

FIGURE 1. Distorted circle with holes, $\mathcal{D}_1(\lambda)$.

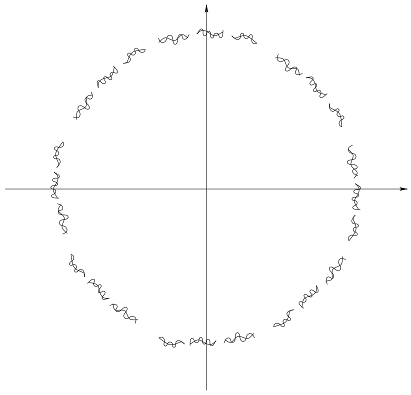

FIGURE 2. Distorted circle with holes, $\mathcal{D}_2(\lambda)$.

∞. We show that there is a λ_*, $\lambda_* = \lambda_*(V)$, such that the semiaxis $[\lambda_*, \infty)$ is contained in the spectra of **all** operators $H^{(n)}$. For every operator $H^{(n)}$ there is a set of eigenfunctions (corresponding to the semiaxis) being close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_n$ of $\mathbb{R}^2$, there is a solution $\Psi_n(\vec{k}, \vec{x})$ of the differential equation $H^{(n)}\Psi_n = \lambda_n \Psi_n$, which can be described by the formula:

$$\Psi_n(\vec{k}, \vec{x}) = e^{i\langle \vec{k}, \vec{x}\rangle}\left(1 + \tilde{u}_n(\vec{k}, \vec{x})\right), \quad \|\tilde{u}_n\|_{L_\infty(\mathbb{R}^2)} \underset{|\vec{k}|\to\infty}{=} O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1 > 0, \tag{1.12}$$

where $\tilde{u}_n(\vec{k}, \vec{x})$ has periods $2^{M_n - 1}\vec{d}_1, 2^{M_n - 1}\vec{d}_2$.[1] The corresponding eigenvalue $\lambda^{(n)}(\vec{k})$ is close to $|\vec{k}|^2$:

$$\lambda^{(n)}(\vec{k}) =_{|\vec{k}|\to\infty} |\vec{k}|^2 + O\left(|\vec{k}|^{-\gamma_2}\right), \quad \gamma_2 > 0. \tag{1.13}$$

The non-resonant set $\mathcal{G}_n$ for which (1.13) holds, is proven to be extensive in $\mathbb{R}^2$:

$$\frac{|\mathcal{G}_n \cap \mathbf{B_R}|}{|\mathbf{B_R}|} =_{R\to\infty} 1 + O(R^{-\gamma_3}). \tag{1.14}$$

Estimates (1.12) – (1.14) are uniform in n. The set $\mathcal{D}_n(\lambda)$ is defined as the level (isoenergetic) set for non-resonant eigenvalue $\lambda^{(n)}(\vec{k})$:

$$\mathcal{D}_n(\lambda) = \left\{\vec{k} \in \mathcal{G}_n : \lambda^{(n)}(\vec{k}) = \lambda\right\}.$$

This set is proven to be a slightly distorted circle with a finite number of holes (Fig. 1, 2). The set $\mathcal{D}_n(\lambda)$ can be described by the formula:

$$\mathcal{D}_n(\lambda) = \{\vec{k} : \vec{k} = \varkappa_n(\lambda, \vec{\nu})\vec{\nu},\ \vec{\nu} \in \mathcal{B}_n(\lambda)\}, \tag{1.15}$$

where $\mathcal{B}_n(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_n(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (1.12). It is shown that $\{\mathcal{B}_n(\lambda)\}_{n=1}^{\infty}$ is a decreasing sequence of sets, since on each step more and more

[1] Obviously, $\tilde{u}_n(\vec{k}, \vec{x})$ is simply related to functions $u_r(\vec{k}, \vec{x})$ used in (1.5): $\tilde{u}_n(\vec{k}, \vec{x}) = \sum_{r=M_{n-1}+1}^{M_n} u_r(\vec{k}, \vec{x})$.

directions are excluded. Each $\mathcal{B}_n(\lambda)$ has an asymptotically full measure on S_1 as $\lambda \to \infty$:

$$L\left(\mathcal{B}_n(\lambda)\right) =_{\lambda\to\infty} 2\pi + O\left(\lambda^{-\gamma_3/2}\right), \tag{1.16}$$

the estimate being uniform in n. The set $\mathcal{B}_n$ has only a finite number of holes, however their number is growing with n. More and more holes of a smaller and smaller size are added at each step. The value $\varkappa_n(\lambda, \vec{\nu}) - \lambda^{1/2}$ gives the deviation of $\mathcal{D}_n(\lambda)$ from the perfect circle of the radius $\lambda^{1/2}$ in the direction $\vec{\nu}$. It is proven that the deviation is asymptotically small uniformly in n:

$$\varkappa_n(\lambda, \vec{\nu}) = \lambda^{1/2} + O\left(\lambda^{-\gamma_4}\right), \quad \frac{\partial \varkappa_n(\lambda, \vec{\nu})}{\partial \varphi} = O\left(\lambda^{-\gamma_5}\right), \quad \gamma_4, \gamma_5 > 0, \tag{1.17}$$

φ being an angle variable $\vec{\nu} = (\cos\varphi, \sin\varphi)$.

On each step more and more points are excluded from the non-resonant sets $\mathcal{G}_n$, thus $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets. The set $\mathcal{G}_\infty$ is defined as the limit set: $\mathcal{G}_\infty = \cap_{n=1}^{\infty}\mathcal{G}_n$. It has the infinite number of holes at each bounded region, but nevertheless satisfies the relation (1.7). For every $\vec{k} \in \mathcal{G}_\infty$ and every n, there is a generalized eigenfunction of $H^{(n)}$ of the type (1.12). It is proven that the sequence of $\Psi_n(\vec{k}, \vec{x})$ has a limit in $L_\infty(\mathbb{R}^2)$ as $n \to \infty$, when $\vec{k} \in \mathcal{G}_\infty$. The function $\Psi_\infty(\vec{k}, \vec{x}) = \lim_{n\to\infty} \Psi_n(\vec{k}, \vec{x})$ is a generalized eigenfunction of H. It can be written in the form (1.3) – (1.5). Naturally, the corresponding eigenvalue $\lambda_\infty(\vec{k})$ is the limit of $\lambda^{(n)}(\vec{k})$ as $n \to \infty$.

We consider the limit $\mathcal{B}_\infty(\lambda)$ of $\mathcal{B}_n(\lambda)$:

$$\mathcal{B}_\infty(\lambda) = \bigcap_{n=1}^{\infty} \mathcal{B}_n(\lambda), \quad \mathcal{B}_n \subset \mathcal{B}_{n-1}.$$

This set has a Cantor type structure on the unit circle. It is proven that $\mathcal{B}_\infty(\lambda)$ has an asymptotically full measure on the unit circle (see (1.9)). We prove that the sequence $\varkappa_n(\lambda, \vec{\nu})$, $n = 1, 2, ...,$, describing the isoenergetic curves $\mathcal{D}_n$, quickly converges as $n \to \infty$. Hence, $\mathcal{D}_\infty(\lambda)$ can be described as the limit of $\mathcal{D}_n(\lambda)$ in the sense (1.8), where $\varkappa_\infty(\lambda, \vec{\nu}) = \lim_{n\to\infty} \varkappa_n(\lambda, \vec{\nu})$ for every $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. It is shown that the derivatives of the functions $\varkappa_n(\lambda, \vec{\nu})$ (with respect to the angle variable φ on the unit circle) have a limit as $n \to \infty$ for every $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. We denote this limit by $\frac{\partial \varkappa_\infty(\lambda, \vec{\nu})}{\partial \varphi}$. Using (1.17) we prove that

$$\frac{\partial \varkappa_\infty(\lambda, \vec{\nu})}{\partial \varphi} = O\left(\lambda^{-\gamma_5}\right). \tag{1.18}$$

Thus, the limit curve $\mathcal{D}_\infty(\lambda)$ has a tangent vector in spite of its Cantor type structure, the tangent vector being the limit of the corresponding tangent vectors for $\mathcal{D}_n(\lambda)$ as $n \to \infty$. The curve $\mathcal{D}_\infty(\lambda)$ looks as a slightly distorted circle with the infinite number of holes.

Absolute continuity of the branch of the spectrum $[\lambda_*(V), \infty)$, corresponding to the functions $\Psi_\infty(\vec{k}, \vec{x})$, $\vec{k} \in \mathcal{G}_\infty$, follows from continuity properties of level curves $\mathcal{D}_\infty(\lambda)$ with respect to λ, and from convergence of spectral projections corresponding to $\Psi_n(\vec k,\vec x)$, $\vec k\in\mathcal{G}_\infty$, to spectral projections of H in the strong sense and uniformly in λ, $\lambda>\lambda_*$.

The limit-periodic operator $H=(-\Delta)^l+V$, $l\geq 6$, $d=2$ is considered in [24, 25]. The results proved in [24, 25] are analogous to 1-4 on pages 2, 3. The main difficulty of the case $l=1$ comparing with $l\geq 6$ is in starting the recurrent procedure. Step 1 here is really cumbersome comparing with the case $l\geq 6$. This technical difficulty is related to the fact that perturbation theory for a periodic operator $(-\Delta)^l+V_{per}$ is much simpler for $l>1$ then for $l=1$, since Bloch eigenvalues are well-spaced for $l>1$ at high energies.

Further steps of approximation procedure are similar to those in [24, 25] up to some technical modifications. The main technical difficulty to overcome is construction of non-resonance sets $\mathcal{B}_n(\lambda)$ for every sufficiently large λ, $\lambda>\lambda_*(V)$, the last bound being uniform in n. The set $\mathcal{B}_n(\lambda)$ is obtained by deleting a "resonant" part from $\mathcal{B}_{n-1}(\lambda)$. Definition of $\mathcal{B}_{n-1}\setminus\mathcal{B}_n$, naturally, includes Bloch eigenvalues of $H^{(n-1)}$. To describe $\mathcal{B}_{n-1}\setminus\mathcal{B}_n$ one needs both non-resonant eigenvalues (1.13) and resonant eigenvalues. No suitable formulae are known for resonant eigenvalues. Absence of formulae causes difficulties in estimating the size of $\mathcal{B}_n\setminus\mathcal{B}_{n-1}$. Let us describe shortly how we treat this problem in the second and higher steps of approximation. Indeed, on n-th step of approximation we start with the operator $H^{(n-1)}$. It has elementary cell of periods Q_{n-1}. The corresponding Bloch decomposition is denoted by $H^{(n-1)}(t^{(n-1)})$, quasimomentum $t^{(n-1)}$ belonging to the dual elementary cell K_{n-1}. We take $t^{(n-1)}(\varphi)$ being equal to $\vec\varkappa_{n-1}(\varphi)$ modulo K_{n-1}, where $\vec\varkappa_{n-1}(\varphi)$ describes $\mathcal{D}_{n-1}(\lambda)$ (Fig. 1,2), $\vec\varkappa_{n-1}(\varphi)=\varkappa_{n-1}(\lambda,\vec\nu)\vec\nu$, $\vec\nu=(\cos\varphi,\sin\varphi)$. When $\vec\varkappa_{n-1}(\varphi)$ is a point on $\mathcal{D}_{n-1}(\lambda)$, vector $\vec\nu=(\cos\varphi,\sin\varphi)$ belongs to $\mathcal{B}_{n-1}(\lambda)$. The operator $H^{(n-1)}(t^{(n-1)})$ has a simple eigenvalue equal to λ, formulas (1.12)–(1.17) with $n-1$ instead of n being valid. All other eigenvalues are separated from λ by the distance much greater than $\|W_{n-1}\|$, see (1.11). It is convenient to denote such $H^{(n-1)}(t^{(n-1)})$ by $H^{(n-1)}(\vec\varkappa_{n-1}(\varphi))$. Next, perturbation W_n of operator $H^{(n-1)}$ has bigger periods than $H^{(n-1)}$. We assign these bigger periods to $H^{(n-1)}$. The corresponding Bloch decomposition we denote by $\tilde H^{(n-1)}(t^{(n)})$, where $t^{(n)}$ is quasimomentum in the dual elementary cell K_n. According to Bloch's theory, for any $t^{(n)}$ the spectrum of $\tilde H^{(n-1)}(t^{(n)})$ is the union of the spectra of $H^{(n-1)}(t^{(n)}+\vec b)$, where $\vec b$ belongs to the lattice $P^{(n)}$ generated by K_n in K_{n-1}, see Fig.4, page 35. We take $t^{(n)}(\varphi)$ being equal to $\vec\varkappa_{n-1}(\varphi)$ modulo K_n. This means that $t^{(n-1)}(\varphi)$ and $t^{(n)}(\varphi)$ satisfy the relation $t^{(n-1)}(\varphi)=t^{(n)}(\varphi)+\vec b_*$ for some $\vec b_*\in P^{(n)}$ where $t^{(n-1)}(\varphi)$ is $\vec\varkappa_{n-1}(\varphi)$ modulo K_{n-1}. It is convenient to denote such operator $\tilde H^{(n-1)}(t^{(n)})$ by $\tilde H^{(n-1)}(\vec\varkappa_{n-1}(\varphi))$. Obviously, $\tilde H^{(n-1)}(\vec\varkappa_{n-1}(\varphi))$ has an eigenvalue equal to λ, since $H^{(n-1)}(\vec\varkappa_{n-1}(\varphi)+\vec b_*)$ does, all other eigenvalues of $H^{(n-1)}(\vec\varkappa_{n-1}(\varphi)+\vec b_*)$ being separated from λ. We say that φ is resonant if $H^{(n-1)}\left(\vec\varkappa_{n-1}(\varphi)+\vec b\right)$ has an eigenvalue close to λ for some $\vec b\neq\vec b_*$. Note that it happens if and only if the operator

$$I+S_n(\varphi)=\Big(H^{(n-1)}\big(\vec\varkappa_{n-1}(\varphi)+\vec b\big)-\lambda-\epsilon\Big)\Big(H_0\big(\vec\varkappa_{n-1}(\varphi)+\vec b\big)+\lambda\Big)^{-1} \tag{1.19}$$

has an eigenvalue equal to zero for some small ϵ and $\vec{b} \neq \vec{b}_*$, the operator H_0 being the free operator with the same periods as $H^{(n-1)}$. Assume for a moment we consider a polyharmonic operator $H = (-\Delta)^l + V$, $l > 1$. Then, S_n is in the trace class and the determinant of $I + S_n$ exists. The resonant set $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$ can be described (in terms of φ) as the set of solutions of the equation $Det\big(I + S_n(\varphi)\big) = 0$. To obtain $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$ we take all values of ϵ in a small interval and values of $\vec{b}$ in a finite set, $\vec{b} \neq 0$. To estimate the size of $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$ we introduce a complex angle variable φ, i.e., we extend our considerations to a complex neighborhood $\mathit{\Phi}_0$ of $[0, 2\pi)$. We show that the determinant (1.19) is an analytic function of φ in $\mathit{\Phi}_0$, and, by this, reduce the problem of estimating the size of $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$ to a problem in complex analysis. We use Rouche's theorem to count zeros of the determinants and to investigate how far the zeros move when ε changes or W_n is added to $H^{(n-1)}$. It enables us to estimate the size of the zero set of the determinants, and, hence, the size of the non-resonance set $\mathit{\Phi}_n \subset \mathit{\Phi}_0$, which is defined as a non-zero set for the determinants. Proving that the non-resonance set $\mathit{\Phi}_n$ is sufficiently large, we obtain estimates (1.16) for $\mathcal{B}_n$, the set $\mathcal{B}_n$ being defined by the real part of $\mathit{\Phi}_n$.

To obtain $\mathit{\Phi}_n$ we delete from $\mathit{\Phi}_0$ more and more holes of smaller and smaller radii at each step. Thus, the non-resonance set $\mathit{\Phi}_n \subset \mathit{\Phi}_0$ has a structure of Swiss Cheese (Fig. 4, 6), pages 11, 38. Deleting resonance set from $\mathit{\Phi}_0$ at each step of the recurrent procedure we call a "Swiss Cheese Method" . The essential difference of our method from those applied in similar situations before (see e.g. [18]–[21]) is that we construct a non-resonance set not only in the whole space of a parameter ($\vec{k} \in \mathbb{R}^2$ here), but also on all isoenergetic curves $\mathcal{D}_n(\lambda)$ in the space of parameter, corresponding to sufficiently large λ. Estimates for the size of non-resonance sets on a curve require more subtle technical considerations ("Swiss Cheese construction") than those sufficient for description of a non-resonant set in the whole space of the parameter.

When $l = 1$ (the present case) the determinant of $I + S_n(\varphi)$ (see (1.19)) does not exist, since $S_n(\varphi)$ is not from a trace class. We approximate $S_n(\varphi)$ by finite dimensional operators $S_n^{(N)}$ and consider the solutions of $Det\big(I + S_n^{(N)}(\varphi)\big) = 0$. The accumulation points of these solutions as $N \to \infty$ we call the solutions of "Det"$\big(I + S_n(\varphi)\big) = 0$. Swiss Cheese method is applied here with such a modification.

The requirement for super exponential decay of $\|V_r\|$ as $r \to \infty$ is essential, since it is needed to ensure convergence of the recurrent procedure. At every step we use the upper bounds on $\|V_r\|$ to prove perturbation formulae for Bloch eigenvalues and eigenfunctions when $\lambda > \lambda_*(V)$, λ_* being the same for all steps. It is not important that potentials V_r have doubling periods, in the sense that the periods of the type $q^{r-1}\vec{d}_1,\ q^{r-1}\vec{d}_2, q \in \mathbb{N}$, can be treated in the same way as the doubling.

The plan of the paper is the following. Section 2 (page 8) is devoted to the first step of the recurrent procedure. Sections 3 (page 34), 4 (page 50) and 5 (page 59) describe second, third and n-th steps of the recurrent procedure, respectively. Discussion of convergence of the procedure and proofs of the results 1 – 3, listed above, are in Section 6 (page 65). Absolute continuity is proved in Section 7 (page 69). Proofs of geometric lemmas and appendices are in Sections 8 (page 72) and 9 (page 85), respectively.

Further, we denote by c, C absolute constants, by $C(\|V\|)$ a value depending only on $\|V\|$, etc.

2. The first approximation

We fix $0 < s_1 < 10^{-4}$. Let $k > 0$ and be large enough so that $k^{s_1} > 500R_0, d_1, d_2, d_1^{-1}, d_2^{-1}$. We define the first operator $H^{(1)}$ by

$$H^{(1)} := -\Delta + W_1, \quad W_1 := \sum_{r=1}^{M_1} V_r, \quad 2^{M_1} \approx k^{s_1}.\ ^{2} \tag{2.1}$$

Obviously, operator $H^{(1)}$ has a periodic potential. We denote its periods by $(a_1, 0)$, $(0, a_2)$, $a_1 = 2^{M_1-1}d_1$, $a_2 = 2^{M_1-1}d_2$. We employ Bloch theory (see e.g. [22]) for this operator, i.e., consider a family of operators $H^{(1)}(t)$, $t \in K_1$, where K_1 is the elementary cell of the dual lattice, $K_1 = [0, 2\pi a_1^{-1}) \times [0, 2\pi a_2^{-1})$. Vector t is called quasi-momentum. Each operator $H^{(1)}(t)$ acts in $L_2(Q_1)$, $Q_1 = [0, a_1] \times [0, a_2]$. The operator $H^{(1)}(t)$ is defined by (2.1) and quasi-periodic boundary conditions:

$$\begin{aligned} u(a_1, x_2) = \exp(it_1a_1)u(0, x_2), \quad u(x_1, a_2) = \exp(it_2a_2)u(x_1, 0),\\ u_{x_1}(a_1, x_2) = \exp(it_1a_1)u_{x_1}(0, x_2), \quad u_{x_2}(x_1, a_2) = \exp(it_2a_2)u_{x_2}(x_1, 0). \end{aligned} \tag{2.2}$$

Each operator $H^{(1)}(t)$, $t \in K_1$, has a discrete bounded below spectrum $\Lambda^{(1)}(t)$,

$$\Lambda^{(1)}(t) = \bigcup_{n=1}^{\infty} \{\lambda_n^{(1)}(t)\}, \quad \lambda_n^{(1)}(t) \to \infty \text{ as } n \to \infty.$$

The spectrum $\Lambda^{(1)}$ of the original operator $H^{(1)}$ is the union of the spectra of the operators $H^{(1)}(t)$ over all $t \in K_1$: $\Lambda^{(1)} = \cup_{t\in K_1}\Lambda^{(1)}(t)$. The functions $\lambda_n^{(1)}(t)$ are continuous, so $\Lambda^{(1)}$ has a band structure. Extending all the eigenfunctions of $H^{(1)}(t)$ for all $t \in K_1$ by the quasi-periodic boundary conditions to $\mathbb{R}^2$ yields a complete system of generalized eigenfunctions of $H^{(1)}$.

Let $H_0^{(1)}$ be the operator (2.1) corresponding to $W_1 = 0$. We consider that it has periods a_1, a_2 and that operators $H_0^{(1)}(t)$ are defined in $L_2(Q_1)$. The eigenfunctions of the operator $H_0^{(1)}(t)$ are plane waves satisfying (2.2). They are naturally indexed by points in $\mathbb{Z}^2$:

$$\Psi_j^0(t, x) = |Q_1|^{-1/2} \exp i\langle \vec{p}_j(t), x\rangle, \quad |Q_1| = a_1a_2,$$

where here and below $\vec{p}_j(t) := (2\pi j_1/a_1 + t_1, 2\pi j_2/a_2 + t_2)$. The eigenvalue corresponding to $\Psi_j^0(t, x)$ is equal to $p_j^2(t)$, $p_j(t) := |\vec{p}_j(t)|$.

Next, we introduce an isoenergetic surface [3] $S_0(\lambda)$ of the free operator $H_0^{(1)}$. A point $t \in K_1$ belongs to $S_0(\lambda)$ if and only if $H_0^{(1)}(t)$ has an eigenvalue equal to λ, i.e., there exists $j \in \mathbb{Z}^2$ such that $p_j^2(t) = \lambda$. This surface can be obtained as follows: the circle of radius $k = \sqrt{\lambda}$ centered at the origin is divided into pieces by the dual lattice $\{\vec{p}_q(0)\}_{q\in\mathbb{Z}^2}$, and then all pieces are translated in a parallel manner into the

[2] We write $a(k) \approx b(k)$ when the inequalities $\frac{1}{2}b(k) \le a(k) \le 2b(k)$ hold.

[3] "surface" is a traditional term. In our case, it is a curve.

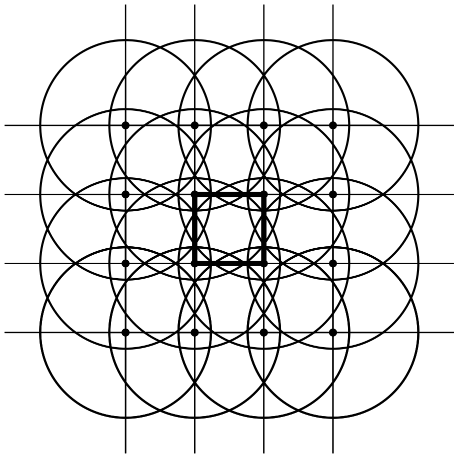

FIGURE 3. The isoenergetic surface $S_0(\lambda)$ of the free operator $H_0^{(1)}$

cell K_1 of the dual lattice. We also can get $S_0(\lambda)$ by drawing sufficiently many circles of radii k centered at the dual lattice $\{\vec{p}_q(0)\}_{q\in\mathbb{Z}^2}$ and by looking at the figure in the cell K_1. As the result of either of these two procedures, we obtain a circle of radius k "packed into the bag K_1," as is shown in the Fig. 3. Note that each piece of $S_0(\lambda)$ can be described by an equation $p_j^2(t)=\lambda$ for a fixed j. If $t\in S_0(\lambda)$, then j can be uniquely defined from the last equation, unless t is a point of self-intersection of the isoenergetic surface. A point t is a self-intersection of $S_0(\lambda)$ if and only if $p_q^2(t)=p_j^2(t)=\lambda$ for at least one pair of indices $q,\ j,\ q\neq j$.

Note that any vector $\vec{\varkappa}$ in $\mathbb{R}^2$ can be uniquely represented in the form $\vec{\varkappa}=\vec{p}_j(t)$, where $j\in\mathbb{Z}^2$ and $t\in K_1$. Let $\mathcal{K}_1$ be the parallel shift into K_1:

$$\mathcal{K}_1:\mathbb{R}^2\to K_1,\quad \mathcal{K}_1\vec{p}_j(t)=t.$$

Suppose $\Omega\subset\mathbb{R}^2$. In order to obtain $\mathcal{K}_1\Omega$, it is necessary to partition Ω by the lattice with nodes at the points $\vec{p}_q(0),\ q\in\mathbb{Z}^2$ and to shift all parts in a parallel manner into a single cell. It is obvious that $|\mathcal{K}_1\Omega|\leq|\Omega|$ for any Ω. If Ω is a smooth curve, then $L(\mathcal{K}_1\Omega)\leq L(\Omega)$. For any pair of sets Ω_1 and Ω_2, $\mathcal{K}_1\big(\Omega_1\cup\Omega_2\big)=\mathcal{K}_1\Omega_1\cup\mathcal{K}_1\Omega_2$. Obviously, $\mathcal{K}_1S_k=S_0(\lambda)$ and $L\big(S_0(\lambda)\big)=L(S_k)=2\pi k$, $k=\sqrt{\lambda}$, S_k being the circle of radius k centered at the origin.

The operator $H^{(1)}(t)$, $t\in K_1$, has the following matrix representation in the basis of plane waves $\Psi_j^0(t,x),\ j\in\mathbb{Z}^2$:

$$H^{(1)}(t)_{mq}=p_m^2(t)\delta_{mq}+w_{m-q},\quad m,q\in\mathbb{Z}^2.$$

Here and below, δ_{mq} is the Kronecker symbol, w_{m-q} are Fourier coefficients of W_1:

$$w_q=\frac{1}{|Q_1|}\int_{Q_1}W_1(x)\exp -i\langle\vec{p}_q,x\rangle dx,\quad \overline{w}_q=w_{-q}.$$

Using assumptions on potentials V_r, we easily obtain:

$$w_0 = 0, \quad |w_q| \le \|W_1\|, \quad w_q = 0, \text{ when } p_q > R_0. \tag{2.3}$$

The matrix $H^{(1)}(t)_{mq}$ describes an operator in $l_2(\mathbb{Z}^2)$ unitary equivalent to $H^{(1)}(t)$ in $L_2(Q_1)$. From now on, we denote the operator in $l_2(\mathbb{Z}^2)$ also by $H^{(1)}(t)$. Since the canonical basis in $l_2(\mathbb{Z}^2)$ does not depend on t and all dependence on t is in the matrix, the matrix $H^{(1)}(t)_{mq}$ and, hence, the operator $H^{(1)}(t) : l_2(\mathbb{Z}^2) \to l_2(\mathbb{Z}^2)$ can be analytically extended in t from K_1 to $\mathbb{C}^2$. From now on, when we refer to $H^{(1)}(t)$ for $t \in \mathbb{C}^2$, we mean the operator in $l_2(\mathbb{Z}^2)$.

2.1. **Perturbation theory.** We formulate the main results of the perturbation theory considering $H^{(1)}(t)$ as a perturbed operator of the free operator $H_0^{(1)}$, i.e., we construct the perturbation series for eigenvalues and spectral projections for t in a small neighborhood of a non-resonant set χ_1.

Lemma 2.1 (Geometric Lemma)**.** *For arbitrarily small positive $\delta < s_1$ and β : $4s_1 < 2\beta \le 1 - 15s_1 - 8\delta$ and sufficiently large λ, $\lambda > \lambda_0(\beta, s_1, \delta)$, there exists a non-resonant set $\chi_1(\lambda, \beta, s_1, \delta) \subset S_0(\lambda)$ satisfying*

(i) *For any $t \in \chi_1(\lambda, \beta, s_1, \delta)$, there exists a unique $j \in \mathbb{Z}^2$ such that $p_j(t) = k$, $k := \sqrt{\lambda}$. The following inequalities hold:*

$$4 \min_{\substack{i, i+q \neq j \\ 0 < p_q < k^{s_1}}} |p_i^2(t) - p_j^2(t)||p_{i+q}^2(t) - p_j^2(t)| > k^{2\beta}, \tag{2.4}$$

$$2 \min_{0 < p_q < k^{s_1}} |p_{j+q}^2(t) - p_j^2(t)| > k^{1 - 3s_1 - \delta}. \tag{2.5}$$

(ii) *For any t in the complex $(k^{2\beta - 2 - s_1 - 2\delta})$-neighborhood of the non-resonance set in $\mathbb{C}^2$, there exists a unique $j \in \mathbb{Z}^2$ such that*

$$|p_j^2(t) - k^2| < 3k^{2\beta - 1 - s_1 - 2\delta}. \tag{2.6}$$

Estimates (2.4) *and* (2.5) *hold.*

(iii) *The non-resonance set $\chi_1(\lambda, \beta, s_1, \delta)$ has an asymptotically full measure on $S_0(\lambda)$ in the sense of*

$$\frac{L\left(S_0(\lambda) \setminus \chi_1(\lambda, \beta, s_1, \delta)\right)}{L\left(S_0(\lambda)\right)} \underset{\lambda \to \infty}{=} O(k^{-\delta}). \tag{2.7}$$

Corollary 2.2. *If t belongs to the complex $(k^{2\beta - 2 - s_1 - 2\delta})$-neighborhood of the non-resonant set $\chi_1(\lambda, \beta, s_1, \delta)$ in $\mathbb{C}^2$, then for any $z \in \mathbb{C}$ lying on the circle*

$$C_1 = \{z : |z - k^2| = k^{2\beta - 1 - s_1 - \delta}\} \tag{2.8}$$

and for any i and q in $\mathbb{Z}^2$ with $0 < p_q < k^{s_1}$

$$2|p_i^2(t) - z| \ge k^{2\beta - 1 - s_1 - \delta} \tag{2.9}$$

$$16|p_i^2(t) - z||p_{i+q}^2(t) - z| > k^{2\beta - 4s_1 - 2\delta} \tag{2.10}$$

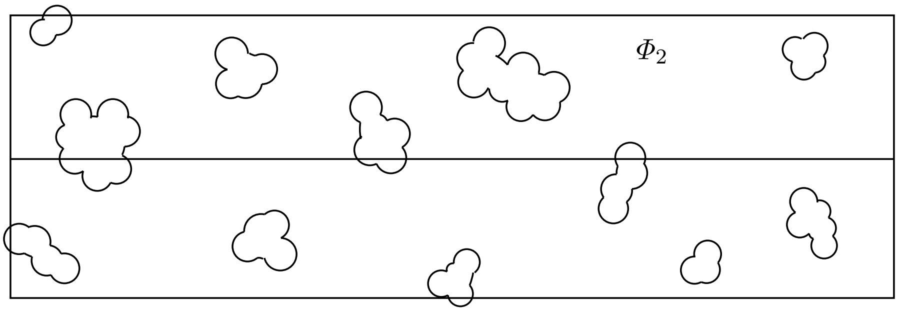

FIGURE 4. The set Φ_2.

The set $\chi_1(\lambda,\beta,s_1,\delta)$ is defined below. The lemma and the corollary are proven in Section 8.1. Section 8.1.1 is introductory. Properties (i), (ii) and Corollary 2.2 are proven in Section 8.1.2 (Corollary 8.5). Section 8.1.3 is devoted to the proof of (iii). Here we just note the following. An analog of the lemma and corollary is proven earlier in [23]. However, in this work we need more detailed description of $\chi_1(\lambda,\beta,s_1,\delta)$ than that in [23]. Moreover, we need a complex analog of $\chi_1(\lambda,\beta,s_1,\delta)$.

We define $\chi_1(\lambda,\beta,s_1,\delta)$ by the formula:

$$\chi_1(\lambda,\beta,s_1,\delta)=\mathcal{K}_1\mathcal{D}_0(\lambda)_{nonres}, \tag{2.11}$$

where $\mathcal{D}_0(\lambda)_{nonres}$ is a subset of the sphere S_k,

$$\mathcal{D}_0(\lambda)_{nonres}=\{k\vec{\nu},\ \vec{\nu}=(\cos\varphi,\sin\varphi),\ \varphi\in\Theta_1\}, \tag{2.12}$$

the set Θ_1 being defined as the real part of a set $\Phi_1\subset\mathbb{C}$:

$$\Theta_1=\Phi_1\cap[0,2\pi), \tag{2.13}$$

$$\Phi_1=\Phi_0\setminus\mathcal{O}^{(1)}, \tag{2.14}$$

where Φ_0 is the complex $k^{-\delta}$ neighborhood of $[0,2\pi)$,

$$\Phi_0=\{\varphi\in\mathbb{C}:\Re\varphi\in[0,2\pi),|\Im\varphi|<k^{-\delta}\}, \tag{2.15}$$

$\mathcal{O}^{(1)}$ is the union of discs $\mathcal{O}_m^{\pm}$:

$$\mathcal{O}^{(1)}=\cup_{m\in\mathbb{Z}^2,\ 0<p_m<4k,\ \pm}\ \mathcal{O}_m^{\pm}. \tag{2.16}$$

Discs $\mathcal{O}_m^{\pm}$ are centered at the points $\varphi_m^{\pm}$, which are solutions in $\mathbb{C}$ (two for each m) of the equations:

$$|\vec{k}(\varphi)+\vec{p}_m|_*^2=k^2,\quad \vec{k}(\varphi)=k(\cos\varphi,\sin\varphi), \tag{2.17}$$

here and below $|\vec{x}|_*^2=x_1^2+x_2^2$ for any $\vec{x}=(x_1,x_2)$, $x_1,x_2\in\mathbb{C}$. The radii r_m of the discs are chosen to ensure the properties of the set $\chi_1(\lambda,\beta,s_1,\delta)$ to be valid. They are given by the formula (8.4).

Let $E_j(t)$ be the spectral projection of the free operator corresponding to the eigenvalue $p_j^2(t)$, $(E_j)_{rm}=\delta_{jr}\delta_{jm}$. In the $(k^{2\beta-2-s_1-2\delta})$-neighborhood of $\chi_1(\lambda,\beta,s_1,\delta)$, we define functions $g_r^{(1)}(k,t)$ and operator-valued functions $G_r^{(1)}(k,t)$, $r=1,2,\dots$ as follows.

$$g_r^{(1)}(k,t)=\frac{(-1)^r}{2\pi i r}\mathrm{Tr}\oint_{C_1}((H_0^{(1)}(t)-z)^{-1}W_1)^r dz, \tag{2.18}$$

$$G_r^{(1)}(k,t)=\frac{(-1)^{r+1}}{2\pi i}\oint_{C_1}((H_0^{(1)}(t)-z)^{-1}W_1)^r(H_0^{(1)}(t)-z)^{-1}dz. \tag{2.19}$$

To find $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$, it is necessary to compute the residues of a rational function of a simple structure, whose numerator does not depend on z, while the denominator is a product of factors of the type $(p_i^2(t)-z)$. For all t in the non-resonance set the integrand has a single pole within C_1 at the point $z=k^2=p_j^2(t)$. By computing the residue at this point, we obtain explicit expressions for $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$. For example, $g_1^{(1)}(k,t)=0$,

$$\begin{aligned} g_2^{(1)}(k,t)&=\sum_{q\in\mathbb{Z}^2,q\neq0}|w_q|^2(p_j^2(t)-p_{j+q}^2(t))^{-1}\\ &=-\sum_{q\in\mathbb{Z}^2,q\neq0}\frac{|w_q|^2p_q^2(0)}{(p_j^2(t)-p_{j+q}^2(t))(p_j^2(t)-p_{j-q}^2(t))}, \end{aligned} \tag{2.20}$$

$$G_1^{(1)}(k,t)_{rm}=\frac{w_{j-m}}{p_j^2(t)-p_m^2(t)}\delta_{rj}+\frac{w_{r-j}}{p_j^2(t)-p_r^2(t)}\delta_{mj},\quad G_1^{(1)}(k,t)_{jj}=0. \tag{2.21}$$

For technical reasons, we introduce parameter α in front of the potential W_1, $H_\alpha^{(1)}=(-\Delta)^l+\alpha W_1$, $0\le\alpha\le1$. We denote the operator $H_\alpha^{(1)}$ with $\alpha=1$ simply by $H^{(1)}$.

Theorem 2.3. *Suppose t belongs to the $(k^{2\beta-2-s_1-2\delta})$-neighborhood in K_1 of the non-resonant set $\chi_1(\lambda,\beta,s_1,\delta)$, $0<\delta<s_1$, $8s_1+6\delta<2\beta<1-15s_1-8\delta$. Then for sufficiently large k, $k>k_0(\|W_1\|,s_1,\beta,\delta)$ and for all α, $-1\le\alpha\le1$, there exists a single eigenvalue of the operator $H_\alpha^{(1)}(t)$ in the interval $\varepsilon_1(k,\delta):=(k^2-k^{2\beta-1-s_1-\delta},k^2+k^{2\beta-1-s_1-\delta})$. It is given by the series*

$$\lambda_j^{(1)}(\alpha,t)=p_j^2(t)+\sum_{r=2}^{\infty}\alpha^r g_r^{(1)}(k,t), \tag{2.22}$$

converging absolutely, where the index j is defined as in Lemma 2.1. The spectral projection corresponding to $\lambda_j^{(1)}(\alpha,t)$ is given by the series

$$E_j^{(1)}(\alpha,t)=E_j+\sum_{r=1}^{\infty}\alpha^r G_r^{(1)}(k,t), \tag{2.23}$$

which converges in the trace class $\mathbf{S_1}$.

Moreover, $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$ satisfy the estimates:

$$|g_r^{(1)}(k,t)|<k^{2\beta-1-s_1-\delta}\big(c\|W_1\|k^{-\beta+4s_1+2\delta}\big)^r, \tag{2.24}$$

$$\|G_r^{(1)}(k,t)\|_1<r\big(ck^{-\beta+4s_1+2\delta}\big)^r, \tag{2.25}$$

$\|\cdot\|_1$ *being the norm in the trace class* $\mathbf{S_1}$*. In addition,*

$$|g_2^{(1)}(k,t)| < c\|W_1\|^2 k^{-2+10s_1+4\delta}, \quad |g_3^{(1)}(k,t)| < c\|W_1\|^3 k^{-2+10s_1+4\delta}, \tag{2.26}$$

$$g_2^{(1)}(k,t) > 0, \text{ when } \|W_1\| \neq 0, \tag{2.27}$$

Operators $G_r^{(1)}(k,t)$ *are finite dimensional:*

$$G_r^{(1)}(k,t)_{il} = 0 \quad \text{when } |i-j| > rR_0 \text{ or } |l-j| > rR_0. \tag{2.28}$$

The series (2.22) and (2.23) converge uniformly with respect to α *in the complex disk* $|\alpha| \leq 1$*.*

Corollary 2.4. *The perturbed eigenvalue and its spectral projection satisfy*

$$|\lambda_j^{(1)}(\alpha,t) - p_j^2(t)| \leq \alpha^2 C(\|W_1\|) k^{-1-2\beta+15s_1+11\delta}, \tag{2.29}$$

$$\|E_j^{(1)}(\alpha,t) - E_j\|_1 \leq c|\alpha|\|W_1\| k^{-\beta+4s_1+3\delta}. \tag{2.30}$$

Proof. The proof of the theorem is based on expanding the resolvent $(H_\alpha^{(1)}(t)-z)^{-1}$ in a perturbation series for z belonging to the contour C_1 about the unperturbed eigenvalue $p_j^2(t)$. It is completely analogous to the proofs of Theorems 2.1 and 3.1 in [23]. Indeed, it is obvious that

$$(H_\alpha^{(1)}(t)-z)^{-1} = (H_0^{(1)}(t)-z)^{-1/2}(I-\alpha A_1)^{-1}(H_0^{(1)}(t)-z)^{-1/2}, \tag{2.31}$$

$$A_1 = A_1(z,t) := -(H_0^{(1)}(t)-z)^{-1/2} W_1 (H_0^{(1)}(t)-z)^{-1/2}. \tag{2.32}$$

It follows from Corollary 2.2, estimates (2.3) and $R_0 < k^{s_1}$, that

$$\|(H_0^{(1)}(t)-z)^{-1}\| < 2k^{-2\beta+1+s_1+\delta}, \quad z \in C_1, \tag{2.33}$$

$$\|A_1\| < 16\|W_1\| k^{-\beta+4s_1+2\delta}. \tag{2.34}$$

Thus, $\|A_1\| < 1$ for sufficiently large k, $k > k_0(\|W_1\|, \beta, s_1, \delta)$. Expanding $(I-\alpha A_1)^{-1}$ in powers of αA_1, we obtain

$$(H_\alpha^{(1)}(t)-z)^{-1} - (H_0^{(1)}(t)-z)^{-1} = \sum_{r=1}^{\infty} \alpha^r (H_0^{(1)}(t)-z)^{-1/2} A_1^r (H_0^{(1)}(t)-z)^{-1/2}. \tag{2.35}$$

Note that $(H_0^{(1)}(t)-z)^{-1} \in \mathbf{S_2}$. Taking into account estimates (2.34), we see that the series (2.35) converges in the class $\mathbf{S_1}$ uniformly with respect to α in the whole complex disk $|\alpha| \leq 1$. For real α-s we substitute the series into the following formula for a spectral projection

$$E_j^{(1)}(\alpha,t) = -\frac{1}{2\pi i}\oint_{C_1} (H_\alpha^{(1)}(t)-z)^{-1} dz. \tag{2.36}$$

Integrating termwise, we arrive at (2.23). Next, we prove estimate (2.25) for $G_r^{(1)}(k,t)$. Indeed, it is easy to see that

$$G_r^{(1)}(k,t) = -\frac{1}{2\pi i}\oint_{C_1} (H_0^{(1)}(t)-z)^{-1/2} A_1^r (H_0^{(1)}(t)-z)^{-1/2} dz. \tag{2.37}$$

We introduce the operator $A_0=(I-E_j)A_1(I-E_j)$. It is obvious that $\|A_0\|\leq\|A_1\|$. In addition

$$\oint_{C_1}(H_0^{(1)}(t)-z)^{-1/2}A_0^r(H_0^{(1)}(t)-z)^{-1/2}dz=0, \tag{2.38}$$

since the integrand is holomorphic inside the circle. Thus,

$$G_r^{(1)}(k,t)=-\frac{1}{2\pi i}\oint_{C_1}(H_0^{(1)}(t)-z)^{-1/2}(A_1^r-A_0^r)(H_0^{(1)}(t)-z)^{-1/2}dz. \tag{2.39}$$

Since $A_1-A_0=E_jA_1(I-E_j)+(I-E_j)A_1E_j$, $E_jA_1E_j=0$ and E_j is a one-dimensional projection, we get

$$\|A_1-A_0\|_1\leq\|A_1\|. \tag{2.40}$$

Since

$$\|A_1^r-A_0^r\|_1\leq r\|A_1-A_0\|_1\|A_1\|^{r-1}, \tag{2.41}$$

we obtain from relation (2.34) that

$$\|A_1^r-A_0^r\|_1\leq r\left(c\|W_1\|k^{-\beta+4s_1+2\delta}\right)^r. \tag{2.42}$$

Noting that the length of C_1 is equal to $2\pi k^{2\beta-1-s_1-\delta}$, we obtain from formula (2.39):

$$\|G_r^{(1)}(k,t)\|_1\leq k^{2\beta-1-s_1-\delta}\|(H_0^{(1)}(t)-z)^{-1/2}\|^2\|A_1^r-A_0^r\|_1 \tag{2.43}$$

Using inequalities (2.33) and (2.42), we get (2.25). Convergence of the series in the complex disk $|\alpha|<1$ easily follows. Note that $(A_1)_{il}=0$ if $|i-l|>R_0$. Now it is not difficult to check (2.28), for details see [23].

We show now that operator $H_\alpha^{(1)}$, $-1<\alpha<1$, has a single eigenvalue ($n_\alpha=1$) in the interval $\varepsilon(k,\delta)$. It is easy to see that n_α is determined from the formula $n_\alpha-n_0=\mathrm{Tr}(E_j^{(1)}-E_j)$. Considering formula (2.23), we obtain

$$|\ n_\alpha-n_0\ |<\sum_{r=1}^{\infty}\|G_r^{(1)}(k,t)\|_1=o(1).$$

Since n_α and n_0 are integers, and $n_0=1$ by the hypothesis of the theorem, the operator $H_\alpha^{(1)}(t)$ for all α, $-1\leq\alpha\leq1$, has a single eigenvalue in $\varepsilon(k,\delta)$.

Further, we use the well-known formula:

$$\frac{\partial\lambda^{(1)}(\alpha,t)}{\partial\alpha}=-\frac{1}{2\pi i}\mathrm{Tr}\oint_{C_1}W_1(H_\alpha^{(1)}(t)-z)^{-1}dz \tag{2.44}$$

Using formula (2.35) and considering that $\mathrm{Tr}(W_1(H_0^{(1)}(t)-z)^{-1})=0$, we obtain:

$$\frac{\partial\lambda^{(1)}(\alpha,t)}{\partial\alpha}=\sum_{r=2}^{\infty}r\alpha^{r-1}g_r^{(1)}(k,t). \tag{2.45}$$

Integrating the last relation with respect to α and noting that $\lambda^{(1)}(0,t)=p_j^2(t)$, we get formula (2.22). To prove estimate (2.24) we note that $g_r^{(1)}=-(2\pi ir)^{-1}\mathrm{Tr}\oint_{C_1}A_1^rdz$, and, therefore,

$$|\ g_r^{(1)}(k,t)\ |\leq r^{-1}k^{2\beta-1-s_1-\delta}\|A_1^r-A_0^r\|_1, \tag{2.46}$$

Using estimate (2.42), we obtain inequality (2.24). Estimates (2.26) are better than those provided for g_2, g_3 by the general estimate (2.24). We obtain estimate (2.26) for $g_2(k,t)$ using the second part of formula (2.20) and the estimate (2.5). Estimate (2.27) easily follows from the same formulas. The estimate for $g_3(k,t)$ can be obtained in the analogous way. For details, see [23], Theorem 3.2. ∎

The series (2.22) and (2.23) can be extended as holomorphic functions of t in a complex neighborhood of $\chi_1(k,\beta,s_1,\delta)$. They can be differentiated with respect to t any number of times with retaining their asymptotic character. Before presenting exact statements, we introduce the following notations:

$$T(m):=\frac{\partial^{|m|}}{\partial t_1^{m_1}\partial t_2^{m_2}},\quad T(0)f:=f,$$
$$m=(m_1,m_2),\quad |m|:=m_1+m_2,\quad m!:=m_1!m_2!.$$

Lemma 2.5. *Coefficients $g_r^{(1)}(k,t)$, and $G_r^{(1)}(k,t)$ can be continued as holomorphic functions of two variables from the real $(k^{2\beta-2-s_1-2\delta})$-neighborhood of the non-resonance set $\chi_1(k,\beta,s_1,\delta)$ to its complex $(k^{2\beta-2-s_1-2\delta})$-neighborhood. In this complex neighborhood the following estimates hold.*

$$|T(m)g_r^{(1)}(k,t)|<m!k^{2\beta-1-s_1-\delta}\big(c\|W_1\|k^{-\beta+4s_1+2\delta}\big)^rk^{|m|(2-2\beta+s_1+2\delta)},\tag{2.47}$$

$$\|T(m)G_r^{(1)}(k,t)\|<m!r\big(c\|W_1\|k^{-\beta+4s_1+2\delta}\big)^rk^{|m|(2-2\beta+s_1+2\delta)}.\tag{2.48}$$

The coefficients $g_r(k,t)$, $r<k^{s_1}R_0^{-1}$ can be continued to the complex $(k^{-3s_1-2\delta})$-neighborhood of $\chi_1(k,\beta,s_1,\delta)$. They obey the estimates:

$$|T(m)g_2^{(1)}(k,t)|<cm!\|W_1\|^2k^{-2+10s_1+4\delta}k^{|m|(3s_1+2\delta)},\tag{2.49}$$

$$|T(m)g_3^{(1)}(k,t)|<cm!\|W_1\|^3k^{-2+10s_1+4\delta}k^{|m|(3s_1+2\delta)}.\tag{2.50}$$

$$|T(m)g_r^{(1)}(k,t)|<m!k^{2\beta-1-s_1-\delta}\big(c\|W_1\|k^{-\beta+4s_1+2\delta}\big)^rk^{|m|(3s_1+2\delta)},\quad 4\leq r<k^{s_1}R_0^{-1}.\tag{2.51}$$

Proof. Coefficients $g_r^{(1)}(k,t)$, and $G_r^{(1)}(k,t)$ can be extended analytically to the $(k^{2\beta-2-s_1-2\delta})$-neighborhood of $\chi_1(k,\beta,s_1,\delta)$, since estimates (2.9), (2.10) are stable in this neighborhood. The coefficients $g_r(k,t)$, $r<k^{s_1}R_0^{-1}$, can be extended to a bigger neighborhood, since only (2.5) is required to estimate them. The estimates for the derivatives are obtained by means of Cauchy integrals. For details, see Theorem 3.3 and Corollary 3.3 in [23]. ∎

Lemma 2.5 implies the following theorem.

Theorem 2.6. *The series (2.22) and (2.23) can be continued as holomorphic functions of two variables from the real $(k^{2\beta-2-s_1-2\delta})$-neighborhood of the non-resonance set $\chi_1(k,\beta,s_1,\delta)$ to its complex $(k^{2\beta-2-s_1-2\delta})$-neighborhood. The following estimates hold in the complex neighborhood:*

$$|T(m)(\lambda_j^{(1)}(\alpha,t)-p_j^2(t))|<m!C(W_1)\alpha^2k^{-1-2\beta+15s_1+11\delta+|m|(2-2\beta+s_1+2\delta)},\tag{2.52}$$

$$\|T(m)(E_j^{(1)}(\alpha,t)-E_j)\|<cm!\|W_1\|\alpha k^{-\beta+4s_1+3\delta+|m|(2-2\beta+s_1+2\delta))}.\tag{2.53}$$

There are stronger estimates for $|m|=1,2$:

$$\left|\nabla\lambda_j^{(1)}(\alpha,t)-2\vec{p}_j(t)\right|<C(W_1)\alpha^2k^{-1-2\beta+18s_1+13\delta}, \tag{2.54}$$

$$\left|T(m)\left(\lambda_j^{(1)}(\alpha,t)-p_j^2\right)\right|<C(W_1)\alpha^2k^{-1-2\beta+21s_1+15\delta},\ \textit{if } |m|=2. \tag{2.55}$$

The next lemma is used in the second step of approximation, where the operator $H^{(1)}(t)$ plays a role of the initial (unperturbed) operator.

Lemma 2.7. *For any z on the circle C_1 given in* (2.8) *and t in the $(k^{2\beta-2-s_1-2\delta})$-neighborhood of $\chi_1(k,\delta)$ in K_1,*

$$\|(H^{(1)}(t)-z)^{-1}\|\leq 2k^{-2\beta+1+s_1+\delta}. \tag{2.56}$$

Proof. The estimate follows from (2.31), (2.33) and (2.34). ∎

2.2. **Non-resonant part of the Isoenergetic Set of $H^{(1)}$.** In this subsection we choose β to have the biggest possible value $\beta=\beta_0$, $2\beta_0=1-15s_1-9\delta$. Let $S_1(\lambda)$ be the isoenergetic surface of the perturbed operator $H_\alpha^{(1)}$, i.e.,

$$S_1(\lambda)=\{t\in K_1:\exists n\in\mathbb{N}\text{ s.t. }\lambda_n^{(1)}(\alpha,t)=\lambda\}, \tag{2.57}$$

where $\{\lambda_n^{(1)}(\alpha,t)\}_{n=1}^{\infty}$ is the complete set of eigenvalues of $H_\alpha^{(1)}(t)$. We construct a "non-resonance" subset $\chi_1^*(\lambda)$ of $S_1(\lambda)$, which corresponds to the non-resonance eigenvalues $\lambda_j^{(1)}(\alpha,t)$ given by the perturbation series. By Lemma 2.1, for every t belonging to the non-resonant set $\chi_1(\lambda,\beta,s_1,\delta)$, there is a single $j\in\mathbb{Z}^2$ such that $p_j(t)=k$, $k=\sqrt{\lambda}$. This means that formula (2.11) establishes a one-to-one correspondence between $\chi_1(\lambda,\beta,s_1,\delta)$ and $\mathcal{D}_0(\lambda)_{nonres}$. Let $\vec{\varkappa}\in\mathcal{D}_0(\lambda)_{nonres}$ and $j\in\mathbb{Z}^2$, $t\in\chi_1(\lambda,\beta,s_1,\delta)$ are defined by the relation $\vec{\varkappa}=\vec{p}_j(t)$.[4] According to Theorem 2.3, for sufficiently large k, there exists an eigenvalue of the operator $H_\alpha^{(1)}(t)$, $t=\mathcal{K}_1\vec{\varkappa}$, $0\leq\alpha\leq 1$, given by (2.22). It is convenient here to denote $\lambda_j^{(1)}(\alpha,t)$ by $\lambda^{(1)}(\alpha,\vec{\varkappa})$; we can do this since there is a one-to-one correspondence between $\vec{\varkappa}$ and the pair (t,j). We rewrite (2.22) in the form

$$\lambda^{(1)}(\alpha,\vec{\varkappa})=\varkappa^2+f_1(\alpha,\vec{\varkappa}),\quad \varkappa=|\vec{\varkappa}|,\quad f_1(\alpha,\vec{\varkappa})=\sum_{r=2}^{\infty}\alpha^r g_r^{(1)}(\vec{\varkappa}), \tag{2.58}$$

$g_r^{(1)}(\vec{\varkappa})$ being defined by (2.18) with j and t such that $\vec{p}_j(t)=\vec{\varkappa}$. By Theorem 2.6, $f_1(\alpha,\vec{\varkappa})$ satisfies the following estimates when $2\beta=1-15s_1-9\delta$:

$$|f_1(\alpha,\vec{\varkappa})|\leq\alpha^2C(W_1)k^{-2+30s_1+20\delta}, \tag{2.59}$$

$$|T(m)f_1(\alpha,\vec{\varkappa})|\leq\alpha^2C(W_1)k^{-2+30s_1+20\delta+|m|(1+16s_1+11\delta)}. \tag{2.60}$$

Estimates (2.54), (2.55) yield:

$$\nabla f_1(\alpha,\vec{\varkappa})=O\left(k^{-2+33s_1+22\delta}\right), \tag{2.61}$$

[4]Usually the vector $\vec{p}_j(t)$ is denoted by $\vec{k}$, the corresponding plane wave being $e^{\langle\vec{k},x\rangle}$. We use the less common notation $\vec{\varkappa}$, since we already have other k's in the text.

$$\nabla\lambda^{(1)}(\alpha,\vec{\varkappa}) = 2\vec{\varkappa} + O\left(k^{-2+33s_1+22\delta}\right), \tag{2.62}$$

$$T(m)f_1(\alpha,\vec{\varkappa}) = O\left(k^{-2+36s_1+24\delta}\right), \quad T(m)\lambda^{(1)}(\alpha,\vec{\varkappa}) = O(1), \quad \text{if } |m| = 2. \tag{2.63}$$

By Theorem 2.6, the series (2.58) converges in the $(k^{-1-16s_1-11\delta})$-neighborhood of $\mathcal{D}_0(\lambda)_{nonres}$, the estimate (2.59)–(2.63) hold.

Let us recall that operator $H^{(1)}(t)$ is defined in l^2, i.e., just by the matrix depending on t. We define $H^{(1)}(\vec{\varkappa})$ by substituting $\vec{\varkappa}$ instead of $\vec{p}_j(t)$ into $H^{(1)}(t)$. For any $t \in \chi_1(k,\beta_0,s_1,\delta)$ matrices $H^{(1)}(t)$ and $H^{(1)}(\vec{\varkappa})$ are the same, up to the shift of indices by j. Further we consider $H^{(1)}(\vec{\varkappa})$ for $\vec{\varkappa} = \varkappa\vec{\nu}$, when φ is in the complex $2k^{-2-16s_1-11\delta}$ neighborhood of Φ_1 and $\varkappa$ is in the complex $2k^{-1-16s_1-11\delta}$ neighborhood of k. Using Lemma 8.9, we easily obtain the estimates analogous to (2.33), (2.34), which provide the convergence of the series for the resolvent. Operator $H^{(1)}(\vec{\varkappa})$ is not self-adjoint for complex φ and we do not try to make spectral analysis of it. However, we notice that the series (2.58) converges when φ is in the complex $2k^{-2-16s_1-11\delta}$ neighborhood of Φ_1 and $\varkappa$ is in the complex $2k^{-1-16s_1-11\delta}$ neighborhood of k. Lemma 2.7 admits the following generalization.

Lemma 2.8. *For any z on the circle C_1 given in (2.8) and $\vec{\varkappa} = \varkappa\vec{\nu}$, φ being in the complex $2k^{-2-16s_1-11\delta}$ neighborhood of Φ_1, $\varkappa$ being in the complex $2k^{-1-16s_1-11\delta}$ neighborhood of k,*

$$\|(H^{(1)}(\vec{\varkappa}) - z)^{-1}\| \le 2k^{-2\beta+1+s_1+\delta}. \tag{2.64}$$

Let $\mathcal{B}_1(\lambda)$ be a set of unit vectors corresponding to $\mathcal{D}_0(\lambda)_{nonres}$:

$$\mathcal{B}_1(\lambda) = \{\vec{\nu} \in S_1 : k\vec{\nu} \in \mathcal{D}_0(\lambda)_{nonres}\} = \{\vec{\nu} \in S_1 : \vec{\nu} = (\cos\varphi,\sin\varphi),\ \varphi \in \Theta_1\}.$$

It is easy to see that $\mathcal{B}_1(\lambda)$ is a unit circle with holes, centered at the origin. Since formulas (2.58)–(2.63) hold in the $(k^{-1-16s_1-11\delta})$-neighborhood of $\mathcal{D}_0(\lambda)_{nonres}$, they hold for any $\varkappa\vec{\nu}$ such that $\vec{\nu} \in \mathcal{B}_1(\lambda)$, $|\varkappa - k| < k^{-1-16s_1-11\delta}$. We define $\mathcal{D}_1(\lambda)$ as the level set of the function $\lambda^{(1)}(\alpha,\vec{\varkappa})$ in this neighborhood:

$$\mathcal{D}_1(\lambda) := \{\vec{\varkappa} = \varkappa\vec{\nu} : \vec{\nu} \in \mathcal{B}_1(\lambda),\ |\varkappa - k| < k^{-1-16s_1-11\delta},\ \lambda^{(1)}(\alpha,\vec{\varkappa}) = \lambda\}. \tag{2.65}$$

We prove in Lemma 2.10 that $\mathcal{D}_1(\lambda)$ is a distorted circle with holes, which is close to the circle of radius k; see Fig. 1. First, we prove that the equation $\lambda^{(1)}(\alpha,\vec{\varkappa}) = \lambda$ is solvable with respect to $\varkappa = |\vec{\varkappa}|$ for any $\vec{\nu} = \frac{\vec{\varkappa}}{\varkappa} \in \mathcal{B}_1(\lambda)$.

Lemma 2.9. *For every $\vec{\nu} \in \mathcal{B}_1(\lambda)$ and every α, $0 \le \alpha \le 1$, and sufficiently large λ, there is a unique $\varkappa_1 = \varkappa_1(\lambda,\vec{\nu})$ in the interval*

$$I_1 := [k - k^{-1-16s_1-11\delta}, k + k^{-1-16s_1-11\delta}], \quad k^2 = \lambda,$$

such that

$$\lambda^{(1)}(\alpha,\varkappa_1\vec{\nu}) = \lambda. \tag{2.66}$$

Furthermore, $|\varkappa_1 - k| \le C(W_1)k^{-3+30s_1+20\delta}$.

Proof. Formula (2.62) yields

$$\frac{\partial\lambda^{(1)}(\alpha,\varkappa)}{\partial\varkappa} = 2\varkappa, \tag{2.67}$$

when $|\varkappa - k| < k^{-1-16s_1-11\delta}$. Now the lemma easily follows from (2.59). For details see Lemma 2.10 in [24]. ∎

Let us introduce new notations. Let $\hat{\Phi}_1$ be the $(k^{-2-16s_1-11\delta})$-neighborhood of Φ_1. Note that (2.58)–(2.63) hold for φ in $\hat{\Phi}_1$ and even its $(k^{-2-16s_1-11\delta})$-neighborhood when $\vec{\varkappa} = \varkappa\vec{\nu}$, $|\varkappa - k| < k^{-1-16s_1-11\delta}$. Let

$$\varkappa_1(\varphi) := \varkappa_1(\lambda, \vec{\nu}), \quad \vec{\nu} = (\cos\varphi, \sin\varphi), \quad \vec{\varkappa}_1(\varphi) = \varkappa_1(\varphi)\vec{\nu}, \quad h_1(\varphi) = \varkappa_1(\varphi) - k, \tag{2.68}$$

$\varkappa_1(\lambda, \vec{\nu})$ being defined by Lemma 2.9.

Lemma 2.10. (1) *For sufficiently large λ, the set $\mathcal{D}_1(\lambda)$ is a distorted circle with holes; it can be described by the formula*

$$\mathcal{D}_1(\lambda) = \left\{\vec{\varkappa} \in \mathbb{R}^2 : \vec{\varkappa} = \varkappa_1(\varphi)\vec{\nu}, \quad \vec{\nu} \in \mathcal{B}_1(\lambda)\right\}, \tag{2.69}$$

where $\varkappa_1(\varphi) = k + h_1(\varphi)$ and $h_1(\varphi)$ obeys the inequalities

$$|h_1| < C(W_1)k^{-3+30s_1+20\delta}, \qquad \left|\frac{\partial h_1}{\partial\varphi}\right| < C(W_1)k^{-1+33s_1+22\delta}. \tag{2.70}$$

(2) *The total length of $\mathcal{B}_1(\lambda)$ satisfies the estimate*

$$L\left(\mathcal{B}_1\right) = 2\pi(1 + O(k^{-\delta})). \tag{2.71}$$

(3) *The function $h_1(\varphi)$ can be extended as a holomorphic function of φ from Θ_2 to $\hat{\Phi}_1$. Estimates (2.70) hold in $\hat{\Phi}_1$ too.*

(4) *The curve $\mathcal{D}_1(\lambda)$ has a length which is asymptotically close to that of the whole circle in the sense that*

$$L\big(\mathcal{D}_1(\lambda)\big) \underset{\lambda\to\infty}{=} 2\pi k\big(1 + O(k^{-\delta})\big), \quad \lambda = k^2. \tag{2.72}$$

Proof. Here are the main points of the proof, for details see Lemma 2.11 in [24].

(1) Inequalities (2.70) easily follow from (2.58) – (2.60) and the definition of $\mathcal{D}_1(\lambda)$, Implicit function theorem being applied.

(2) By definition, $\mathcal{B}_1(\lambda)$ is the set of directions corresponding to $\mathcal{D}_0(\lambda)_{nonres}$, the latter set being a subset of the sphere of radius k. Formula (2.11) establishes a one-to-one correspondence between $\chi_1(\lambda, \beta, s_1, \delta)$ and $\mathcal{D}_0(\lambda)_{nonres}$, their lengths being equal. Considering (2.7), we obtain $L\left(\mathcal{D}_0(\lambda)_{nonres}\right) = 2\pi k\big(1 + O(k^{-\delta})\big)$. Hence, (2.71) holds.

(3) The series (2.58) converges for $\vec{\varkappa} = \varkappa\vec{\nu}$, when φ in $\hat{\Phi}_1$ and even its $(k^{-2-16s_1-11\delta})$-neighborhood and $\varkappa \in \omega$, ω being the complex $(k^{-1-16s_1-11\delta})$-neighborhood of k, $\omega = \{\varkappa \in \mathbb{C} : |\varkappa - k| < k^{-1-16s_1-11\delta}\}$. Function $\lambda^{(1)}(\alpha, \vec{\varkappa})$ is analytic in $\varkappa$ and φ. The estimate (2.59) holds. Thus, $|\lambda^{(1)}(\alpha, \vec{\varkappa}\nu) - \varkappa^2| < C(W_1)k^{-2+30s_1+20\delta}$, and $|\varkappa^2 - k^2| \approx k^{-16s_1-11\delta}$ when $\varkappa \in \partial\omega$. Applying Rouche's theorem in ω, we obtain that the equations $\lambda^{(1)}(\vec{\varkappa}\nu) = k^2$ and $\varkappa^2 = k^2$ have the same number of solutions $\varkappa$ in ω for every $\varphi \in \hat{\Phi}_1$. Obviously, $\varkappa^2 = k^2$ has just one solution. Therefore, $\lambda^{(1)}(\alpha, \vec{\varkappa}\nu) = k^2$ also has just one solution $\varkappa_1(\varphi)$. Using (2.67) and Implicit function Theorem, we obtain that $\varkappa_1(\varphi)$ is an analytic function of φ in $\hat{\Phi}_1$ and estimates (2.70) hold.

(4) Estimate (2.72) follows from (2.70) and (2.71).

■

Next, we define the non-resonance subset $\chi_1^*(\lambda)$ of the isoenergetic set $S_1(\lambda)$ as the parallel shift of $\mathcal{D}_1(\lambda)$ into K_1:

$$\chi_1^*(\lambda) := \mathcal{K}_1\mathcal{D}_1(\lambda). \tag{2.73}$$

Lemma 2.11. *The set $\chi_1^*(\lambda)$ belongs to the $(k^{-3+30s_1+20\delta})$-neighborhood of $\chi_1(\lambda)$ in K_1. If $t \in \chi_1^*(\lambda)$, then the operator $H_\alpha^{(1)}(t)$ has a simple eigenvalue $\lambda_n^{(1)}(\alpha, t)$, $n \in \mathbb{N}$, equal to λ, no other eigenvalues being in the interval $\varepsilon_1(k, \delta)$, $\varepsilon_1(k, \delta) := (k^2 - k^{-16s_1-11\delta}, k^2 + k^{-16s_1-11\delta})$. This eigenvalue is given by the perturbation series (2.22), where j is uniquely defined by t from the relation $p_j^2(t) \in \varepsilon_1(k, \delta)$.*

Proof. By Lemma 2.10, $\mathcal{D}_1(\lambda)$ is in the $\left(k^{-3+30s_1+20\delta}\right)$-neighborhood of $\mathcal{D}_0(\lambda)$. Since $\chi_1(\lambda) = \mathcal{K}_1\mathcal{D}_0(\lambda)$ and $\chi_1^*(\lambda) = \mathcal{K}_1\mathcal{D}_1(\lambda)$, we immediately obtain that $\chi_1^*(\lambda)$ is in the $(ck^{-3+30s_1+20\delta})$-neighborhood of $\chi_1(\lambda)$. The size of this neighborhood is less than $k^{-1-16s_1-11\delta}$, here $-1-16s_1-11\delta = 2\beta_0 - 2 - s_1 - 2\delta$ for $2\beta_0 = 1 - 15s_1 - 9\delta$. Hence, Theorem 2.3 holds for any $t \in \chi_1^*(\lambda)$: there is a single eigenvalue of $H_\alpha^{(1)}(t)$ in the interval $\varepsilon_1(k, \delta)$. Since $\chi_1^*(\lambda) \subset S_1(\lambda)$, this eigenvalue is equal to λ. By the theorem, the eigenvalue is given by the series (2.22), j being uniquely defined by t from the relation $p_j^2(t) \in \varepsilon_1(k, \delta)$. ■

Lemma 2.12. *Formula* (2.73) *establishes a one-to-one correspondence between $\chi_1^*(\lambda)$ and $\mathcal{D}_1(\lambda)$.*

Remark 1. *From the geometric point of view, this means that $\chi_1^*(\lambda)$ does not have self-intersections.*

Proof. Suppose there is a pair $\vec{\varkappa}_{1,1}, \vec{\varkappa}_{1,2} \in \mathcal{D}_1(\lambda)$, such that $\mathcal{K}_1\vec{\varkappa}_{1,1} = \mathcal{K}_1\vec{\varkappa}_{1,2} = t$, $t \in \chi_1^*(\lambda)$. By the definition (2.65) of $\mathcal{D}_1(\lambda)$, we have $\lambda^{(1)}(\alpha, \vec{\varkappa}_{1,1}) = \lambda^{(1)}(\alpha, \vec{\varkappa}_{1,2}) = \lambda$, i.e., the eigenvalue λ of $H_\alpha^{(1)}(t)$ is not simple. This contradicts the previous lemma. ■

2.3. **Preparation for the Next Approximation.**

In the next steps of approximations we will need estimates for the resolvent $\left(H^{(1)}\left(\vec{\varkappa}_1(\varphi) + \vec{b}\right)^{-1} - k^2\right)^{-1}$, $\vec{b} \in K_1 \setminus \{0\}$, $\varphi \in \Phi_1$. Let b_0 be the distance of the point $\vec{b}$ to the nearest corner of K_1:

$$b_0 = \min_{m=(0,0),(0,1),(1,0),(1,1)} |\vec{b} - 2\pi m/a|. \tag{2.74}$$

We will consider separately two cases: $b_0 \geq k^{-1-16s_1-12\delta}$ and $0 < b_0 < k^{-1-16s_1-12\delta}$. Our goal is to prove Lemmas 2.29 and 2.34, which give the estimates for the resolvent on a subset of Φ_1 for the cases $b_0 \geq k^{-1-16s_1-12\delta}$ and $0 < b_0 < k^{-1-16s_1-12\delta}$, respectively. In this subsection we choose β to be relatively small, $100s_1 < \beta < 1/12 - 28s_1 - 14\delta$. We denote such β by β_1.

2.3.1. *The case* $b_0 \geq k^{-1-16s_1-12\delta}$. The goal is to construct a set $\mathcal{O}_s^{(1)}(\vec{b})$ such that Lemma 2.29 holds. The general scheme of considerations is similar to that for the case $H = (-\Delta)^l + V$, $l \geq 6$, see [24], Section 3.5.1. First, we describe shortly the scheme in [24]. Further, we explain what changes have to be made to adjust it for the present case. Indeed, in [24] we constructed a set $\mathcal{O}(\vec{b})$, which consists of small discs centered at the poles of the resolvent

$$\left(H_0^{(1)}(\vec{k}(\varphi)+\vec{b})-k^2\right)^{-1}. \tag{2.75}$$

However, the size of discs was chosen sufficiently large to ensure that

$$\left\|\left(H_0^{(1)}(\vec{k}(\varphi)+\vec{b})-k^2\right)^{-1}\right\| < k^{-\mu},\ \mu>0,\ \text{when } \varphi \notin \mathcal{O}(\vec{b}). \tag{2.76}$$

It was shown that the number of poles of (2.75) in Φ_0 does not exceed $c_0k^{2+2s_1}$. Using perturbation arguments we showed that the estimate analogous to (2.75) holds for the perturbed resolvent

$$\left(H^{(1)}(\vec{\varkappa}_1(\varphi)+\vec{b})-k^2\right)^{-1} \tag{2.77}$$

in $\Phi_1 \setminus \mathcal{O}(\vec{b})$:

$$\left\|\left(H^{(1)}(\vec{\varkappa}_1(\varphi)+\vec{b})-k^2\right)^{-1}\right\| < k^{-\mu},\ \mu>0. \tag{2.78}$$

Next, suppose $\mathcal{O}_c(\vec{b})$ is a connected component of $\mathcal{O}(\vec{b})$, which is entirely in Φ_1: $\mathcal{O}_c(\vec{b}) \subset \Phi_1$. Then (2.77) is an analytic function in $\mathcal{O}_c(\vec{b})$. Using determinants and Rouche's theorem, we showed that (2.75) and (2.77) have the same number of poles inside each connected component of $\mathcal{O}(\vec{b})$. Therefore, the total number of poles of (2.77) in such components does not exceed $c_0k^{2+2s_1}$. Next, we replaced the discs in $\mathcal{O}(\vec{b})$ by much smaller discs around the poles of (2.77). By doing this, we obtained a much smaller set $\mathcal{O}_s(\vec{b})$, $\mathcal{O}_s(\vec{b}) \subset \mathcal{O}(\vec{b})$. Using analyticity of (2.77) inside $\mathcal{O}(\vec{b})$ and the maximum principle, we arrived at the estimate:

$$\left\|\left(H^{(1)}(\vec{\varkappa}_1(\varphi)+\vec{b})-k^2\right)^{-1}\right\| < ck^{-\mu}\left(\frac{R_b}{r^{(1)}}\right)^J,\quad J=c_0k^{2+2s_1}, \tag{2.79}$$

where R_b is the size of the biggest component of $\mathcal{O}(\vec{b})$ and $r^{(1)}$ is the radius of the discs constituting $\mathcal{O}_s(\vec{b})$. Thus, we shrank the set $\mathcal{O}(\vec{b})$, but still obtained a bound for the norm of the resolvent, which is essential for the next step of approximation in [24]. It may happen, however, that $\mathcal{O}_c(\vec{b}) \cap \Phi_1 \neq \emptyset$, but $\mathcal{O}_c(\vec{b}) \not\subset \Phi_1$. In this case we showed that the radii of the discs in $\mathcal{O}_c(\vec{b})$ were sufficiently small to make sure that the function $\varkappa_1(\varphi)$ can be analytically extended from Φ_1 to the interior each $\mathcal{O}_c(\vec{b})$ when $\mathcal{O}_c(\vec{b}) \cap \Phi_1 \neq \emptyset$ even if $\mathcal{O}_c(\vec{b}) \not\subset \Phi_1$. For $l \geq 6$ this condition on radii does not contradict to (2.76).

In the present case $l = 1$ the plan is basically the same. However, we have essential technical complications. The set $\mathcal{O}(\vec{b})$, constructed for $l \geq 6$, is now too small to provide convergence of perturbation series for (2.77) when $\varphi \notin \mathcal{O}(\vec{b})$. Therefore, we need to construct a bigger set $\mathcal{O}^{(1)}(\vec{b})$, such that the series converges when $\varphi \notin \mathcal{O}^{(1)}(\vec{b})$.

It can be constructed by analogy with (2.16). Let $\mathcal{O}_c^{(1)}(\vec{b})$ be a connected component of $\mathcal{O}^{(1)}(\vec{b})$. If $\mathcal{O}_c^{(1)}(\vec{b}) \subset \Phi_1$, then considering as in [24], we obtain estimates similar to (2.79). A difficult case is $\mathcal{O}_c^{(1)}(\vec{b}) \cap \Phi_1 \neq \emptyset$, but $\mathcal{O}_c^{(1)}(\vec{b}) \not\subset \Phi_1$. On one hand, the function $\vec{\varkappa}_1(\varphi)$, included in (2.16), is defined on the set Φ_1, which has holes as small as $\varepsilon = k^{-2-12s_1-9\delta}$. These holes are essential in the construction of Φ_1. Hence, the function $\vec{\varkappa}_1(\varphi)$ can be analytically extended no further than $c\varepsilon$, $c<1$, neighborhood of Φ_1. Thus, to extend $\vec{\varkappa}_1(\varphi)$ analytically into a connected component $\mathcal{O}_c^{(1)}(\vec{b})$ of a set $\mathcal{O}^{(1)}(\vec{b})$, such that $\mathcal{O}_c^{(1)}(\vec{b}) \not\subset \Phi_1$, we have to make sure that the size of $\mathcal{O}_c^{(1)}(\vec{b})$ is smaller than ε. On the other hand, deleting only such a small set from Φ_1 cannot provide convergence of perturbations series for (2.77) in $\Phi_1 \setminus \mathcal{O}^{(1)}(\vec{b})$. The properties of the smallest set $\mathcal{O}^{(1)}(\vec{b})$, which we can construct to provide convergence of the perturbation series, is given in Lemma 2.16. Its connected component still can have a size up to $k^{-11/6+\beta_1+12s_1+4\delta}$, at least this is the strongest estimate we can prove. Obviously, the size of such a component is still greater then the size ε of the neighborhood of Φ_1 where $\varkappa_1(\varphi)$ is holomorphic. Thus, we can not construct the set $\mathcal{O}^{(1)}(\vec{b})$ which fits into ε-neighborhood of Φ_1, even though it is much smaller then (2.16). To overcome this difficulty we need additional considerations. We represent $\vec{\varkappa}_1$ in the form $\vec{\varkappa}_1(\varphi) = \vec{\varkappa}_*(\varphi) + O\left(k^{-ck^{3/4}}\right)$, where $\vec{\varkappa}_*(\varphi)$ is an analytic function in a set $\tilde{\Phi}_0$ larger then Φ_1, $\tilde{\Phi}_0 \supset \Phi_1$, see Lemma 2.21. The smallest hole in $\tilde{\Phi}_0$ has the size $k^{-7/4-12s_1-9\delta}$. Since the size of each connected component of $\mathcal{O}^{(1)}(\vec{b})$ is smaller than $k^{-11/6+\beta_1+12s_1+4\delta}$, $\vec{\varkappa}_*(\varphi)$ is holomorphic in every $\mathcal{O}_c^{(1)}(\vec{b})$, such that $\mathcal{O}_c^{(1)}(\vec{b}) \cap \Phi_1 \neq \emptyset$. We prove an estimate for the norm of

$$\left(H^{(1)}(\vec{\varkappa}_*(\varphi) + \vec{b}) - k^2\right)^{-1} \tag{2.80}$$

outside $\mathcal{O}_c^{(1)}(\vec{b})$. Since, the set $\mathcal{O}^{(1)}(\vec{b})$ is rather small, the proof of an estimate for the norm is quite technical (Lemmas 2.16, 2.20, 2.23, 2.25). Further we construct a set $\mathcal{O}_s^{(1)}(\vec{b})$ taking small discs around the poles of (2.80). Using the analyticity of the resolvent and the maximum principle, we obtain (Lemma 2.25) an estimate of the type (2.79) for (2.80):

$$\left\|\left(H^{(1)}(\vec{\varkappa}_*(\varphi) + \vec{b}) - k^2\right)^{-1}\right\| < ck^{\mu_*}\left(\frac{R_b}{r^{(1)}}\right)^J, \quad J = c_0 k^{2+2s_1},, \quad \mu_* > 0. \tag{2.81}$$

The size of each connected component of $\mathcal{O}_s^{(1)}(\vec{b})$ is much less than $k^{-2-12s_1-9\delta}$. Therefore, the function $\varkappa_1(\varphi)$ can be analytically extended into interior of every connected component of $\mathcal{O}_s^{(1)}(\vec{b})$ intersecting with Φ_1. Using the estimate $\vec{\varkappa}_1(\varphi) = \vec{\varkappa}_*(\varphi) + O\left(k^{-ck^{3/4}}\right)$ in the ε neighborhood of Φ_1, we prove that (2.77) obeys an estimate similar to (2.81), see Lemma 2.29. This lemma is used in the second and step of approximation.

One more technical difficulty is related to the fact that the resolvent (2.75) is not from the trace class when $l=1$, while it is from the trace class when $l>1$. This means that the determinant of the operator $\left(H^{(1)}(\vec{k}(\varphi)+\vec{b})-k^2\right)\left(H_0^{(1)}(\vec{k}(\varphi)+\vec{b})-k^2\right)^{-1}$, which we considered in [24] as a complex function of φ for $l\geq 6$, is not defined for the present case $l=1$. To overcome this difficulty, we take a family of finite dimensional projections P_N, $P_N\to_s I$ as N goes to infinity. We consider a finite dimensional operator $H_N^{(1)}=P_N H^{(1)}P_N$ and the determinant of $\left(H_N^{(1)}(\vec{k}(\varphi)+\vec{b})-k^2\right)\left(H_{0,N}^{(1)}(\vec{k}(\varphi)+\vec{b})-k^2\right)^{-1}$. We prove all necessary results for $H_N^{(1)}$ and, then, the analogous results for $H^{(1)}$ by sending N to infinity.

Definition 2.13. Let

$$\Phi_0'=\Phi_0\setminus\left(\cup_{q:\ 0<p_q<4k^{s_1},\pm}\ \mathcal{O}_q^{\pm}\right), \tag{2.82}$$

Obviously, $\Phi_1\subset\Phi_0'\subset\Phi_0$. Note that the circles $\mathcal{O}_q^{\pm}$ deleted from Φ_0 to obtain Φ_0' are relatively large. Their radius is larger that $\frac{1}{4}k^{-4s_1-\delta}$, see (8.4). These circles are essentially bigger that other $\mathcal{O}_m^{\pm}$ constituting $\mathcal{O}^{(1)}$. Properties of Φ_0' are stable in its $k^{-4s_1-2\delta}$ neighborhood. We denote such neighborhood of Φ_0' by $\hat{\Phi}_0'$.

Definition 2.14. The set $\mathcal{O}^{(1)}(\vec{b})$ is defined by the formula:

$$\mathcal{O}^{(1)}(\vec{b})=\cup_{0<p_m<4k,\pm}\ \mathcal{O}_m^{\pm}(\vec{b}), \tag{2.83}$$

where $\mathcal{O}_m^{\pm}(\vec{b})$ are discs in complex plane centered at $\varphi_m^{\pm}(\vec{b})$, which are zeros of $|\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2-k^2$. The radii of the discs are given by Definition 8.8.

Definition 2.15. The total size of $\mathcal{O}^{(1)}(\vec{b})$ is the sum of the sizes of its connected components.

Lemma 2.16. *Let* $100s_1<\beta_1<1/12-28s_1-14\delta$.

(1) *If* $\varphi\in\hat{\Phi}_0'\setminus\mathcal{O}^{(1)}(\vec{b})$ *and* m *is such that* $\left||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2-k^2\right|<k^{\beta_1}$, *then*

$$\min_{0<p_q<k^{s_1}}\left||\vec{k}(\varphi)+\vec{p}_{m+q}(\vec{b})|_*^2-k^2\right|>k^{\beta_1}, \tag{2.84}$$

$$\left||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2-k^2\right|\left||\vec{k}(\varphi)+\vec{p}_{m+q_1}(\vec{b})|_*^2-k^2\right|\left||\vec{k}(\varphi)+\vec{p}_{m+q_2}(\vec{b})|_*^2-k^2\right|>\frac{1}{64}k^{2\beta_1} \tag{2.85}$$

when $0<p_{q_1},p_{q_2}<k^{s_1}$. *This property is preserved in the* $k^{-4+2\beta_1-2s_1-\delta}$ *neighborhood of* $\hat{\Phi}_0'\setminus\mathcal{O}^{(1)}(\vec{b})$.

(2) *The size of each connected component* $\mathcal{O}_c^{(1)}(\vec{b})$ *of* $\mathcal{O}^{(1)}(\vec{b})$ *is less than* $k^{-\gamma}$, $\gamma=11/6-\beta_1-12s_1-4\delta$. *Each component contains no more than* $c_1k^{2/3+s_1}$, $c_1=c_1(d_1,d_2)$ *discs. The total size of* $\mathcal{O}^{(1)}(\vec{b})$ *does not exceed* $2\pi c_1k^{-5/6+\beta_1+12s_1+4\delta}$. *The set* $\mathcal{O}^{(1)}(\vec{b})$ *contains less that* $c_0k^{2+2s_1}$ *discs.*

Corollary 2.17. *For every* φ *in the* $k^{-4+2\beta_1-2s_1-\delta}$ *neighborhood of* $\hat{\Phi}_0'\setminus\mathcal{O}^{(1)}(\vec{b})$ *and for every* $m\in Z^2$

$$\left||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|^2-k^2\right|>k^{-2+2\beta_1-2s_1}. \tag{2.86}$$

Proof of Corollary 2.17. Obviously,

$$\left||\vec{k}(\varphi)+\vec{p}_{m+q_{1,2}}(\vec{b})|_*-k^2\right| \le \left||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*-k^2\right| + 2\left|\langle\vec{k}(\varphi)+\vec{p}_m(\vec{b}),\vec{p}_{q_{1,2}}\rangle_*\right| + p_q^2.$$

If $\left||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|-k^2\right| < 1$, then, clearly, $\left||\vec{k}(\varphi)+\vec{p}_{m+q_{1,2}}(\vec{b})|-k^2\right| \le ck^{1+s_1}$. Now the corollary follows from (2.85) and the last estimate. ∎

Corollary 2.18. *If φ in the $k^{-4+2\beta_1-2s_1-\delta}$ neighborhood of $\hat{\Phi}_0'\setminus\mathcal{O}^{(1)}(\vec{b})$, then the following estimates holds:*

$$\left\|\left(H_0^{(1)}\left(\vec{k}(\varphi)+\vec{b}\right)^{-1}-k^2\right)\right)^{-1}\right\| < ck^{2-2\beta_1+2s_1}. \tag{2.87}$$

The resolvent has no more than $c_1k^{2/3+s_1}$ poles in each connected component of $O^{(1)}(\vec{b})$. The total number of poles in Φ_0 is less than $c_0k^{2+2s_1}$.

Corollary 2.19. *If $\mathcal{O}_c^{(1)}(\vec{b})\cap\Phi_0'\neq\emptyset$, then $\mathcal{O}_c^{(1)}(\vec{b})\subset\hat{\Phi}_0'$.*

This corollary follows from the statement that the size of $\mathcal{O}_c^{(1)}(\vec{b})$ does not exceed $k^{-\gamma}$ and the definition of $\hat{\Phi}_0'$ as the $k^{-4s_1-2\delta}$ neighborhood of Φ_0'.

The lemma is proven is Section 8.2. Note that the total size $k^{-5/6+\beta_1+12s_1+4\delta}$ of the set $\mathcal{O}^{(1)}(\vec{b})\cap\hat{\Phi}_0'$ is small comparing even with the smallest of the circle $\mathcal{O}_q$, $p_q<k^{s_1}$.

Lemma 2.20. *If $\varphi\in\hat{\Phi}_0'\setminus\mathcal{O}^{(1)}(\vec{b})$, then*

$$\left\|\left(H(\vec{k}(\varphi)+\vec{b})-k^2\right)^{-1}\right\| < ck^4. \tag{2.88}$$

The only possible singularities of the resolvent $\left(H(\vec{k}(\varphi)+\vec{b})-k^2\right)^{-1}$ in each connected $\mathcal{O}_c^{(1)}(\vec{b})$ component of $\mathcal{O}^{(1)}(\vec{b})$, such that $\mathcal{O}_c^{(1)}(\vec{b})\cap\Phi_0'\neq\emptyset$, are poles. The number of poles (counting multiplicity) inside each component does not exceed $c_1k^{2/3+s_1}$. The total number of such poles is less then $c_0k^{2+2s_1}$.

Remark 2. *In Lemma 2.20 we could prove convergence of perturbation series and the estimate for the resolvent analogous to (2.87) by somewhat longer considerations. However, (2.88) is good enough for our purposes. The main reason that convergence holds, in spite of weaker conditions on φ are formulas of the type* (2.20).

Proof. Let $\varphi\in\hat{\Phi}_0'\setminus\mathcal{O}^{(1)}(\vec{b})$. We define the set $\Omega(\varphi)\subset\mathbb{Z}^2$ as follows:

$$\Omega=\left\{m\in\mathbb{Z}^2: \left||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|^2-k^2\right|<k^{\beta_1}\right\}.$$

Let P_0 be the diagonal projection, corresponding to Ω: $P_{0mm}=1$ if and only if $m\in\Omega$. Let $P_1=I-P_0$.

It follows from Lemma 2.16 that $P_0W_1P_0=0$. Indeed, suppose it is not so. Then, there is a pair $m, m+q\in\Omega$ such that $w_q\neq 0$. Function W_1 is a trigonometric polynomial, see (2.3). Hence $0<p_q<k^{s_1}$. This contradicts to (2.84).

Next, we consider Hilbert identity:

$$(H^{(1)}-k^2)^{-1}=(H_0^{(1)}-k^2)^{-1}-(H_0^{(1)}-k^2)^{-1}W_1(H^{(1)}-k^2)^{-1},$$

abbreviations $H^{(1)}=H^{(1)}\left(\vec k(\varphi)+\vec b\right)$, $H_0^{(1)}=H_0^{(1)}\left(\vec k(\varphi)+\vec b\right)$ being used. Applying P_1 from two sides and solving for $P_1(H^{(1)}-k^2)^{-1}P_1$, we obtain:

$$P_1(H^{(1)}-k^2)^{-1}P_1=(I+\mathcal{E})^{-1}P_1(H_0^{(1)}-k^2)^{-1}\left(I-W_1P_0(H^{(1)}-k^2)^{-1}P_1\right), \quad (2.89)$$

where $\mathcal{E}:P_1l^2\to P_1l^2$, $\mathcal{E}=P_1(H_0^{(1)}-k^2)^{-1}W_1P_1$. By the definition of P_1,

$$\|P_1(H_0^{(1)}-k^2)^{-1}\|\le k^{-\beta_1}, \quad (2.90)$$

$$\|\mathcal{E}\|\le\|W_1\|k^{-\beta_1}. \quad (2.91)$$

Next, we apply P_0 from the left and P_1 from the right to the Hilbert identity. Considering that $P_0W_1P_0=0$ and $P_0(H_0^{(1)}-k^2)^{-1}P_1=0$, we obtain:

$$P_0(H^{(1)}-k^2)^{-1}P_1=-P_0(H_0^{(1)}-k^2)^{-1}W_1P_1(H^{(1)}-k^2)^{-1}P_1. \quad (2.92)$$

Substituting (2.89) into (2.92) gives:

$$P_0(H^{(1)}-k^2)^{-1}P_1=BP_0(H^{(1)}-k^2)^{-1}P_1-C, \quad (2.93)$$

$$B=P_0QP_1(I+\mathcal{E})^{-1}P_1QP_0,\quad C=P_0QP_1(I+\mathcal{E})^{-1}P_1(H_0^{(1)}-k^2)^{-1},$$
$$Q=\left(H_0^{(1)}-k^2\right)^{-1}W_1.$$

It follows from (2.87) that

$$\|Q\|<C(W_1)k^{2-2\beta_1+2s_1}. \quad (2.94)$$

Using also (2.90) and (2.91) we get:

$$\|C\|<C(W_1)k^{2-3\beta_1+2s_1}. \quad (2.95)$$

Let us prove that

$$\|B\|<C(W_1)k^{-2\beta_1+8s_1}. \quad (2.96)$$

Using $(I+\mathcal{E})^{-1}=I+\sum_{r=1}^N(-\mathcal{E})^r+(-\mathcal{E})^{N+1}(I+\mathcal{E})^{-1}$, $N=[40\beta_1^{-1}]$, we expand B into sum of $N+2$ operators:

$$B=\sum_{r=0}^{N+1}B_r,\quad B_0=P_0QP_1QP_0,\ \ B_r=(-1)^rP_0QP_1(QP_1)^rQP_0,\ r=1,...,N,$$

$$B_{N+1}=(-1)^{N+1}P_0QP_1(QP_1)^{N+1}(I+\mathcal{E})^{-1}P_1QP_0.$$

Let us estimate $\|B_r\|$, $r=0,...,N$. First, we prove that B_0 is diagonal and

$$\|B_0\|<k^{-2\beta_1+6s_1}. \quad (2.97)$$

Indeed, suppose it is not diagonal. Then there is a pair $m,m+q\in\Omega$, $q\neq0$, such that $B_{0m,m+q}=0$. Considering that W_1 is a trigonometric polynomial, we obtain that $p_q<2R_0<k^{s_1}$. Now we easily arrive to contradiction with (2.84), Lemma 2.16. Thus, $\|B_0\|=\sup_{m\in\Omega}|B_{0mm}|$,

$$B_{0mm}=\sum_{q\in\mathbb{Z}^2,0<p_q<R_0}\frac{|w_q|^2}{\left(|\vec k(\varphi)+\vec p_m(\vec b)|^2-k^2\right)\left(|\vec k(\varphi)+\vec p_{m+q}(\vec b)|^2-k^2\right)}.$$

Considering that $w_q = \bar{w}_{-q}$, we easily show that the right-hand side is equal to

$$\sum_{\substack{q\in\mathbb{Z}^2,\\ 0<p_q<R_0}} \frac{|w_q|^2\left(|\vec{k}(\varphi)+\vec{p}_m(\vec{b})|^2-k^2+p_q^2\right)}{\left(|\vec{k}(\varphi)+\vec{p}_m(\vec{b})|^2-k^2\right)\left(|\vec{k}(\varphi)+\vec{p}_{m+q}(\vec{b})|^2-k^2\right)\left(|\vec{k}(\varphi)+\vec{p}_{m-q}(\vec{b})|^2-k^2\right)}.$$

Using (2.84) and (2.85), we obtain $B_{0mm} = O(k^{-2\beta_1+6s_1})$. Therefore, (2.97) holds. Similar argument yields that B_1 is diagonal and $\|B_1\| = \sup_{m\in\Omega}|B_{1mm}|$,

$$B_{1mm} = \sum_{\substack{q_1,q_2\in\mathbb{Z}^2,\\ 0<|p_{q_{1,2}}|<R_0,\\ q_1+q_2\neq 0}} \frac{-w_{q_1}w_{q_2}w_{-q_1-q_2}}{\left(|\vec{k}(\varphi)+\vec{p}_m(\vec{b})|^2-k^2\right)\left(|\vec{k}(\varphi)+\vec{p}_{m+q_1}(\vec{b})|^2-k^2\right)\left(|\vec{k}(\varphi)+\vec{p}_{m+q_1+q_2}(\vec{b})|^2-k^2\right)}.$$

Applying (2.85) in Lemma 2.16, we obtain

$$\|B_1\| < \|W_1\|^3 k^{-2\beta_1+8s_1}. \tag{2.98}$$

Similar estimates hold for all B_r, $r = 2, \dots N$:

$$\|B_r\| \leq (c\|W_1\|)^{r+2} k^{-(\beta_1-4s_1)(r+1)}. \tag{2.99}$$

To estimate B_{N+1}, we note that $\|P_1 Q\| < \|W_1\| k^{-\beta_1}$. Considering that $\|P_0 Q\| < ck^{2-2\beta_1+2s_1}$ and N is sufficiently large, we arrive to $\|B_{N+1}\| < ck^{-2\beta_1+8s_1}$. Hence, (2.96) is proven. Next, considering (2.93) and (2.95), (2.96), we obtain:

$$\left\|P_0(H^{(1)}-k^2)^{-1}P_1\right\| < ck^{2-3\beta_1+2s_1}. \tag{2.100}$$

Using the last estimate in (2.89), we get

$$\left\|P_1(H^{(1)}-k^2)^{-1}P_1\right\| < ck^{2-4\beta_1+2s_1}. \tag{2.101}$$

Taking into account that $P_1(H^{(1)}-k^2)^{-1}P_0$ corresponding to a φ is the adjoint of $P_0(H^{(1)}-k^2)^{-1}P_1$ corresponding to $\bar{\varphi}$ and the set $\mathcal{O}^{(1)}(\vec{b})$ is symmetric with respect to real axis, we obtain:

$$\left\|P_1(H^{(1)}-k^2)^{-1}P_0\right\| < ck^{2-3\beta_1+2s_1}. \tag{2.102}$$

Applying P_0 to both parts of Hilbert equations, using $P_0W_1P_0 = 0$ and (2.87), (2.102), we obtain:

$$\left\|P_0(H^{(1)}-k^2)^{-1}P_0\right\| < ck^{4-5\beta_1+4s_1} < ck^4. \tag{2.103}$$

Combining (2.100) – (2.103), we obtain (2.88). Note that (2.88) holds on the boundary $\partial\mathcal{O}_c^{(1)}(\vec{b})$ of each connected component $\mathcal{O}_c^{(1)}(\vec{b})$, such that $\mathcal{O}_c^{(1)}(\vec{b})\cap\Phi_0' \neq \emptyset$. It is true, since the size of $\mathcal{O}_c^{(1)}(\vec{b})$ is less than $ck^{-\gamma}$, i.e. much smaller than $k^{-4s_1-2\delta}$.

It remains to show that all singularities of $(H^{(1)}-k^2)^{-1}$ inside each connected component $\mathcal{O}_c^{(1)}(\vec{b})$, such that $\mathcal{O}_c^{(1)}(\vec{b})\cap\Phi_0' \neq \emptyset$, are poles and the number of poles, counting multiplicity, does not exceed $c_1k^{2/3+s_1}$. We follow here the approach developed in [24] for $(-\Delta)^l + V$, $l > 6$, The plan in [24] is the following. We consider the operator $I+A:=(H^{(1)}-k^2)(H_0^{(1)}-k^2)^{-1}$ and , use a well-known relation (see e.g. [22]):

$$\Big|\det(I+A)-1\Big|\leq \|A\|_1 e^{\|A\|_1+2},\ A\in \mathbf{S}_1. \tag{2.104}$$

on the boundary of $\mathcal{O}_c^{(1)}(\vec{b})$. Using Rouche's theorem, we arrive at the conclusion that $I+A$ has the same number of zeros and poles inside $\mathcal{O}_c^{(1)}(\vec{b})$. Therefore, the resolvents $(H^{(1)}-k^2)^{-1}$ and $(H_0^{(1)}-k^2)^{-1}$ have the same number of poles. Using Corollary 2.18, we obtain that the number of poles of $(H_0^{(1)}-k^2)^{-1}$ does not exceed $c_1k^{2/3+s_1}$. However, there is a technical obstacle on the way of this proof in the present case $l=1$: the determinant of $(H^{(1)}-k^2)(H_0^{(1)}-k^2)^{-1}$ is not defined, since the resolvent of H_0 is not from $\mathbf{S}_1$. To overcome this obstacle, we introduce a family of expanding diagonal finite dimensional projections $P_N: P_N\to I$. We consider a finite dimensional analog of $H^{(1)}$ with a multiplier α in front of W_1:

$$H_{\alpha,N}^{(1)}=H_0^{(1)}P_N+\alpha P_NW_1P_N,\quad 0\leq\alpha\leq 1. \tag{2.105}$$

First, we show that all $(H_{\alpha,N}^{(1)}-k^2)^{-1}$, $0\leq\alpha\leq 1$, have the same number of poles inside each $\mathcal{O}_c^{(1)}(\vec{b})$. Indeed, let $\alpha,\alpha_0\in[0,1]$. We introduce operator A by the formula: $I+(\alpha-\alpha_0)A=(H_{\alpha,N}^{(1)}-k^2)(H_{\alpha_0,N}^{(1)}-k^2)^{-1}$. Obviously, $A=P_NW_1P_N(H_{\alpha_0,N}^{(1)}-k^2)^{-1}$. Applying formula (2.104) to $I+(\alpha-\alpha_0)A$, we obtain:

$$\Big|\det(H_{\alpha,N}^{(1)}-k^2)(H_{\alpha_0,N}^{(1)}-k^2)^{-1}-1\Big|\leq|\alpha-\alpha_0|\|A\|_1e^{\||A\|_1+2}. \tag{2.106}$$

Clearly, $\|A\|\leq\|W_1\|\|(H_{\alpha_0,N}^{(1)}-k^2)^{-1}\|$. Considering as in the proof of (2.88), we obtain $\|(H_{\alpha_0,N}^{(1)}(\varphi)-k^2)^{-1}\|<ck^4$, when φ is on the boundary of $\mathcal{O}c^{(1)}(\vec{b})$. Since A is an N dimensional operator: $\|A\|_1\leq\|A\|N$. Combining the last three inequalities, we get: $\|A\|_1<ck^4N$. Using this estimate in (2.106), we obtain:

$$\Big|\det(H_{\alpha,N}^{(1)}(\varphi)-k^2)(H_{\alpha_0,N}^{(1)}(\varphi)-k^2)^{-1}-1\Big|\leq c|\alpha-\alpha_0|Nk^4, \tag{2.107}$$

when φ is on the boundary of $\mathcal{O}_c^{(1)}(\vec{b})$. If $|\alpha-\alpha_0|$ is sufficiently small, then the right-hand side of (2.107) is less than 1. By Rouche's theorem $\det(H_{\alpha,N}^{(1)}-k^2)(H_{\alpha_0,N}^{(1)}-k^2)^{-1}$ has the same number of poles and zeros inside $\mathcal{O}_c^{(1)}(\vec{b})$. Therefore, $\det(H_{\alpha,N}^{(1)}-k^2)$ and $\det(H_{\alpha_0,N}^{(1)}-k^2)$ have the same number of zeros. Covering $[0,1]$ by small intervals and using a finite step induction, we obtain that $\det(H_{\alpha,N}^{(1)}-k^2)$ has the same number of zeros as $\det(H_{0,N}^{(1)}-k^2)$ for any $\alpha\in[0,1]$.

Obviously the zeros of $\det(H_{0,N}^{(1)}-k^2)$ in $\mathcal{O}_c^{(1)}(\vec{b})$ stay the same after N surpasses a certain number. They are solutions of the equations $|\vec{k}(\varphi)+\vec{p}_m|^2=k^2$ in $\mathcal{O}_c^{(1)}(\vec{b})$. Hence, the number of zeros of $\det(H_{1,N}^{(1)}-k^2)$ is the same for all sufficiently large N. It is equal to the number of discs in $\mathcal{O}_c^{(1)}(\vec{b})$.We denote this number by M. By Lemma

2.16, $M < c_1 k^{2/3+s_1}$. We denote the zeros of $\det(H_{1,N}^{(1)}-k^2)$ as $\varphi_1^{(N)},...\varphi_M^{(N)}$, multiplicity being taking into account. Obviously, the function $\prod_{n=1}^{M}\left(\varphi-\varphi_n^{(N)}\right)(H_{1,N}^{(1)}-k^2)^{-1}$ is holomorphic in $\mathcal{O}^{(1)}(\vec{b})$. For a fixed n the sequence $\left\{\varphi_n^{(N)}\right\}_{N=1}^{\infty}$ has an accumulation point. We choose a subsequence N_i such that each $\left\{\varphi_n^{(N_i)}\right\}_{i=1}^{\infty}$ converges. With slight abuse of notations we drop the index i and consider that each $\left\{\varphi_n^{(N)}\right\}_{N=1}^{\infty}$ has a limit φ_n. Considering as in the proof of (2.88), we obtain:

$$\left\|(H_{1,N}^{(1)}(\varphi)-k^2)^{-1}\right\| < ck^4,\ \varphi\in\partial\mathcal{O}_c^{(1)}(\vec{b}) \tag{2.108}$$

Therefore,

$$\left\|\prod_{n=1}^{M}\left(\varphi-\varphi_n^{(N)}\right)(H_{1,N}^{(1)}(\varphi)-k^2)^{-1}\right\| < cr^M k^4, \tag{2.109}$$

when $\varphi\in\partial\mathcal{O}^{(1)}(\vec{b})$, r being the size of $\mathcal{O}^{(1)}(\vec{b})$, $r<k^{-11/6+\beta_1+12s_1+4\delta}$. By the maximum principle, the above estimate holds inside $\mathcal{O}_c^{(1)}(\vec{b})$ too. Suppose $|\varphi-\varphi_n|>2\varepsilon$ for all n and some $\varepsilon>0$. Then $|\varphi-\varphi_{n,N}|>\varepsilon$ for all n and sufficiently large N. Using (2.109), we obtain:

$$\left\|(H_{1,N}^{(1)}(\varphi)-k^2)^{-1}\right\| < cr^M k^4\varepsilon^{-M}. \tag{2.110}$$

Thus, all resolvents are bounded uniformly in N, when $\varphi\neq\varphi_n$, $n=1,...M$. Now, it is easy to show now that $(H_{1,N}^{(1)}(\varphi)-k^2)^{-1}$ tends to $(H_1^{(1)}(\varphi)-k^2)^{-1}$ in the class of bounded operators when $\varphi\neq\varphi_n$, $n=1,...M$. Taking the limit in (2.109), we obtain:

$$\left\|\prod_{n=1}^{M}(\varphi-\varphi_n)(H_1^{(1)}(\varphi)-k^2)^{-1}\right\| < cr^M k^4. \tag{2.111}$$

This means that the only possible singularities of of $(H_1^{(1)}(\varphi)-k^2)^{-1}$ are poles at the points $\varphi=\varphi_n$, $n=1,...M$. The number of poles, counting multiplicity does not exceed $M=c_1k^{2/3+s_1}$. ∎

Let us recall that function $\varkappa_1(\varphi)$, defined by Lemma 2.9, is holomorphic in $\hat{\Phi}_1$, see Lemma 2.10. Now we prove existence of a function $\varkappa_*(\varphi)$, which is a good approximation of $\varkappa_1(\varphi)$ in $\hat{\Phi}_1$ and defined on a larger set $\hat{\Phi}_*$.. Let

$$\Phi_*=\Phi_0\setminus\tilde{\mathcal{O}}^{(1)},\quad \tilde{\mathcal{O}}^{(1)}=\cup_{i\in Z^2,0<p_i<k^{3/4}}\mathcal{O}_i.$$

The size of the discs constituting $\tilde{\mathcal{O}}^{(1)}$ is at least $k^{-7/4-16s_1-9\delta}$. Let $\hat{\Phi}_*$ be the $k^{-7/4-16s_1-10\delta}$–neighborhood of Φ_*. Since the size of the neighborhood is much smaller than the size of the discs, Φ_* and $\hat{\Phi}_*$ are the sets of the same type. The following relations hold: , $\Phi_1\subset\Phi_*\subset\Phi_0'\subset\Phi_0$, $\hat{\Phi}_1\subset\hat{\Phi}_*\subset\hat{\Phi}_0'\subset\Phi_0$.

Lemma 2.21. *Let $100s_1<\beta_1<1/12-28s_1-14\delta$. There is a function $\varkappa_*(\varphi)$ holomorphic in $\hat{\Phi}_*$, which satisfies the estimates analogous to (2.70) on this set:*

$$|\varkappa_*-k|<C(W_1)k^{-3+30s_1+20\delta},\qquad \left|\frac{\partial\varkappa_*}{\partial\varphi}\right|<C(W_1)k^{-1+33s_1+22\delta}. \tag{2.112}$$

It satisfies the following estimate when $\varphi \in \hat{\Phi}_1$:

$$|\varkappa_1(\varphi) - \varkappa_{1*}(\varphi)| < ck^{-\frac{1}{4}k^{3/4}} \tag{2.113}$$

Corollary 2.22. *Function $\varkappa_*$ is holomorphic and obeys (2.112) inside each connected component $\mathcal{O}_c^{(1)}(\vec{b})$ of $\mathcal{O}^{(1)}(\vec{b})$, such that $\mathcal{O}_c^{(1)}(\vec{b}) \cap \Phi_* \neq \emptyset$.*

Proof of the corollary. By Lemma 2.16, $\mathcal{O}_c^{(1)}(\vec{b})$ is in the $ck^{-\gamma}$-neighborhood of Φ_*. Considering that $-7/4 - 16s_1 - 10\delta > -\gamma$, we obtain $\mathcal{O}_c^{(1)}(\vec{b}) \subset \hat{\Phi}_*$.

Proof. Let us consider the function:

$$\lambda_*^{(1)}(\alpha, \vec{\varkappa}) = \varkappa^2 + f_{1*}(\alpha, \vec{\varkappa}), \quad \varkappa = |\vec{\varkappa}|, \ \ f_{1*}(\alpha, \vec{\varkappa}) = \sum_{r=2}^{\lfloor k^{3/4} \rfloor + 1} \alpha^r g_r^{(1)}(\vec{\varkappa}). \tag{2.114}$$

Obviously $f_{1*}(\alpha, \vec{\varkappa})$ is a finite sum for the series $f_1(\alpha, \vec{\varkappa})$, see (2.58). It satisfies the estimates analogous to (2.59), when $\varphi \in \Phi_1$

$$|f_{1*}(\alpha, \vec{\varkappa})| \leq 2\alpha^2 k^{-2+30s_1+20\delta}, \quad |T(m) f_{1*}(\alpha, \vec{\varkappa})| \leq 2\alpha^2 k^{-2+30s_1+20\delta+|m|(1+16s_1+11\delta)}. \tag{2.115}$$

By the construction of Φ_*, the estimates (2.4) and (2.5) hold for all $\varphi \in \Phi_*$ and $p_i \leq k^{3/4}$. Now it is easy to see that all coefficients $g_r^{(1)}(\vec{\varkappa})$, $2 \leq r \leq k^{3/4}$, are holomorphic functions in Φ_* and even to its $2k^{-7/4-16s_1-10\delta}$ neighborhood. The estimates (2.24), (2.26), (2.47), (2.49)–(2.51) hold when $r \leq k^{3/4}$. Therefore, the finite sum can be analytically extended to $2k^{-7/4-16s_1-10\delta}$ neighborhood of Φ_*, the estimates (2.115) being valid. Estimating the tail of the series, we get

$$|f_1 - f_{1*}| < k^{-\frac{1}{4}k^{3/4}}.$$

Solving the equation $\lambda_*^{(1)}(\alpha, \vec{\varkappa}) = k^2$ for $\varkappa$, we obtain that there is a function $\varkappa_*(\varphi)$ defined in the $\hat{\Phi}_*$, such that (2.113) holds. This function obviously obeys estimates (2.112). ∎

Lemma 2.23. (1) *Let $100s_1 < \beta_1 < 1/12 - 28s_1 - 14\delta$. If $\varphi \in \hat{\Phi}_* \setminus \mathcal{O}^{(1)}(\vec{b})$, and m is such that $\big||\vec{\varkappa}_*(\varphi) + \vec{p}_m(\vec{b})|_*^2 - k^2\big| < \frac{1}{2}k^{\beta_1}$, then*

$$\min_{0 < p_q < k^{s_1}} \big||\vec{\varkappa}_*(\varphi) + \vec{p}_{m+q}(\vec{b})|_*^2 - k^2\big| > \frac{1}{2}k^{\beta_1}, \tag{2.116}$$

$$\big||\vec{\varkappa}_*(\varphi)+\vec{p}_m(\vec{b})|_*^2 - k^2\big| \, \big||\vec{\varkappa}_*(\varphi)+\vec{p}_{m+q_1}(\vec{b})|_*^2 - k^2\big| \, \big||\vec{\varkappa}_*(\varphi)+\vec{p}_{m+q_2}(\vec{b})|_*^2 - k^2\big| > \frac{1}{128}k^{2\beta_1} \tag{2.117}$$

when $0 < p_{q_1}, p_{q_2} < k^{s_1}$. This property is preserved in the $\frac{1}{2}k^{-4+2\beta_1-2s_1-\delta}$ neighborhood of $\hat{\Phi}_* \setminus \mathcal{O}^{(1)}(\vec{b})$.

(2) *Each equation $|\vec{\varkappa}_*(\varphi) + \vec{p}_m(\vec{b})|_*^2 = k^2$ has the same number of solutions as the "unperturbed" equation $|\vec{k}(\varphi) + \vec{p}_m(\vec{b})|_*^2 = k^2$ inside every $\mathcal{O}_c^{(1)}(\vec{b})$, $\mathcal{O}_c^{(1)}(\vec{b}) \cap \Phi_* \neq \emptyset$.*

Corollary 2.24. *The number of points φ satisfying one of the equations $|\vec{\varkappa}_*(\varphi)+\vec{p}_m(\vec{b})|_*^2=k^2$, $m\in Z^2$, in $\mathcal{O}_c^{(1)}(\vec{b})$, $\mathcal{O}_c^{(1)}(\vec{b})\cap\Phi_*\neq\emptyset$, does not exceed $c_1k^{2/3+s_1}$. The total number of such points is less then $c_0k^{2+2s_1}$.*

The corollary follows from the above lemma and the last statement of Lemma 2.16.

Proof. Let $\varphi\in\hat{\Phi}_*\setminus\mathcal{O}^{(1)}(\vec{b})$. Noticing that $\hat{\Phi}_*\setminus\mathcal{O}^{(1)}(\vec{b})\subset\hat{\Phi}_0'\setminus\mathcal{O}^{(1)}(\vec{b})$, we obtain that φ satisfies the conditions of Lemma 2.16. Considering the first inequality in (2.112), we conclude:

$$\big||\vec{\varkappa}_*(\varphi)+\vec{p}_m(\vec{b})|_*^2-|\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2\big|=O(k^{-2+30s_1+20\delta})\ \text{ for any } m\in Z^2. \quad (2.118)$$

Taking into account that $30s_1+20\delta<2\beta_1-2s_1$ and (2.86), we get

$$\big||\vec{\varkappa}_*(\varphi)+\vec{p}_m(\vec{b})|_*^2-|\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2\big|<\frac{1}{2}\left||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2-k^2\right|\ \text{ for any } m\in Z^2. \quad (2.119)$$

Therefore, $2\big||\vec{\varkappa}_*(\varphi)+\vec{p}_m(\vec{b})|_*^2-k^2\big|>\big||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2-k^2\big|$ for all $m\in Z^2$. Now, the first statement of the lemma easily follows from the first statement of Lemma 2.16.

In particular, (2.119) holds on the boundary of each $\mathcal{O}_c^{(1)}(\vec{b})$, $\mathcal{O}_c^{(1)}(\vec{b})\cap\Phi_*\neq\emptyset$, see Corollary 2.22. By Rouche's theorem, each equation $|\vec{\varkappa}_*(\varphi)+\vec{p}_m(\vec{b})|_*^2=k^2$ has the same number of solutions inside every $\mathcal{O}_c^{(1)}(\vec{b})$ as the "unperturbed" equation $|\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2=k^2$. ∎

Lemma 2.25. *Let $100s_1<\beta_1<1/12-28s_1-14\delta$. If $\varphi\in\hat{\Phi}_*\setminus\mathcal{O}^{(1)}(\vec{b})$, then*

$$\left\|\left(H(\vec{\varkappa}_*(\varphi)+\vec{b})-k^2\right)^{-1}\right\|<ck^4. \quad (2.120)$$

This estimate is stable in the $\frac{1}{2}k^{-4+2\beta_1-2s_1-\delta}$ neighborhood of $\hat{\Phi}_*\setminus\mathcal{O}^{(1)}(\vec{b})$. The only possible singularities of the resolvent $\left(H(\vec{\varkappa}_*(\varphi)+\vec{b})-k^2\right)^{-1}$ inside each $\mathcal{O}_c^{(1)}(\vec{b})$, $\mathcal{O}_c^{(1)}(\vec{b})\cap\Phi_*\neq\emptyset$, are poles. The number of poles, counting multiplicity, inside each such $\mathcal{O}_c^{(1)}(\vec{b})$ does not exceed $c_1k^{2/3+s_1}$. The total number of such poles is less then $c_0k^{2+2s_1}$.

Proof. The proof of The lemma is analogous to that of Lemma 2.20 up to replacement of Lemma 2.16 by Lemma 2.23. ∎

Definition 2.26. Let us numerate all components $\mathcal{O}_c^{(1)}(\vec{b})$, $\mathcal{O}_c^{(1)}(\vec{b})\cap\Phi_*\neq\emptyset$, by index i, $i=1,...,I$. We denote the poles of $\left(H^{(1)}(\vec{\varkappa}_*(\varphi)+\vec{b})-k^2\right)^{-1}$ in a component $\mathcal{O}_c^{(1)}(\vec{b})_i$ by φ_{i,n_i}, $n_i=1,...,M_i$, $M_i<c_1k^{2/3+s_1}$. Let us consider the discs $\mathcal{O}_s(\vec{b})_{i,n_i}$ of the radius $r^{(1)}=k^{-4-6s_1-3\delta}$ around these poles. This radius is much less then the size of discs constituting $\mathcal{O}_c^{(1)}(\vec{b})_i$. Let $\mathcal{O}_s^{(1)}(\vec{b})_i$ be the union of all small discs corresponding to a component $\mathcal{O}_c^{(1)}(\vec{b})_i$:

$$\mathcal{O}_s^{(1)}(\vec{b})_i=\cup_{n_i=1}^{M_i}\mathcal{O}_s^{(1)}(\vec{b})_{i,n_i}. \quad (2.121)$$

Obviously, $\mathcal{O}_s^{(1)}(\vec{b})_i\subset\mathcal{O}_c^{(1)}(\vec{b})_i$ and, therefore, $\mathcal{O}_s^{(1)}(\vec{b})_i\cap\mathcal{O}_s^{(1)}(\vec{b})_{i'}=\emptyset$ if $i\neq i'$. Let

$$\mathcal{O}_s^{(1)}(\vec{b})=\cup_{i=1}^{I}\mathcal{O}_s^{(1)}(\vec{b})_i. \quad (2.122)$$

Lemma 2.27. *The size of each connected component of $\mathcal{O}_s^{(1)}(\vec b)$ does not exceed $c_1k^{-3\frac{1}{3}-5s_1-3\delta}$. The total size of $\mathcal{O}_s^{(1)}(\vec b)$ does not exceed $c_0k^{-2-4s_1-3\delta}$.*

Proof. Since each disc has the radius $r^{(1)}=k^{-4-6s_1-3\delta}$ and $M_i<c_1k^{2/3+s_1}$, the size of a connected component of $\mathcal{O}_s^{(1)}(\vec b)$ does not exceed $c_1k^{-3\frac{1}{3}-5s_1-3\delta}$. The total number of discs in $\mathcal{O}_s^{(1)}(\vec b)$ does not exceed $c_0k^{2+2s_1}$. Now the second statement of the lemma easily follows. ∎

Further we consider only those connected components $\mathcal{O}_{sc}^{(1)}(\vec b)$ of $\mathcal{O}_s^{(1)}(\vec b)$ who have non-empty intersection with Φ_1: $\mathcal{O}_{sc}^{(1)}(\vec b)_i\cap\Phi_1\neq\emptyset$.

Lemma 2.28. *Function $\varkappa_1(\varphi)$ is analytic in every $\mathcal{O}_{sc}^{(1)}(\vec b)$ such that $\mathcal{O}_{sc}^{(1)}(\vec b)\cap\Phi_1\neq\emptyset$. The estimates* (2.112) *hold inside $\mathcal{O}_{sc}^{(1)}(\vec b)$.*

Proof. By Lemma 2.27, the length of each connected component of $\mathcal{O}_s^{(1)}(\vec b)$ is less than $c_1k^{-3\frac{1}{3}-5s_1-\delta}$. Thus, each connected component of $\mathcal{O}_s^{(1)}(\vec b)$, which has a non-empty intersection with Φ_1, is, in fact, in $\hat\Phi_1$. By Lemma 2.10(3), $\varkappa_1(\varphi)$ is analytic in in $\hat\Phi_1$ and, therefore, in $\mathcal{O}_s^{(1)}(\vec b)$. ∎

Lemma 2.29. *If $100s_1<\beta_1<1/12-28s_1-14\delta$ and $\varphi\in\hat\Phi_1\setminus\mathcal{O}_s^{(1)}(\vec b)$, then*

$$\left\|\left(H(\vec\varkappa_1(\varphi)+\vec b)-k^2\right)^{-1}\right\|<k^{J^{(1)}},\quad J^{(1)}=5c_1k^{2/3+s_1}. \tag{2.123}$$

This estimate is stable in the $k^{-4-6s_1-4\delta}$ neighborhood of $\hat\Phi_1\setminus\mathcal{O}_s^{(1)}(\vec b)$. The resolvent $\left(H(\vec\varkappa_1(\varphi)+\vec b)-k^2\right)^{-1}$ is an analytic function of φ in every component of $\mathcal{O}_s^{(1)}(\vec b)$, whose intersection with Φ_1 is not empty. The only singularities of the resolvent are poles. The number of poles in each connected component does not exceed $c_1k^{2/3+s_1}$. The total number of poles in all components of $\mathcal{O}_s^{(1)}(\vec b)$, whose intersection with Φ_1 is not empty, is less then $c_0k^{2+2s_1}$.

Proof. Suppose $\varphi\in\mathcal{O}_c^{(1)}(\vec b)_i$. Using Lemma 2.25, we obtain

$$\left\|\prod_{n_i=1}^{M_i}(\varphi-\varphi_{i,n_i})\left(H(\vec\varkappa_*(\varphi)+\vec b)-k^2\right)^{-1}\right\|<cr^{c_1k^{2/3}}k^4, \tag{2.124}$$

r being the maximal size of the components $\mathcal{O}_c^{(1)}(\vec b)_i$, $r=k^{-\gamma}$. Let $\varphi\in\mathcal{O}_c^{(1)}(\vec b)_i\setminus\mathcal{O}_s^{(1)}(\vec b)$. Considering that $|\varphi-\varphi_{i,n_i}|>k^{-4-6s_1-3\delta}$, we obtain

$$2\left\|\left(H(\vec\varkappa_*(\varphi)+\vec b)-k^2\right)^{-1}\right\|<k^{5c_1k^{2/3+s_1}}. \tag{2.125}$$

Combining the last estimate with (2.120), we obtain that

$$2\left\|\left(H(\vec\varkappa_*(\varphi)+\vec b)-k^2\right)^{-1}\right\|<k^{5c_1k^{2/3+s_1}}\quad\text{when }\varphi\in\hat\Phi_1\setminus\mathcal{O}_s^{(1)}(\vec b). \tag{2.126}$$

Hilbert identity yields:

$$\left(H^{(1)}(\vec\varkappa_1(\varphi)+\vec b)-k^2\right)^{-1}-\left(H^{(1)}(\vec\varkappa_*(\varphi)+\vec b)-k^2\right)^{-1}=\left(H^{(1)}(\vec\varkappa_1(\varphi)+\vec b)-k^2\right)^{-1}\mathcal{E}, \tag{2.127}$$

$$\mathcal{E}=\big(H_0(\vec{\varkappa}_*(\varphi)+\vec{b})-H_0(\vec{\varkappa}_1(\varphi)+\vec{b})\big)\big(H^{(1)}(\vec{\varkappa}_*(\varphi)+\vec{b})-k^2\big)^{-1}.$$

Let us show that $\|\mathcal{E}\|<\frac{1}{2}$. Indeed, we write $\mathcal{E}$ in the form $\mathcal{E}=\mathcal{E}_1A$,

$$\mathcal{E}_1=\Big(H_0(\vec{\varkappa}_*(\varphi)+\vec{b})-H_0(\vec{\varkappa}_1(\varphi)+\vec{b})\Big)\big(H_0(\vec{\varkappa}_*(\varphi)+\vec{b})+k^2\big)^{-1},$$

$$A=\big(H_0(\vec{\varkappa}_*(\varphi)+\vec{b})+k^2\big)\big(H^{(1)}(\vec{\varkappa}_*(\varphi)+\vec{b})-k^2\big)^{-1}.$$

It is easy to show that $\mathcal{E}_1$ is a diagonal operator and $\|\mathcal{E}_1\|\leq|\varkappa_*(\varphi)-\varkappa_1(\varphi)|$. Using (2.113), we obtain:

$$\|\mathcal{E}_1\|\leq ck^{-1-(\beta_0-4s_1-4\delta)k^{3/4}}.$$

It easily follows from formula for A that $\|A\|\leq 1+2k^2\|(\big(H^{(1)}(\vec{\varkappa}_*(\varphi)+\vec{b})-k^2\big)^{-1}\|$. Using (2.125), we get $\|A\|\leq k^{6c_1k^{2/3+s_1}}$. Multiplying the estimates for the norms $\mathcal{E}_1$ and $\|A\|$, we get $\|\mathcal{E}\|=o(1)$. Now (2.127) yields:

$$\Big\|\big(H^{(1)}(\vec{\varkappa}_1(\varphi)+\vec{b})-k^2\big)^{-1}\Big\|\leq 2\Big\|\big(H^{(1)}(\vec{\varkappa}_*(\varphi)+\vec{b})-k^2\big)^{-1}\Big\|<k^{5c_1k^{2/3+s_1}}. \quad (2.128)$$

Considering that the discs in $\mathcal{O}^{(1)}(\vec{b})$ have the radius $k^{-4-6s_1-3\delta}$, we easily get that the estimate (2.123) is stable in the $k^{-4-6s_1-4\delta}$ neighborhood of $\hat{\Phi}_*\setminus\mathcal{O}_s^{(1)}(\vec{b})$.

Next, we use determinants to estimate the number of poles of $\big(H^{(1)}(\vec{\varkappa}_1(\varphi)+\vec{b})-k^2\big)^{-1}$. We will follow the scheme established in Lemma 2.20. First, we note that (2.127) and (2.128) hold not only for $\vec{\varkappa}_1(\varphi)$, but also for all points in the segment between $\vec{\varkappa}_*(\varphi)$ and $\vec{\varkappa}_1(\varphi)$, i.e., for $\vec{\varkappa}_*(\varphi)+\alpha\big(\vec{\varkappa}_1(\varphi)-\vec{\varkappa}_*(\varphi)\big)$, $\alpha\in[0,1]$. Second, we consider the family of projectors P_N and the operator $H_N^{(1)}=P_NH^{(1)}P_N$. We use analogs of (2.125), (2.127) for operators H_N and $\alpha\in[0,1]$. Using a a multi-step procedure similar to that in Lemma 2.20, see (2.105)–(2.107), we obtain that $\big(H_N^{(1)}(\vec{\varkappa}_*(\varphi)+\vec{b})-k^2\big)^{-1}$ and $\big(H_N^{(1)}(\vec{\varkappa}_1(\varphi)+\vec{b})-k^2\big)^{-1}$ have the same number of poles inside each connected component of $\mathcal{O}_s^{(1)}(\vec{b})$, whose intersection with Φ_1 is not empty. Considering exactly as in Lemma 2.20, see page 26, we show that all operators $\big(H_N^{(1)}(\vec{\varkappa}_*(\varphi)+\vec{b})-k^2\big)^{-1}$ with sufficiently large N have the same number of poles M, $M<c_1k^{2/3+s_1}$, inside a component of of $\mathcal{O}_s^{(1)}(\vec{b})$, whose intersection with Φ_1 is not empty. Hence, the same is true for $\big(H_N^{(1)}(\vec{\varkappa}_1(\varphi)+\vec{b})-k^2\big)^{-1}$. Considering further as in the proof of Lemma 2.20, we obtain that $\big(H^{(1)}(\vec{\varkappa}_1(\varphi)+\vec{b})-k^2\big)^{-1}$ has no more that $c_1k^{2/3}$ poles. ∎

2.3.2. *The set $\mathcal{O}_s^{(1)}(\vec{b})$ for small b_0.* Everything we considered so far is valid for $\vec{b}$ obeying the inequality $b_0\geq k^{-1-16s_1-12\delta}$, here b_0 is the distance from $\vec{b}$ to the nearest vertex of K_1. However, in the next section and later, b_0 will be taken smaller, since the reciprocal lattice is getting finer with each step. To prepare for this, let us consider $\vec{b}$ being close to a vertex of K_1:

$$0<b_0<k^{-1-16s_1-12\delta}. \quad (2.129)$$

We show that for such $\vec{b}$ the resolvent $\left(H^{(1)}(\vec{y}(\varphi)) - k^2\right)^{-1}$, $\vec{y}(\varphi) = \vec{\varkappa}_1(\varphi) + \vec{b}$ has no more than two poles $\varphi^{\pm}$ in $\hat{\Phi}_1$. We surround these poles by two contours $\gamma^{(1)\pm}$ and obtain estimate (2.134) for $\left(H^{(1)}(\vec{y}(\varphi)) - k^2\right)^{-1}$ when φ is outside $\gamma^{(1)\pm}$.

Suppose $|\vec{b}| = b_0$, i.e., the closest vertex of K_1 for $\vec{b}$ is $(0,0)$. The perturbation series (2.58) converge for both $\lambda^{(1)}\left(\vec{\varkappa}_1(\varphi)\right)$ and $\lambda^{(1)}\left(\vec{y}(\varphi)\right)$ when $\varphi \in \hat{\Phi}_1$, and both functions are holomorphic in $\hat{\Phi}_1$ (Lemma 2.10). Note that $\lambda^{(1)}\left(\vec{\varkappa}_1(\varphi)\right) = k^2$ for all $\varphi \in \hat{\Phi}_1$. We base our further considerations on these perturbation series expansions. For $\vec{b}$ being close to a vertex $\vec{e}$ other than $(0,0)$, we take $\vec{y}(\varphi) = \vec{\varkappa}_1(\varphi) + \vec{b} - \vec{e}$.

We define $\varphi_b \in [0, 2\pi)$ by the formula $\vec{b} = b_0(\cos\varphi_b, \sin\varphi_b)$ when $|\vec{b}| = b_0$, and by the analogous formula $\vec{b} - \vec{e} = b_0(\cos\varphi_b, \sin\varphi_b)$ when $\vec{b}$ is close to a vertex $\vec{e}$ other than $(0,0)$.

Lemma 2.30. *If $\vec{b}$ satisfies (2.129) and $|\epsilon_0| < b_0 k^{-1-16s_1-12\delta}$, then the equation*

$$\lambda^{(1)}\left(\vec{y}(\varphi)\right) = k^2 + \epsilon_0 \tag{2.130}$$

has no more than two solutions, $\varphi^{\pm}_{\epsilon_0}$, in $\hat{\Phi}_1$. They satisfy the inequality

$$\left|\varphi^{\pm}_{\epsilon_0} - (\varphi_b \pm \pi/2)\right| < \frac{1}{8} k^{-2-16s_1-11\delta}. \tag{2.131}$$

Proof. Suppose $W_1 = 0$ and $|\vec{b}| = b_0$, i.e., the closest vertex of K_1 for $\vec{b}$ is $(0,0)$. Then the equation (2.130) has the form $|k\vec{\nu} + \vec{b}|_*^2 = k^2 + \epsilon_0$, $\vec{\nu} = (\cos\varphi, \sin\varphi)$. It is easy to show that it has two solutions $\varphi^{\pm}_{\epsilon_0}$ satisfying (2.131). Applying perturbative arguments and Rouché's theorem, we prove the lemma for nonzero W_1. A detailed proof can be found in Appendix 1. In the case when $\vec{b}$ is close to a vertex other than $(0,0)$, the considerations are the same up to a parallel shift. ∎

Lemma 2.31. *Suppose $\vec{b}$ satisfies (2.129), $\varphi \in \hat{\Phi}_1$ and obeys the inequality analogous to (2.131): $|\varphi - (\varphi_b \pm \pi/2)| < k^{-2-16s_1-11\delta}$. Then,*

$$\frac{\partial}{\partial\varphi}\lambda^{(1)}\left(\vec{y}(\varphi)\right) =_{k\to\infty} \pm 2b_0 k\big(1 + o(1)\big). \tag{2.132}$$

Proof. Let $W_1 = 0$ and $|\vec{b}| = b_0$. Then $\lambda^{(1)}(\vec{y}(\varphi)) = |k\vec{\nu} + \vec{b}|_*^2$ and

$$\frac{\partial}{\partial\varphi}|k\vec{\nu} + \vec{b}|_*^2 = 2\langle k\vec{\nu} + \vec{b}, k\vec{\mu}\rangle_*, \quad \vec{\mu} = \frac{\partial\vec{\nu}}{\partial\varphi} = (-\sin\varphi, \cos\varphi).$$

For φ close to $\varphi_b \pm \pi/2$, we have $\langle\vec{b}, \vec{\mu}\rangle = \pm b_0(1 + o(1))$. Considering also that $\langle\vec{\mu}, \vec{\nu}\rangle = 0$, we obtain $\frac{\partial}{\partial\varphi}|k\vec{\nu} + b|^2 = \pm 2b_0 k(1 + o(1))$. Applying perturbative arguments, we get a similar formula for nonzero W_1. For a detailed proof see Appendix 2. In the case when $\vec{b}$ is close to a vertex other than $(0,0)$, the considerations are the same up to a parallel shift. ∎

Definition 2.32. Let $\Gamma^{(1)\pm}(\vec{b})$ be two open disks centered at $\varphi^{\pm}_0 \in \hat{\Phi}_1$ [5] with the radius $r^{(1)} = k^{-4-6s_1-3\delta}$; $\gamma^{(1)\pm}(\vec{b})$ be their boundary circles and $\mathcal{O}^{(1)}_s(\vec{b}) = \Gamma^{(1)+} \cup \Gamma^{(1)-}$.

[5] $\varphi^{\pm}_0$ is $\varphi^{\pm}_{\epsilon_0}$ for $\epsilon_0 = 0$.

Lemma 2.33. *For any φ in $\hat{\Phi}_1 \setminus \mathcal{O}_s^{(1)}(\vec{b})$,*

$$\left|\lambda^{(1)}\big(\vec{y}(\varphi)\big) - k^2\right| \geq b_0 k r^{(1)}. \tag{2.133}$$

This estimate is stable in the $k^{-4-6s_1-4\delta}$ neighborhood of $\hat{\Phi}_1 \setminus \mathcal{O}_s^{(1)}(\vec{b})$.

Proof. Suppose (2.133) does not hold for some φ in $\hat{\Phi}_1 \setminus \mathcal{O}_s^{(1)}(\vec{b})$. This means that φ satisfies equation (2.130) with some ε_0: $|\varepsilon_0| < b_0 k r^{(1)}$. By Lemma 2.30, φ obeys (2.131). Thus φ could be either $\varphi_{\epsilon_0}^+$ or $\varphi_{\epsilon_0}^-$. Without loss of generality, assume $\varphi = \varphi_{\epsilon_0}^+$. Obviously, the $r^{(1)}$-neighborhood of $\varphi_{\epsilon_0}^+$ also satisfies conditions of Lemma 2.31, since $r^{(1)}$ is small considering with the size $k^{-2-16s_1-11\delta}$ of the neighborhood $\hat{\Phi}_1$. Using (2.132), we obtain that $\left|\lambda^{(1)}\big(\vec{y}(\varphi)\big) - k^2 - \varepsilon_0\right| = 2kb_0 r^{(1)}(1+o(1))$ on the boundary of this neighborhood. Applying Rouche's Theorem, we obtain that there is a point φ_0^+ in this neighborhood. It immediately follows that $\left|\varphi - \varphi_0^+\right| < r^{(1)}$, i.e., $\varphi \in \Gamma^{(1)+} \subset O_s(\vec{b})$, which contradicts the assumption $\varphi \in \Phi_1 \setminus \mathcal{O}_s^{(1)}(\vec{b})$. ∎

Lemma 2.34. *Let $0 < b_0 < k^{-1-16s_1-12\delta}$. For any φ in $\hat{\Phi}_1 \setminus \mathcal{O}_s^{(1)}(\vec{b})$ or its $k^{-4-6s_1-4\delta}$ neighborhood*

$$\left\|\Big(H^{(1)}\big(\vec{y}(\varphi)\big) - k^2\Big)^{-1}\right\| < \frac{4k^{3+6s_1+4\delta}}{b_0}. \tag{2.134}$$

The resolvent is an analytic function of φ in every component of $\mathcal{O}_s^{(1)}(\vec{b})$, whose intersection with Φ_1 is not empty. The only singularities of the resolvent are poles. The number of poles in each connected component does not exceed two.

Proof. Suppose we have proven that

$$\left\|\Big(\lambda^{(1)}\big(\vec{y}(\varphi)\big) - k^2\Big)\Big(H^{(1)}\big(\vec{y}(\varphi)\big) - k^2\Big)^{-1}\right\| < 2 \tag{2.135}$$

for all φ in $\hat{\Phi}_1$. Then, using (2.133) with $r^{(1)} = k^{-4-6s_1-3\delta}$, we easily get (2.134). Let us prove (2.135). By Lemma 2.10, $\vec{\varkappa}_1(\varphi)$ is a holomorphic function in $\hat{\Phi}_1$ and estimates (2.70) hold. Therefore, $\vec{y}(\varphi) = \vec{k}(\varphi) + O(k^{-1-16s_1-12\delta})$. Using (2.64) with $2\beta = 1 - 15s_1 - 9\delta$, we obtain

$$\left\|\Big(H_\alpha^{(1)}(\vec{y}(\varphi)) - z\Big)^{-1}\right\| < k^{16s_1+10\delta}, \quad z \in C_1, \quad \alpha \in [0,1], \tag{2.136}$$

C_1 being given by (2.8). Using considerations similar to those in Lemma 2.20 (see (2.105) and further), we prove that $\Big(H_\alpha^{(1)}\big(\vec{y}(\varphi)\big) - z\Big)^{-1}$ has at most the same number of poles z inside C_1 as $\Big(H_0^{(1)}\big(\vec{y}(\varphi)\big) - z\Big)^{-1}$, i.e., at most one pole. The pole obviously exists and is located at the point $z = \lambda^{(1)}\big(\vec{y}(\varphi)\big)$. Hence, $\Big(\lambda^{(1)}(\vec{y}(\varphi)) - z\Big)\Big(H^{(1)}(\vec{y}(\varphi)) - z\Big)^{-1}$ is a holomorphic function of z inside C_1 for a fixed φ. From the definition of C_1 and estimate (2.29) it follows that $\left|\lambda^{(1)}(\vec{y}(\varphi)) - z\right| < 2k^{-16s_1-10\delta}$. Multiplying (2.136) and the last estimate, we get

$$\left\|\Big(\lambda^{(1)}(\vec{y}(\varphi)) - z\Big)\Big(H^{(1)}(\vec{y}(\varphi)) - z\Big)^{-1}\right\| < 2, \quad z \in C_1.$$

Using the maximum principle, we obtain the same estimate inside the circle and, therefore, for $z = k^2$. Thus, we have proved (2.134) for $\varphi \in \hat{\Phi}_1 \setminus \mathcal{O}_s^{(1)}(\vec{b})$. It is easy to see that estimate (2.134) is stable with respect to a perturbation of φ of order $k^{-4-6s_1-4\delta}$.

Suppose $\mathcal{O}_s^{(1)}(\vec{b}) \cap \Phi_1 \neq \emptyset$. This means that $O_s^{(1)}(\vec{b}) \subset \hat{\Phi}_1$. Considering (2.135), we see that the only poles of the resolvent are the zeros of the function $\lambda^{(1)}\big(\vec{y}(\varphi)\big) - k^2$. By Lemma 2.30, the number of zeros does not exceed two. ∎

3. The Second Approximation

Let us start with establishing a lower bound for k. Let $\eta > 3 \cdot 10^4$. Since $\eta s_1 > 2 + 4s_1$, there is a number $k_* > e$ such that

$$C_*(1+s_1)k^{2+4s_1} \ln k < k^{\eta s_1}, \; C_* = 400(c_0+c_1+1)^2, \; c_0 = 32 d_1 d_2. \tag{3.1}$$

for any $k > k_*$. Assume also, that k_* is sufficiently large to ensure validity of all estimates in the first step for any $k > k_*$. In particular we assume that all $o(1)$ in the first step satisfy the estimate $|o(1)| < 10^{-2}$ when $k > k_*$.

3.1. **Operator $H_\alpha^{(2)}$.**

Choosing $s_2 = 2s_1$, we define the second operator $H_\alpha^{(2)}$ by the formula:

$$H_\alpha^{(2)} = H^{(1)} + \alpha W_2, \quad (0 \le \alpha \le 1), \qquad W_2 = \sum_{r=M_1+1}^{M_2} V_r, \tag{3.1}$$

where $H^{(1)}$ is defined by (2.1), M_2 is chosen in such a way that $2^{M_2} \approx k^{s_2}$. Obviously, the periods of W_2 are $2^{M_2-1}(d_1,0)$ and $2^{M_2-1}(0,d_2)$. We will write them in the form: $N_1(a_1,0)$ and $N_1(0,a_2)$, where a_1, a_2 are the periods of W_1 and $N_1 = 2^{M_2-M_1}$, $\frac{1}{4}k^{s_2-s_1} < N_1 < 4k^{s_2-s_1}$. Note that

$$\|W_2\|_\infty \le \sum_{r=M_1+1}^{M_2} \|V_r\|_\infty \le \sum_{r=M_1+1}^{M_2} \exp(-2^{\eta r}) < \exp(-k^{\eta s_1}). \tag{3.2}$$

3.1.1. *Multiple Periods of* $W_1(x)$. The operator $H^{(1)} = H_0 + W_1(x)$ has the periods a_1, a_2. The corresponding family of operators, $\{H^{(1)}(t)\}_{t \in K_1}$, acts in $L_2(Q_1)$, where $Q_1 = [0,a_1] \times [0,a_2]$ and $K_1 = [0, 2\pi/a_1) \times [0, 2\pi/a_2)$. Eigenvalues of $H^{(1)}(t)$ are denoted by $\lambda_n^{(1)}(t)$, $n \in \mathbb{N}$, and its spectrum by $\Lambda^{(1)}(t)$. Now let us consider the same $W_1(x)$ as a periodic function with the periods $N_1 a_1, N_1 a_2$. Obviously, the definition of the operator $H^{(1)}$ does not depend on the way how we define the periods of W_1. However, the family of operators $\{H^{(1)}(t)\}_{t \in K_1}$ does change, when we replace the periods a_1, a_2 by $N_1 a_1, N_1 a_2$. The family of operators $\{H^{(1)}(t)\}_{t \in K_1}$ has to be replaced by a family of operators $\{\tilde{H}^{(1)}(\tau)\}_{\tau \in K_2}$ acting in $L_2(Q_2)$, where $Q_2 = [0, N_1 a_1] \times [0, N_1 a_2]$ and $K_2 = [0, 2\pi/N_1 a_1) \times [0, 2\pi/N_1 a_2)$. We denote eigenvalues of $\tilde{H}^{(1)}(\tau)$ by $\tilde{\lambda}_n^{(1)}(\tau)$, $n \in \mathbb{N}$ and its spectrum by $\tilde{\Lambda}^{(1)}(\tau)$. The next lemma establishes a connection between spectra of operators $H^{(1)}(t)$ and $\tilde{H}^{(1)}(\tau)$. It easily follows from Bloch theory (see e.g. [22]).

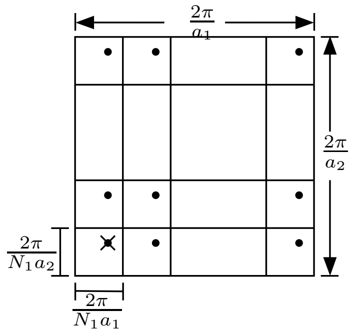

FIGURE 5. Relation between τ $(\times)$ and t_p $(\cdot)$

Lemma 3.1. *For any* $\tau \in K_2$,

$$\tilde{\Lambda}^{(1)}(\tau) = \bigcup_{p \in P} \Lambda^{(1)}(t_p), \tag{3.3}$$

where

$$P = \{p = (p_1, p_2) \in \mathbb{Z}^2 : 0 \leq p_1 \leq N_1 - 1,\ 0 \leq p_2 \leq N_1 - 1\} \tag{3.4}$$

and $t_p = (t_{p,1}, t_{p,2}) = (\tau_1 + 2\pi p_1/N_1 a_1, \tau_2 + 2\pi p_2/N_1 a_2) \in K_1$*, see Fig. 6.*

We defined isoenergetic set $S_1(\lambda) \subset K_1$ of $H^{(1)}$ by formula (2.57). Obviously, this definition is directly associated with the family of operators $H^{(1)}(t)$ and, therefore, with periods a_1, a_2, which we assigned to $W_1(x)$. Now, assuming that the periods are equal to $N_1 a_1, N_1 a_2$, we give an analogous definition of the isoenergetic set $\tilde{S}_1(\lambda)$ in K_2:

$$\tilde{S}_1(\lambda) := \{\tau \in K_2 : \exists n \in \mathbb{N} : \quad \tilde{\lambda}_n^{(1)}(\tau) = \lambda\}. \tag{3.5}$$

By Lemma 3.1, $\tilde{S}_1(\lambda)$ can be expressed as follows:

$$\tilde{S}_1(\lambda) = \Big\{\tau \in K_2 : \exists n \in \mathbb{N},\ p \in P :\ \lambda_n^{(1)}\big(\tau + 2\pi p/N_1 a\big) = \lambda\Big\}, \tag{3.6}$$

$$2\pi p/N_1 a = \left(\frac{2\pi p_1}{N_1 a_1}, \frac{2\pi p_2}{N_1 a_2}\right).$$

The relation between $S_1(\lambda)$ and $\tilde{S}_1(\lambda)$ can be easily understood from the geometric point of view as

$$\tilde{S}_1(\lambda) = \mathcal{K}_2 S_1(\lambda), \tag{3.7}$$

where $\mathcal{K}_2$ is the parallel shift into K_2, i.e.,

$$\mathcal{K}_2 : \mathbb{R}^2 \to K_2,\ \mathcal{K}_2(\tau + 2\pi m/N_1 a) = \tau,\ m \in \mathbb{Z}^2,\ \tau \in K_2. \tag{3.8}$$

Thus, $\tilde{S}_1(\lambda)$ is obtained from $S_1(\lambda)$ by cutting $S_1(\lambda)$ into pieces of the size K_2 and shifting them together in K_2.

Definition 3.2. We say that τ is a point of self-intersection of $\tilde{S}_1(\lambda)$, if there is a pair $m, \hat{m} \in N$, $m \neq \hat{m}$ such that $\tilde{\lambda}_m^{(1)}(\tau) = \tilde{\lambda}_{\hat{m}}^{(1)}(\tau) = \lambda$.

Remark 3.3. By Lemma 3.1, τ is a point of self-intersection of $\tilde{S}_1(\lambda)$, if there is a pair $p, \hat{p} \in P$ and a pair $n, \hat{n} \in N$ such that $|p-\hat{p}|+|n-\hat{n}| \neq 0$ and $\lambda_n^{(1)}(\tau+2\pi p/N_1a) = \lambda_{\hat{n}}^{(1)}(\tau + 2\pi\hat{p}/N_1a) = \lambda$.

Now let us recall that the isoenergetic set $S_1(\lambda)$ consists of two parts: $S_1(\lambda) = \chi_1^*(\lambda) \cup \big(S_1(\lambda) \setminus \chi_1^*(\lambda)\big)$, where $\chi_1^*(\lambda)$ is the first non-resonance set given by (2.73). Obviously $\mathcal{K}_2\chi_1^*(\lambda) \subset \mathcal{K}_2S_1(\lambda) = \tilde{S}_1(\lambda)$ and can be described by the formula:

$$\mathcal{K}_2\chi_1^*(\lambda) = \{\tau \in K_2 : \ \exists p \in P : \tau + 2\pi p/N_1a \in \chi_1^*(\lambda)\}.$$

Let us consider only those self-intersections of $\tilde{S}_1$ which belong to $\mathcal{K}_2\chi_1^*(\lambda)$, i.e., we consider the points of intersection of $\mathcal{K}_2\chi_1^*(\lambda)$ both with itself and with $\tilde{S}_1(\lambda) \setminus \mathcal{K}_2\chi_1^*(\lambda)$.

Lemma 3.4. *A self-intersection τ of $\tilde{S}_1(\lambda)$ belongs to $\mathcal{K}_2\chi_1^*(\lambda)$ if and only if there are a pair $p, \hat{p} \in P$, $p \neq \hat{p}$ and a pair $n, \hat{n} \in N$ such that $\tau + 2\pi p/N_1a \in \chi_1^*(\lambda)$ and $\lambda_n^{(1)}(\tau + 2\pi p/N_1a) = \lambda_{\hat{n}}^{(1)}(\tau + 2\pi\hat{p}/N_1a) = \lambda$, the eigenvalue $\lambda_n^{(1)}(\tau + 2\pi p/N_1a)$ being given by the series (2.22) with $t = \tau + 2\pi p/N_1a$ and j uniquely defined by t from the relation $p_j^2(t) \in \varepsilon_1$,*

$$\varepsilon_1 = (k^2 - 3k^{-16s_1-11\delta}, k^2 + 3k^{-16s_1-11\delta}). \tag{3.9}$$

Proof. Suppose τ is a point of self-intersection of $\tilde{S}_1(\lambda)$ belonging to $\mathcal{K}_2\chi_1^*(\lambda)$. Since, $\tau \in \mathcal{K}_2\chi_1^*(\lambda)$, there is a $p \in P$ such that $\tau + 2\pi p/N_1a \in \chi_1^*(\lambda)$. By Lemma 2.11, there is a single eigenvalue $\lambda_n^{(1)}(\tau + 2\pi p/N_1a) = \lambda$ of $H^{(1)}(\tau + 2\pi p/N_1a)$ in ε_1. It is given by the series (2.22) with $t = \tau + 2\pi p/N_1a$ and j uniquely defined by t from the relation $p_j^2(t) \in \varepsilon_1$. Uniqueness means: $\lambda_n^{(1)}(\tau + 2\pi p/N_1a) \neq \lambda_{\hat{n}}^{(1)}(\tau + 2\pi p/N_1a)$ when $\hat{n} \neq n$. Since τ is a point of self-intersection of $\tilde{S}_1(\lambda)$, $\lambda_n^{(1)}(\tau + 2\pi p/N_1a) = \lambda_{\hat{n}}^{(1)}(\tau + 2\pi\hat{p}/N_1a) = \lambda$ for some $\hat{p} \neq p$. The converse part of the lemma is trivial. ∎

To obtain a new non-resonance set $\chi_2(\lambda)$ we remove from $\mathcal{K}_2\chi_1^*(\lambda)$ a neighborhood of its intersections (quasi-intersections) with the whole isoenergetic surface $\tilde{S}_1(\lambda)$ given by (3.5)–(3.7). More precisely, we remove from $\mathcal{K}_2\chi_1^*(\lambda)$ the following set:

$$\begin{aligned}\Omega_1(\lambda) = \{\tau \in \mathcal{K}_2\chi_1^*(\lambda) : \exists n, \hat{n} \in \mathbb{N},\ p, \hat{p} \in P,\ p \neq \hat{p} :\ \lambda_n^{(1)}(\tau + 2\pi p/N_1a) = \lambda,\\ \tau + 2\pi p/N_1a \in \chi_1^*(\lambda),\ |\lambda_n^{(1)}(\tau + 2\pi p/N_1a) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi\hat{p}/N_1a)| \leq \epsilon_1\},\end{aligned} \tag{3.10}$$

where [6]

$$\epsilon_1 = e^{-\frac{1}{4}k^{\eta s_1}}. \tag{3.11}$$

We define $\chi_2(\lambda)$ by the formula:

$$\chi_2(\lambda) = \mathcal{K}_2\chi_1^*(\lambda) \ \setminus \Omega_1(\lambda). \tag{3.12}$$

[6]Note that ε_1 is an interval, while ϵ_1 is a number.

3.2. **Perturbation Formulae.** Before proving the main result, we formulate Geometric Lemma:

Lemma 3.5 (Geometric Lemma)**.** *If $\lambda > k_*^2$, there exists a non-resonance set $\chi_2(\lambda, \delta) \subset \mathcal{K}_2\chi_1^*$ such that:*

(1) *For any $\tau \in \chi_2$, the following conditions hold:*
   (a) *There exists a unique $p \in P$ such that $\tau + 2\pi p/N_1 a \in \chi_1^*$.* [7]
   (b) *The following relation holds:*

$$\lambda_j^{(1)}(\tau + 2\pi p/N_1 a) = k^2,$$

   where $\lambda_j^{(1)}(\tau + 2\pi p/N_1 a)$ is given by the perturbation series (2.22) with $\alpha = 1$, j being uniquely defined by $t = \tau + 2\pi p/N_1 a$ and the relation $p_j^2(t) \in \varepsilon_1$.
   (c) *The eigenvalue $\lambda_j^{(1)}(\tau + 2\pi p/N_1 a)$ is a simple eigenvalue of $\tilde{H}^{(1)}(\tau)$ and its distance from all other eigenvalues $\lambda_{\hat{n}}^{(1)}(\tau + 2\pi\hat{p}/N_1 a)$, $\hat{n} \in \mathbb{N}$ of $\tilde{H}_1(\tau)$ is greater than $\epsilon_1 = e^{-\frac{1}{4}k^{\eta s_1}}$:*

$$|\lambda_j^{(1)}(\tau + 2\pi p/N_1 a) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi\hat{p}/N_1 a)| > \epsilon_1. \tag{3.13}$$

(2) *For any τ in the real $(\epsilon_1 k^{-1-\delta})$-neighborhood of χ_2, there exists a unique $p \in P$ such that $\tau + 2\pi p/N_1 a$ is in the $(\epsilon_1 k^{-1-\delta})$-neighborhood of χ_1^* and*

$$|\lambda_j^{(1)}(\tau + 2\pi p/N_1 a) - k^2| < 2\epsilon_1 k^{-\delta}, \tag{3.14}$$

   j being uniquely defined by $t = \tau + 2\pi p/N_1 a$ j and the relation $p_j^2(t) \in \varepsilon_1$. An estimate analogous to (3.13) *holds:*

$$2|\lambda_j^{(1)}(\tau + 2\pi p/N_1 a) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi\hat{p}/N_1 a)| > \epsilon_1. \tag{3.15}$$

(3) *The second non-resonance set χ_2 has an asymptotically full measure in χ_1^* in the following sense:*

$$\frac{L(\mathcal{K}_2\chi_1^* \setminus \chi_2))}{L(\chi_1^*)} < c_0 k^{-2-2s_1}. \tag{3.16}$$

Corollary 3.6. *If τ belongs to the complex $(\epsilon_1 k^{-1-\delta})-$neighborhood of the second non-resonance set $\chi_2(\lambda, \delta)$, then for any $z \in \mathbb{C}$ lying on the circle*

$$C_2 = \{z : |z - k^2| = \epsilon_1/2\}, \tag{3.17}$$

the following inequalities hold:

$$\|(\tilde{H}^{(1)}(\tau) - z)^{-1}\| \leq \frac{4}{\epsilon_1}, \tag{3.18}$$

Corollary is proven in Appendix 3.

[7] From geometric point of view this means that $\chi_2(\lambda)$ does not have self-intersections.

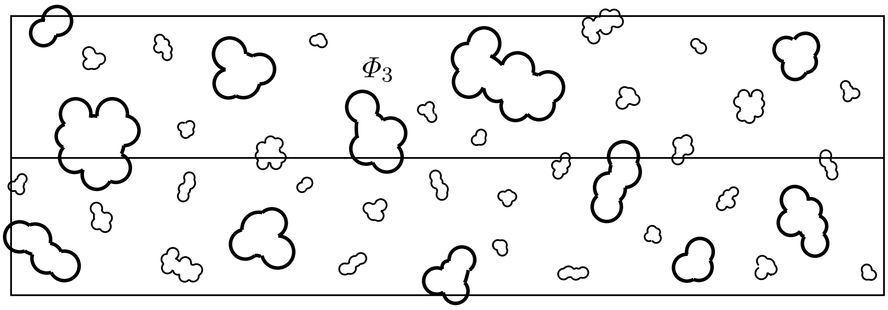

FIGURE 6. The set Φ_3.

Proof. Let us consider

$$\mathcal{O}^{(2)} = \cup_{p' \in P \setminus \{0\}} \mathcal{O}_s \left(\frac{2\pi p'}{N_1 a} \right), \tag{3.19}$$

where $\mathcal{O}_s(\cdot)$ is defined by (2.122) in Definition 2.26, $\vec{b} = \frac{2\pi p'}{N_1 a}$. Note, that the definition make sense, since $|\frac{2\pi p'}{N_1 a}| > d_{max}^{-1} k^{-s_2} > k^{-1-16s_1-12\delta}$, $d_{max} = \max\{d_1, d_2\}$. Let

$$\Phi_2 = \Phi_1 \setminus \mathcal{O}^{(2)}, \quad \Theta_2 = \Phi_2 \cap [0, 2\pi). \tag{3.20}$$

By Lemma 2.29,

$$\left\| \left(H^{(1)} \left(\vec{\varkappa}_1(\varphi) + \frac{2\pi p'}{N_1 a} \right) - k^2 \right)^{-1} \right\| < k^{J^{(1)}}, J^{(1)} = c_1 k^{2/3+s_1}. \tag{3.21}$$

for all $p' \in P \setminus \{0\}$ and $\varphi \in \Phi_2$. Estimate (3.1) yields $k^{J^{(1)}} < \epsilon_1^{-1}$. We consider $\mathcal{D}_{1,nonres} \subset \mathcal{D}_1$:

$$\mathcal{D}_{1,nonres} = \{ \vec{\varkappa}_1(\varphi), \ \varphi \in \Theta_2 \}. \tag{3.22}$$

We define χ_2 by the formula:

$$\chi_2 = \mathcal{K}_2 \mathcal{D}_{1,nonres}. \tag{3.23}$$

By definition of $\mathcal{K}_2$, for every τ in χ_2, there are $p \in P$ and $j \in Z^2$ such that

$$\tau + \frac{2\pi p}{N_1 a} + \frac{2\pi j}{a} = \vec{\varkappa}_1(\varphi), \quad \vec{\varkappa}_1(\varphi) \in \mathcal{D}_{1,nonres}. \tag{3.24}$$

Considering (3.21), the estimate $k^{J^{(1)}} < \epsilon_1^{-1}$ and the definition of $\mathcal{D}_{1,nonres}$, we obtain:

$$\left\| \left(H^{(1)} \left(\tau + \frac{2\pi \hat{p}}{N_1 a} \right) - k^2 \right)^{-1} \right\| < \epsilon_1^{-1}, \quad \hat{p} = p + p'. \tag{3.25}$$

Note that the index j does not play a role, since it just produces the shift j of indices of the matrix elements of the resolvent. Considering that p' can be any but zero, we obtain that (3.25) holds for all $\hat{p} \in P \setminus \{p\}$. Taking into account that $\lambda_j \left(\tau + \frac{2\pi p}{N_1 a} \right) = k^2$ and inequality (3.25), we arrive at (3.13) for all $\hat{p} \neq p$. It remains to check (3.13) for $p = \hat{p}$. Let $t = \tau + \frac{2\pi p}{N_1 a}$. By (3.24), $t \in \mathcal{K}_1 \mathcal{D}_1 = \chi_1$. By Theorem 2.3

$\lambda_j\left(\tau+\frac{2\pi p}{N_1 a}\right)$ is a holomorphic function of τ in $(\epsilon_1 k^{-1-\delta})-$neighborhood of $\chi_2(\lambda,\delta)$ and

$$|\lambda_j^{(1)}(\tau+2\pi p/N_1 a)-\lambda_{\hat{n}}^{(1)}(\tau+2\pi p/N_1 a)|>k^{2\beta-2-2s_1}>\epsilon_1.$$

Part 2 follows from stability of all estimates with respect to perturbation of τ smaller then $(\epsilon_1 k^{-1-\delta})$. Indeed, suppose τ is in the real $(\epsilon_1 k^{-1-\delta})-$neighborhood of $\chi_2(\lambda,\delta)$. Then, there is a $\tau_0\in\chi_2(\lambda,\delta)$, such that $|\tau-\tau_0|<\epsilon_1 k^{-1-\delta}$. Let p,j be defined by τ_0 as in Part 1. Obviously, perturbation series for $\lambda_j^{(1)}(\tau+2\pi p/N_1 a)$ converges and (3.14) holds. Using (3.13), we easily obtain (3.15). Therefore, p, j are defined uniquely by (3.14).

Let us estimate the size of $\mathcal{O}_2$. According to Lemma 2.27, the total size of each $\mathcal{O}_s\left(\frac{2\pi p'}{N_1 a}\right)$ is less then $c_0 k^{-2-4s_1}$. Considering that the number of p does not exceed $4k^{2s_1}$, we obtain that the total size of $\mathcal{O}_2$ is less then $c_0k^{-2-2s_1}$. Using (2.70), we arrive at (3.16). ∎

Remark 3.7. Note that every point $\frac{2\pi m}{N_1 a}$ $(m\in\mathbb{Z}^2)$ of a dual lattice corresponding to the larger periods N_1a_1, N_1a_2 can be uniquely represented in the form $\frac{2\pi m}{N_1 a}=\frac{2\pi j}{a}+\frac{2\pi p}{N_1 a}$, where $m=N_1 j+p$ and $\frac{2\pi j}{a}$ is a point of a dual lattice for periods a_1, a_2, while $p\in P$ is responsible for refining the lattice.

Let us consider a normalized eigenfunction $\psi_n(t,x)$ of $H^{(1)}(t)$ in $L_2(Q_1)$. We extend it quasiperiodically to Q_2, renormalize in $L_2(Q_2)$ and denote the new function by $\tilde{\psi}_n(\tau,x)$, $\tau=\mathcal{K}_2 t$. The Fourier representations of $\psi_n(t,x)$ in $L_2(Q_1)$ and $\tilde{\psi}_n(\tau,x)$ in $L_2(Q_2)$ are simply related. If we denote Fourier coefficients of $\psi_n(t,x)$ with respect to the basis of exponential functions $\frac{1}{|Q_1|^{1/2}}e^{i\langle t+\frac{2\pi j}{a},x\rangle}$, $j\in\mathbb{Z}^2$, in $L_2(Q_1)$ by C_{nj}, then, the Fourier coefficients $\tilde{C}_{nm}$ of $\tilde{\psi}_n(\tau,x)$ with respect to the basis of exponential functions $\frac{1}{|Q_2|^{1/2}}e^{i\langle \tau+\frac{2\pi m}{N_1 a},x\rangle}$, $m\in\mathbb{Z}^2$, in $L_2(Q_2)$ are given by the formula:

$$\tilde{C}_{nm}=\begin{cases}C_{nj}, & \text{if } m=jN_1+p;\\ 0, & \text{otherwise,}\end{cases}$$

p being defined from the relation $t=\tau+\frac{2\pi p}{N_1 a}$, $p\in P$. Hence, matrices of the projections on $\psi_n(t,x)$ and $\tilde{\psi}_n(\tau,x)$ with respect to the above bases are simply related:

$$(\tilde{E}_n)_{j\hat{j}}=\begin{cases}(E_n)_{m\hat{m}}, & \text{if } m=jN_1+p,\ \hat{m}=\hat{j}N_1+p;\\ 0, & \text{otherwise,}\end{cases}$$

$\tilde{E}_n$ and E_n being projections in $L_2(Q_2)$ and $L_2(Q_1)$, respectively.

Let us denote by $\tilde{E}_j^{(1)}\left(\tau+\frac{2\pi p}{N_1 a}\right)$ the spectral projection $E_j^{(1)}(\alpha,t)$ (see (2.23)) with $\alpha=1$ and $t=\tau+\frac{2\pi p}{N_1 a}$, "extended" from $L_2(Q_1)$ to $L_2(Q_2)$.

By analogy with (2.18), (2.19), we define functions $g_r^{(2)}(k,\tau)$ and operator-valued functions $G_r^{(2)}(k,\tau)$, $r=1,2,\cdots$, as follows:

$$g_r^{(2)}(k,\tau)=\frac{(-1)^r}{2\pi i r}\mathrm{Tr}\oint_{C_2}\left(\left(\tilde{H}^{(1)}(\tau)-z\right)^{-1}W_2\right)^r dz, \tag{3.26}$$

$$G_r^{(2)}(k,\tau)=\frac{(-1)^{r+1}}{2\pi i}\oint_{C_2}\Big(\big(\tilde H^{(1)}(\tau)-z\big)^{-1}W_2\Big)^r\big(\tilde H^{(1)}(\tau)-z\big)^{-1}dz, \tag{3.27}$$

$\tilde H^{(1)}(\tau)$ being defined at the beginning of Section 3.1.1, C_2 being given by (3.17). We consider the operators $H_\alpha^{(2)}=H^{(1)}+\alpha W_2$ and the family $H_\alpha^{(2)}(\tau)$, $\tau\in K_2$, acting in $L_2(Q_2)$. By (3.2) and (3.11)

$$\|W_2\|<\epsilon_1^4, \tag{3.28}$$

$\|W_2\|$ being the norm of the operator here.

Theorem 3.8. *Suppose τ belongs to the $(\epsilon_1 k^{-1-\delta})$-neighborhood in K_2 of the second non-resonance set $\chi_2(\lambda,\delta)$, $0<\delta<s_1$, $\epsilon_1=e^{-\frac14 k^{\eta s_1}}$. Then, for sufficiently large λ, $\lambda>k_*^2$ and for all α, $0\le\alpha\le1$, there exists a unique eigenvalue of the operator $H_\alpha^{(2)}(\tau)$ in the interval $\varepsilon_2(k):=(k^2-\epsilon_1/2,k^2+\epsilon_1/2)$. It is given by the series:*

$$\lambda_{\tilde j}^{(2)}(\alpha,\tau)=\lambda_j^{(1)}\big(\tau+2\pi p/N_1a\big)+\sum_{r=1}^{\infty}\alpha^r g_r^{(2)}(k,\tau),\qquad \tilde j=j+p/N_1, \tag{3.29}$$

converging absolutely in the disk $|\alpha|\le1$, where $p\in P$ and $j\in\mathbb{Z}^2$ are described as in Geometric Lemma 3.5. The spectral projection corresponding to $\lambda_{\tilde j}^{(2)}(\alpha,\tau)$ is given by the series:

$$E_{\tilde j}^{(2)}(\alpha,\tau)=\tilde E_j^{(1)}\big(\tau+2\pi p/N_1a\big)+\sum_{r=1}^{\infty}\alpha^r G_r^{(2)}(k,\tau), \tag{3.30}$$

which converges in the trace class $\mathbf{S_1}$ uniformly with respect to α in the disk $|\alpha|\le1$.

The following estimates hold for coefficients $g_r^{(2)}(k,\tau)$, $G_r^{(2)}(k,\tau)$, $r\ge1$:

$$\big|g_r^{(2)}(k,\tau)\big|<\frac{3\epsilon_1}{2}(4\epsilon_1^3)^r,\qquad \big\|G_r^{(2)}(k,\tau)\big\|_1<6r(4\epsilon_1^3)^r. \tag{3.31}$$

Corollary 3.9. *The following estimates hold for the perturbed eigenvalue and its spectral projection:*

$$\Big|\lambda_{\tilde j}^{(2)}(\alpha,\tau)-\lambda_j^{(1)}\big(\tau+2\pi p/N_1a\big)\Big|\le12\alpha\epsilon_1^4, \tag{3.32}$$

$$\Big\|E_{\tilde j}^{(2)}(\alpha,\tau)-\tilde E_j^{(1)}\big(\tau+2\pi p/N_1a\big)\Big\|_1\le48\alpha\epsilon_1^3. \tag{3.33}$$

Remark 3.10. The theorem states that $\lambda_{\tilde j}^{(2)}(\alpha,\tau)$ is a single eigenvalue in the interval $\varepsilon_2(k,\delta)$. This means that $\big|\lambda_{\tilde j}^{(2)}(\alpha,\tau)-k^2\big|<\epsilon_1/2$. Formula (3.32) provides a stronger estimate on the location of $\lambda_{\tilde j}^{(2)}(\alpha,\tau)$.

Proof. The proof of the theorem is based on expanding the resolvent $(H_\alpha^{(2)}(\tau)-z)^{-1}$ in a perturbation series for $z\in C_2$. Integrating the resolvent yields the formulae for an eigenvalue of $H_\alpha^{(2)}$ and its spectral projection. In fact, it is obvious that

$$(H_\alpha^{(2)}(\tau)-z)^{-1}=(\tilde H^{(1)}(\tau)-z)^{-1}(I-\alpha A_2)^{-1},\ \ A_2:=-W_2(\tilde H^{(1)}(\tau)-z)^{-1}. \tag{3.34}$$

Suppose $z \in C_2$. Using Corollary 3.6 and estimate (3.28), we obtain:

$$\|(\tilde{H}^{(1)}(\tau) - z)^{-1}\| \le \frac{4}{\epsilon_1}, \quad \|A_2\| \le \frac{4\|W_2\|}{\epsilon_1} < 4\epsilon_1^3 < 1. \tag{3.35}$$

The last inequality makes it possible to expand $(I - \alpha A_2)^{-1}$ in the series in powers of αA_2. Integrating the series for the resolvent and considering as in the proof of Theorem 2.3 we obtain formulae (3.29), (3.30). Estimates (3.31) follow from the estimates (3.35). ∎

Next, we show that the series (3.29), (3.30) can be extended as holomorphic functions of τ in a complex neighborhood of χ_2; they can be differentiated any number of times with respect to τ and retain their asymptotic character.

Lemma 3.11. *The following estimates hold for the coefficients $g_r^{(2)}(k,\tau)$ and $G_r^{(2)}(k,\tau)$ in the complex $(\frac{1}{2}\epsilon_1 k^{-1-\delta})$-neighborhood of the non-resonance set χ_2:*

$$|T(m) g_r^{(2)}(k,\tau)| < m! \cdot 3 \cdot 2^{2r-1+|m|} \epsilon_1^{3r+1-|m|} k^{|m|(1+\delta)}, \tag{3.36}$$

$$\|T(m) G_r^{(2)}(k,\tau)\| < m! \cdot 3r \cdot 2^{2r+1+|m|} \epsilon_1^{3r-|m|} k^{|m|(1+\delta)}. \tag{3.37}$$

Proof. Since (3.18) is valid in the complex $(\epsilon_1 k^{-1-\delta})$- neighborhood of the second non-resonance set, it is not hard to see that the coefficients $g_r^{(2)}(k,\tau)$ and $G_r^{(2)}(k,\tau)$ can be continued from the real $(\epsilon_1 k^{-1-\delta})$-neighborhood of τ to the complex $(\epsilon_1 k^{-1-\delta})$-neighborhood as holomorphic functions of two variables and inequalities (3.31) are hereby preserved. Estimating, by means of the Cauchy integral formula, the value of the derivative with respect to τ in terms of the value of the function itself on the boundary of the $(\frac{1}{2}\epsilon_1 k^{-1-\delta})$-neighborhood of τ (formulas (3.31)), we obtain (3.36) and (3.37). ∎

From this lemma the following theorem easily follows.

Theorem 3.12. *Under the conditions of Theorem 3.8 the series (3.29), (3.30) can be continued as holomorphic functions of two variables from the real $(\epsilon_1 k^{-1-\delta})$-neighborhood of the non-resonance set χ_2 to its complex $(\epsilon_1 k^{-1-\delta})$-neighborhood and the following estimates hold in the complex neighborhood:*

$$\left|T(m)\left(\lambda_{\tilde{j}}^{(2)}(\alpha,\tau) - \lambda_j^{(1)}(\tau + 2\pi p/N_1 a)\right)\right| < \alpha C_m \epsilon_1^{4-|m|} k^{|m|(1+\delta)}, \tag{3.38}$$

$$\left\|T(m)\left(E_{\tilde{j}}^{(2)}(\alpha,\tau) - \tilde{E}_j^{(1)}(\tau + 2\pi p/N_1 a)\right)\right\| < \alpha C_m \epsilon_1^{3-|m|} k^{|m|(1+\delta)}, \tag{3.39}$$

here and below $C_m = 48 m! 2^{|m|}$.

Corollary 3.13.

$$\left|\nabla \lambda_j^{(2)}(\alpha,\tau) - 2\vec{k}\right| < 2C(W_1) k^{-1-2\beta+15s_1+11\delta}, \quad \vec{k} = \vec{p}_j(\tau + 2\pi p/N_1 a) \tag{3.40}$$

$$\left|T(m)\lambda_j^{(2)}(\alpha,\tau)\right| < 2 + 2C(W_1) k^{-1-2\beta+21s_1+15\delta}, \text{ if } |m| = 2. \tag{3.41}$$

The next lemma will be used in the third step of approximation. The operator $H^{(2)}(\tau)$ is $H^{(2)}_\alpha(\tau)$ with $\alpha=1$. It will play a role of the initial (unperturbed) operator in the third step.

Lemma 3.14. *For any z on the circle C_2 and τ in the complex $(\epsilon_1 k^{-1-\delta})-$ neighborhood of χ_2,*

$$\|(H^{(2)}(\tau)-z)^{-1}\| \leq \frac{8}{\epsilon_1}. \tag{3.42}$$

Proof. Considering the Hilbert relation

$$(H^{(2)}(\tau)-z)^{-1}=(\tilde{H}^{(1)}(\tau)-z)^{-1}+(\tilde{H}^{(1)}(\tau)-z)^{-1}(-W_2)(H^{(2)}(\tau)-z)^{-1},$$

and the estimate (3.18), together with the estimate (3.28), we obtain:

$$\|(H^{(2)}(\tau)-z)^{-1}\| \leq \frac{\|(\tilde{H}^{(1)}(\tau)-z)^{-1}\|}{1-\|(\tilde{H}^{(1)}(\tau)-z)^{-1}W_2\|} \leq 2\|(\tilde{H}^{(1)}(\tau)-z)^{-1}\| \leq \frac{8}{\epsilon_1}. \tag{3.43}$$

∎

3.3. **Non-resonant part of the isoenergetic set of $H^{(2)}_\alpha$.** Let $S_2(\lambda)$ be an isoenergetic set of the operator $H^{(2)}_\alpha$: $S_2(\lambda)=\{\tau\in K_2 : \exists n\in\mathbb{N} : \ \lambda^{(2)}_n(\alpha,\tau)=\lambda\}$, here $\{\lambda^{(2)}_n(\alpha,\tau)\}_{n=1}^{\infty}$ is the spectrum of $H^{(2)}_\alpha(\tau)$. Now we construct a non-resonance subset $\chi^*_2(\lambda)$ of $S_2(\lambda)$. It corresponds to non-resonance eigenvalues $\lambda^{(2)}_{\tilde{j}}(\tau)$ given by the perturbation series (3.29). Recall that $\mathcal{D}_1(\lambda)_{nonres}$ and $\chi_2(\lambda)$ are defined by the formulae (3.22) and (3.23) respectively. Recall also that $\chi_2\subset\mathcal{K}_2\chi^*_1(\lambda)$ (see the Geometric Lemma) and $\chi^*_1(\lambda)=\mathcal{K}_1\mathcal{D}_1(\lambda)$, see (2.73). Hence, $\chi_2\subset\mathcal{K}_2\mathcal{D}_1(\lambda)$.

Lemma 3.15. *The formula $\mathcal{K}_2\mathcal{D}_1(\lambda)_{nonres}=\chi_2$ establishes one-to-one correspondence between $\mathcal{D}_1(\lambda)_{nonres}$ and χ_2.*

Proof. Suppose there is a pair $\vec{\varkappa}_1,\vec{\varkappa}_{1*}\in\mathcal{D}_1(\lambda)_{nonres}$ such that $\mathcal{K}_2\vec{\varkappa}_1=\mathcal{K}_2\vec{\varkappa}_{1*}=\tau$, $\tau\in\chi_2$. We introduce also $t_1=\mathcal{K}_1\vec{\varkappa}_1$ and $t_{1*}=\mathcal{K}_1\vec{\varkappa}_{1*}$. The definition (2.73) of $\chi^*_1(\lambda)$ implies that $t_1,t_{1*}\in\chi^*_1(\lambda)$, since $\vec{\varkappa}_1,\vec{\varkappa}_{1*}\in\mathcal{D}_1(\lambda)_{nonres}\subset\mathcal{D}_1(\lambda)$. Clearly, $\mathcal{K}_2t_1=\mathcal{K}_2t_{1*}=\tau$ and, hence, $t_1=\tau+2\pi p_1/N_1a$, $t_{1*}=\tau+2\pi p_2/N_1a$ for some $p_1,p_2\in P$. Now, by Part 1a of Geometric Lemma 3.5, $p_1=p_2$, and, therefore, $t_1=t_{1*}$. Next, by Lemma 2.12, $\vec{\varkappa}_1=\vec{\varkappa}_{1*}$. ∎

We define $\mathcal{B}_2(\lambda)$ as the set of directions corresponding to the set Θ_2 given by (3.20):

$$\mathcal{B}_2(\lambda)=\{\vec{\nu}\in\mathcal{B}_1(\lambda):\varphi\in\Theta_2\}.$$

Note that $\mathcal{B}_2(\lambda)$ is a unit circle with holes, centered at the origin, and $\mathcal{B}_2(\lambda)\subset\mathcal{B}_1(\lambda)$.

Let $\vec{\varkappa}\in\mathcal{D}_1(\lambda)_{nonres}$. By (3.23), $\tau\equiv\mathcal{K}_2\vec{\varkappa}\in\chi_2(\lambda)$. According to Theorem 3.8, for sufficiently large λ, there exists an eigenvalue of the operator $H^{(2)}_\alpha(\tau)$, given by (3.29). It is convenient here to denote $\lambda^{(2)}_{\tilde{j}}(\alpha,\tau)$ by $\lambda^{(2)}(\alpha,\vec{\varkappa})$. We can do this, since, by Lemma 3.15, there is one-to-one correspondence between $\vec{\varkappa}\in\mathcal{D}_1(\lambda)_{nonres}$ and the pair $(\tau,\tilde{j})$, $\vec{\varkappa}=2\pi\tilde{j}/a+\tau$. We rewrite (3.29) in the form:

$$\lambda^{(2)}(\alpha,\vec{\varkappa})=\lambda^{(1)}(\vec{\varkappa})+f_2(\alpha,\vec{\varkappa}),\quad f_2(\alpha,\vec{\varkappa})=\sum_{r=1}^{\infty}\alpha^r g^{(2)}_r(\vec{\varkappa}), \tag{3.44}$$

here $g_r^{(2)}(\vec{\varkappa})$ is given by (3.26). The function $f_2(\alpha,\vec{\varkappa})$ satisfies the estimates:

$$|f_2(\alpha,\vec{\varkappa})| \le 12\alpha\epsilon_1^4, \tag{3.45}$$

$$|\nabla f_2(\alpha,\vec{\varkappa})| \le 96\alpha\epsilon_1^3 k^{1+\delta}. \tag{3.46}$$

By Theorems 3.8 and 3.12, the formulas (3.44) – (3.46) hold even in the real $(\epsilon_1 k^{-1-\delta})$-neighborhood of $\mathcal{D}_1(\lambda)_{nonres}$, i.e., for any $\vec{\varkappa} = \varkappa\vec{\nu}$ such that $\vec{\nu} \in \mathcal{B}_2(\lambda)$ and $|\varkappa - \varkappa_1(\lambda,\vec{\nu})| < \epsilon_1 k^{-1-\delta}$. We define $\mathcal{D}_2(\lambda)$ as a level set for $\lambda^{(2)}(\alpha,\vec{\varkappa})$ in this neighborhood:

$$\mathcal{D}_2(\lambda) := \left\{\vec{\varkappa} = \varkappa\vec{\nu} : \vec{\nu} \in \mathcal{B}_2,\ \ \big|\varkappa - \varkappa_1(\lambda,\vec{\nu})\big| < \epsilon_1 k^{-1-\delta},\ \lambda^{(2)}(\alpha,\vec{\varkappa}) = \lambda\right\}. \tag{3.47}$$

Next two lemmas are to prove that $\mathcal{D}_2(\lambda)$ is a distorted circle with holes.

Lemma 3.16. *For every $\vec{\nu} \in \mathcal{B}_2$ and every α, $0 \le \alpha \le 1$, there is a unique $\varkappa = \varkappa_2(\lambda,\vec{\nu})$ in the interval $I_2 := \left[\varkappa_1(\lambda,\vec{\nu}) - \epsilon_1 k^{-1-\delta}, \varkappa_1(\lambda,\vec{\nu}) + \epsilon_1 k^{-1-\delta}\right]$ such that*

$$\lambda^{(2)}(\alpha,\varkappa_2\vec{\nu}) = \lambda. \tag{3.48}$$

Furthermore,

$$|\varkappa_2(\lambda,\vec{\nu}) - \varkappa_1(\lambda,\vec{\nu})| \le 12\alpha\epsilon_1^4 k^{-1}. \tag{3.49}$$

The proof is based on (3.44), (3.45), (3.40) and completely analogous to that of Lemma 3.41 in [24], set $l = 1$.

Further, we use the notations:

$$\varkappa_2(\varphi) \equiv \varkappa_2(\lambda,\vec{\nu}), \quad h_2(\varphi) = \varkappa_2(\varphi) - \varkappa_1(\varphi), \quad \vec{\varkappa}_2(\varphi) = \varkappa_2(\varphi)\vec{\nu}. \tag{3.50}$$

Lemma 3.17. *Let $10^{-4} < s_1 < 10^{-3}$, $\eta > 10^4$. Then the following statements hold for $\lambda > k_*^2$:*

1. *The set $\mathcal{D}_2(\lambda)$ is a distorted circle with holes: it can be described by the formula:*

$$\mathcal{D}_2(\lambda) = \left\{\vec{\varkappa} \in \mathbb{R}^2 : \vec{\varkappa} = \vec{\varkappa}_2(\varphi),\quad \varphi \in \Theta_2(\lambda)\right\}, \tag{3.51}$$

where $\varkappa_2(\varphi) = \varkappa_1(\varphi) + h_2(\varphi)$, $\varkappa_1(\varphi)$ is the "radius" of $\mathcal{D}_1(\lambda)$ and $h_2(\varphi)$ satisfies the estimates

$$|h_2| \le 12\alpha\epsilon_1^4 k^{-1}, \qquad \left|\frac{\partial h_2}{\partial\varphi}\right| \le 96\alpha\epsilon_1^3 k^{1+\delta}. \tag{3.52}$$

2. *The total length of $\mathcal{B}_2(\lambda)$ satisfies the estimate:*

$$L\left(\mathcal{B}_1 \setminus \mathcal{B}_2\right) < c_0 k^{-2-2s_1}. \tag{3.53}$$

3. *The function $h_2(\varphi)$ can be extended as a holomorphic function of φ into the complex non-resonance set Φ_2 and its $k^{-4-6s_1-4\delta}$ neighborhood $\hat{\Phi}_2$, estimates (3.52) being preserved.*
4. *The curve $\mathcal{D}_2(\lambda)$ has a length which is asymptotically close to that of $\mathcal{D}_1(\lambda)$ in the following sense:*

$$L\Big(\mathcal{D}_2(\lambda)\Big) \underset{\lambda\to\infty}{=} L\Big(\mathcal{D}_1(\lambda)\Big)\Big(1 + O\big(k^{-2-2s_1}\big)\Big), \tag{3.54}$$

where $O\big(k^{-2-2s_1}\big) = (1+o(1))c_0 k^{-2-2s_1}$, $|o(1)| < 10^{-2}$ when $k > k_*$.

Proof. The proof is completely analogous to that of Lemma 3.42 in [24], set $l = 1$. Here we give just a short version. Indeed, the first inequality in (3.52) is equivalent to (3.49). Differentiating the identity $\lambda^{(2)}\left(\vec{\varkappa}_2(\varphi)\right) = \lambda^{(1)}\left(\vec{\varkappa}_1(\varphi)\right) = k^2$ with respect to φ and using (3.40), (3.44), (3.46), (3.49), we easily obtain the second estimate in (3.52). Estimate (3.53) valid, since the total size of $O^{(2)}$ is less than $c_0 k^{-2-2s_1}$. To prove the analyticity of $h_2(\varphi)$ in Φ_2 we check the convergence of the perturbation series for $\lambda^{(2)}\left(\vec{\varkappa}_1(\varphi)\right)$, $\varphi \in \Phi_2$. It is enough to show that

$$\left\|\left(\tilde{H}^{(1)}(\vec{\varkappa}_1(\varphi)) - z\right)^{-1}\right\| \leq \frac{4}{\varepsilon_1}, \quad z \in C_2. \tag{3.55}$$

This inequality immediately follows from

$$\left\|\left(H^{(1)}\left(\vec{\varkappa}_1(\varphi) + \frac{2\pi p}{N_1 a}\right) - z\right)^{-1}\right\| \leq \frac{4}{\varepsilon_1} \tag{3.56}$$

proven for all $p \in P$. Let $p = 0$. By Lemma 2.8,

$$\|(H^{(1)}(\vec{\varkappa}_1(\varphi)) - z)^{-1}\| \leq 2k^{-2\beta+1+s_1+\delta}, \quad \text{when } z \in C_1, \quad 2\beta = 1 - 15s_1 - 9\delta.$$

It is not difficult to show that the resolvent has a single pole inside C_1 at the point $z = k^2$. The circle C_2 has the same centrum and a smaller radius $\epsilon_1/2$. Hence

$$\|(H^{(1)}(\vec{\varkappa}_1(\varphi)) - z)^{-1}\| \leq \frac{4}{\epsilon_1}, \quad \text{when } z \in C_2. \tag{3.57}$$

Next we use (2.123) with $\vec{b} = \frac{2\pi p}{N_1 a}$. The right-hand side of (2.123) is less that ϵ_1^{-1}, since $s_1\eta > 1$. Hence, (3.56) holds for $p \neq 0$ too. Convergence of perturbation series follows from (3.55) and (3.28). Note that the smallest circle in $\mathcal{O}^{(2)}$ has the size $k^{-4-6s_1-3\delta}$. Since $k^{-4-6s_1-4\delta}$ is much smaller than the radius of circles, all estimates are stable in the $k^{-4-6s_1-4\delta}$ of Φ_2. Now, using Rouche's and Implicit function theorems we easily show the equation $\lambda^{(2)}(\varkappa\vec{\nu}) = k^2$ has a solution $\varkappa = \varkappa_2(\varphi)$ which is holomorphic in Φ_2 and coincides with $\varkappa_1(\varphi) + h_2(\varphi)$ for real φ. Estimate (3.54) follows from (3.52) and (3.53).

Let us record a remark for the sequel. Convergence of the series for the resolvent $\left(H^{(2)}\left(\vec{\varkappa}_1(\varphi)\right) - z\right)^{-1}$, $z \in C_2$, means that the resolvent has a single pole $z = \lambda^{(2)}(\vec{\varkappa}_1(\varphi)\vec{\nu})$ inside C_2. Similar result holds when we replace $\vec{\varkappa}_1(\varphi)$ by $\vec{\varkappa}_2(\varphi)$, since $\vec{\varkappa}_1(\varphi)$ and $\vec{\varkappa}_2(\varphi)$ are close: $\left|\vec{\varkappa}_2(\varphi) - \vec{\varkappa}_1(\varphi)\right| = o(\epsilon_1)$. Considering that $\lambda^{(2)}(\vec{\varkappa}_2(\varphi)) = \lambda$, we obtain that $(z-\lambda)\left(H^{(2)}\left(\vec{\varkappa}_2(\varphi)\right) - z\right)^{-1}$ is holomorphic inside C_2 and the estimate following holds:

$$\left\|(z-\lambda)\left(H^{(2)}\left(\vec{\varkappa}_2(\varphi)\right) - z\right)^{-1}\right\| < 32. \tag{3.58}$$

∎

Now define the non-resonance set $\chi_2^*(\lambda)$ in $S_2(\lambda)$ by

$$\chi_2^*(\lambda) := \mathcal{K}_2\mathcal{D}_2(\lambda). \tag{3.59}$$

Lemma 3.18. *The set $\chi_2^*(\lambda)$ belongs to the $\left(12\alpha\epsilon_1^4 k^{-1}\right)$-neighborhood of $\chi_2(\lambda)$ in K_2. If $\tau \in \chi_2^*(\lambda)$, then the operator $H_\alpha^{(2)}(\tau)$ has a simple eigenvalue equal to λ. This* *eigenvalue is given by the perturbation series (3.29), where $p \in P, j \in \mathbb{Z}^2$ are uniquely defined by τ as it is described in Geometric Lemma 3.5, part 2.*

Proof. By Lemma 3.17, $\mathcal{D}_2(\lambda)$ is in the $\left(12\alpha\epsilon_1^4 k^{-1}\right)$ neighborhood of $\mathcal{D}_1(\lambda)_{nonres}$. Considering that $\chi_2^*(\lambda) = \mathcal{K}_2\mathcal{D}_2(\lambda)$ and $\chi_2(\lambda) = \mathcal{K}_2\mathcal{D}_1(\lambda)_{nonres}$ (see (3.23)), we immediately obtain that $\chi_2^*(\lambda)$ is in the $(12\alpha\epsilon_1^4 k^{-1})$-neighborhood of $\chi_2(\lambda)$. The size of this neighborhood is less than $\epsilon_1 k^{-1-\delta}$, hence Theorem 3.8 holds in it, i.e., for any $\tau \in \chi_2^*(\lambda)$ there is a single eigenvalue of $H_\alpha^{(2)}(\tau)$ in the interval $\varepsilon_2(k,\delta)$. Since $\chi_2^*(\lambda) \subset S_2(\lambda)$, this eigenvalue is equal to λ. By the theorem, the eigenvalue is given by the series (3.29), where $p \in P, j \in \mathbb{Z}^2$ are are uniquely defined by τ as it is described in Geometric Lemma 3.5, part 2. ∎

Lemma 3.19. *Formula (3.59) establishes one-to-one correspondence between $\chi_2^*(\lambda)$ and $\mathcal{D}_2(\lambda)$.*

Remark 3.20. From geometric point of view this means that $\chi_2^*(\lambda)$ does not have self-intersections.

Proof. Suppose there is a pair $\vec{\varkappa}, \vec{\varkappa}^* \in \mathcal{D}_2(\lambda)$ such that $\mathcal{K}_2\vec{\varkappa} = \mathcal{K}_2\vec{\varkappa}^* = \tau$, $\tau \in \chi_2^*(\lambda)$. By the definition (3.47) of $\mathcal{D}_2(\lambda)$, we have $\lambda^{(2)}(\alpha,\vec{\varkappa}) = \lambda^{(2)}(\alpha,\vec{\varkappa}^*) = \lambda$, i.e., the eigenvalue λ of $H_\alpha^{(2)}(\tau)$ is not simple. This contradicts to the previous lemma. ∎

3.4. **Preparation for the Next Approximation.** Let $\vec{b}^{(2)} \in K_2$ and $b_0^{(2)}$ be the distance of the point $\vec{b}^{(2)}$ to the nearest corner of K_2:

$$b_0^{(2)} = \min_{m=(0,0),(0,1),(1,0),(1,1)} |\vec{b} - 2\pi m/N_1 a|. \tag{3.60}$$

We assume $b_0^{(2)} = |\vec{b}^{(2)}|$. In the case when $\vec{b}$ is closer to a vertex other than $(0,0)$, the considerations are the same up to a parallel shift. We consider two cases: $b_0^{(2)} \geq \epsilon_1 k^{-1-2\delta}$ and $0 < b_0^{(2)} < \epsilon_1 k^{-1-2\delta}$. Let

$$\vec{y}^{(1)}(\varphi) = \vec{\varkappa}_1(\varphi) + \vec{b}^{(2)}. \tag{3.61}$$

3.4.1. *The case $b_0^{(2)} \geq \epsilon_1 k^{-1-2\delta}$.*

Definition 3.21. We define the set $\mathcal{O}^{(2)}(\vec{b}^{(2)})$ by the formula:

$$\mathcal{O}^{(2)}(\vec{b}^{(2)}) = \cup_{p\in P}\mathcal{O}_s^{(1)}(2\pi p/N_1 a + \vec{b}^{(2)}). \tag{3.62}$$

In the above formula we assume $\mathcal{O}_s^{(1)} \cap \hat{\Phi}_1 \neq \emptyset$.

Lemma 3.22. *If $\varphi \in \hat{\Phi}_1 \setminus \mathcal{O}^{(2)}(\vec{b}^{(2)})$ or its $k^{-4-6s_1-4\delta}$ neighborhood, then*

$$\left\| \left(\tilde{H}^{(1)}(\vec{y}^{(1)}(\varphi)) - k^2 \right)^{-1} \right\| \leq \frac{k^{4+6s_1+5\delta}}{\epsilon_1} \tag{3.63}$$

The resolvent is an analytic function of φ in every connected component $\mathcal{O}_c^{(2)}(\vec{b}^{(2)})$ of $\mathcal{O}^{(2)}(\vec{b}^{(2)})$, whose intersection with Φ_1 is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside $\mathcal{O}_c^{(2)}(\vec{b}^{(2)})$ is less that $c_0 k^{1+2s_2}$. The size of each connected *component is less than* $k^{-3-4s_1-2\delta}$. *The total number of poles (counting multiplicity) of the resolvent inside* $\mathcal{O}^{(2)}(\vec{b}^{(2)})$ *is less than* $c_0k^{2+2s_2}$. *The total size of* $\mathcal{O}_2(\vec{b}^{(2)})$ *is less then* $c_0k^{-2-2s_1}$.

Proof. Combining Lemmas 2.29 and 2.34 for $\vec{b}^{(1)} = 2\pi p/N_1a + \vec{b}^{(2)}$, we obtain:

$$\left\| \left(H^{(1)}\big(\vec{\varkappa}_1(\varphi) + 2\pi p/N_1a + \vec{b}^{(2)}\big) - k^2 \right)^{-1} \right\| \leq \max\left\{ k^{J^{(1)}}, \frac{4k^{3+6s_1+4\delta}}{b_0^{(2)}} \right\}, \tag{3.64}$$

$$J^{(1)} = 5c_1k^{2/3+s_1} < c_0k^{2+2s_1},\ p \in P,$$

for any $\varphi \in \hat{\Phi}_1 \setminus \mathcal{O}_2(\vec{b}^{(2)})$. Using (3.1), we easily obtain $k^{J^{(1)}} < \epsilon_1^{-1}$. Considering the condition $b_0^{(2)} \geq \epsilon_1 k^{-1-\delta}$ and the definition of $\tilde{H}^{(1)}$, we obtain (3.63) in $\hat{\Phi}_1 \setminus \mathcal{O}^{(2)}(\vec{b}^{(2)})$ The estimate is stable with respect of the perturbation of order $k^{-4-6s_1-4\delta}$, since the size of the discs in $\mathcal{O}_2(\vec{b}^{(2)})$ is $k^{-4-6s_1-3\delta}$.

Now we prove the second part of the lemma. Let Δ_* be a rectangle in $\mathbb{C}$: $\Delta_* = \{\varphi : |\Re\varphi - \varphi_*| \leq k^{-1},\ |\Im\varphi| < k^{-\delta}\}$ for some $\varphi_* \in [0, 2\pi)$. It is shown in the proof of Lemma 8.10 (see the estimate for the number of points in I_*) that the total number of points $\varphi_m^{\pm}(\vec{b})$ in Δ_* does not exceed $5c_0k^{1+2s_1}$ for any $\vec{b} \in K_1$. Therefore the number of discs of $\mathcal{O}^{(1)}(\vec{b})$ which intersect Δ_* does not exceed $5c_0k^{1+2s_1}$. Therefore, the number of discs of $\mathcal{O}_s^{(1)}(\vec{b})$ which intersect Δ_* does not exceed $5c_0k^{1+2s_1}$ for any $\vec{b} \in K_1$. Hence, the number of discs of $\mathcal{O}^{(2)}(\vec{b}^{(2)})$ which intersect Δ_* does not exceed $5c_0k^{1+2s_2}$. Considering that the size of each disc is less then $k^{-4-6s_1-3\delta}$, we obtain that the total size of $\mathcal{O}^{(2)}(\vec{b}^{(2)}) \cap \Delta_*$ is less than $k^{-3-4s_1-2\delta}$. It is obvious now that the size of each connected component of $\mathcal{O}^{(2)}(\vec{b}^{(2)})$ does not exceed $k^{-3-4s_1-2\delta}$. If $\mathcal{O}_c^{(2)}(\vec{b}^{(2)}) \cap \Phi_1 \neq \emptyset$, than $\mathcal{O}_c^{(2)}(\vec{b}^{(2)}) \subset \hat{\Phi}_1$. Therefore, the resolvent is an analytic function in this $\mathcal{O}_c^{(2)}(\vec{b}^{(2)})$. By construction, the poles of the resolvent are in the centra of the discs in $\mathcal{O}^{(2)}(\vec{b}^{(2)})$. Therefore, the number of poles in each connected component does not exceed $6c_0k^{1+2s_2}$.

By Lemmas 2.29 and 2.34 the number of poles of the resolvent $\Big(H^{(1)}\big(\vec{\varkappa}_1(\varphi) + 2\pi p/N_1a + \vec{b}^{(2)}\big) - k^2\Big)^{-1}$ inside $\mathcal{O}_s(2\pi p/N_1a + \vec{b}^{(2)})$ is less than $c_0k^{2+2s_1}$ for any $p \in P$. Considering that there are no poles in Φ_1 outside $\mathcal{O}_s(2\pi p/N_1a + \vec{b}^{(2)})$, we obtain that the number of poles of the resolvent $\Big(H^{(1)}\big(\vec{\varkappa}_1(\varphi) + 2\pi p/N_1a + \vec{b}^{(2)}\big) - k^2\Big)^{-1}$ inside $\mathcal{O}^{(2)}(\vec{b}^{(2)})$ is less than $c_0k^{2+2s_1}$. Taking into account that the number of $p \in P$ does not exceed $ck^{2(s_2-s_1)}$, $s_2 = 2s_1$, we obtain the estimate for the number of poles for the resolvent $\Big(\tilde{H}^{(1)}\big(\vec{y}^{(1)}((\varphi)\big) - k^2\Big)^{-1}$.

We estimate the size of $\mathcal{O}^{(2)}(\vec{b}^{(2)})$ the same way as we estimated the size of $\mathcal{O}^{(2)}$. Indeed, according to Lemma 2.27, the total size of each $\mathcal{O}_s\left(\frac{2\pi p}{N_1a} + \vec{b}^{(2)}\right)$ is less then $c_0k^{-2-4s_1}$. Considering that the number of p does not exceed $4k^{2s_1}$, we obtain that the total size of $\mathcal{O}_2(\vec{b}^{(2)})$ is less then $c_0k^{-2-2s_1}$.

∎

Definition 3.23. We denote the poles of the resolvent $\left(\tilde{H}^{(1)}(\vec{y}^{(1)}(\varphi)) - k^2\right)^{-1}$ in $\mathcal{O}^{(2)}(\vec{b}^{(2)})$ as $\varphi_n^{(2)}$, $n = 1, ..., M^{(2)}$, $M^{(2)} < c_0 k^{2+2s_2}$. Let us consider the discs $\mathcal{O}_n^{(2)}(\vec{b}^{(2)})$ of the radius $r^{(2)} = k^{-2-4s_2-\delta} r^{(1)} = k^{-6-7s_2-4\delta}$ centered at these poles. Let

$$\mathcal{O}_s^{(2)}(\vec{b}^{(2)}) = \cup_{n=1}^{M^{(2)}} \mathcal{O}_n^{(2)}(\vec{b}^{(2)}). \tag{3.65}$$

Lemma 3.24. *The total size of $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$ is less then $c_0 k^{-2s_2-\delta} r^{(1)} = c_0 k^{-4-5s_2-4\delta}$.*

Proof. The lemma easily follows from the formula $r^{(2)} = k^{-2-4s_2-\delta} r^{(1)} = k^{-6-7s_2-4\delta}$ and the estimate $M^{(2)} < c_0 k^{2+2s_2}$. ∎

Lemma 3.25. *If $\varphi \in \hat{\Phi}_1 \setminus \mathcal{O}_s^{(2)}(\vec{b}^{(2)})$, then*

$$\left\| \left(\tilde{H}^{(1)}(\vec{y}^{(1)}(\varphi)) - k^2\right)^{-1} \right\| \leq \frac{1}{\epsilon_1^2} \tag{3.66}$$

The estimate is stable in the $\left(r^{(2)} k^{-\delta}\right)$-neighborhood of $\Phi_1 \setminus \mathcal{O}_s^{(2)}(\vec{b}^{(2)})$. The resolvent is an analytic function of φ in every component of $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$, whose intersection with Φ_1 is not empty. The only singularities of the resolvent are poles. The number of poles (counting multiplicity) of the resolvent inside $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$ is less than ck^{2+2s_2}.

Proof. By the definition of $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$, the number of poles (counting multiplicity) of the resolvent inside this set is less than ck^{2+2s_2}. Considering as in Lemma 2.29 we obtain

$$\left\| \left(\tilde{H}^{(1)}(\vec{y}^{(1)}(\varphi)) - k^2\right)^{-1} \right\| \leq \nu^{-M^{(2)}} \frac{k^{4+\delta}}{\epsilon_1}, \tag{3.67}$$

where ν is the coefficient of contraction, when we reduce $\mathcal{O}^{(2)}(\vec{b}^{(2)})$ to $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$. Namely ν is the ratio of $r^{(2)}$ to the maximal size of $\mathcal{O}^{(2)}(\vec{b}^{(2)})$. By Lemma 3.22 and the definition of $r^{(2)}$, $\nu = k^{-3-5s_2-2\delta}$. It is not difficult to show that that $\nu^{-M^{(2)}} k^{4+\delta} < \epsilon_1^{-1}$, when $k > k_*$, see (3.1). The estimate (3.66) easily follows. ∎

Obviously the total size of $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$ is much smaller the smallest disc in $O^{(2)}$. Therefore, the function $\vec{\varkappa}_2(\varphi)$ is holomorphic inside each connected component of $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$ which has a non-empty intersection with Φ_2. Let

$$\vec{y}^{(2)}(\varphi) = \vec{\varkappa}_2(\varphi) + \vec{b}^{(2)}. \tag{3.68}$$

Lemma 3.26. *If $\varphi \in \hat{\Phi}_2 \setminus \mathcal{O}_s^{(2)}(\vec{b}^{(2)})$, then*

$$\left\| \left(H^{(2)}(\vec{y}^{(2)}(\varphi)) - k^2\right)^{-1} \right\| \leq \frac{2}{\epsilon_1^2} \tag{3.69}$$

The estimate is stable in the $\left(r^{(2)} k^{-\delta}\right)$-neighborhood of $\Phi_2 \setminus \mathcal{O}_s^{(2)}(\vec{b}^{(2)})$. The resolvent is an analytic function of φ in every component of $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$, whose intersection with Φ_2 is not empty. The only singularities of the resolvent are poles. The number of poles (counting multiplicity) of the resolvent inside $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$ is less than ck^{2+2s_2}.

Proof. We use the Hilbert identity

$$\left(H^{(2)}(\vec{y}^{(2)})-k^2\right)^{-1}-\left(\tilde{H}^{(1)}(\vec{y}^{(1)})-k^2\right)^{-1}=\mathcal{E}\left(H^{(2)}(\vec{y}^{(2)})-k^2\right)^{-1},$$

where $y^{(1)}, y^{(2)}$ are given by (3.61), (3.68), $\mathcal{E}=\mathcal{E}_1+\mathcal{E}_2\mathcal{E}_3$,

$$\mathcal{E}_1=-\left(\tilde{H}^{(1)}(\vec{y}^{(1)})-k^2\right)^{-1}W_2,\quad \mathcal{E}_2=\left(\tilde{H}^{(1)}(\vec{y}^{(1)})-k^2\right)^{-1}\left(\tilde{H}_0^{(1)}(\vec{y}^{(1)})+k^2\right),$$

$$\mathcal{E}_3=\left(\tilde{H}_0^{(1)}(\vec{y}^{(1)})+k^2\right)^{-1}\left(\tilde{H}_0^{(1)}(\vec{y}^{(2)})-\tilde{H}_0^{(1)}(\vec{y}^{(2)})\right).$$

Using the estimates (3.28) and (3.66), we see that $\|\mathcal{E}_1\|<\epsilon_1^2$. Using (3.66), we get $\|\mathcal{E}_2\|<1+\|W_1-2k^2\|\varepsilon_1^{-2}<3k^2\varepsilon_1^{-2}$. It is easy to show that $\|\mathcal{E}_3\|<k^{-1}|\varkappa_1-\varkappa_2|$. Considering (3.49), we get $\|\mathcal{E}_3\|<12\epsilon_1^4k^{-2}$. Therefore, $\|\mathcal{E}\|<4\epsilon_1^2<1/2$ when $k>k_*$. It is easy to see now that

$$\left\|\left(H^{(2)}(\vec{y}^{(2)})-k^2\right)^{-1}\right\|<2\left\|\left(\tilde{H}^{(1)}(\vec{y}^{(1)})-k^2\right)^{-1}\right\|<\frac{2}{\epsilon_1^2}.$$

Thus, we have obtained (3.69). Introducing operators P_N and using the same technique as in the proof of Lemma 2.20 (see (2.105) and further) we prove that the number of poles of $\left(H^{(2)}(\vec{y}^{(2)(\varphi)})-k^2\right)^{-1}$ obeys the same estimate as the number of poles of $\left(\tilde{H}^{(1)}(\vec{y}^{(1)}(\varphi))-k^2\right)^{-1}$, i.e., it does not exceed $c_0k^{2+2s_2}$. ∎

3.4.2. *The set $\mathcal{O}_s^{(2)}\left(\vec{b}^{(2)}\right)$ for small $\vec{b}^{(2)}$.* Everything we considered so far is valid if $b_0^{(2)}>\epsilon_1k^{-1-2\delta}$. However, in the next section and later, $b_0^{(2)}$ is taken smaller, since the reciprocal lattice is getting finer with each step. To prepare for this, let us consider $\vec{b}^{(2)}$ being close to a vertex of K_2: $0<b_0^{(2)}\leq\epsilon_1k^{-1-2\delta}$. We show that for such $\vec{b}^{(2)}$, the resolvent $\left(H^{(2)}(\vec{y}^{(2)}(\varphi))-k^2\right)^{-1}$ has no more than two poles in $\hat{\Phi}_2$. We surround these poles by two contours $\gamma^{\pm\,(2)}$ and prove an estimate for the norm of $\left(H^{(2)}\left(\vec{y}^{(2)}(\varphi)\right)-k^2\right)^{-1}$ when φ is outside these contours.

Suppose $|\vec{b}^{(2)}|=b_0^{(2)}$, i.e., the closest vertex of K_2 for $\vec{b}^{(2)}$ is $(0,0)$. We consider the functions $\lambda^{(1)}(\vec{y}^{(1)}(\varphi))$ and $\lambda^{(2)}(\vec{y}^{(2)}(\varphi))$ defined by perturbation series (2.58) and (3.44) for $\varphi\in\hat{\Phi}_2$. The convergence of these series can be easily justified. In fact, by Lemmas 2.10 and 3.17, $\vec{\varkappa}_1(\varphi)$ and $\vec{\varkappa}_2(\varphi)$ are holomorphic functions of φ in $\hat{\Phi}_2$. The perturbation series (2.58) and (3.44) converge for $\lambda^{(1)}(\vec{\varkappa}_1(\varphi))$ and $\lambda^{(2)}(\vec{\varkappa}_2(\varphi))$, respectively. Since the estimates involved are stable with respect to a change of $\vec{\varkappa}_{1,2}$ not exceeding $\epsilon_1k^{-1-\delta}$, the perturbation series for $\lambda^{(1)}\left(\vec{y}^{(1)}((\varphi)\right)$ and $\lambda^{(2)}\left(\vec{y}^{(2)}(\varphi)\right)$ also converge, both functions being holomorphic in $\hat{\Phi}_2$. We base our further considerations on these perturbation series expansions. For $\vec{b}^{(2)}$ being close to a vertex $\vec{e}$ other than $(0,0)$, we take $\vec{y}^{(2)}(\varphi)=\vec{\varkappa}_2(\varphi)+\vec{b}^{(2)}-\vec{e}$.

From now on, we denote the solutions $\varphi_{\epsilon_0}^{\pm}$ of the equation $\lambda^{(1)}\left(\vec{y}^{(1)}((\varphi)\right)=k^2+\varepsilon_0$, introduced in Lemma 2.30, by $\varphi_{\epsilon_0}^{(1)\pm}$. We set $r^{(2)}=r^{(1)}k^{-2-4s_2-\delta}$.

Lemma 3.27. *If* $0 < b_0^{(2)} \le \epsilon_1 k^{-1-2\delta}$ *and* $|\epsilon_0| < b_0^{(2)} k^{1-\delta} r^{(2)}$*, then the equation*

$$\lambda^{(2)}\left(\vec{y}^{(2)}(\varphi)\right) = k^2 + \epsilon_0 \tag{3.70}$$

has no more than two solutions $\varphi_{\epsilon_0}^{(2)\pm}$ *in* $\hat{\Phi}_2$*. For any* $\varphi_{\epsilon_0}^{(2)\pm} \in \hat{\Phi}_2$ *there is* $\varphi_0^{(1)\pm} \in \hat{\Phi}_1$ *such that*

$$\left|\varphi_{\epsilon_0}^{(2)\pm} - \varphi_0^{(1)\pm}\right| < r^{(2)}/4, \tag{3.71}$$

here and below $\varphi_0^{(1)\pm}$ *is* $\varphi_{\epsilon_0}^{(1)\pm}$ *for* $\epsilon_0 = 0$*.*

Proof. First, we expand $\lambda^{(2)}(\vec{y}^{(2)}(\varphi)) - \lambda^{(1)}(\vec{y}^{(1)}(\varphi))$ near the point $\vec{b}^{(2)} = 0$ and consider that $\lambda^{(2)}(\vec{\varkappa}_2(\varphi)) = \lambda^{(1)}(\vec{\varkappa}_1(\varphi)) = \lambda$. Then, using (3.46), (2.63), (3.49) and (3.41), we check that

$$\left|\lambda^{(2)}(\vec{y}^{(2)}(\varphi)) - \lambda^{(1)}(\vec{y}^{(1)}(\varphi))\right| < b_0^{(2)}\epsilon_1 \tag{3.72}$$

in $\hat{\Phi}_2$ and even in its $r^{(2)}$-neighborhood, the neighborhood being a subset of $\hat{\Phi}_1$. Suppose (3.70) holds for some $\varphi \in \hat{\Phi}_2$. By (3.72), $\lambda^{(1)}(\vec{y}^{(1)}(\varphi)) = k^2 + \varepsilon_0'$, $\varepsilon_0' < \varepsilon_0 + b_0^{(2)}\epsilon_1 < 2b_0^{(2)}k^{1-\delta}r^{(2)}$ when $k > k_*$. Hence, φ satisfies conditions of Lemmas 2.30 and 2.31. Surrounding φ by a circle C of the radius $r^{(2)}/4$ and using (2.132), we see that $|\lambda^{(1)}(\vec{y}^{(1)}(\varphi)) - k^2 - \varepsilon_0'| \approx \frac{1}{4}kb_0^{(2)}r^{(2)} >> |\varepsilon_0'|$ on this circle. Applying Rouché's theorem, we obtain that there is a solution of $\lambda^{(1)}(\vec{y}^{(1)}(\varphi)) = k^2$ inside this circle. Thus, any solution of (3.70) is in the circle of the radius $r^{(2)}/4$ surrounding $\varphi_0^{\pm(1)}$, the point $\varphi_0^{\pm(1)}$ being in the $(r^{(2)}/4)$-neighborhood of $\hat{\Phi}_2$. It remains to check that (3.70) has no more than two solutions and no more than one in a vicinity of each $\varphi_0^{\pm(1)}$. We construct the disk of the radius $r^{(2)}/4$ centered at $\varphi_0^{\pm(1)}$ and note that $|\lambda^{(1)}(\vec{y}^{(1)}(\varphi)) - k^2| > \frac{1}{4}kb_0^{(2)}r^{(2)}$ outside the circle. Using (3.72) and Rouché's Theorem, we obtain that there is only one solution of (3.70) in the disk. ∎

Lemma 3.28. *Suppose* $0 < b_0^{(2)} \le \epsilon_1 k^{-1-2\delta}$ *and* $\varphi \in \hat{\Phi}_2$ *obeys the inequality analogous to (3.71):* $\left|\varphi - \varphi_0^{\pm(1)}\right| < r^{(2)}$*. Then,*

$$\frac{\partial}{\partial\varphi}\lambda^{(2)}\left(\vec{y}^{(2)}(\varphi)\right) =_{k\to\infty} \pm 2b_0^{(2)}k\big(1 + o(1)\big), \tag{3.73}$$

where $|o(1)| < 10^{-2} + \epsilon_1$ *when* $k > k_*$*.*

The proof is completely analogous to that of Lemma 2.31.

Definition 3.29. Let $\Gamma^{(2)\pm}\left(\vec{b}^{(2)}\right)$ be the open disks centered at $\varphi_0^{(2)\pm}$ with radius $r^{(2)}$; $\gamma^{(2)\pm}\big(\vec{b}^{(2)}\big)$ be their boundary circles and $\mathcal{O}_s^{(2)}\left(\vec{b}^{(2)}\right) = \Gamma^{(2)+} \cup \Gamma^{(2)-}$.

Lemma 3.30. *For any* φ *in* $\hat{\Phi}_2 \setminus \mathcal{O}_s^{(2)}(\vec{b}^{(2)})$*,*

$$|\lambda^{(2)}(\vec{y}^{(2)}(\varphi)) - k^2| \ge b_0^{(2)} k^{1-\delta} r^{(2)}. \tag{3.74}$$

The proofs of this and the next lemma are analogous to those of Lemma 2.33, 2.34.

Lemma 3.31. *For any $\varphi \in \hat{\Phi}_2 \setminus \mathcal{O}_s^{(2)}\left(\vec{b}^{(2)}\right)$,*

$$\left\| \left(H^{(2)}\left(\vec{y}^{(2)}(\varphi)\right) - k^2 \right)^{-1} \right\| < \frac{16}{b_0^{(2)} r^{(2)} k^{1-\delta}}, \tag{3.75}$$

The estimate is stable in the $\left(r^{(2)}k^{-\delta}\right)$-neighborhood of $\Phi_2 \setminus \mathcal{O}_s^{(2)}(\vec{b}^{(2)})$. The resolvent is an analytic function of φ in every component of $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$, whose intersection with Φ_2 is not empty. The only singularities of the resolvent are poles. The resolvent has at most two poles inside $\mathcal{O}_s^{(2)}(\vec{b}^{(2)})$.

4. The Third Approximation

Squaring both sides of (3.1), we easily obtain the relation which coincide with (3.1) up to the substitution of s_2 instead of s_1:

$$C_*(1+s_2)k^{2+4s_2}\ln k < k^{\eta s_2}, \quad C_* = 400(c_0+1)^2, \quad c_0 = 32d_1d_2, \quad s_2 = 2s_1. \tag{4.1}$$

for any $k > k_*$. We will use (4.1) in the next step.

4.1. Operator $H_\alpha^{(3)}$.

Choosing $s_3 = 2s_2$, we define the third operator $H_\alpha^{(3)}$ by the formula:

$$H_\alpha^{(3)} = H^{(2)} + \alpha W_3, \quad (0 \le \alpha \le 1), \qquad W_3 = \sum_{r=M_2+1}^{M_3} V_r,$$

where M_3 is chosen in such a way that $2^{M_3} \approx k^{s_3}$. Obviously, the periods of W_3 are $2^{M_3-1}(d_1, 0)$ and $2^{M_3-1}(0, d_2)$. We write them in the form: $N_2N_1(a_1, 0)$ and $N_2N_1(0, a_2)$, here $N_2 = 2^{M_3-M_2}$, $\frac{1}{4}k^{s_3-s_2} < N_2 < 4k^{s_3-s_2}$. Note that

$$\|W_3\|_\infty \le \sum_{r=M_2+1}^{M_3} \|V_r\|_\infty \le \sum_{r=M_2+1}^{M_3} \exp(-2^{\eta r}) < \exp(-k^{\eta s_2}).$$

4.2. Multiple Periods of $W_2(x)$.

The operator, $H^{(2)} = H_1 + W_2(x)$, has the periods N_1a_1, N_1a_2. The corresponding family of operators, $\{H^{(2)}(\tau)\}_{\tau \in K_2}$, acts in $L_2(Q_2)$, where $Q_2 = [0, N_1a_1] \times [0, N_1a_2]$ and $K_2 = [0, 2\pi/N_1a_1) \times [0, 2\pi/N_1a_2)$. Since now on we denote quasimomentum t from the first step by $t^{(1)}$, quasimomentum τ from the second step by $t^{(2)}$. Correspondingly, quasimomentum for $H_\alpha^{(3)}$ we denote by $t^{(3)}$. Eigenvalues of $H^{(2)}\left(t^{(2)}\right)$ are denoted by $\lambda_n^{(2)}\left(t^{(2)}\right)$, $n \in \mathbb{N}$ and its spectrum by $\Lambda^{(2)}\left(t^{(2)}\right)$.

Next, let us consider $W_2(x)$ as a periodic function with the periods $N_2N_1a_1$, $N_2N_1a_2$. When changing the periods, the family of operators $\left\{H^{(2)}\left(t^{(2)}\right)\right\}_{t^{(2)} \in K_2}$ is replaced by the family of operators $\left\{\tilde{H}^{(2)}\left(t^{(3)}\right)\right\}_{t^{(3)} \in K_3}$, acting in $L_2(Q_3)$, where $Q_3 = [0, N_2N_1a_1] \times [0, N_2N_1a_2]$ and $K_3 = [0, 2\pi/N_2N_1a_1) \times [0, 2\pi/N_2N_1a_2)$. We denote eigenvalues of $\tilde{H}^{(2)}\left(t^{(3)}\right)$ by $\tilde{\lambda}_n^{(2)}\left(t^{(3)}\right)$, $n \in \mathbb{N}$, and its spectrum by $\tilde{\Lambda}^{(2)}\left(t^{(3)}\right)$.

We denote now by $P^{(1)}$ the set P, introduced by (3.4), its elements being $p^{(1)}$. By Bloch theory (see e.g.[22]), for any $t^{(3)} \in K_3$,

$$\tilde{\Lambda}^{(2)}\left(t^{(3)}\right) = \bigcup_{p^{(2)} \in P^{(2)}} \Lambda^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right), \tag{4.2}$$

where $P^{(2)} = \left\{p^{(2)} = \left(p_1^{(2)}, p_2^{(2)}\right) \in \mathbb{Z}^2 : 0 \le p_1^{(2)} \le N_2 - 1,\ 0 \le p_2^{(2)} \le N_2 - 1\right\}$,

$$2\pi p^{(2)}/N_2N_1a = \left(\frac{2\pi p_1^{(2)}}{N_2N_1a_1}, \frac{2\pi p_2^{(2)}}{N_2N_1a_2}\right).$$

An isoenergetic set $\tilde{S}_2(\lambda_0) \subset K_3$ of the operator $\tilde{H}^{(2)}$ is defined by the formula:

$$\begin{aligned}\tilde{S}_2(\lambda) &= \left\{t^{(3)} \in K_3 : \exists n \in \mathbb{N} :\ \tilde{\lambda}_n^{(2)}\left(t^{(3)}\right) = \lambda\right\} \\ &= \left\{t^{(3)} \in K_3 : \exists n \in \mathbb{N},\ p^{(2)} \in P^{(2)} : \lambda_n^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right) = \lambda\right\}.\end{aligned}$$

Obviously, $\tilde{S}_2 = \mathcal{K}_3 S_2$, where $\mathcal{K}_3$ is the parallel shift into K_3, that is,

$$\mathcal{K}_3 : \mathbb{R}^2 \to K_3, \quad \mathcal{K}_3\left(t^{(3)} + 2\pi m/N_2N_1a\right) = t^{(3)}, \quad m \in \mathbb{Z}^2, \quad t^{(3)} \in K_3.$$

We denote index j, introduced in Part 1 of Lemma 2.1 (Geometric Lemma for the First approximation), by $j^{(1)}$ and $\tilde{j}$, introduced in Part 1 of Lemma 3.5 (Geometric Lemma for the Second approximation), by $j^{(2)}$, $j^{(2)} = j^{(1)} + p^{(1)}/N_1$.

4.3. **Perturbation Formulae.**

Lemma 4.1 (Geometric Lemma)**.** *For a sufficiently large λ, $\lambda > k_*^2$, there exists a non-resonance set $\chi_3(\lambda, \delta) \subset \mathcal{K}_3\chi_2^*$ such that:*

(1) *For any point $t^{(3)} \in \chi_3$, the following conditions hold:*
   (a) *There exists a unique $p^{(2)} \in P^{(2)}$ such that $t^{(3)} + 2\pi p^{(2)}/N_2N_1a \in \chi_2^*$.*
   (b) *The following relation holds:*

$$\lambda_{j^{(2)}}^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right) = \lambda,$$

   where $\lambda_{j^{(2)}}^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right)$ is given by the perturbation series (3.29) with $\alpha = 1$ and $j^{(2)} = j + p/N_1$, here j and p are defined by the point $\tau = t^{(3)} + 2\pi p^{(2)}/N_2N_1a$ as it is described in Part 1(b) of the Geometric Lemma for the previous step.
   (c) *The eigenvalue $\lambda_{j^{(2)}}^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right)$ is a simple eigenvalue of $\tilde{H}^{(2)}\left(t^{(3)}\right)$ and its distance to all other eigenvalues $\lambda_n^{(2)}\left(t^{(3)} + 2\pi \hat{p}^{(2)}/N_2N_1a\right)$ of $\tilde{H}^{(2)}\left(t^{(3)}\right)$ is greater than $\epsilon_2 = e^{-\frac{1}{4}k^{\eta s_2}}$:*

$$\left|\lambda_{j^{(2)}}^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right) - \lambda_n^{(2)}\left(t^{(3)} + 2\pi \hat{p}^{(2)}/N_2N_1a\right)\right| > \epsilon_2. \tag{4.3}$$

(2) *For any $t^{(3)}$ in the real $(\epsilon_2 k^{-1-\delta})$-neighborhood of χ_3, there exists a unique $p^{(2)} \in P^{(2)}$ such that $t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a$ is in the $(\epsilon_2 k^{-1-\delta})$-neighborhood of χ_2^* and*

$$\left|\lambda_{j^{(2)}}^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a\right) - k^2\right| < \epsilon_2 k^{-\delta}, \tag{4.4}$$

$j^{(2)} = j + p/N_1$, here j and p are defined by the point $\tau = t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a$ as it is described in Part 2 of the Geometric Lemma for the previous step. An estimate analogous to (4.3) holds:

$$|\lambda_j^{(2)}(t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a) - \lambda_{\hat{n}}^{(1)}(t^{(3)} + 2\pi \hat{p}^{(2)}/N_2 N_1 a)| > \big(1 + o(1)\big)\epsilon_2. \tag{4.5}$$

(3) *The third nonresonance set χ_3 has an asymptotically full measure on χ_2^* in the following sense:*

$$\frac{L\left(\mathcal{K}_3 \chi_2^* \setminus \chi_3)\right)}{L\left(\chi_2^*\right)} < c_0 k^{-4-2s_1-2s_2}. \tag{4.6}$$

Proof. The proof of the lemma is analogous to that for Geometric Lemma in the second step. Indeed, let us consider

$$\mathcal{O}^{(3)} = \cup_{\tilde{p}^{(2)} \in P^{(2)} \setminus \{0\}} \mathcal{O}_s^{(2)}\left(\frac{2\pi \tilde{p}^{(2)}}{N_2 N_1 a}\right), \tag{4.7}$$

where $\mathcal{O}_s^{(2)}\left(\frac{2\pi\tilde{p}^{(2)}}{N_2 N_1 a}\right)$ is defined by (3.65) with $\vec{b}^{(2)} = \frac{2\pi\tilde{p}^{(2)}}{N_2 N_1 a}$. Note, that the definition make sense, since $|\frac{2\pi\tilde{p}^{(2)}}{N_2 N_1 a}| > d_{max}^{-1} k^{-s_3} > \epsilon_2 k^{-1-2\delta}$. Let

$$\mathit{\Phi}_3 = \mathit{\Phi}_2 \setminus \mathcal{O}^{(3)}, \quad \Theta_3 = \mathit{\Phi}_3 \cap [0, 2\pi). \tag{4.8}$$

By Lemma 3.26,

$$\left\|\left(H^{(2)}\Big(\vec{\varkappa}_2(\varphi) + \frac{2\pi\tilde{p}^{(2)}}{N_2 N_1 a}\Big) - k^2\right)^{-1}\right\| < \frac{2}{\epsilon_1^2} \tag{4.9}$$

for all $\tilde{p}^{(2)} \in P^{(2)} \setminus \{0\}$ and $\varphi \in \hat{\mathit{\Phi}}_3$, here and below $\hat{\mathit{\Phi}}_3$ is the $\big(r^{(2)} k^{-\delta}\big)$-neighborhood of $\mathit{\Phi}_3$. We consider $\mathcal{D}_{2,nonres} \subset \mathcal{D}_2$:

$$\mathcal{D}_{2,nonres} = \left\{\vec{\varkappa}_2(\varphi),\ \varphi \in \Theta_3\right\}. \tag{4.10}$$

We define χ_3 by the formula:

$$\chi_3 = \mathcal{K}_3 \mathcal{D}_{2,nonres}. \tag{4.11}$$

By the definition of $\mathcal{K}_3$, for every $t^{(3)}$ in χ_3, there are $p^{(1)} \in P^{(1)}$, $p^{(2)} \in P^{(2)}$ and $j \in Z^2$ such that

$$t^{(3)} + \frac{2\pi p^{(2)}}{N_2 N_1 a} + \frac{2\pi p^{(1)}}{N_1 a} + \frac{2\pi j}{a} = \vec{\varkappa}_2(\varphi), \quad \vec{\varkappa}_2(\varphi) \in \mathcal{D}_{2,nonres}. \tag{4.12}$$

Considering the definition of $\mathcal{D}_{2,nonres}$, we obtain:

$$\left\|\left(H^{(2)}\Big(t^{(3)} + \frac{2\pi\hat{p}^{(2)}}{N_2 N_1 a}\Big) - k^2\right)^{-1}\right\| < \frac{2}{\epsilon_1^2}, \tag{4.13}$$

$\hat{p}^{(2)} = p^{(2)} + \tilde{p}^{(2)}$. Note that the indices j, $p^{(1)}$ do not play a role, since they just produce a shift of indices of the matrix elements of the resolvent. Considering that $\tilde{p}^{(2)}$ can be any but zero, we obtain that (4.13) holds for all $\hat{p}^{(2)} \in P^{(2)} \setminus \{p^{(2)}\}$. Taking into account that $\lambda_j^{(2)}(t^{(3)} + 2\pi p^{(2)}/N_2N_1a) = k^2$ and $\epsilon_2 < \epsilon_1^2$, we arrive at (4.5) for all $\hat{p} \neq p$. It remains to check (4.5) for $p = \hat{p}$. Let $t^{(2)} = t^{(3)} + 2\pi p^{(2)}/N_2N_1a$. By (4.12), $t^{(2)} \in \mathcal{K}_2\mathcal{D}_2$. Using (3.59), we get $t^{(2)} \in \chi_2^*$. By Theorem 3.8, $\lambda_j^{(2)}(t^{(3)}+2\pi p^{(2)}/N_2N_1a)$ is the only eigenvalue of $H^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2N_1a\right)$ in the interval ε_2. Hence,

$$|\lambda_j^{(2)}(t^{(3)} + 2\pi p^{(2)}/N_2N_1a) - \lambda_{\hat{n}}^{(2)}(t^{(3)} + 2\pi p^{(2)}/N_2N_1a)| > \epsilon_1.$$

Part 2 holds, since all estimates are stable with respect to the perturbation of $t^{(3)}$ less then $\epsilon_2 k^{-1-\delta}$.

Let us estimate the size of $\mathcal{O}^{(3)}$. According to Lemma 3.24 the total size of each $\mathcal{O}_s^{(2)}(2\pi p^{(2)}/N_2N_1a)$ is less then $c_0k^{-4-5s_2-4\delta}$. Considering that the number of $p^{(2)}$ does not exceed $k^{2(s_3-s_2)} = k^{2s_2}$, $s_2 = 2s_1$, we arrive at (4.6).

∎

Corollary 4.2. *If $t^{(3)}$ belongs to the complex $(\epsilon_2k^{-1-\delta})-$neighborhood of $\chi_3(\lambda, \delta)$, then for any z lying on the circle $C_3 = \{z : |z - k^2| = \epsilon_2/2\}$, the following inequality holds:*

$$\left\|\left(\tilde{H}^{(2)}(t^{(3)}) - z\right)^{-1}\right\| < \frac{4}{\epsilon_2}. \tag{4.14}$$

Remark 4.3. Every point $2\pi q/N_2N_1a$ ($q \in \mathbb{Z}^2$) of the dual lattice for periods $N_2N_1a_1$, $N_2N_1a_2$ can be uniquely represented in the form: $2\pi q/N_2N_1a = 2\pi m/N_1a + 2\pi p^{(2)}/N_2N_1a$, where $m \in \mathbb{Z}^2$, $p^{(2)} \in P^{(2)}$. Note that $2\pi m/N_1a$ is a point of a dual lattice for periods N_1a_1, N_1a_2 and $p^{(2)} \in P^{(2)}$ is responsible for refining the lattice. By Remark 3.7, $2\pi q/N_2N_1a$ also can be uniquely represented as $2\pi q/N_2N_1a = 2\pi j/a + 2\pi p^{(1)}/N_1a + 2\pi p^{(2)}/N_2N_1a$, here $j \in \mathbb{Z}^2$, $p^{(1)} \in P^{(1)}$, $p^{(2)} \in P^{(2)}$.

Let us consider a normalized eigenfunction $\psi_n\left(t^{(2)}, x\right)$ of $H^{(2)}\left(t^{(2)}\right)$ in $L_2(Q_2)$. We extend it quasiperiodically to $L_2(Q_3)$, renormalize and denote the new function by $\tilde{\psi}_n\left(t^{(3)}, x\right)$, $t^{(3)} = \mathcal{K}_3t^{(2)}$. The Fourier representations of $\psi_n\left(t^{(2)}, x\right)$ in $L_2(Q_2)$ and $\tilde{\psi}_n(t^{(3)}, x)$ in $L_2(Q_3)$ are simply related. If we denote Fourier coefficients of $\psi_n\left(t^{(2)}, x\right)$ with respect to the basis $|Q_2|^{-1/2}e^{i\left(2\pi m/N_1a + t^{(2)}, x\right)}$, $m \in \mathbb{Z}^2$, in $L_2(Q_2)$ by C_{nm}, then, obviously, the Fourier coefficients $\tilde{C}_{nq}$ of $\tilde{\psi}_n\left(t^{(3)}, x\right)$ with respect to the basis $|Q_3|^{-1/2}e^{i\left(2\pi q/N_2N_1a + t^{(3)}, x\right)}$, $q \in \mathbb{Z}^2$, in $L_2(Q_3)$ are given by the formula:

$$\tilde{C}_{nq} = \begin{cases} C_{nm}, & \text{if } q = mN_2 + p^{(2)}; \\ 0, & \text{otherwise,} \end{cases}$$

$p^{(2)}$ being defined from the relation $t^{(2)} = t^{(3)} + 2\pi p^{(2)}/N_1N_2a$. Correspondingly, matrices of the projections on $\psi_n(\tau, x)$ and $\tilde{\psi}_n(t^{(3)}, x)$ with respect to the above bases are simply related:

$$(\tilde{E}_n)_{q\hat{q}} = \begin{cases} (E_n)_{m\hat{m}}, & \text{if } q = mN_2 + p^{(2)},\ \hat{q} = \hat{m}N_2 + p^{(2)}; \\ 0, & \text{otherwise,} \end{cases}$$

$\tilde{E}_n$ and E_n being projections in $L_2(Q_3)$ and $L_2(Q_2)$, respectively.

We define functions $g_r^{(3)}(k, t^{(3)})$ and operator-valued functions $G_r^{(3)}(k, t^{(3)})$, $r = 1, 2, \cdots$, as follows:

$$g_r^{(3)}\big(k, t^{(3)}\big) = \frac{(-1)^r}{2\pi i r} \mathrm{Tr} \oint_{C_3} \left(\left(\tilde{H}^{(2)}\big(t^{(3)}\big) - z \right)^{-1} W_3 \right)^r dz,$$

$$G_r^{(3)}\big(k, t^{(3)}\big) = \frac{(-1)^{r+1}}{2\pi i} \oint_{C_3} \left(\left(\tilde{H}^{(2)}\big(t^{(3)}\big) - z \right)^{-1} W_3 \right)^r \left(\tilde{H}^{(2)}\big(t^{(3)}\big) - z \right)^{-1} dz.$$

Theorem 4.4. *Suppose $t^{(3)}$ belongs to the $(\epsilon_2 k^{-1-\delta})$-neighborhood in K_3 of the third nonresonance set $\chi_3(\lambda, \delta)$, $0 < \delta < s_1$, $\epsilon_2 = e^{-\frac{1}{4}k^{\eta s_2}}$. Then, $\lambda > k_*^2$ and for all α, $0 \le \alpha \le 1$, there exists a unique eigenvalue of the operator $H_\alpha^{(3)}\big(t^{(3)}\big)$ in the interval $\varepsilon_3(k) := (k^2 - \epsilon_2/2, k^2 + \epsilon_2/2)$. It is given by the series:*

$$\lambda_{j^{(3)}}^{(3)}\big(\alpha, t^{(3)}\big) = \lambda_{j^{(2)}}^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a\right) + \sum_{r=1}^{\infty} \alpha^r g_r^{(3)}\big(k, t^{(3)}\big), \tag{4.15}$$

converging absolutely in the disk $|\alpha| \le 1$, where $j^{(3)} := j^{(2)} + p^{(2)}/N_2 N_1$, $p^{(2)}$, $j^{(2)}$ being described in Geometric Lemma 4.1. The spectral projection, corresponding to $\lambda_{j^{(3)}}^{(3)}(\alpha, t^{(3)})$, is given by the series:

$$E_{j^{(3)}}^{(3)}\big(\alpha, t^{(3)}\big) = \tilde{E}_{j^{(2)}}^{(2)}\left(t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a\right) + \sum_{r=1}^{\infty} \alpha^r G_r^{(3)}\big(k, t^{(3)}\big), \tag{4.16}$$

which converges in the trace class $\mathbf{S_1}$ uniformly with respect to α in the disk $|\alpha| \le 1$.

The following estimates hold for coefficients $g_r^{(3)}(k, t^{(3)})$, $G_r^{(3)}(k, t^{(3)})$:

$$\left| g_r^{(3)}\big(k, t^{(3)}\big) \right| < \frac{3\epsilon_2}{2} \left(4\epsilon_2^3\right)^r, \qquad \left\| G_r^{(3)}\big(k, t^{(3)}\big) \right\|_1 < 6r \left(4\epsilon_2^3\right)^r. \tag{4.17}$$

Corollary 4.5. *The following estimates hold for the perturbed eigenvalue and its spectral projection:*

$$\left| \lambda_{j^{(3)}}^{(3)}\big(\alpha, t^{(3)}\big) - \lambda_{j^{(2)}}^{(2)}\big(t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a\big) \right| \le 12\alpha\epsilon_2^4, \tag{4.18}$$

$$\left\| E_{j^{(3)}}^{(3)}\big(\alpha, t^{(3)}\big) - \tilde{E}_{j^{(2)}}^{(2)}\big(t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a\big) \right\|_1 \le 48\alpha\epsilon_2^3. \tag{4.19}$$

Proof of the theorem is analogous to that of the Theorem 3.8. The series (4.15), (4.16) can be extended as holomorphic functions of $t^{(3)}$ in the complex $\left(\epsilon_2 k^{-1-\delta}\right)$-neighborhood of χ_3; they can be differentiated any number of times with respect to $t^{(3)}$ and retain their asymptotic character. The results analogous to Lemma 3.11, Theorem 3.12, Corollary 3.13 and Lemma 3.14 hold.

4.4. **Nonresonant part of the isoenergetic set of $H_\alpha^{(3)}$.** This section is analogous to Section 3.3 for the second step. Indeed, let $S_3(\lambda)$ be an isoenergetic set of the operator $H_\alpha^{(3)}$: $S_3(\lambda) = \{t^{(3)} \in K_3 : \exists n \in \mathbb{N} : \lambda_n^{(3)}(\alpha, t^{(3)}) = \lambda\}$, here $\{\lambda_n^{(3)}(\alpha, t^{(3)})\}_{n=1}^{\infty}$ is the spectrum of $H_\alpha^{(3)}(t^{(3)})$. Now we construct a non-resonance subset $\chi_3^*(\lambda)$ of $S_3(\lambda)$. It corresponds to non-resonance eigenvalues $\lambda_{j^{(3)}}^{(3)}(t^{(3)})$ given by the perturbation series (4.15). Recall that $\mathcal{D}_2(\lambda)_{nonres}$ and χ_3 are defined by formulas (4.10) and (4.11). Recall also that $\chi_3 \subset \mathcal{K}_3\chi_2^*(\lambda)$ (see Geometric Lemma 4.1) and $\chi_2^*(\lambda) = \mathcal{K}_2\mathcal{D}_2(\lambda)$, see (3.59). Hence, $\chi_3 \subset \mathcal{K}_3\mathcal{D}_2(\lambda)$. The following lemma is analogous to Lemma 3.15 in the second step.

Lemma 4.6. *The formula $\mathcal{K}_3\mathcal{D}_2(\lambda)_{nonres} = \chi_3$ establishes one-to-one correspondence between $\mathcal{D}_2(\lambda)_{nonres}$ and χ_3.*

Proof. The proof is analogous to that of Lemma 3.15 up to the shift of indices by 1, i.e., $\chi_2 \to \chi_3$, $\chi_1^*(\lambda) \to \chi_2^*(\lambda)$, $\tau = t^{(2)} \to t^{(3)}$; we use formula (3.59) instead of (2.73), Part 1a of the Geometric Lemma for the third step instead of Part 1a of the Geometric Lemma for the second step, and Lemma 3.19 instead of 2.12. ∎

We define $\mathcal{B}_3(\lambda)$ as the set of directions corresponding to Θ_3, Θ_3 being given by (4.8):

$$\mathcal{B}_3(\lambda) = \{\vec{\nu} \in \mathcal{B}_2(\lambda) : \varphi \in \Theta_3\}.$$

Note that $\mathcal{B}_3(\lambda)$ is a unit circle with holes, centered at the origin, and $\mathcal{B}_3(\lambda) \subset \mathcal{B}_2(\lambda) \subset \mathcal{B}_1(\lambda)$. We define $\mathcal{D}_3(\lambda)$ as a level set for $\lambda^{(3)}(\alpha, \vec{\varkappa})$ in the $(\epsilon_2 k^{-1-\delta})$-neighborhood of $\mathcal{D}_2(\lambda)_{nonres}$:

$$\mathcal{D}_3(\lambda) := \left\{\vec{\varkappa} = \varkappa\vec{\nu} : \vec{\nu} \in \mathcal{B}_3(\lambda), \quad \left|\varkappa - \varkappa_2(\lambda, \vec{\nu})\right| < \epsilon_2 k^{-1-\delta}, \ \lambda^{(3)}(\alpha, \vec{\varkappa}) = \lambda\right\}.$$

Next two lemmas are to prove that $\mathcal{D}_3(\lambda)$ is a distorted circle with holes. Their formulations and proofs are analogous to those of Lemmas 3.16 and 3.17.

Lemma 4.7. *For every $\vec{\nu} \in \mathcal{B}_3(\lambda)$ and every α, $0 \le \alpha \le 1$, there is a unique $\varkappa = \varkappa_3(\lambda, \vec{\nu})$ in the interval $I_3 := \left[\varkappa_2(\lambda, \vec{\nu}) - \epsilon_2 k^{-1-\delta}, \varkappa_2(\lambda, \vec{\nu}) + \epsilon_2 k^{-1-\delta}\right]$ such that*

$$\lambda^{(3)}(\alpha, \varkappa_3\vec{\nu}) = \lambda. \tag{4.20}$$

Furthermore,

$$|\varkappa_3(\lambda, \vec{\nu}) - \varkappa_2(\lambda, \vec{\nu})| \le 2\alpha\epsilon_2^4 k^{-1}. \tag{4.21}$$

Further we use the notations $\varkappa_3(\varphi) \equiv \varkappa_3(\lambda, \vec{\nu})$, $h_3(\varphi) \equiv \varkappa_3(\varphi) - \varkappa_2(\varphi)$, $\vec{\varkappa}_3(\varphi) = \varkappa_3(\varphi)\vec{\nu}$.

Lemma 4.8. *The following statements hold for $\lambda > k_*^2$:*

(1) *The set $\mathcal{D}_3(\lambda)$ is a distorted circle with holes: it can be described by the formula:*

$$\mathcal{D}_3(\lambda) = \left\{\vec{\varkappa} \in \mathbb{R}^2 : \vec{\varkappa} = \vec{\varkappa}_3(\varphi), \quad \varphi \in \Theta_3(\lambda)\right\}, \tag{4.22}$$

where $\varkappa_3(\varphi) = \varkappa_2(\varphi) + h_3(\varphi)$, $\varkappa_2(\varphi)$ *is the "radius" of* $\mathcal{D}_2(\lambda)$ *and* $h_3(\varphi)$ *satisfies the estimates*

$$|h_3| < 12\alpha\epsilon_2^4 k^{-1}, \quad \left|\frac{\partial h_3}{\partial \varphi}\right| \le 96\alpha\epsilon_2^3 k^{1+\delta}. \tag{4.23}$$

(2) *The total length of* $\mathcal{B}_3(\lambda)$ *satisfies the estimate:*

$$L\left(\mathcal{B}_2 \setminus \mathcal{B}_3\right) < 4\pi k^{-4-2s_1-2s_2}. \tag{4.24}$$

(3) *The function* $h_3(\varphi)$ *can be extended as a holomorphic function of* φ *to the complex non-resonce set* Φ_3 *and its* $\big(k^{-\delta}r^{(2)}\big)$*-neighborhood* $\hat{\Phi}_3$*, estimates (4.23) being preserved.*

(4) *The curve* $\mathcal{D}_3(\lambda)$ *has a length which is asymptotically close to that of* $\mathcal{D}_2(\lambda)$ *in the following sense:*

$$L\Big(\mathcal{D}_3(\lambda)\Big) \underset{\lambda\to\infty}{=} L\Big(\mathcal{D}_2(\lambda)\Big)\Big(1 + O(k^{-4-2s_1-2s_2})\Big), \tag{4.25}$$

where $O\big(k^{-2-2s_1-2s_2}\big) = (1+o(1))c_0 k^{-2-2s_1-2s_2}$, $|o(1)| < 10^{-2}$ *when* $k > k_*$.

Proof. The proof is analogous to that of Lemma 3.17. Note only that, in Part 2, when proving convergence of the series for the resolvent $\big(H^{(3)}(\vec{\varkappa}_2(\varphi)) - z\big)^{-1}$, we use the estimate

$$\left\|\Big(\tilde{H}^{(2)}(\vec{\varkappa}_2(\varphi)) - z\Big)^{-1}\right\| < \frac{4}{\epsilon_2}, \quad z \in C_3, \tag{4.26}$$

analogous to (3.55), the operator $\tilde{H}^{(2)}$ acting in $L_2(Q_3)$. The estimate (4.26) follows from (4.9) and (3.42). As a side result of these considerations, we obtain an estimate analogous to (3.58) for the new resolvent and z being inside C_3. ∎

We define the non-resonance set, $\chi_3^*(\lambda)$ in $S_3(\lambda)$ by the formula analogous to (3.59):

$$\chi_3^*(\lambda) := \mathcal{K}_3\mathcal{D}_3(\lambda). \tag{4.27}$$

The following lemmas are analogous to Lemmas 3.18 and 3.19.

Lemma 4.9. *The set* $\chi_3^*(\lambda)$ *belongs to the* $\big(12\alpha\epsilon_2^4 k^{-1}\big)$*-neighborhood of* $\chi_3(\lambda)$ *in* K_3*. If* $t^{(3)} \in \chi_3^*(\lambda)$*, then the operator* $H_\alpha^{(3)}(t^{(3)})$ *has a simple eigenvalue equal to* λ*. This eigenvalue is given by the perturbation series (4.15).*

Lemma 4.10. *Formula (4.27) establishes one-to-one correspondence between* $\chi_3^*(\lambda)$ *and* $\mathcal{D}_3(\lambda)$*.*

Remark 4.11. From geometric point of view this means that $\chi_3^*(\lambda)$ does not have self-intersections.

4.5. **Preparation for the Next Approximation.** Let $\vec{b}^{(3)} \in K_3$ and $b_0^{(3)}$ be the distance of the point $\vec{b}^{(3)}$ to the nearest corner of K_3:

$$b_0^{(3)} = \min_{m=(0,0),(0,1),(1,0),(1,1)} |\vec{b}^{(3)} - 2\pi m/N_2N_1a|. \tag{4.28}$$

We assume $b_0^{(3)} = |\vec{b}^{(3)}|$. In the case when $\vec{b}^{(3)}$ is closer to a vertex other than $(0,0)$, the considerations are the same up to a parallel shift. We consider two cases: $b_0^{(3)} \geq \epsilon_2 k^{-1-2\delta}$ and $0 < b_0^{(3)} < \epsilon_2 k^{-1-2\delta}$. Let

$$\vec{y}^{(2)}(\varphi) = \vec{\varkappa}_2(\varphi) + \vec{b}^{(3)}. \tag{4.29}$$

4.5.1. *The case $b_0^{(3)} \geq \epsilon_2 k^{-1-2\delta}$.*

Definition 4.12. We define the set $\mathcal{O}^{(3)}(\vec{b}^{(3)})$ by the formula:

$$\mathcal{O}^{(3)}(\vec{b}^{(3)}) = \cup_{p^{(2)} \in P^{(2)}} \mathcal{O}_s^{(2)}(2\pi p^{(2)}/N_2 N_1 a + \vec{b}^{(3)}), \tag{4.30}$$

set $\mathcal{O}_s^{(2)}$ being defined by formula (3.65). We assume $\mathcal{O}_s^{(2)} \cap \hat{\Phi}_2 \neq \emptyset$.

The set $\mathcal{O}^{(3)}(\vec{b}^{(3)})$ consists the disks with the radius $r^{(2)} = r^{(1)} k^{-2-4s_2-\delta}$ centered at poles of the resolvent $\left(\tilde{H}_2(\vec{y}^{(2)}(\varphi)) - k^2\right)^{-1}$.

Lemma 4.13. *If $\varphi \in \hat{\Phi}_2 \setminus \mathcal{O}^{(3)}(\vec{b}^{(3)})$, then*

$$\left\| \left(\tilde{H}^{(2)}(\vec{y}^{(2)}(\varphi)) - k^2\right)^{-1} \right\| \leq \frac{17 r^{(2)} k^{3\delta}}{\epsilon_2} \tag{4.31}$$

The estimate is stable in the $\left(r^{(2)}k^{-\delta}\right)$-neighborhood of $\hat{\Phi}_2 \setminus \mathcal{O}^{(3)}(\vec{b}^{(3)})$. The resolvent is an analytic function of φ in every connected component $\mathcal{O}_c^{(3)}(\vec{b}^{(3)})$ of $\mathcal{O}^{(3)}(\vec{b}^{(3)})$, whose intersection with Φ_2 is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside $\mathcal{O}^{(3)}(\vec{b}^{(3)})$ is less than ck^{2+2s_3}. The total size of $\mathcal{O}^{(3)}(\vec{b}^{(3)})$ does not exceed $c_0 k^{-4-2s_1-2s_2}$.

Proof. The proof is analogous to that of Lemma 3.22. To obtain (4.31) we combine Lemmas 3.26 and 3.31 for $\vec{b}^{(2)} = 2\pi p^{(2)}/N_2 N_1 a + \vec{b}^{(3)}$, $p^{(2)} \in P^{(2)}$, and take into account that $\epsilon_1^2 > \epsilon_2$. By the same lemmas the number of poles of $\Big(H^{(2)}\big(\vec{\varkappa}_2(\varphi) + 2\pi p^{(2)}/N_2 N_1 a + \vec{b}^{(3)}\big) - k^2\Big)^{-1}$ inside $\mathcal{O}_s^{(2)}(2\pi p^{(2)}/N_2 N_1 a + \vec{b}^{(3)})$ is less than ck^{2+2s_2}. Considering that this resolvent does not have poles outside $\mathcal{O}_s^{(2)}(2\pi p^{(2)}/N_2 N_1 a + \vec{b}^{(3)})$, we obtain that the total number of poles of $\Big(H^{(2)}\big(\vec{\varkappa}_2(\varphi) + 2\pi p^{(2)}/N_2 N_1 a + \vec{b}^{(3)}\big) - k^2\Big)^{-1}$ inside $\mathcal{O}^{(3)}(\vec{b}^{(3)})$ is less than $c_0 k^{2+2s_2}$. Taking into account that the number of $p^{(2)} \in P^{(2)}$ does not exceed $ck^{2(s_3-s_2)}$, $s_3 = 2s_2$, we obtain the estimate for the number of poles for the resolvent $\left(\tilde{H}^{(2)}(\vec{y}^{(2)}((\varphi)) - k^2\right)^{-1}$.

We estimate the size of $\mathcal{O}^{(3)}(\vec{b}^{(3)})$ the same way as we estimated the size of $\mathcal{O}^{(3)}$. Indeed, according to Lemma 3.24 the total size of each $\mathcal{O}_s^{(2)}(2\pi p^{(2)}/N_2 N_1 a)$ is less then $c_0 k^{-4-5s_2-4\delta}$. Considering that the number of $p^{(2)}$ does not exceed $k^{2(s_3-s_2)} = k^{2s_2}$, $s_2 = 2s_1$, we arrive at the estimate for $\mathcal{O}^{(3)}(\vec{b}^{(3)})$. ∎

Definition 4.14. We denote the poles of the resolvent $\left(\tilde{H}^{(2)}(\vec{y}^{(2)}((\varphi)) - k^2\right)^{-1}$ in $\mathcal{O}^{(3)}(\vec{b}^{(3)})$ as $\varphi_n^{(3)}$, $n = 1, ..., M^{(3)}$ $M^{(3)} < c_0 k^{2+2s_3}$. Let us consider the circles $\mathcal{O}_n^{(3)}(\vec{b}^{(3)})$ of the radius $r^{(3)} = k^{-2-4s_3-\delta} r^{(2)}$ around these poles. Let

$$\mathcal{O}_s^{(3)}(\vec{b}^{(3)}) = \cup_{n=1}^{M^{(3)}} \mathcal{O}_n^{(3)}(\vec{b}^{(3)}). \tag{4.32}$$

Lemma 4.15. *The total size of $\mathcal{O}_s^{(3)}(\vec{b}^{(3)})$ is less then $c_0 k^{-2s_3-\delta} r^{(2)}$.*

Proof. The lemma easily follows from the formula $r^{(3)} = k^{-2-4s_3-\delta} r^{(2)}$ and the estimate $M^{(3)} < c_0 k^{2+2s_3}$. ∎

Lemma 4.16. *If $\varphi \in \hat{\Phi}_2 \setminus \mathcal{O}_s^{(3)}(\vec{b}^{(3)})$, then*

$$\left\| \left(\tilde{H}^{(2)}(\vec{y}^{(2)}(\varphi)) - k^2\right)^{-1} \right\| \leq \frac{1}{\epsilon_2^2} \tag{4.33}$$

The estimate is stable in the $\left(r^{(3)} k^{-\delta}\right)$-neighborhood of $\hat{\Phi}_2 \setminus \mathcal{O}_s^{(3)}(\vec{b}^{(3)})$. The resolvent is an analytic function of φ in every connected component $\mathcal{O}_{sc}^{(3)}(\vec{b}^{(3)})$ of $\mathcal{O}_s^{(3)}(\vec{b}^{(3)})$, whose intersection with Φ_2 is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside $\mathcal{O}_s^{(3)}(\vec{b}^{(3)})$ is less than $c_0 k^{2+2s_3}$.

Proof. By the definition of $\mathcal{O}_s^{(3)}(\vec{b}^{(3)})$, the number of poles (counting multiplicity) of the resolvent inside this set is less than ck^{2+2s_3}. Considering as in Lemma 2.29 we obtain

$$\left\| \left(\tilde{H}^{(2)}(\vec{y}^{(2)}(\varphi)) - k^2\right)^{-1} \right\| \leq \nu^{-M^{(3)}} \frac{17 r^{(2)} k^{3\delta}}{\epsilon_2}, \tag{4.34}$$

where ν is the coefficient of contraction, when we reduce $\mathcal{O}^{(3)}(\vec{b}^{(3)})$ to $\mathcal{O}_s^{(3)}(\vec{b}^{(3)})$. Namely ν is the ratio of $r^{(3)}$ to the maximal size of $\mathcal{O}^{(3)}(\vec{b}^{(3)})$. By Lemma 4.13 and the definition of $r^{(3)}$, $\nu = k^{-4-4s_3-4s_2-2\delta}$. Considering that $\nu^{-M^{(3)}} r^{(2)} k^{\delta} < \epsilon_2^{-1}$, we obtain (4.33). ∎

By Lemma 4.15, the total size of $\mathcal{O}_s^{(3)}(\vec{b}^{(3)})$ is less then $r^{(2)}$. Since the smallest circle in $\mathcal{O}^{(3)}$ has the size $r^{(2)}$, the function $\vec{\varkappa}_3(\varphi)$ is holomorphic inside each connected component of $\mathcal{O}_s^{(3)}(\vec{b}^{(3)})$ which has non-empty intersection with Φ_3. Let

$$\vec{y}^{(3)}(\varphi) = \vec{\varkappa}_3(\varphi) + \vec{b}^{(3)}. \tag{4.35}$$

Lemma 4.17. *If $\varphi \in \hat{\Phi}_3 \setminus \mathcal{O}_s^{(3)}(\vec{b}^{(3)})$, then*

$$\left\| \left(H^{(3)}(\vec{y}^{(3)}(\varphi)) - k^2\right)^{-1} \right\| \leq \frac{2}{\epsilon_2^2} \tag{4.36}$$

The estimate is stable in the $\left(r^{(3)} k^{-\delta}\right)$-neighborhood of $\hat{\Phi}_3 \setminus \mathcal{O}_s^{(3)}(\vec{b}^{(3)})$. The resolvent is an analytic function of φ in every component of $\mathcal{O}_s^{(3)}(\vec{b}^{(3)})$, whose intersection with Φ_3 is not empty. The only singularities of the resolvent are poles. The number of poles (counting multiplicity) of the resolvent inside $\mathcal{O}^{(3)}(\vec{b}^{(3)})$ is less than $c_0 k^{2+2s_3}$.

Proof. The proof is analogous to that of Lemma 3.26 up to the shift of indices by one. We use Lemma 4.16, $\|W_3\|_\infty < \epsilon_2^4$, and the first estimate in (4.23). ∎

4.5.2. *The set* $\mathcal{O}_s^{(3)}\left(\vec{b}^{(3)}\right)$ *for small* $\vec{b}^{(3)}$. The considerations of this section is analogous to those of Section 3.4.2 up to the shift of indices by one. The following lemmas and the definitions are completely analogous to 3.27 – 3.31.

Lemma 4.18. *If* $0 < b_0^{(3)} \leq \epsilon_2 k^{-1-2\delta}$ *and* $|\epsilon_0| < b_0^{(3)} k^{1-\delta} r^{(3)}$, *then the equation*

$$\lambda^{(3)}\left(\vec{y}^{(3)}(\varphi)\right) = k^2 + \epsilon_0 \tag{4.37}$$

has no more than two solutions $\varphi_{\epsilon_0}^{\pm\ (3)}$ *in* $\hat{\Phi}_3$. *For any* $\varphi_{\epsilon_0}^{\pm\ (3)}$ *there is* $\varphi_0^{\pm(2)} \in \hat{\Phi}_2$ *such that*

$$\left|\varphi_{\epsilon_0}^{\pm(3)} - \varphi_0^{\pm(2)}\right| < r^{(3)}/4, \tag{4.38}$$

here and below $\varphi_0^{\pm(2)}$ *is* $\varphi_{\epsilon_0}^{\pm(2)}$ *for* $\epsilon_0 = 0$.

Lemma 4.19. *Suppose* $0 < b_0^{(3)} \leq \epsilon_2 k^{-1-2\delta}$ *and* $\varphi \in \hat{\Phi}_3$ *obeys the inequality analogous to (4.38):* $\left|\varphi - \varphi_0^{\pm(2)}\right| < r^{(3)}$. *Then,* $\frac{\partial}{\partial\varphi}\lambda^{(3)}\left(\vec{y}^{(3)}(\varphi)\right) =_{k\to\infty} \pm 2b_0^{(3)} k\big(1 + o(1)\big)$, *where* $|o(1)| < 10^{-2} + \epsilon_1 + \epsilon_2$ *when* $k > k_*$.

Definition 4.20. Let $\Gamma^{(3)\pm}\left(\vec{b}^{(3)}\right)$ be the open disks centered at $\varphi_0^{(3)\pm}$ with radius $r^{(3)}$; $\gamma_s^{(3)\pm}\big(\vec{b}^{(3)}\big)$ be their boundary circles and $\mathcal{O}_s^{(3)}\big(\vec{b}^{(2)}\big) = \Gamma^{(3)+} \cup \Gamma^{(3)-}$.

Lemma 4.21. *For any* φ *in* $\hat{\Phi}_3 \setminus \mathcal{O}_s^{(3)}(\vec{b}^{(3)})$, $|\lambda^{(3)}(\vec{y}^{(3)}(\varphi)) - k^2| \geq b_0^{(3)} k^{1-\delta} r^{(3)}$.

Lemma 4.22. *For any* $\varphi \in \hat{\Phi}_3 \setminus \mathcal{O}_s^{(3)}\left(\vec{b}^{(3)}\right)$,

$$\left\|\left(H^{(3)}\left(\vec{y}^{(3)}(\varphi)\right) - k^2\right)^{-1}\right\| < \frac{16}{b_0^{(3)} r^{(3)} k^{1-\delta}}. \tag{4.39}$$

The estimate is stable in the $\big(r^{(3)}k^{-\delta}\big)$*-neighborhood of* $\Phi_3 \setminus \mathcal{O}_s^{(3)}(\vec{b}^{(3)})$. *The resolvent is an analytic function of* φ *in every component of* $\mathcal{O}_s^{(3)}(\vec{b}^{(3)})$, *whose intersection with* Φ_3 *is not empty. The only singularities of the resolvent are poles. The resolvent has at most two poles inside* $\mathcal{O}_s^{(3)}(\vec{b}^{(3)})$.

5. The n-th Step of Approximation. Swiss Cheese Method.

5.1. **Introduction.** On the n-th step, $n \geq 4$, we choose $s_n = 2s_{n-1}$ and define the operator $H_\alpha^{(n)}$ by the formula:

$$H_\alpha^{(n)} = H^{(n-1)} + \alpha W_n, \quad (0 \leq \alpha \leq 1), \qquad W_n = \sum_{r=M_{n-1}+1}^{M_n} V_r,$$

where M_n is chosen in such a way that $2^{M_n} \approx k^{s_n}$. Obviously, the periods of W_n are $2^{M_n-1}(d_1, 0)$ and $2^{M_n-1}(0, d_2)$. We write them in the form: $N_{n-1} \cdot ... \cdot N_1(a_1, 0)$ and $N_{n-1} \cdot ... \cdot N_1(0, a_2)$, here $N_{n-1} = 2^{M_n - M_{n-1}}$, $\frac{1}{4}k^{s_n - s_{n-1}} < N_{n-1} < 4k^{s_n - s_{n-1}}$. Note that $\|W_n\|_\infty \leq \sum_{r=M_{n-1}+1}^{M_n} \|V_r\|_\infty < \exp(-k^{\eta s_{n-1}})$. By analogy with the definition of ϵ_1, ϵ_2 we introduse the notation $\epsilon_n = \exp(-k^{\eta s_n})$.

The n-th step is analogous to the second step up to replacement the indices 3 by n, 3 by $n-1$, the producr N_2N_1 by $N_{n-1}\cdot ... \cdot N_1$, etc.

We note that k, satisfying (3.1), obeys the analogous condition for with any s_n instead of s_1

$$C_*(1+s_n)k^{2+4s_n}\ln k < k^{\eta s_n} \tag{5.1}$$

with the same constant C_*. The inequality (5.1) can be obtained from (3.1) by induction. This is an important fact: it means that the lower bound for k does not grow with n, i.e., all steps hold uniformly in k for $k > k_*$, k_* being introduced by (3.1). Further we assume $k > k_*$.

The formulation of the geometric lemma for n-th step is the same as that for Step 2 up to a shift of indices, we skip it here for shortness. Note only that in the lemma we use the set $\chi^*_{n-1}(\lambda)$ to define $\chi_n(\lambda)$. In fact, we started with the definition of $\chi_1(\lambda)$ and then use it to define $\chi^*_1(\lambda)$ (Step 1). Considering $\chi^*_1(\lambda)$, we constructed $\chi_2(\lambda)$ (Lemma 3.5) and later used it to define $\chi^*_2(\lambda)$ (Section 3.3). Using $\chi^*_2(\lambda)$, we introduced $\chi_3(\lambda)$ (Lemma 4.1) and, then $\chi^*_3(\lambda)$ (Section 4.4). Thus, the process goes like $\chi_1 \to \chi^*_1 \to \chi_2 \to \chi^*_2 \to \chi_3 \to \chi^*_3$. Here we start with the set $\chi^*_{n-1}(\lambda)$ defined by (4.27) for $n = 4$ and by (5.15) for $n > 4$. [8] The estimate (4.6) for n-th step takes the form:

$$\frac{L\left(\mathcal{K}_n\chi^*_{n-1} \setminus \chi_n)\right)}{L\left(\chi^*_{n-1}\right)} < k^{-S_n}, \qquad S_n = 2\sum_{i=1}^{n-1}(1+s_i). \tag{5.2}$$

The formulation of the main results (perturbation formulae) for n-th step is the same as for the second and third step: Theorems analogous up to the shift of indices to Theorem 3.8/4.4, Lemma 3.11, Theorem 3.12, Corollary 3.13 and Lemma 3.14 hold.

5.2. **Proof of Geometric Lemma.** The proof of the lemma is analogous to that for Geometric Lemma in the second step. Let us consider

$$\mathcal{O}^{(n)} = \cup_{\tilde{p}^{(n-1)}\in P^{(n-1)}\setminus\{0\}}\mathcal{O}_s^{(n-1)}\left(\frac{2\pi\tilde{p}^{(n-1)}}{N_{n-1}...N_1a}\right), \tag{5.3}$$

where $\mathcal{O}_s^{(n-1)}\left(\frac{2\pi\tilde{p}^{(2)}}{N_{n-1}...N_1a}\right)$ is defined by Definition 4.14 with $\vec{b}^{(3)} = \frac{2\pi\tilde{p}^{(3)}}{N_3N_2N_1a}$ when $n = 4$. When $n > 4$ we use Definition 5.6 with $n-1$ instead of n and take $b^{(n-1)} = \frac{2\pi\tilde{p}^{(n-1)}}{N_{n-1}...N_1a}$. The radius $r^{(n-1)}$ of $\mathcal{O}_s^{(n-1)}$ is defined by the recurrent formula: $r^{(n-1)} = k^{-2-4s_{n-1}-\delta}r^{(n-2)}$. This means

$$r^{(n-1)} = k^{-2n-4\sum_{k=1}^{n-1}s_n-2s_1-2\delta} = k^{-2n-2\sum_{k=1}^{n}s_n-2\delta} = k^{-S_{n+1}-2\delta}, \tag{5.4}$$

[8]Strictly speaking we assume that there is a subset $\chi^*_{n-1}(\lambda)$ of the isoenergetic surface $S_{n-1}(\lambda)$ of $H^{(n-1)}$ such that perturbation series of the type (4.15), (4.16) converges for $t^{(n-1)} \in \chi^*_{n-1}(\lambda)$ and $\chi^*_{n-1}(\lambda)$ has properties described in Section 4.4 up to replacement of 3 by $n-1$. In particular, we assume that $\chi^*_{n-1}(\lambda) = \mathcal{K}_{n-1}\mathcal{D}_{n-1}(\lambda)$, where $\mathcal{D}_{n-1}(\lambda)$ satisfies the analog of Lemma 4.8 and that the analogs of Lemmas 4.9 and 4.10 hold too. Here, $\mathcal{K}_{n-1}$ is the parallel shift into K_{n-1}. Further in this section we describe the next set $\chi^*_n(\lambda)$ which has analogous properties.

S_n being defined by (5.2). Note, that the definition make sense, since $|\frac{2\pi\tilde{p}^{(n-1)}}{N_{n-1}...N_1a}| > d_{max}^{-1}k^{-s_n} > \epsilon_{n-1}k^{-1-2\delta}$, when $k > k_*$, the estimate (5.1) has been used. The last inequality can be easily proved by induction. Let

$$\Phi_n = \Phi_{n-1} \setminus \mathcal{O}^{(n)}, \quad \Theta_n = \Phi_n \cap [0, 2\pi), \tag{5.5}$$

Φ_3, Θ_3 are given by (4.8). By Lemmas 4.17, 5.9

$$\left\| \left(H^{(n-1)}\Big(\vec{\varkappa}_{n-1}(\varphi) + \frac{2\pi\tilde{p}^{(n-1)}}{N_{n-1}...N_1a}\Big) - k^2 \right)^{-1} \right\| < \frac{2}{\epsilon_{n-2}^2} \tag{5.6}$$

for all $\tilde{p}^{(n-1)} \in P^{(n-1)} \setminus \{0\}$ and $\varphi \in \hat{\Phi}_n$, here and below $\hat{\Phi}_n$ is $r^{(n-1)}k^{-\delta}$-neighborhood of Φ_n. +We consider $\mathcal{D}_{n-1,nonres} \subset \mathcal{D}_{n-1}$:

$$\mathcal{D}_{n-1,nonres} = \{\vec{\varkappa}_{n-1}(\varphi),\ \varphi \in \Theta_n\}. \tag{5.7}$$

We define χ_n by the formula:

$$\chi_n = \mathcal{K}_n \mathcal{D}_{n-1,nonres}. \tag{5.8}$$

By the definition of $\mathcal{K}_n$, for every $t^{(n)}$ in χ_n, there are $p^{(1)} \in P^{(1)}, \ldots ,p^{(n-1)} \in P^{(n-1)}$ and $j \in Z^2$ such that

$$t^{(n)} + \frac{2\pi p^{(n-1)}}{N_{n-1}...N_1a} + ... + \frac{2\pi p^{(1)}}{N_1a} + \frac{2\pi j}{a} = \vec{\varkappa}_{n-1}(\varphi), \quad \vec{\varkappa}_{n-1}(\varphi) \in \mathcal{D}_{n-1,nonres}. \tag{5.9}$$

Considering the definition of $\mathcal{D}_{n-1,nonres}$ and (5.6), we obtain:

$$\left\| \left(H^{(n-1)}\Big(t^{(n)} + \frac{2\pi\hat{p}^{(n-1)}}{N_{n-1}...N_1a}\Big) - k^2 \right)^{-1} \right\| < \frac{2}{\epsilon_{n-2}^2}, \tag{5.10}$$

$\hat{p}^{(n-1)} = p^{(n-1)} + \tilde{p}^{(n-1)}$. Note that the indices j, $p^{(1)}$,...,$p^{(n-1)}$ do not play a role, since they just produce a shift of indices of the matrix elements of the resolvent. Considering that $\tilde{p}^{(n-1)}$ can be any but zero, we obtain that (5.10) holds for all $\hat{p}^{(n-1)} \in P^{(n-1)} \setminus \{p^{(n-1)}\}$. Taking into account that $\lambda_j^{(n-1)}(t^{(n)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a) = k^2$ and $\epsilon_{n-1} < \epsilon_{n-2}^2$. We arrive at the analog of (4.5) for all $\hat{p}^{(n-1)} \neq p^{(n-1)}$.This also proves that $p^{(n-1)}$ is uniquely defined by (5.9). It remains to check the analog of (4.5) for $p^{(n-1)} = \hat{p}^{(n-1)}$. Let $t^{(n-1)} := t^{(n)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a$. By (5.9), $t^{(n-1)} \in \mathcal{K}_{n-1}\mathcal{D}_{n-1}$. Using (4.27) for $n = 4$ and (5.15) with $n-1$ instead of n for $n > 4$, we get $t^{(n-1)} \in \chi_{n-1}^*$. By the analog of Theorem 4.4 for step $n-1$, $\lambda_j^{(n-1)}(t^{(n)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a)$ is the only eigenvalue of $H^{(n-1)}\big(t^{(n)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a\big)$ in the interval ε_{n-1}. Hence,

$$|\lambda_j^{(n-1)}(t^{(3)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a) - \lambda_{\hat{m}}^{(n-1)}(t^{(n)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a)| > \epsilon_{n-2}.$$

Thus, the analog of (4.3) holds for all $p^{(n-1)} \in P^{(n-1)}$. Part 2 holds, since all estimates are stable with respect to the perturbation of $t^{(3)}$ less then $\epsilon_{n-1}k^{-1-\delta}$.

Let us estimate the size of $\mathcal{O}_n$. According to Lemma 4.15, 5.7, the total size of each $\mathcal{O}_s^{(n-1)}(2\pi p^{(n-1)}/N_{n-1}...N_1a)$ is less then $c_0k^{-2s_{n-1}-\delta}r^{(n-2)}$. Considering that the number of $p^{(n-1)}$ does not exceed $k^{2(s_n-s_{n-1})}=k^{2s_{n-1}}$, $s_n=2s_{n-1}$, we obtain that the size of $\mathcal{O}_n$ is less than $k^{-\delta}r^{(n-2)}$. Using the formula for $r^{(n-2)}$, we obtain that the total size of $\mathcal{O}_n$ does not exceed $k^{-2(n-1)-2\sum_{k=1}^{n-1}s_k}$. Using this estimate we easily arrive at (5.2). It is easy to see that $S_n=2(n-1)+(2^n-2)\,s_1$ and $S_n\approx 2^n s_1\approx s_n$ for large n. The lemma is proved.

5.3. **Nonresonant part of the isoenergetic set of $H_\alpha^{(n)}$.** Now we construct a nonresonance subset $\chi_n^*(\lambda)$ of the isoenergetic surface $S_n(\lambda)$ of $H_\alpha^{(n)}$ in K_n, $S_n(\lambda)\subset K_n$. It corresponds to nonresonance eigenvalues given by perturbation series. The sets $\chi_1^*(\lambda)$, $\chi_2^*(\lambda)$, $\chi_3^*(\lambda)$ are defined in the previous steps as well as the non-resonance sets $\chi_1(\lambda)$, $\chi_2(\lambda)$, $\chi_3(\lambda)$. Let us recall that we started with the definition of $\chi_1(\lambda)$ and then use it to define $\chi_1^*(\lambda)$. Considering $\chi_1^*(\lambda)$, we constructed $\chi_2(\lambda)$ (Step 2). Next, we defined $\chi_2^*(\lambda)$. Using $\chi_2^*(\lambda)$, we introduced $\chi_3(\lambda)$ and, then $\chi_3^*(\lambda)$. Thus, the process looks like $\chi_1\to\chi_1^*\to\chi_2\to\chi_2^*\to\chi_3\to\chi_3^*$. The geometric lemma in this section gives us χ_4 and every next χ_n if χ_{n-1}^* is defined. To ensure the reccurent procedure we show now how to define $\chi_n^*(\lambda)$ using $\chi_n(\lambda)$.

We define $\mathcal{B}_n$ as the set of directions corresponding to Θ_n:

$$\mathcal{B}_n(\lambda)=\{\vec{\nu}\in S_1:\varphi\in\Theta_n\}.$$

Note that $\mathcal{B}_n$ is a unit circle with holes centered at the origin and $\mathcal{B}_n(\lambda)\subset\mathcal{B}_{n-1}(\lambda)$. We define $\mathcal{D}_n(\lambda)$ as a level set for $\lambda^{(n)}(\alpha,\vec{\varkappa})$ in the $(\epsilon_{n-1}k^{-1-\delta})$-neighborhood of $\mathcal{D}_{n-1,nonres}(\lambda)$:

$$\mathcal{D}_n(\lambda)=\left\{\vec{\varkappa}=\varkappa\vec{\nu}:\vec{\nu}\in\mathcal{B}_n,\left|\varkappa-\varkappa_{n-1}(\lambda,\vec{\nu})\right|<\epsilon_{n-1}k^{-1-\delta},\lambda^{(n)}(\alpha,\vec{\varkappa})=\lambda\right\}.$$

Considering as in the previous step, we prove the analogs of Lemmas 4.7 and 4.8. For shortness, we provide here only the second lemma. By analogy with previous sections, we shorten notations here: $\varkappa_{n-1}(\varphi)\equiv\varkappa_{n-1}(\lambda,\vec{\nu})$, $\vec{\varkappa}_{n-1}(\varphi)\equiv\varkappa_{n-1}(\lambda,\vec{\nu})\vec{\nu}$.

Lemma 5.1. *For $\lambda>k_*^2$:*

(1) *The set $\mathcal{D}_n(\lambda)$ is a distorted circle with holes: it can be described by the formula:*

$$\mathcal{D}_n(\lambda)=\left\{\vec{\varkappa}\in\mathbb{R}^2:\vec{\varkappa}=\vec{\varkappa}_n(\varphi),\quad\varphi\in\Theta_n(\lambda)\right\},\tag{5.11}$$

where $\varkappa_n(\varphi)=\varkappa_{n-1}(\varphi)+h_n(\varphi)$, and $h_n(\varphi)$ satisfies the estimates

$$|h_n|<12\alpha\epsilon_{n-1}^4k^{-1},\quad\left|\frac{\partial h_n}{\partial\varphi}\right|\le 4\alpha\epsilon_{n-1}^3k^{1+\delta}.\tag{5.12}$$

(2) *The total length of $\mathcal{B}_n(\lambda)$ satisfies the estimate:*

$$L\left(\mathcal{B}_{n-1}\setminus\mathcal{B}_n\right)<4\pi k^{-S_n}.\tag{5.13}$$

(3) *Function $\varkappa_n(\varphi)$ can be extended as a holomorphic function of φ to $\hat{\Phi}_n$, estimates (5.12) being preserved.*

(4) *The curve $\mathcal{D}_n(\lambda)$ has a length which is asymptotically close to that of $\mathcal{D}_{n-1}(\lambda)$ in the following sense:*

$$L\Big(\mathcal{D}_n(\lambda)\Big)\underset{\lambda\to\infty}{=}L\Big(\mathcal{D}_{n-1}(\lambda)\Big)\Big(1+O(k^{-S_n})\Big).\tag{5.14}$$

Now define the nonresonance set, $\chi_n^*(\lambda)$ in $S_n(\lambda)$ by the formula analogous to (4.27). Indeed,

$$\chi_n^*(\lambda) := \mathcal{K}_n\mathcal{D}_n(\lambda). \tag{5.15}$$

The following lemmas are analogous to Lemmas 4.9 and 4.10.

Lemma 5.2. *The set $\chi_n^*(\lambda)$ belongs to the $\left(2\alpha\epsilon_{n-1}^4k^{-1}\right)$-neighborhood of $\chi_n(\lambda)$ in K_n. If $t^{(n)} \in \chi_n^*(\lambda)$, then the operator $H_\alpha^{(n)}(t^{(n)})$ has a simple eigenvalue equal to λ. This eigenvalue is given by the perturbation series analogous to (4.15).*

Lemma 5.3. *Formula (5.15) establishes one-to-one correspondence between $\chi_n^*(\lambda)$ and $\mathcal{D}_n(\lambda)$.*

5.4. **Preparation for the Next Approximation.** Let $\vec{b}^{(n)} \in K_n$ and $b_0^{(n)}$ be the distance of the point $\vec{b}^{(n)}$ to the nearest corner of K_n. We assume $b_0^{(n)} = |\vec{b}^{(n)}|$. We consider two cases: $b_0^{(n)} \geq \epsilon_{n-1}k^{-1-2\delta}$ and $0 < b_0^{(n)} < \epsilon_{n-1}k^{-1-2\delta}$. Let

$$\vec{y}^{(n-1)}(\varphi) = \vec{\varkappa}_{n-1}(\varphi) + \vec{b}^{(n)}. \tag{5.16}$$

5.4.1. *The case $b_0^{(n)} \geq \epsilon_{n-1}k^{-1-2\delta}$.*

Definition 5.4. We define the set $\mathcal{O}^{(n)}(\vec{b}^{(n)})$ by the formula:

$$\mathcal{O}^{(n)}(\vec{b}^{(n)}) = \cup_{p^{(n-1)}\in P^{(n-1)}}\mathcal{O}_s^{(n-1)}\left(2\pi p^{(n-1)}/N_{n-1}...N_1a + \vec{b}^{(n)}\right), \tag{5.17}$$

set $\mathcal{O}_s^{(n-1)}$ being defined by Definition 5.4 for $n = 4$ and by 5.6 for $n > 4$.

The set $\mathcal{O}^{(n)}(\vec{b}^{(n)})$ consists the disks with the radius $r^{(n-1)} = r^{(n-2)}k^{-2-4s_{n-1}-\delta}$ centered at poles of the resolvent $\left(\tilde{H}_{n-1}(\vec{y}^{(n-1)}(\varphi)) - k^2\right)^{-1}$.

Lemma 5.5. *If $\varphi \in \hat{\Phi}_{n-1} \setminus \mathcal{O}^{(n)}(\vec{b}^{(n)})$, then*

$$\left\|\left(\tilde{H}^{(n-1)}(\vec{y}^{(n-1)}(\varphi)) - k^2\right)^{-1}\right\| \leq \frac{17r^{(n-1)}k^{3\delta}}{\epsilon_{n-1}} \tag{5.18}$$

The estimate is stable in the $\left(r^{(n-1)}k^{-\delta}\right)$-neighborhood of $\hat{\Phi}_{n-1} \setminus \mathcal{O}^{(n)}(\vec{b}^{(n)})$. The resolvent is an analytic function of φ in every connected component $\mathcal{O}_c^{(n)}(\vec{b}^{(n)})$ of $\mathcal{O}^{(n)}(\vec{b}^{(n)})$, whose intersection with Φ_{n-1} is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside $\mathcal{O}^{(n)}(\vec{b}^{(n)})$ is less than ck^{2+2s_n}. The total size of $\mathcal{O}^{(n)}(\vec{b}^{(n)})$ does not exceed $c_0k^{-S_n}$.

Proof. The proof is analogous to that of Lemma 3.22. Let $n = 4$. To obtain (5.18) we combine Lemmas 4.17 and 4.22 for $\vec{b}^{(n-1)} = 2\pi p^{(n-1)}/N_{n-1}...N_1a + \vec{b}^{(n)}$ taking into account $\epsilon_{n-2}^2 > \epsilon_{n-1}$ and the estimate for $b_0^{(n)}$. If $n > 4$, then using the recurrent procedure, we apply Lemmas 5.9 and 5.14 with $n-1$ instead of n. Moreover, the number of poles of $\left(H^{(n-1)}(\vec{\varkappa}_{n-1}(\varphi) + \vec{b}^{(n-1)}) - k^2\right)^{-1}$ inside $\mathcal{O}^{(n)}(\vec{b}_n)$ is less than $ck^{2+2s_{n-1}}$. Considering that the number of $p^{(n-1)} \in P^{(n-1)}$ does not exceed

$ck^{2(s_n-s_{n-1})}$, $s_n = 2s_{n-1}$, we obtain the estimate for the number of poles for the resolvent $\left(\tilde{H}^{(n-1)}\big(\vec{y}^{(n-1)}((\varphi)\big) - k^2\right)^{-1}$. We estimate the size of $\mathcal{O}^{(n)}(\vec{b}^{(n)})$ the same way as we estimated the size of $\mathcal{O}^{(n)}$. ∎

Definition 5.6. We denote the poles of the resolvent $\left(\tilde{H}^{(n-1)}\big(\vec{y}^{(n-1)}((\varphi)\big) - k^2\right)^{-1}$ in $\mathcal{O}^{(n)}(\vec{b}^{(n)})$ as $\varphi_m^{(n)}$, $m = 1, ..., M^{(n)}$ $M^{(n)} < c_0k^{2+2s_n}$. Let us consider the circles $\mathcal{O}_m^{(n)}(\vec{b}^{(n)})$ of the radius $r^{(n)} = k^{-2-4s_n-\delta}r^{(n-1)}$ around these poles. Let

$$\mathcal{O}_s^{(n)}(\vec{b}^{(n)}) = \cup_{m=1}^{M^{(n)}}\mathcal{O}_m^{(n)}(\vec{b}^{(n)}). \tag{5.19}$$

Lemma 5.7. *The total size of $\mathcal{O}_s^{(n)}(\vec{b}^{(n)})$ is less then $c_0k^{-2s_n-\delta}r^{(n-1)}$.*

Proof. The lemma easily follows from the formula $r^{(n)} = k^{-2-4s_n-\delta}r^{(n-1)}$ and the estimate $M^{(n)} < c_0k^{2+2s_n}$. ∎

Lemma 5.8. *If $\varphi \in \hat{\Phi}_{n-1} \setminus \mathcal{O}_s^{(n)}(\vec{b}^{(n)})$, then*

$$\left\|\left(\tilde{H}^{(n-1)}\big(\vec{y}^{(n-1)}(\varphi)\big) - k^2\right)^{-1}\right\| \leq \frac{1}{\epsilon_{n-1}^2} \tag{5.20}$$

The estimate is stable in the $\left(r^{(n)}k^{-\delta}\right)$-neighborhood of $\hat{\Phi}_{n-1} \setminus \mathcal{O}_s^{(n)}(\vec{b}^{(n)})$. The resolvent is an analytic function of φ in every connected component $\mathcal{O}_{sc}^{(n)}(\vec{b}^{(n)})$ of $\mathcal{O}_s^{(n)}(\vec{b}^{(n)})$, whose intersection with Φ_{n-1} is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside $\mathcal{O}_s^{(n)}(\vec{b}^{(n)})$ is less than ck^{2+2s_n}.

Proof. By the definition of $\mathcal{O}_s^{(n)}(\vec{b}^{(n)})$, the number of poles (counting multiplicity) of the resolvent inside this set is less than $M^{(n)} = c_0k^{2+2s_n}$. Considering as in Lemma 2.29 we obtain

$$\left\|\left(\tilde{H}^{(n-1)}\big(\vec{y}^{(n-1)}(\varphi)\big) - k^2\right)^{-1}\right\| \leq \nu^{-M^{(n)}}\frac{17r^{(n-1)}k^{3\delta}}{\epsilon_{n-1}}, \tag{5.21}$$

where ν is the coefficient of contraction, when we reduce $\mathcal{O}^{(n)}(\vec{b}^{(n)})$ to $\mathcal{O}_s^{(n)}(\vec{b}^{(n)})$. Namely ν is the ratio of $r^{(n)}$ to the maximal size of $\mathcal{O}^{(n)}(\vec{b}^{(n)})$. By Lemma 5.5 and formula (5.4), $\nu = k^{-4-4s_n-4s_{n-1}}$. Considering that $r^{(n-1)}k^{\delta} < 1$ and **using** (5.1) , we obtain $\nu^{-M^{(n)}}r^{(n-1)}k^{\delta} < \epsilon_{n-1}^{-1}$ when $k > k_*$. Thus, we obtain (5.20). ∎

By Lemma 5.7, the total size of $\mathcal{O}_s^{(n)}(\vec{b}^{(n)})$ is less then $r^{(n-1)}$. Therefore, the function $\vec{\varkappa}_n(\varphi)$ is holomorphic inside each connected component of $\mathcal{O}_s^{(n)}(\vec{b}^{(n)})$ which has non-empty intersection with Φ_n. Let

$$\vec{y}^{(n)}(\varphi) = \vec{\varkappa}_n(\varphi) + \vec{b}^{(n)}. \tag{5.22}$$

Lemma 5.9. *If $\varphi \in \hat{\Phi}_n \setminus \mathcal{O}_s^{(n)}(\vec{b}^{(n)})$, then*

$$\left\|\left(H^{(n)}\big(\vec{y}^{(n)}(\varphi)\big) - k^2\right)^{-1}\right\| \leq \frac{2}{\epsilon_{n-1}^2} \tag{5.23}$$

The estimate is stable in the $\left(r^{(n)}k^{-\delta}\right)$*-neighborhood of* $\hat{\Phi}_n \setminus \mathcal{O}_s^{(n)}(\vec{b}^{(n)})$. *The resolvent is an analytic function of* φ *in every connected component* $\mathcal{O}_{sc}^{(n)}(\vec{b}^{(n)})$ *of* $\mathcal{O}_s^{(n)}(\vec{b}^{(n)})$, *whose intersection with* Φ_n *is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside* $\mathcal{O}^{(n)}(\vec{b}^{(n)})$ *is less than* $c_0k^{2+2s_n}$.

Proof. The proof is analogous to that of Lemma 3.26 up to the shift of indices by two. We use Lemma 5.8, $\|W_n\|_\infty < \epsilon_{n-1}^4$, and the first estimate in (5.12). ∎

5.4.2. *The set* $\mathcal{O}_s^{(n)}\left(\vec{b}^{(n)}\right)$ *for small* $\vec{b}^{(n)}$. The considerations of this section is analogous to those of By analogy with the previous subsection, we choose $r^{(n)} = k^{-2-2s_n-\delta}r^{(n-1)}$ here. The following Lemmas 5.10 and 5.11 are identical to 4.18 and 4.19 up to replacement of indices (3) by (n), (2) by $(n-1)$, etc. Next, Definition 5.12 is analogous to 4.20. Lemmas 5.13, 5.14 are analogous to 4.21, 4.22.

Lemma 5.10. *If* $0 < b_0^{(n)} \leq \epsilon_{n-1}k^{-1-2\delta}$ *and* $|\epsilon_0| < b_0^{(n)}k^{1-\delta}r^{(n)}$, *then the equation*

$$\lambda^{(n)}\left(\vec{y}^{(n)}(\varphi)\right) = k^2 + \epsilon_0 \tag{5.24}$$

has no more than two solutions $\varphi_{\epsilon_0}^{\pm\,(n)}$ *in* $\hat{\Phi}_n$. *For any* $\varphi_{\epsilon_0}^{\pm\,(n)}$ *there is* $\varphi_0^{\pm(n-1)} \in \hat{\Phi}_{n-1}$ *such that*

$$\left|\varphi_{\epsilon_0}^{\pm(n)} - \varphi_0^{\pm(n-1)}\right| < r^{(n)}/4, \tag{5.25}$$

here and below $\varphi_0^{\pm(n-1)}$ *is* $\varphi_{\epsilon_0}^{\pm(n-1)}$ *for* $\epsilon_0 = 0$.

Lemma 5.11. *Suppose* $0 < b_0^{(n)} \leq \epsilon_{n-1}k^{-1-2\delta}$ *and* $\varphi \in \hat{\Phi}_n$ *obeys the inequality analogous to (5.25):* $\left|\varphi - \varphi_0^{\pm(n-1)}\right| < r^{(n)}$. *Then,* $\frac{\partial}{\partial\varphi}\lambda^{(n)}\left(\vec{y}^{(n)}(\varphi)\right) =_{k\to\infty} \pm 2b_0^{(n)}k\big(1 + o(1)\big)$, *where* $|o(1)| < 10^{-2} + \epsilon_1 + ... + \epsilon_{n-1}$ *when* $k > k_*$.

Definition 5.12. Let $\Gamma^{(n)\pm}\left(\vec{b}^{(n)}\right)$ be the open disks centered at $\varphi_0^{(n)\pm}$ with radius $r^{(n)}$; $\gamma_s^{(n)\pm}\big(\vec{b}^{(n)}\big)$ be their boundary circles and $\mathcal{O}_s^{(n)}\big(\vec{b}^{(n-1)}\big) = \Gamma^{(n)+} \cup \Gamma^{(n)-}$.

Lemma 5.13. *For any* φ *in* $\hat{\Phi}_n \setminus \mathcal{O}_s^{(n)}(\vec{b}^{(n)})$, $|\lambda^{(n)}(\vec{y}^{(n)}(\varphi)) - k^2| \geq b_0^{(n)}k^{1-\delta}r^{(n)}$.

Lemma 5.14. *For any* $\varphi \in \hat{\Phi}_{n-1} \setminus \mathcal{O}_s^{(n)}\left(\vec{b}^{(n)}\right)$,

$$\left\|\left(H^{(n)}\left(\vec{y}^{(n)}(\varphi)\right) - k^2\right)^{-1}\right\| < \frac{16}{b_0^{(n)}r^{(n)}k^{1-\delta}}, \tag{5.26}$$

The estimate is stable in the $\left(r^{(n)}k^{-\delta}\right)$*-neighborhood of* $\Phi_n \setminus \mathcal{O}_s^{(n)}(\vec{b}^{(n)})$. *The resolvent is an analytic function of* φ *in every component of* $\mathcal{O}_s^{(n)}(\vec{b}^{(n)})$, *whose intersection with* Φ_n *is not empty. The only singularities of the resolvent are poles. The resolvent has at most two poles inside* $\mathcal{O}_s^{(n)}(\vec{b}^{(n)})$.

6. Limit-Isoenergetic Set and Eigenfunctions

6.1. **Limit-Isoenergetic Set and Proof of the Bethe-Sommerfeld Conjecture.** At every step n we constructed a subset $\mathcal{B}_n(\lambda)$ of the unit circle, and a function $\varkappa_n(\lambda,\vec{\nu})$, $\vec{\nu}\in\mathcal{B}_n(\lambda)$, with the following properties. The sequence $\mathcal{B}_n(\lambda)$ is decreasing: $\mathcal{B}_n(\lambda)\subset\mathcal{B}_{n-1}(\lambda)$. The set $\mathcal{D}_n(\lambda)$ of vectors $\vec{\varkappa}=\varkappa_n(\lambda,\vec{\nu})\vec{\nu}$, $\vec{\nu}\in\mathcal{B}_n(\lambda)$, is a slightly distorted circle with holes, see Fig.1, Fig.2, formula (1.15) and Lemmas 2.10, 3.17, 4.8, 5.1. For any $\vec{\varkappa}_n(\lambda,\vec{\nu})\in\mathcal{D}_n(\lambda)$ there is a simple eigenvalue of $H^{(n)}(\vec{\varkappa}_n)$ equal to λ and given by a perturbation series. [9] Let $\mathcal{B}_\infty(\lambda)=\bigcap_{n=1}^{\infty}\mathcal{B}_n(\lambda)$. Since $\mathcal{B}_{n+1}\subset\mathcal{B}_n$ for every n, $\mathcal{B}_\infty(\lambda)$ is a unit circle with the infinite number of holes, more and more holes of smaller and smaller size appearing at each step.

Lemma 6.1. *The length of $\mathcal{B}_\infty(\lambda)$ satisfies estimate (1.9) with $\gamma_3=\delta$.*

Proof. Using (2.71), (3.53), (4.24) and (5.13) and consideirng that $S_n\approx 2^n s_1$, we easily conclude that $L\left(\mathcal{B}_n\right)=\left(1+O(k^{-\delta})\right)$, $k=\lambda^{1/2}$ uniformly in n. Since $\mathcal{B}_n$ is a decreasing sequence of sets, (1.9) holds. ∎

Let us consider $\varkappa_\infty(\lambda,\vec{\nu})=\lim_{n\to\infty}\varkappa_n(\lambda,\vec{\nu}),\quad \vec{\nu}\in\mathcal{B}_\infty(\lambda)$.

Lemma 6.2. *The limit $\varkappa_\infty(\lambda,\vec{\nu})$ exists for any $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$ and the following estimates hold when $n\geq 1$:*

$$|\varkappa_\infty(\lambda,\vec{\nu})-\varkappa_n(\lambda,\vec{\nu})|<14\epsilon_n^4k^{-1},\quad \epsilon_n=\exp(-\frac{1}{4}k^{\eta s_n}),\quad s_n=2^{n-1}s_1. \tag{6.1}$$

Corollary 6.3. *For every $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$ estimate (1.10) holds, where $\gamma_4=3-30s_1-20\delta>0$.*

Proof. The lemma easily follows from the estimates (3.52), (4.23) and (5.12). To obtain corollary we use (2.70). ∎

Estimates (3.52), (4.23) and (5.12) justify convergence of the series $\sum_{m=1}^{\infty}\frac{\partial h_n}{\partial\varphi}$, and hence, of the sequence $\frac{\partial\varkappa_n}{\partial\varphi}$. We denote the limit of this sequence by $\frac{\partial\varkappa_\infty}{\partial\varphi}$.

Lemma 6.4. *The estimate (1.18) with $\gamma_5=1-33s_1-22\delta>0$ holds for any $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$.*

Proof. The lemma easily follows from (2.70), (3.52), (4.23) and (5.12). ∎

We define $\mathcal{D}_\infty(\lambda)$ by (1.8). Clearly, $\mathcal{D}_\infty(\lambda)$ is a slightly distorted circle of radius k with the infinite number of holes. We can assign a tangent vector $\frac{\partial\varkappa_\infty}{\partial\varphi}\vec{\nu}+\varkappa_\infty\vec{\mu}$, $\vec{\mu}=(-\sin\varphi,\cos\varphi)$ to the curve $\mathcal{D}_\infty(\lambda)$, this tangent vector being the limit of corresponding tangent vectors for curves $\mathcal{D}_n(\lambda)$ at points $\vec{\varkappa}_n(\lambda,\vec{\nu})$ as $n\to\infty$.

Remark 6.5. We easily see from (6.1), that any $\vec{\varkappa}\in\mathcal{D}_\infty(\lambda)$ belongs to the $\left(14\epsilon_n^4k^{-1}\right)$-neighborhood of $\mathcal{D}_n(\lambda)$. Applying perturbation formulae for n-th step, we easily obtain that there is an eigenvalue $\lambda^{(n)}(\vec{\varkappa})$ of $H^{(n)}(\vec{\varkappa})$ satisfying the estimate

[9] The operator $H^{(n)}(\vec{\varkappa})$ is defined for every $\vec{\varkappa}\in\mathbb{R}^2$. The perturbation series is given by a formula analogous to (3.44), which coincides with (3.29) up to a shift of indices corresponding to the parallel shift of $\vec{\varkappa}$ into K_n.

$\lambda^{(n)}(\vec{\varkappa}) = \lambda + \delta_n$, $\delta_n = O\left(\epsilon_n^4\right)$, the eigenvalue $\lambda^{(n)}(\vec{\varkappa})$ being given by a perturbation series of the type (3.44). Hence, for every $\vec{\varkappa} \in \mathcal{D}_\infty(\lambda)$ there is a limit:

$$\lim_{n\to\infty} \lambda^{(n)}(\vec{\varkappa}) = \lambda. \tag{6.2}$$

Theorem 6.6 (Bethe-Sommerfeld Conjecture)**.** *The spectrum of operator H contains a semi-axis.*

Proof. By Remark 6.5, there is a point of the spectrum of H_n in the δ_n-neighborhood of λ for every $\lambda > k_*^2$, k_* being introduced by (3.1). Since $\|H_n - H\| < \epsilon_n^4$, there is a point of the spectrum of H in the δ_n^*-neighborhood of λ, $\delta_n^* = \delta_n + \epsilon_n^4$. Since it is true for every n and the spectrum of H is closed, λ is in the spectrum of H. ∎

6.2. **Generalized Eigenfunctions of H.** A plane wave is usually denoted by $e^{i\langle \vec{k}, x\rangle}$, $\vec{k} \in \mathbb{R}^2$. Here we use $\vec{\varkappa}$ instead of $\vec{k}$ to comply with our previous notations. We show that for every $\vec{\varkappa}$ in a set

$$\mathcal{G}_\infty = \cup_{\lambda \geq \lambda_*} \mathcal{D}_\infty(\lambda), \quad \lambda_* = k_*^2, \tag{6.3}$$

there is a solution $\Psi_\infty(\vec{\varkappa}, x)$ of the equation for eigenfunctions:

$$-\Delta\Psi_\infty(\vec{\varkappa}, x) + V(x)\Psi_\infty(\vec{\varkappa}, x) = \lambda_\infty(\vec{\varkappa})\Psi_\infty(\vec{\varkappa}, x), \tag{6.4}$$

which can be represented in the form

$$\Psi_\infty(\vec{\varkappa}, x) = e^{i\langle \vec{\varkappa}, x\rangle}\Big(1 + u_\infty(\vec{\varkappa}, x)\Big), \qquad \big\|u_\infty(\vec{\varkappa}, x))\big\|_{L_\infty(\mathbb{R}^2)} < c(V)|\vec{\varkappa}|^{-\gamma_1}, \tag{6.5}$$

where $u_\infty(\vec{\varkappa}, x)$ is a limit-periodic function, $\gamma_1 = 1/2 - 15s_1 - 8\delta$; the eigenvalue $\lambda_\infty(\vec{\varkappa})$ satisfies the asymptotic formula:

$$\lambda_\infty(\vec{\varkappa}) = |\vec{\varkappa}|^2 + O(|\vec{\varkappa}|^{-\gamma_2}), \quad \gamma_2 = 2 - 30s_1 - 20\delta. \tag{6.6}$$

We also show that the set $\mathcal{G}_\infty$ satisfies (1.7).

In fact, by (6.1), any $\vec{\varkappa} \in \mathcal{D}_\infty(\lambda)$ belongs to the $(\epsilon_n k^{-1-\delta})$-neighborhood of $\mathcal{D}_n(\lambda)$. Applying (2.29), (2.30) with $2\beta = 2\beta_0 = 1 - 15s_1 - 9\delta$ and the perturbation formulae proved for next steps, we obtain the following inequalities:

$$\big\|E^{(1)}(\vec{\varkappa}) - E^{(0)}(\vec{\varkappa})\big\|_1 < c\|W_1\| k^{-1/2+12s_1+8\delta}, \quad \big\|E^{(n+1)}(\vec{\varkappa}) - \tilde{E}^{(n)}(\vec{\varkappa})\big\|_1 < 48\epsilon_n^3, \quad n \geq 1, \tag{6.7}$$

$$\big|\lambda^{(1)}(\vec{\varkappa}) - |\vec{\varkappa}|^2\big| < C(W_1)k^{-2+30s_1+20\delta}, \quad \big|\lambda^{(n+1)}(\vec{\varkappa}) - \lambda^{(n)}(\vec{\varkappa})\big| < 12\epsilon_n^4, \quad n \geq 1, \tag{6.8}$$

where $E^{(n+1)}$, $\tilde{E}^{(n)}$ are one-dimensional spectral projectors in $L_2(Q_{n+1})$ corresponding to potentials W_{n+1} and W_n, respectively; $\lambda^{(n+1)}(\vec{\varkappa})$ is the eigenvalue corresponding to $E^{(n+1)}(\vec{\varkappa})$, $E^{(0)}(\vec{\varkappa})$ corresponds to $V = 0$ and the periods a_1, a_2. This means that for properly chosen eigenfunctions $\Psi_{n+1}(\vec{\varkappa}, x)$:

$$\|\Psi_1 - \Psi_0\|_{L_2(Q_1)} < c\|W_1\| k^{-1/2+12s_1+8\delta}|Q_1|^{1/2}, \quad \Psi_0(x) = e^{i\langle \vec{\varkappa}, x\rangle}, \tag{6.9}$$

$$\|\Psi_{n+1} - \tilde{\Psi}_n\|_{L_2(Q_{n+1})} < 100\epsilon_n^3 |Q_{n+1}|^{1/2}, \tag{6.10}$$

where $\tilde{\Psi}_n$ is Ψ_n extended quasi-periodically from Q_n to Q_{n+1}. Eigenfunctions Ψ_n, $n \geq 1$, are chosen to obey two conditions. First, $\|\Psi_n\|_{L_2(Q_n)} = |Q_n|^{1/2}$; [10] second

[10] The condition $\|\Psi_n\|_{L_2(Q_n)} = |Q_n|^{1/2}$ implies $\|\tilde{\Psi}_n\|_{L_2(Q_{n+1})} = |Q_{n+1}|^{1/2}$.

$(\Psi_n, \tilde{\Psi}_{n-1})_n > 0$, here $(\cdot,\cdot)_n$ is an inner product in $L_2(Q_n)$. These two conditions, obviously, provide a unique choice of each Ψ_n. Considering that Ψ_{n+1} and $\tilde{\Psi}_n$ satisfy equations for eigenfunctions and taking into account (6.8), (6.10) we obtain: $\|\Psi_{n+1} - \tilde{\Psi}_n\|_{W_2^2(Q_{n+1})} < 2k^2\epsilon_n^3|Q_{n+1}|^{1/2}$, $n \geq 1$, and, hence, $\|\Psi_{n+1} - \tilde{\Psi}_n\|_{L_\infty(Q_{n+1})} < 6k^2\epsilon_n^3|Q_{n+1}|^{1/2}$. Since Ψ_{n+1} and $\tilde{\Psi}_n$ obey the same quasiperiodic conditions, the same inequality holds in the whole space $\mathbb{R}^2$:

$$\|\Psi_{n+1} - \Psi_n\|_{L_\infty(\mathbb{R}^2)} < 6k^2\epsilon_n^3|Q_{n+1}|^{1/2}, \quad n \geq 1, \tag{6.11}$$

where Ψ_{n+1}, Ψ_n are quasiperiodically extended to $\mathbb{R}^2$. Obviously, we have a Cauchy sequence in $L_\infty(\mathbb{R}^2)$. Let $\Psi_\infty(\vec{\varkappa}, x) = \lim_{n\to\infty} \Psi_n(\vec{\varkappa}, x)$. This limit is defined pointwise uniformly in x and in $W_{2,loc}^2(\mathbb{R}^2)$.

Theorem 6.7. *For every sufficiently large λ, $\lambda > \lambda_0(\sum_{r=1}^\infty \|V_r\|, \delta)$, and $\vec{\varkappa} \in \mathcal{D}_\infty(\lambda)$ the sequence of functions $\Psi_n(\vec{\varkappa}, x)$ converges in $L_\infty(\mathbb{R}^2)$ and $W_{2,loc}^2(\mathbb{R}^2)$. The limit function $\Psi_\infty(\vec{\varkappa}, x)$, $\Psi_\infty(\vec{\varkappa}, x) = \lim_{n\to\infty} \Psi_n(\vec{\varkappa}, x)$, satisfies the equation*

$$-\Delta\Psi_\infty(\vec{\varkappa}, x) + V(x)\Psi_\infty(\vec{\varkappa}, x) = \lambda\Psi_\infty(\vec{\varkappa}, x). \tag{6.12}$$

It can be represented in the form

$$\Psi_\infty(\vec{\varkappa}, x) = e^{i\langle\vec{\varkappa}, x\rangle}\big(1 + u_\infty(\vec{\varkappa}, x)\big), \tag{6.13}$$

where $u_\infty(\vec{\varkappa}, x)$ is a limit-periodic function:

$$u_\infty(\vec{\varkappa}, x) = \sum_{n=1}^\infty \tilde{u}_n(\vec{\varkappa}, x), \tag{6.14}$$

$\tilde{u}_n(\vec{\varkappa}, x)$ being a periodic function with the periods $2^{M_n-1}d_1, 2^{M_n-1}d_2$ with $2^{M_n} \approx k^{2^{n-1}s_1}$,

$$\|\tilde{u}_1\|_{L_\infty(\mathbb{R}^2)} < c(V)k^{-\gamma_1}, \quad \gamma_1 = 1/2 - 15s_1 - 8\delta, \tag{6.15}$$

$$\|\tilde{u}_n\|_{L_\infty(\mathbb{R}^2)} < 6k^2\epsilon_{n-1}^3|Q_n|^{1/2}, \quad n \geq 2. \tag{6.16}$$

Corollary 6.8. *Function $u_\infty(\vec{\varkappa}, x)$ obeys the estimate (6.5).*

Proof. Let us show that Ψ_∞ is a limit-periodic function. Obviously, $\Psi_\infty = \Psi_0 + \sum_{n=0}^\infty(\Psi_{n+1} - \Psi_n)$, the series converging in $L_\infty(\mathbb{R}^2)$ by (6.11). Introducing the notation $\tilde{u}_{n+1} = e^{-i\langle\vec{\varkappa}, x\rangle}(\Psi_{n+1} - \Psi_n)$, we arrive at (6.13), (6.14). Note that $\tilde{u}_n$ is periodic with the periods $2^{M_n-1}d_1, 2^{M_n-1}d_2$. Estimate (6.16) follows from (6.11). We check (6.15). Indeed, by (6.9), Fourier coefficients $(\tilde{u}_1)_m$, $m \in \mathbb{Z}^2$, satisfy the estimate $|(\tilde{u}_1)_m| < c(V)k^{-1/2+12s_1+8\delta}|Q_1|^{1/2} < c(V)k^{-1/2+13s_1+8\delta}$. We will use this estimate for $m : |m - j| < k^{s_1}$. Next, we obtain a stronger estimate for other m-s. Indeed, the inequality $\big|E^{(1)}(\varkappa)_{mj}\big| < \big(c(V)k^{-(1/2-12s_1-7\delta)}\big)^{|m-j|R_0^{-1}}$ follows from (2.25) and (2.28). Hence, similar estimates holds for Fourier coefficients of Ψ_1: $|(\tilde{u}_1)_m| < \big(c(V)k^{-(1/2-12s_1-7\delta)}\big)^{|m-j|R_0^{-1}}$. Summarizing the inequalities, we obtain that (6.15) holds. It remains to prove (6.12). Indeed, $\Psi_n(\vec{\varkappa}, x)$, $n \geq 1$, satisfy equations for eigenfunctions: $H^{n)}\Psi_n = \lambda^{(n)}(\vec{\varkappa})\Psi_n$. Considering that $\Psi_n(\vec{\varkappa}, x)$ converges to $\Psi(\vec{\varkappa}, x)$ in $W_{2,loc}^2$ and relation (6.2), we arrive at (6.12). ∎

Theorem 6.9. *Formulae (6.4), (6.5) and (6.6) hold for every $\vec{\varkappa} \in \mathcal{G}_\infty$. The set $\mathcal{G}_\infty$ is Lebesgue measurable and satisfies (1.7) with $\gamma_3 = \delta$.*

Proof. By Theorem 6.7, (6.4), (6.5) hold, where $\lambda_\infty(\vec{\varkappa}) = \lambda$ for $\vec{\varkappa} \in \mathcal{D}_\infty(\lambda)$. Using (1.10), which is proven in Corollary 6.3, with $\varkappa_\infty = |\vec{\varkappa}|$, we easily obtain (6.6). It remains to prove (1.7). Let us consider a small region $U_n(\lambda_0)$ around an isoenergetic surface $\mathcal{D}_n(\lambda_0)$, $\lambda_0 > k_*^2$. Namely, $U_n(\lambda_0) = \cup_{|\lambda-\lambda_0|<\rho_n}\mathcal{D}_n(\lambda)$, $\rho_n = \epsilon_{n-1}k^{-2\delta}$, $k = \lambda_0^{1/2}$. Considering an estimate of the type (2.62) for $\lambda^{(n)}(\vec{\varkappa})$, which holds in the $\left(\epsilon_{n-1}k^{-1-2\delta}\right)$-neighborhood of $\mathcal{D}_n(\lambda_0)$, we see that $U_n(\lambda_0)$ is an open set (a distorted ring with holes) and the width of the ring is of order $\epsilon_{n-1}k^{-1-2\delta}$. By Part 4 of Lemmas 2.10, 3.17, 4.8, 5.1, the length of $\mathcal{D}_n(\lambda_0)$ is $2\pi k\left(1 + O(k^{-\delta})\right)$. Hence, $|U_n(\lambda_0)| = 2\pi k\rho_n\left(1 + O(k^{-\delta})\right)$. It easily follows from Lemma 5.1 that $U_{n+1} \subset U_n$. Definition of $\mathcal{D}_\infty(\lambda_0)$ yield: $\mathcal{D}_\infty(\lambda_0) = \cap_{n=1}^\infty U_n(\lambda_0)$. Hence, $\mathcal{G}_\infty = \cap_{n=1}^\infty \mathcal{G}_n$, where

$$\mathcal{G}_n = \cup_{\lambda_0>\lambda_*}U_n(\lambda_0) = \cup_{\lambda>\lambda_*-\rho_n(\lambda_*)}\mathcal{D}_n(\lambda). \tag{6.17}$$

Considering that $U_{n+1} \subset U_n$ for every $\lambda_0 \geq \lambda_*$, we obtain $\mathcal{G}_{n+1} \subset \mathcal{G}_n$. Hence, $|\mathcal{G}_\infty \cap \mathbf{B_R}| = \lim_{n\to\infty}|\mathcal{G}_n \cap \mathbf{B_R}|$. Summarizing volumes of the regions U_n, we easily conclude $|\mathcal{G}_n \cap \mathbf{B_R}| = |\mathbf{B_R}|\left(1 + O(R^{-\delta})\right)$ uniformly in n. Thus, we have obtained (1.7) with $\gamma_3 = \delta$. ∎

7. Proof of Absolute Continuity of the Spectrum

The proof is completely analogous to that for the case $l \geq 6$. Here we give only the list of lemmas and the main result. For details, see [25].

7.1. **Projections $E_n(\mathcal{G}_n')$, $\mathcal{G}_n' \subset \mathcal{G}_n$.** Let us consider the open sets $\mathcal{G}_n$ given by (6.17). There is a family of Bloch eigenfunctions $\Psi_n(\vec{\varkappa}, x)$, $\vec{\varkappa} \in \mathcal{G}_n$, of the operator $H^{(n)}$, which are described by the perturbation formulas (1.12). Let $\mathcal{G}_n'$ be a Lebesgue measurable subset of $\mathcal{G}_n$. We consider the spectral projection $E_n\left(\mathcal{G}_n'\right)$ of $H^{(n)}$, corresponding to functions $\Psi_n(\vec{\varkappa}, x)$, $\vec{\varkappa} \in \mathcal{G}_n'$. By [26], $E_n\left(\mathcal{G}_n'\right) : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ can be presented by the formula:

$$E_n\left(\mathcal{G}_n'\right)F = \frac{1}{4\pi^2}\int_{\mathcal{G}_n'}\big(F, \Psi_n(\vec{\varkappa})\big)\Psi_n(\vec{\varkappa})d\vec{\varkappa} \tag{7.1}$$

for any $F \in C_0^\infty(\mathbb{R}^2)$, here and below $\big(\cdot,\cdot\big)$ is the canonical scalar product in $L_2(\mathbb{R}^2)$, i.e.,

$$\big(F, \Psi_n(\vec{\varkappa})\big) = \int_{\mathbb{R}^2} F(x)\overline{\Psi_n(\vec{\varkappa}, x)}dx.$$

The above formula can be rewritten in the form:

$$E_n\left(\mathcal{G}_n'\right) = S_n\left(\mathcal{G}_n'\right)T_n\left(\mathcal{G}_n'\right), \tag{7.2}$$

$$T_n : C_0^\infty(\mathbb{R}^2) \to L_2\left(\mathcal{G}_n'\right), \qquad S_n : L_\infty\left(\mathcal{G}_n'\right) \to L_2(\mathbb{R}^2),$$

$$T_nF = \frac{1}{2\pi}\big(F, \Psi_n(\vec{\varkappa})\big) \text{ for any } F \in C_0^\infty(\mathbb{R}^2), \tag{7.3}$$

T_nF being in $L_\infty(\mathcal{G}'_n)$, and,

$$S_n f = \frac{1}{2\pi}\int_{\mathcal{G}'_n} f(\vec{\varkappa})\Psi_n(\vec{\varkappa},x)d\vec{\varkappa} \quad \text{for any } f \in L_\infty(\mathcal{G}'_n). \tag{7.4}$$

By [26], $\|T_nF\|_{L_2(\mathcal{G}'_n)} \le \|F\|_{L_2(\mathbb{R}^2)}$ and $\|S_nf\|_{L_2(\mathbb{R}^2)} \le \|f\|_{L_2(\mathcal{G}'_n)}$. Hence, T_n, S_n can be extended by continuity from $C_0^\infty(\mathbb{R}^2)$, $L_\infty(\mathcal{G}'_n)$ to $L_2(\mathbb{R}^2)$ and $L_2(\mathcal{G}'_n)$, respectively. Thus, the operator $E_n(\mathcal{G}'_n)$ is described by (7.2) in the whole space $L_2(\mathbb{R}^2)$.

Obviously, for every $\vec{\varkappa}$ in $\mathcal{G}_n$, there exists a pair (λ_n,φ) such that $\lambda_n = \lambda^{(n)}(\vec{\varkappa})$ and that $(\cos\varphi,\sin\varphi) = \frac{\vec{\varkappa}}{|\vec{\varkappa}|}$. Let us introduce new coordinates (λ_n,φ) in $\mathcal{G}_n$: $\lambda_n = \lambda^{(n)}(\vec{\varkappa})$, $(\cos\varphi,\sin\varphi) = \frac{\vec{\varkappa}}{|\vec{\varkappa}|}$.

Lemma 7.1. *Every point $\vec{\varkappa}$ in $\mathcal{G}_n$ is represented by a unique pair (λ_n,φ), $\lambda_n > \lambda_*$, $\varphi \in [0,2\pi)$, such that*

$$\vec{\varkappa}(\lambda_n,\varphi) = \varkappa_n(\lambda_n,\vec{\nu})\vec{\nu}, \quad \vec{\nu} = (\cos\varphi,\sin\varphi), \tag{7.5}$$

$\varkappa_n(\lambda_n,\vec{\nu})$ being the "radius" of the isoenergetic curve $\mathcal{D}_n(\lambda_n)$ in the direction $\vec{\nu}$. where $\lambda_* = k_*^{2l}$.

For any function $f(\vec{\varkappa})$ integrable on $\mathcal{G}_n$, we use the new coordinates and write

$$\begin{aligned}\int_{\mathcal{G}_n} f(\vec{\varkappa})d\vec{\varkappa} &= \int_{\mathbb{R}^2}\chi(\mathcal{G}_n,\vec{\varkappa})f(\vec{\varkappa})d\vec{\varkappa}\\ &= \int_{\lambda_*}^{\infty}\int_0^{2\pi}\chi\big(\mathcal{G}_n,\vec{\varkappa}(\lambda_n,\varphi)\big)f\big(\vec{\varkappa}(\lambda_n,\varphi)\big)\frac{\varkappa_n(\lambda_n,\vec{\nu})}{\frac{\partial\lambda_n}{\partial\varkappa}}d\varphi\, d\lambda_n,\end{aligned}$$

where $\chi(\mathcal{G}_n,\vec{\varkappa})$ is the characteristic function on $\mathcal{G}_n$, $\vec{\varkappa}(\lambda_n,\varphi)$ is given by (7.5) and $\frac{\partial\lambda_n}{\partial\varkappa} = \big(\nabla\lambda^{(n)}(\vec{\varkappa}),\vec{\nu}\big)\big|_{\vec{\varkappa}=\vec{\varkappa}_n(\lambda_n,\vec{\nu})}$. Let

$$\mathcal{G}_{n,\lambda} = \{\vec{\varkappa}\in\mathcal{G}_n : \lambda^{(n)}(\vec{\varkappa}) < \lambda\}. \tag{7.6}$$

This set is Lebesgue measurable, since $\mathcal{G}_n$ is open and $\lambda^{(n)}(\vec{\varkappa})$ is continuous on $\mathcal{G}_n$.

Lemma 7.2. $|\mathcal{G}_{n,\lambda+\varepsilon}\setminus\mathcal{G}_{n,\lambda}| \le 2\pi\varepsilon$ *when* $0\le\varepsilon\le 1$.

By (7.1), $E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda}) = E_n(\mathcal{G}_{n,\lambda+\varepsilon}\setminus\mathcal{G}_{n,\lambda})$. Let us obtain an estimate for this projection.

Lemma 7.3. *For any $F\in C_0^\infty(\mathbb{R}^2)$ and $0\le\varepsilon\le 1$,*

$$\big\|\big(E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda})\big)F\big\|^2_{L_2(\mathbb{R}^2)} \le C(F)\epsilon, \tag{7.7}$$

where $C(F)$ is uniform with respect to n and λ.

7.2. **Sets $\mathcal{G}_\infty$ and $\mathcal{G}_{\infty,\lambda}$.** The sets $\mathcal{G}_\infty$, $\mathcal{G}_n$ are given by (6.3), (6.17). As it was shown in the proof of Theorem 6.9, $\mathcal{G}_{n+1}\subset\mathcal{G}_n$, $\mathcal{G}_\infty = \bigcap_{n=1}^\infty\mathcal{G}_n$. Therefore, the perturbation formulas for $\lambda^{(n)}(\vec{\varkappa})$ and $\Psi_n(\vec{\varkappa})$ hold in $\mathcal{G}_\infty$ for all n. Moreover, coordinates (λ_n,φ) can be used in $\mathcal{G}_\infty$ for every n. Let

$$\mathcal{G}_{\infty,\lambda} = \{\vec{\varkappa}\in\mathcal{G}_\infty : \lambda_\infty(\vec{\varkappa}) < \lambda\}. \tag{7.8}$$

The function $\lambda_\infty(\vec{\varkappa})$ is a Lebesgue measurable function, since it is a limit of the sequence of measurable functions. Hence, the set $\mathcal{G}_{\infty,\lambda}$ is measurable.

Lemma 7.4. *The measure of the symmetric difference of two sets $\mathcal{G}_{\infty,\lambda}$ and $\mathcal{G}_{n,\lambda}$ converges to zero as $n \to \infty$ uniformly in λ in every bounded interval:*

$$\lim_{n\to\infty} |\mathcal{G}_{\infty,\lambda}\Delta\mathcal{G}_{n,\lambda}| = 0.$$

7.3. **Spectral Projections $E(\mathcal{G}_{\infty,\lambda})$.** In this section, we show that spectral projections $E_n(\mathcal{G}_{\infty,\lambda})$ have a strong limit $E_\infty(\mathcal{G}_{\infty,\lambda})$ in $L_2(\mathbb{R}^2)$ as n tends to infinity. The operator $E_\infty(\mathcal{G}_{\infty,\lambda})$ is a spectral projection of H. It can be represented in the form $E_\infty(\mathcal{G}_{\infty,\lambda}) = S_\infty T_\infty$, where S_∞ and T_∞ are strong limits of $S_n(\mathcal{G}_{\infty,\lambda})$ and $T_n(\mathcal{G}_{\infty,\lambda})$, respectively. For any $F \in C_0^\infty(\mathbb{R}^2)$, we show:

$$E_\infty\left(\mathcal{G}_{\infty,\lambda}\right) F = \frac{1}{4\pi^2}\int_{\mathcal{G}_{\infty,\lambda}} \big(F, \Psi_\infty(\vec{\varkappa})\big)\Psi_\infty(\vec{\varkappa})d\vec{\varkappa}, \tag{7.9}$$

$$HE_\infty\left(\mathcal{G}_{\infty,\lambda}\right) F = \frac{1}{4\pi^2}\int_{\mathcal{G}_{\infty,\lambda}} \lambda_\infty(\vec{\varkappa})\big(F, \Psi_\infty(\vec{\varkappa})\big)\Psi_\infty(\vec{\varkappa})d\vec{\varkappa}. \tag{7.10}$$

Using properties of $E_\infty\left(\mathcal{G}_{\infty,\lambda}\right)$, we prove absolute continuity of the branch of the spectrum corresponding to functions $\Psi_\infty(\vec{\varkappa})$.

Now we consider the sequence of operators $T_n(\mathcal{G}_{\infty,\lambda})$ which are given by (7.3) and act from $L_2(\mathbb{R}^2)$ to $L_2(\mathcal{G}_{\infty,\lambda})$. We prove that the sequence has a strong limit and describe its properties.

Lemma 7.5. *The sequence $T_n(\mathcal{G}_{\infty,\lambda})$ has a strong limit $T_\infty(\mathcal{G}_{\infty,\lambda})$. The operator $T_\infty(\mathcal{G}_{\infty,\lambda})$ satisfies $\|T_\infty\| \le 1$ and can be described by the formula $T_\infty F = \frac{1}{2\pi}\big(F, \Psi_\infty(\vec{\varkappa})\big)$ for any $F \in C_0^\infty(\mathbb{R}^2)$. The convergence of $T_n(\mathcal{G}_{\infty,\lambda})F$ to $T_\infty(\mathcal{G}_{\infty,\lambda})F$ is uniform in λ for every $F \in L_2(\mathbb{R}^2)$.*

Now we consider the sequence of operators $S_n(\mathcal{G}_{\infty,\lambda})$ which are given by (7.4) with $\mathcal{G}_n' = \mathcal{G}_{\infty,\lambda}$:

$$S_n(\mathcal{G}_{\infty,\lambda}):\ L_2(\mathcal{G}_{\infty,\lambda}) \to L_2(\mathbb{R}^2).$$

We prove that the sequence has a strong limit and describe its properties.

Lemma 7.6. *The sequence of operators $S_n(\mathcal{G}_{\infty,\lambda})$ has a strong limit $S_\infty(\mathcal{G}_{\infty,\lambda})$. The operator $S_\infty(\mathcal{G}_{\infty,\lambda})$ satisfies $\|S_\infty\| \le 1$ and can be described by the formula*

$$(S_\infty f)(x) = \frac{1}{2\pi}\int_{\mathcal{G}_{\infty,\lambda}} f(\vec{\varkappa})\Psi_\infty(\vec{\varkappa}, x)d\vec{\varkappa} \tag{7.11}$$

for any $f \in L_\infty\left(\mathcal{G}_{\infty,\lambda}\right)$. The convergence of $S_n(\mathcal{G}_{\infty,\lambda})f$ to $S_\infty(\mathcal{G}_{\infty,\lambda})f$ is uniform in λ for every $f \in L_2\left(\mathcal{G}_\infty\right)$.

Lemma 7.7. *Spectral projections $E_n(\mathcal{G}_{\infty,\lambda})$ have a strong limit $E_\infty(\mathcal{G}_{\infty,\lambda})$ in $L_2(\mathbb{R}^2)$, the convergence being uniform in λ for every element. The operator $E_\infty(\mathcal{G}_{\infty,\lambda})$ is a projection. For any $F \in C_0^\infty(\mathbb{R}^2)$ it is given by (7.9) and formula (7.10) holds.*

Lemma 7.8. *There is a strong limit $E_\infty(\mathcal{G}_\infty)$ of the projections $E_\infty(\mathcal{G}_{\infty,\lambda})$ as λ goes to infinity.*

Corollary 7.9. *The operator $E_\infty(\mathcal{G}_\infty)$ is a projection.*

Lemma 7.10. *Projections $E_\infty(\mathcal{G}_{\infty,\lambda})$, $\lambda \in \mathbb{R}$, and $E_\infty(\mathcal{G}_\infty)$ reduce the operator H.*

Lemma 7.11. *The family of projections $E_\infty(\mathcal{G}_\infty, \lambda)$ is the resolution of the identity of the operator $HE_\infty(\mathcal{G}_\infty)$ acting in $E_\infty(\mathcal{G}_\infty)L_2(\mathbb{R}^2)$.*

7.4. **Proof of Absolute Continuity.** Now we show that the branch of spectrum (semi-axis) corresponding to $\mathcal{G}_\infty$ is absolutely continuous.

Theorem 7.12. *For any $F \in C_0^\infty(\mathbb{R}^2)$ and $0 \le \varepsilon \le 1$,*

$$|(E_\infty(\mathcal{G}_{\infty,\lambda+\varepsilon})F, F) - (E_\infty(\mathcal{G}_{\infty,\lambda})F, F)| \le C_F \varepsilon. \tag{7.12}$$

Corollary 7.13. *The spectrum of the operator $HE_\infty(\mathcal{G}_\infty)$ is absolutely continuous.*

Proof. By formula (7.9),

$$\left| (E_\infty(\mathcal{G}_{\infty,\lambda+\varepsilon})F, F) - (E(\mathcal{G}_{\infty,\lambda})F, F) \right| \le C_F \left|\mathcal{G}_{\infty,\lambda+\varepsilon} \setminus \mathcal{G}_{\infty,\lambda}\right|.$$

Applying Lemmas 7.2 and 7.4, we immediately get (7.12). ∎

8. Geometrical Lemmas.

8.1. **Proof of Lemma 2.1 (Geometric Lemma, step I).** Corollaries 8.5 and 8.7, obtained in Sections 8.1.2 and 8.1.3, together give Lemma 2.1 and its corollary.

8.1.1. *Set $\chi_1(\lambda, \beta, s_1, \delta)$.* The set $\chi_1(\lambda, \beta, s_1, \delta)$ is given by formulas (2.11)–(2.17). It remains to define radii r_m of the discs $\mathcal{O}_m^\pm$. First, we need more notations. Angle φ_m is defined as a polar coordinate of $\vec{p}_m$:

$$\vec{p}_m := \vec{p}_m(0) = p_m(\cos\varphi_m, \sin\varphi_m), \ m = (m_1, m_2) \in \mathbb{Z}^2.$$

In addition, let $\Pi_{m,q,\pm}$ be the distance in $\mathbb{C}$ between $|\vec{k}(\varphi) + \vec{p}_m|_*^2$ and $|\vec{k}(\varphi) + \vec{p}_{m+q}|_*^2$ when $\varphi = \varphi_m^\pm$, i.e., when φ solves $|\vec{k}(\varphi) + \vec{p}_m|_*^2 = k^2$. Obviously,

$$\Pi_{m,q,\pm} := |\langle \vec{k}(\varphi_m^\pm) + \vec{p}_m, \vec{p}_q\rangle_* + p_q^2|. \tag{8.1}$$

We will be interested in values of $\Pi_{m,q,\pm}$ for $q : 0 < p_q < k^{s_1}$. Let

$$\Pi'_{m,q,\pm} := |\langle \vec{k}(\varphi_m^\pm) + \vec{p}_m, \vec{p}_q\rangle_*| + k^{2s_1}. \tag{8.2}$$

Obviously, $\Pi'_{m,q,\pm} = \Pi_{m,q,\pm} + O\left(k^{2s_1}\right)$, $\Pi'_{m,q,\pm} \ge k^{2s_1}$. Next,

$$\Pi_m := \min_{0<p_q<k^{s_1},\ \pm} \Pi'_{m,q,\pm} = \min_{0<p_q<k^{s_1},\ \pm} |\langle \vec{k}(\varphi_m^\pm) + \vec{p}_m, \vec{p}_q\rangle_*| + k^{2s_1}. \tag{8.3}$$

The value Π_m characterizes the smallest distance between $|k(\varphi) + \vec{p}_m|_*^2$ and $|\vec{k}(\varphi) + \vec{p}_{m+q}|_*^2$ when $|k(\varphi) + \vec{p}_m|_*^2 = k^2$, $0 < p_q < k^{s_1}$. It coincides with this distance up to the value $O(k^{2s_1})$. Further we use that $\Pi_m \ge k^{2s_1}$.

Definition 8.1. The radius r_m of the two open disks $\mathcal{O}_m^{\pm}$ in $\mathbb{C}$ centered at $\varphi_m^{\pm}$ is defined by

$$r_m := \begin{cases} \dfrac{k^{-3s_1-\delta}}{p_m} & \text{if } 0 < p_m \le 4k^{s_1}, \\ \dfrac{k^{2\beta-1}}{p_m \Pi_m \sqrt{\left|1-\frac{p_m^2}{4k^2}\right|}} & \text{if } p_m > 4k^{s_1} \text{ and } \left|1-\frac{p_m^2}{4k^2}\right| \ge \frac{k^{2\beta-2}}{\Pi_m}, \\ \dfrac{8k^{\beta}}{p_m\sqrt{\Pi_m}} & \text{if } \left|1-\frac{p_m^2}{4k^2}\right| < \frac{k^{2\beta-2}}{\Pi_m}. \end{cases} \tag{8.4}$$

Remark 8.2. If $p_m \ge 4k$, then the imaginary part of $\varphi_m^{\pm}$ is too large to be in Φ_0 and so we do not need to remove corresponding disks from Φ_0 to construct Φ_1.

Lemma 8.3. *For every m with $0 < p_m < 4k$, $r_m = o(1)$ as $k \to \infty$.*

Proof. Let us consider the following four cases:

If $0 < p_m \le 4k^{s_1}$, then $r_m < \frac{k^{-3s_1-\delta}}{k^{-s_1}} = k^{-2s_1-\delta} = o(1)$, since the periods a_1, a_2 of $H^{(1)}$ are of order k^{s_1}.

If $4k^{s_1} < p_m \le k$, then $r_m^2 < \frac{k^{4\beta-2}}{k^{2s_1}k^{4s_1}3/4} < 2k^{4\beta-2-6s_1} = o(1)$, since $\Pi_m \ge k^{2s_1}$.

If $k < p_m < 4k$ and $\left|1-\frac{p_m^2}{4k^2}\right| \ge \frac{k^{2\beta-2}}{\Pi_m}$, then $r_m^2 < \frac{k^{4\beta-2}}{k^2\Pi_m^2\frac{k^{2\beta-2}}{\Pi_m}} < k^{2\beta-2-2s_1} = o(1)$.

If $\left|1-\frac{p_m^2}{4k^2}\right| < \frac{k^{2\beta-2}}{\Pi_m}$, then of course $p_m > k$ and $r_m < \frac{8k^{\beta}}{k^{1+s_1}} = 8k^{\beta-1-s_1} = o(1)$. ∎

8.1.2. *Proof of the statements (i), (ii) in Lemma 2.1.*

Lemma 8.4. *Let $4s_1 < 2\beta < 1-15s_1-8\delta$, φ is in Φ_1 or its $(k^{2\beta-3-s_1-2\delta})$-neighborhood, $m, q \in Z^2$, $m, q, m+q \ne 0$, $0 < p_q < k^{s_1}$. Then*

$$2\left||\vec{k}(\varphi)+\vec{p}_m|_*^2-k^2\right|\left||\vec{k}(\varphi)+\vec{p}_{m+q}|_*^2-k^2\right| > k^{2\beta}, \tag{8.5}$$

$$\left||\vec{k}(\varphi)+\vec{p}_q|_*^2-k^2\right| > k^{1-3s_1-\delta}. \tag{8.6}$$

Corollary 8.5 (Statements (i), (ii) in Lemma 2.1, Corollary 2.2)**.** *If $t \in \chi_1(\lambda,\beta,s_1,\delta)$, then there is a unique $j \in \mathbb{Z}^2$ such that $p_j(t) = k$ and* (2.4), (2.5) *hold. For any t in the $(k^{2\beta-2-s_1-2\delta})$-neighborhood of the non-resonance set in $\mathbb{C}^2$, there exists a unique $j \in \mathbb{Z}^2$ such that* (2.6) *and* (2.4), (2.5), (2.9) (2.10) *hold.*

Proof of Corollary 8.5. If $t \in \chi_1(\lambda,\beta,s_1,\delta)$, then, by the definition of χ_1, there is $\varphi \in \Theta_1$ such that $t = \mathcal{K}_1\vec{k}(\varphi)$. Hence, there exists a j such that $p_j(t) = k$. Let us show that j is unique. Suppose there is $j' = j+m$, $m \ne 0$ such that $p_{j+m}(t) = k$. Then $\vec{p}_{j+m}(t) = \vec{k}(\varphi)+\vec{p}_m$ and $|\vec{k}(\varphi)+\vec{p}_m| = k$. The last relation contradicts to (8.5). Therefore, $j' = j$.

Substituting $\vec{k}(\varphi) = \vec{p}_j(t)$ into (8.5), (8.6) and using the notation $j+m = i$, we obtain (2.4) and (2.5). Similar arguments work when t is in the $(k^{2\beta-2-s_1-2\delta})$-neighborhood of $\chi_1(\lambda,\beta,s_1,\delta)$ in $\mathbb{C}^2$.

The inequalities (2.9), (2.10) are obvious, when $p_i > 4k$. We assume now $p_i < 4k$. Let us prove (2.9). When $i = j$, (2.9) follows directly from (2.8) and (2.6).

Suppose $i \neq j$. Clearly, $|p_i^2(t) - p_{i+q}^2(t)| = O(k^{1+s_1})$. This together with (2.4) yields: $|p_i^2(t) - p_j^2(t)| > ck^{2\beta-1-s_1}$. Considering (2.8), we obtain

$$2|p_i^2(t) - z| \geq |p_i^2(t) - p_j^2(t)|. \tag{8.7}$$

The estimate (2.9) easily follows. Let us prove (2.10). In the case of $i = j$, using (2.5) and the definition of C_1, we get

$$|p_{j+q}^2(t) - z| \geq |p_j^2(t) - p_{j+q}^2(t)| - k^{2\beta-1-s_1-\delta} > \frac{1}{2}k^{1-3s_1-\delta}. \tag{8.8}$$

Using this together with the definition of C_1, we get (2.10) in the case $i = j$. In the case of $i \neq j$ (2.4) and (8.7) yield (2.10). It is easy to see that all the estimates are stable under a perturbation of t of order $k^{2\beta-2-s_1-2\delta}$. Therefore, the estimate (2.10) can be extended to the complex $(k^{2\beta-2-s_1-2\delta})$-neighborhood of $\chi_1(k, \beta, s_1, \delta)$. ■

Proof of Lemma 8.4. If $p_m > 4k$ then (8.5) is obvious, since both factors are greater than k^2. Assume $0 < p_m < 4k$. Noting that

$$|\vec{k}(\varphi) + \vec{p}_m|_*^2 - k^2 = 2kp_m \cos(\varphi - \varphi_m) + p_m^2 \tag{8.9}$$

and recalling that $\varphi_m^{\pm}$ are the solutions of $|\vec{k}(\varphi) + \vec{p}_m|_*^2 = k^2$, we see:

$$\cos(\varphi_m^{\pm} - \varphi_m) = -\frac{p_m}{2k}, \quad |\sin(\varphi_m^{\pm} - \varphi_m)| = \sqrt{\left|1 - \frac{p_m^2}{4k^2}\right|}. \tag{8.10}$$

Now let φ be on the boundary of $\mathcal{O}_m^+$ or $\mathcal{O}_m^-$. Expanding (8.9) around $\varphi_m^{\pm}$, we get:

$$|\vec{k}(\varphi)+\vec{p}_m|_*^2-k^2 = 2kp_m \sin(\varphi_m^{\pm}-\varphi_m)r_m \left(1 + O(r_m^2)\right) - kp_m \cos(\varphi_m^{\pm}-\varphi_m)r_m^2 \left(1 + O(r_m^2)\right). \tag{8.11}$$

Next, we prove

$$\left||\vec{k}(\varphi) + \vec{p}_m|_*^2 - k^2\right| > k^{2\beta}/\Pi_m. \tag{8.12}$$

in three cases as in Definition 8.1(i). In special, we get (8.6) in the first case.

Case (i) : $r_m = \frac{k^{-3s_1-\delta}}{p_m}$ when $0 < p_m \leq 4k^{s_1}$. The modulus of the first term in (8.11) is $2k^{1-3s_1-\delta}\sqrt{1 - \frac{p_m^2}{4k^2}}(1 + o(1)) > \frac{3}{2}k^{1-3s_1-\delta}$ and that of the second term is $O(k^{-6s_1-2\delta})$, which is much smaller than the first term. Thus, we obtain (8.6). Considering that $k^{1-3s_1-\delta} > \frac{2k^{2\beta}}{\Pi_m}$, we obtain (8.12).

Case (ii) : $r_m = \frac{k^{2\beta-1}}{p_m\Pi_m\sqrt{\left|1-\frac{p_m^2}{4k^2}\right|}}$ when $4k^{s_1} < p_m < 4k$ and $\left|1 - \frac{p_m^2}{4k^2}\right| > \frac{k^{2\beta-2}}{\Pi_m}$. Substituting r_m into (8.11), we get that the modulus of the first term is $\frac{2k^{2\beta}}{\Pi_m}(1 + o(1))$ and that of the second term is $\frac{k^{4\beta-2}}{\Pi_m^2\left|1-\frac{p_m^2}{4k^2}\right|}(1 + o(1))$. Using the condition $\left|1 - \frac{p_m^2}{4k^2}\right| > \frac{k^{2\beta-2}}{\Pi_m}$, one can easily see that the former is at least twice greater than the latter. Thus, we get (8.12).

Case (iii) : $r_m = \frac{8k^{\beta}}{p_m\sqrt{\Pi_m}}$ when $\left|1-\frac{p_m^2}{4k^2}\right| < \frac{k^{2\beta-2}}{\Pi_m}$. This time the modulus of the second term is $\frac{32k^{2\beta}}{\Pi_m}(1+o(1))$ and that of the first is smaller than $\frac{16k^{2\beta}}{\Pi_m}(1+o(1))$. Therefore we again have (8.12).

Now we prove (8.5). If $0 < p_m, p_{m+q} \leq 4k^{s_1}$, then (8.5) easily follows from (8.6) proven in the Case (i) above for such $m, m+q$. Next, if both factors in the left hand side of (8.5) is greater than k^{β}, then (8.5) is obvious. Therefore, without loss of generality, we can assume

$$\left||\vec{k}(\varphi)+\vec{p}_m|_*^2-k^2\right| \leq k^{\beta}, \quad p_m > 4k^{s_1}. \tag{8.13}$$

Suppose we have proved that

$$\left||\vec{k}(\varphi)+\vec{p}_{m+q}|_*^2-k^2\right| > \frac{1}{2}\Pi_m \tag{8.14}$$

for every $m: \ \Pi_m > k^{\beta}$ and φ satisfying (8.13). [11] Then, notice that $\Pi_m > k^{\beta}$ follows from (8.12). Considering (8.12) and (8.14) together, we obtain (8.5).

It remains to prove (8.14). First, we check that

$$\left||\vec{k}(\varphi)+\vec{p}_m|_*^2-|\vec{k}(\varphi)+\vec{p}_{m+q}|_*^2\right| \geq \frac{3}{2}\Pi_m + O(k^{2s_1}) \tag{8.15}$$

for every $m: \ \Pi_m > k^{\beta}$ and φ satisfying (8.13). Indeed, taking into account (8.11) and considering as in the proof of (8.12), we conclude that the set of φ satisfying (8.13) is inside the circle of the radius r'_m around $\varphi_m^{\pm}$, where

$$r'_m = \begin{cases} \frac{k^{\beta-1}}{p_m\sqrt{1-\frac{p_m^2}{4k^2}}}, & \text{when } 1-\frac{p_m^2}{4k^2} > k^{\beta-2} \\ 10k^{\beta/2-1}, & \text{otherwise.} \end{cases} \tag{8.16}$$

It is easy to show that for any φ inside those circles:

$$\left|\left||\vec{k}(\varphi)+\vec{p}_m|_*^2-|\vec{k}(\varphi)+\vec{p}_{m+q}|_*^2\right| - \left||\vec{k}(\varphi_m^{\pm})+\vec{p}_m|_*^2-|\vec{k}(\varphi_m^{\pm})+\vec{p}_{m+q}|_*^2\right|\right| \leq \tag{8.17}$$

$$2kp_q r'_m < \frac{1}{2}k^{\beta}+o(1) < \frac{1}{2}\Pi_m + o(1).$$

Considering the definition (8.3) of Π_m, we easily get:

$$\left||\vec{k}(\varphi_m^{\pm})+\vec{p}_m|_*^2-|\vec{k}(\varphi_m^{\pm})+\vec{p}_{m+q}|_*^2\right| \geq 2\Pi_m + O(k^{2s_1}). \tag{8.18}$$

Combining (8.17) and (8.18), we arrive at (8.15). Now inequalities (8.13) and (8.15) yield (8.14). Thus, (8.5) is proven.

Note that all estimates are stable with respect the perturbation of φ of order of $k^{2\beta-3-s_1-2\delta}$. Therefore, (8.5), (8.6) hold not only in $\mathit{\Phi}_1$, but also in its $(k^{2\beta-3-s_1-2\delta})-$neighborhood. ∎

[11] The only conditions required for (8.14) are $\Pi_m > k^{\beta}$ and (8.13). We will use (8.14) under this condition in Lemma 8.9.

8.1.3. *Proof of the statement (iii) in Lemma 2.1.*

Lemma 8.6.
$$\sum_{\substack{m\in\mathbb{Z}^2\\ 0<p_m\leq 4k^{s_1}}} r_m = O(k^{-\delta}), \tag{8.19}$$
$$\widehat{\sum}_{\substack{m\in\mathbb{Z}^2\\ 4k^{s_1}<p_m<4k}} r_m = O(k^{-\gamma_0}), \tag{8.20}$$
where
$$\gamma_0 = \begin{cases} 1-2\beta-15s_1-7\delta, & \text{if } \beta\geq 1/6;\\ 5/6-\beta-15s_1-7\delta & \text{if } \beta<1/6, \end{cases}$$
the sum $\widehat{\sum}$ *in* (8.20) *includes only such* m *that* $\mathcal{O}_m\not\subset \cup_{0<p_q\leq 4k^{s_1},\pm}\mathcal{O}_q^{\pm}$.

Corollary 8.7. *If* $2\beta<1-15s_1-8\delta$, *then the total length of* $\mathcal{O}^{(1)}\cap[0,2\pi)$ *does not exceed* $O(k^{-\delta})$. *Estimate* (2.7) *holds.*

Proof. We split $m\in\mathbb{Z}^2$ with $0<p_m<4k$ into five sets:
$$I_1 := \left\{m\in\mathbb{Z}^2 : 0<p_m\leq 4k^{s_1}\right\},$$
$$I_2 := \left\{m\in\mathbb{Z}^2 : 4k^{s_1}<p_m\leq k^{1-4s_1-4\delta}\right\},$$
$$I_3 := \left\{m\in\mathbb{Z}^2 : p_m> k^{1-4s_1-4\delta},\ 2k-p_m\geq k^{1-8s_1-4\delta}\right\},$$
$$I_4 := \left\{m\in\mathbb{Z}^2 : 2k-p_m< k^{1-8s_1-4\delta}, \left|1-\frac{p_m^2}{4k^2}\right|\geq \frac{k^{2\beta-2}}{4\Pi_m}\right\},$$
$$I_5 := \left\{m\in\mathbb{Z}^2 : \left|1-\frac{p_m^2}{4k^2}\right|< \frac{k^{2\beta-2}}{4\Pi_m}\right\}.$$
and define $\Sigma_j := \sum_{m\in I_j} r_m,\ j=1,..,5$ and get an upper bound for each.
(i) By (8.4),
$$\Sigma_1 = \sum_{m\in I_1}\frac{k^{-3s_1-\delta}}{p_m}. \tag{8.21}$$
Considering that the size of the lattice formed by $\vec{p}_m$ is ck^{-s_1}, we obtain:
$$\sum_{m\in I_1}\frac{1}{p_m}\leq ck^{2s_1}\int_0^{2\pi}\int_0^{k^{s_1}}\frac{1}{r}\cdot r\,drd\varphi = ck^{3s_1}.$$
Substituting the last estimate into (8.21), we obtain (8.19).
(ii) Now we estimate Σ_2. Let us check that
$$\Pi_m\geq \frac{1}{2}k^{1-3s_1-3\delta},\ \text{when } m\in I_2. \tag{8.22}$$
Indeed, suppose $\Pi_m<\frac{1}{2}k^{1-3s_1-3\delta}$. Then for some $q\in\mathbb{Z}^2$ with $0<p_q<k^{s_1}$: $\left|\langle\vec{k}(\varphi_m^{\pm})+\vec{p}_m,\vec{p}_q\rangle_*\right|+k^{2s_1}<\frac{1}{2}k^{1-3s_1-3\delta}$, where $\pm$ means $+$ or $-$. Hence, $\left|\langle\vec{k}(\varphi_m^{\pm}),\vec{p}_q\rangle_*\right|< k^{1-3s_1-3\delta}$, since $\left|\langle\vec{p}_m,\vec{p}_q\rangle_*\right|\leq k^{1-4s_1-4\delta}\cdot k^{s_1}=k^{1-3s_1-4\delta}$ when $m\in I_2$. Therefore, $\left||\vec{k}(\varphi_m^{\pm})+\vec{p}_q|_*^2-k^2\right| = \left|2\langle\vec{k}(\varphi_m^{\pm}),\vec{p}_q\rangle_*+p_q^2\right|\leq 2k^{1-3s_1-3\delta}+k^{2s_1}<3k^{1-3s_1-3\delta}$, which

means $\varphi_m^\pm \in \mathcal{O}_q^\pm$, see (8.6) in Lemma 8.4. Moreover, $\varphi_m^\pm$ is relatively close to the centrum of $\mathcal{O}_q^\pm$, its distance to the centrum of $\mathcal{O}_q^\pm$ is less than $r_q k^{-2\delta}$. Using (8.4), one can easily check that $r_m << r_q k^{-2\delta}$ when $m \in I_2$. Thus $\mathcal{O}_m^\pm$ is completely contained in $\mathcal{O}_q^\pm$. We do not consider such an m in the sum (8.20). Thus, we can use $\Pi_m \geq \frac{1}{2}k^{1-3s_1-3\delta}$. By (8.4)

$$\Sigma_2 = \sum_{m\in I_2} \frac{k^{2\beta-1}}{p_m \Pi_m \sqrt{\left|1-\frac{p_m^2}{4k^2}\right|}} \leq \frac{ck^{2\beta-1}}{k^{1-3s_1-3\delta}} \sum_{m\in I_2} \frac{1}{p_m}$$

We estimate the sum on the r.h.s. by the integral $k^{2s_1}\int_0^{2\pi}\int_{4k^{s_1}}^{k^{1-4s_1-4\delta}} \frac{1}{r}\cdot r\, dr d\varphi$. Computing the integral, we obtain :

$$\Sigma_2 \leq ck^{2\beta-1+s_1-\delta}. \tag{8.23}$$

(iii) Let $m \in I_3$. First we note that

$$\sqrt{1-\frac{p_m^2}{4k^2}} \geq \frac{1}{2}k^{-4s_1-2\delta} \tag{8.24}$$

since $2k - p_m \geq k^{1-8s_1-4\delta}$. For the moment we assume to have

$$\sum_{m\in I_3} \frac{1}{\Pi_m} \leq ck^{1+7s_1+\delta}. \tag{8.25}$$

Then,

$$\Sigma_3 = \sum_{m\in I_3} \frac{k^{2\beta-1}}{p_m \Pi_m \sqrt{\left|1-\frac{p_m^2}{4k^2}\right|}} \leq \frac{ck^{2\beta-1}}{k^{1-4s_1-4\delta}\cdot k^{-4s_1-2\delta}} \sum_{m\in I_3} \frac{1}{\Pi_m}$$

where we used the definition of I_3 and (8.24). Considering also (8.25), we obtain:

$$\Sigma_3 \leq ck^{2\beta-1+15s_1+7\delta}. \tag{8.26}$$

Now we need to show (8.25). Obviously, $\varphi_m^\pm$ are real, when $m \in I_3$. Therefore, all $\langle\cdot,\cdot\rangle_*$ and $|\cdot|_*^2$ in the following formulas are real too. We note that $|\vec{k}(\varphi_m^\pm)|_*^2 = k^2$, $|\vec{k}(\varphi_m^\pm)+\vec{p}_m|_*^2 = k^2$ and $\left|\langle\vec{k}(\varphi_m^\pm)+\vec{p}_m, \vec{p}_{q_m}\rangle_*\right| + k^{2s_1} = \Pi_m$ for some q_m, $0 < p_{q_m} < k^{s_1}$ and φ_m^+ or/and φ_m^-. Assume for definiteness that the last equality holds for φ_m^+. Denoting $\vec{k}(\varphi_m^+) + \vec{p}_m$ by $\vec{k}_m$ gives us:

$$|\vec{k}_m - \vec{p}_m|_*^2 = k^2, \quad |\vec{k}_m|_*^2 = k^2 \tag{8.27}$$

$$\left|\langle\vec{k}_m, \vec{p}_{q_m}\rangle_*\right| + k^{2s_1} = \Pi_m \tag{8.28}$$

Obviously, $k^{2s_1} \leq \Pi_m \leq k^{1+s_1}$. Define Ω_j as the set of $\vec{p}_m$ in I_3 satisfying the inequality : $k^{2s_1+j\delta} < \Pi_m \leq k^{2s_1+(j+1)\delta}$, $j = 0,\cdots,J$, $J = \lfloor\frac{1-s_1}{\delta}\rfloor$. Here $\lfloor r\rfloor = \max\{z \in \mathbb{Z} : z \leq r\}$. Obviously, $\Omega_j \subset \cup_{0<p_q<k^{s_1}}\Omega_{j,q}$, where $\Omega_{j,q}$ consists of $\vec{p}_m$ in I_3 satisfying (8.27) and

$$k^{2s_1+j\delta} < \left|\langle\vec{k}_m, \vec{p}_q\rangle_*\right| + k^{2s_1} \leq k^{2s_1+(j+1)\delta}. \tag{8.29}$$

Let $\vec{p}_m \in \Omega_{j,q}$. By (8.27), (8.29), $\vec{p}_m$ belongs to a circle of radius k centered at a point $\vec{k}_m$, where $\vec{k}_m$ belongs to a circle of the radius k centered at the origin and satisfies the inequality $k^{2s_1+j\delta} < \left|\langle \vec{k}_m, \vec{p}_q\rangle_*\right| + k^{2s_1} \le k^{2s_1+(j+1)\delta}$, i.e. $\vec{k}_m$ belongs to one of two arcs. Considering that $p_q > k^{-s_1}$, we easily see that all such points $\vec{p}_m$ belong to a couple of rings of the same radii $k - k^{3s_1+(j+1)\delta}, k + k^{3s_1+(j+1)\delta}$ centered at two points $\vec{k}^{q,\pm}$ satisfying the conditions $|\vec{k}^{q,\pm}| = k, \langle \vec{k}^{q,\pm}, \vec{p}_q\rangle = 0$. Estimating the number of points $\vec{p}_m$ at this region by its area, we obtain, that the number of points in $\Omega_{j,q}$ does not exceed $ck^{1+5s_1+(j+1)\delta}$. Therefore, the number of points in Ω_j does not exceed $ck^{1+9s_1+(j+1)\delta}$. Thus, we obtain the estimate

$$\sum_{m\in I_3} \frac{1}{\Pi_m} \le \sum_{j=0}^{J} \sum_{m\in\Omega_j} \frac{1}{\Pi_m} \le \sum_{j=0}^{J} ck^{1+9s_1+(j+1)\delta} \cdot \frac{1}{k^{2s_1+j\delta}} \le ck^{1+7s_1+\delta},$$

which finishes this case.

(iv) Let $m \in I_4$. For the moment, we assume the following two estimates, [12]

$$\Pi_m > \frac{1}{4}k^{1-9s_1-4\delta}, \tag{8.30}$$

$$\omega := \sum_{m\in I_4} \frac{1}{\sqrt{|2k-p_m|}} < \begin{cases} ck^{3/2+3s_1+\delta}, & \text{if } \beta \ge 1/6; \\ ck^{-\beta+5/3+3s_1+\delta}, & \text{if } \beta < 1/6. \end{cases} \tag{8.31}$$

Then,

$$\Sigma_4 = \sum_{m\in I_4} \frac{k^{2\beta-1}}{p_m\Pi_m\sqrt{\left|1-\frac{p_m^2}{4k^2}\right|}} \le \frac{ck^{2\beta-1}}{k\cdot k^{1-9s_1-4\delta}\cdot k^{-1/2}} \sum_{m\in I_4} \frac{1}{\sqrt{|2k-p_m|}}$$

$$\le \begin{cases} ck^{-1+2\beta+12s_1+5\delta}, & \text{if } \beta \ge 1/6; \\ ck^{-5/6+\beta+12s_1+5\delta}, & \text{if } \beta < 1/6. \end{cases} \tag{8.32}$$

Now we show (8.30). We consider two cases: $|2k-p_m| \le k^{1-8s_1-4\delta}$ and $2k-p_m < -k^{1-8s_1-4\delta}$. We start with the former. Suppose, (8.30) is not true, i.e., $\Pi_m \le \frac{1}{4}k^{1-9s_1-4\delta}$. Then, there is a q such that $0 < p_q < k^{s_1}$ and

$$\left|\langle \vec{k}(\varphi_m^{\pm}) + \vec{p}_m, \vec{p}_q\rangle_*\right| \le \frac{1}{4}k^{1-9s_1-4\delta}$$

for φ_m^+ or/and φ_m^-. Note that $\varphi_m^{\pm}$ has a non-zero imaginary part when $p_m > 2k$. Denoting $\vec{k}(\varphi_m^{\pm}) + \vec{p}_m$ by $\vec{k}_m^{\pm}$, $\vec{k}_m^{\pm} \in \mathbb{C}^2$, we get (8.27) for both $\vec{k}_m^{\pm}$. It follows from (8.27) that $\vec{k}_m^{\pm} = \vec{k}(\varphi_{-m}^{\pm})$, we mean here that the pairs of vectors are the same. Therefore, there is at least one $\varphi_{-m}^{\pm}$, such that $\left|\langle \vec{k}(\varphi_{-m}^{\pm}), \vec{p}_q\rangle_*\right| \le \frac{1}{4}k^{1-9s_1-4\delta}$. We denote it by φ_{-m}^{+}. The last inequality means that φ_{-m}^{+} is in $\mathcal{O}_q$, $\mathcal{O}_q = \mathcal{O}_q^+ \cup \mathcal{O}_q^-$, and even relatively close to the center of a disc $\mathcal{O}_q^{\pm}$: its distance to the centrum is less than $r_qk^{-2s_1}$, the radius r_q of $\mathcal{O}_q^{\pm}$ being given by $r_q = k^{-3s_1-\delta}{p_q}^{-1}$, see (8.4). The distance between centers of $\mathcal{O}_q^+$ and $\mathcal{O}_q^-$ is $\pi + O(p_qk^{-1})$, It is easy to see also

[12] To prove (8.30) we use only the following conditions: $2k - p_m \le -k^{1-8s_1-4\delta}$ or $|2k-p_m| \le k^{1-8s_1-4\delta}$ and at least one of $\varphi_m^{\pm}$ is in $\Phi_0' = \Phi_0 \setminus \cup_{0<p_q<4k^{s_1}} \pm\mathcal{O}_q^{\pm}$ or its $2k^{-4s_1-2\delta}$-neighborhood. Under these conditions, we use estimate (8.30) also in Lemmas 8.9, 8.10.

that $\varphi^{\pm}_{-m} = \varphi^{\pm}_m + \pi$ modulo 2π, we mean here that the pairs of angles are the same. Considering the last two relations, we obtain that φ^+_m or/and φ^-_m is also in $\mathcal{O}_q$. Using $|2k - p_m| < k^{1-8s_1-4\delta}$, we get from (8.10): $|\varphi^+_m - \varphi^-_m| < 2k^{-4s_1-2\delta}$. It is much less than r_q. Hence, both φ^+_m and φ^-_m are in either $\mathcal{O}^+_q$ or $\mathcal{O}^-_q$, relatively close to a center. It is easy to show that $r_m = o(k^{\beta-1}) = o(r_q)$. This means $\mathcal{O}_m \subset \mathcal{O}_q$. We don't include include such r_m in the sum (8.20). Thus, (8.30) is proven for the case $|2k - p_m| \leq k^{1-8s_1-4\delta}$. Let $2k - p_m < -k^{1-8s_1-4\delta}$. By (8.10),

$$|\Im\varphi^{\pm}_m| = |\Im\varphi^{\pm}_{-m}| \approx \sqrt{\frac{p_m^2}{4k^2} - 1} \geq \frac{1}{2}k^{-4s_1-2\delta}.$$

It is easy to see that from the definition of $\varphi^{\pm}_m$ that $\vec{k}(\varphi^{\pm}_m) + \vec{p}_m = \vec{k}(\varphi^{\pm}_{-m})$. Hence,

$$\left|\langle \vec{k}(\varphi^{\pm}_m) + \vec{p}_m, \vec{p}_q\rangle_*\right| = \left|\langle \vec{k}(\varphi^{\pm}_{-m}), \vec{p}_q\rangle_*\right| = kp_q|\cos(\varphi^{\pm}_{-m} - \varphi_q)| > \frac{1}{4}kp_qk^{-8s_1-4\delta} \geq \frac{1}{4}k^{1-9s_1-4\delta}.$$

Therefore, $\Pi_m > \frac{1}{4}k^{1-9s_1-4\delta}$. Thus, we proved (8.30).

To show (8.31), we split I_4 into three subsets $I_4 = I_4' + I_4'' + I_4'''$:

$$I_4' = \{m \in I_4 : \ |2k - p_m| < k^{-1/3}\}, \quad I_4'' = \{m \in I_4 : \ k^{-1/3} < |2k - p_m| < k^{\delta}\},$$
$$I_4''' = \{m \in I_4 : \ |2k - p_m| \geq k^{\delta}\}.$$

Correspondingly, $\omega = \omega' + \omega'' + \omega'''$. Let us estimate ω'. We use a well known estimate (see e.g. [27]) for the number $N(k)$ of a rectangular lattice points in the circle of radius k: $N(k) = \pi v_{-1}k^2 + O(k^{2/3})$, where v is the area of the elementary cell of the lattice and the implicit constant depend on the periods. Considering that our lattice has a size of order k^{-s_1}, we rescale the last estimate, so it becomes: $N(k) = \pi v_d^{-1}k^{2+2s_1} + O(k^{2/3+2s_1/3})$, $v_d = 4\pi^2 d_1^{-1}d_2^{-1}$. This means that I_4' contains less than $ck^{2/3+2s_1}$ points. Note also that $|2k - p_m| > \frac{1}{4}k^{2\beta-2-s_1-\delta}$ follows from $|1 - \frac{p_m^2}{4k^2}| \geq \frac{k^{2\beta-2}}{4\Pi_m}$ and $\Pi_m \leq k^{1+s_1}$. Hence,

$$\omega' < cv_d^{-1}k^{2/3+2s_1} \cdot k^{-\beta+1+s_1/2+\delta/2} = o(k^{-\beta+5/3+3s_1+\delta}).$$

Next, considering that the region I_4'' contains no more than $cv_d^{-1}k^{1+\delta+2s_1}$ of points and $|2k - p_m| \geq k^{-1/3}$, we obtain

$$\omega'' < cv_d^{-1}k^{1+\delta+2s_1}k^{1/6} < cv_d^{-1}k^{7/6+2s_1+\delta}.$$

Further, we estimate ω''' by an integral:

$$\omega''' < ck^{2s_1}v_d^{-1}\int_0^{2\pi} \cdot \int_{|2k-r|<2k} \frac{1}{\sqrt{|2k-r|}} r\, dr d\varphi \leq cv_d^{-1}k^{\frac{3}{2}}$$

Combining the last three estimates, we obtain (8.31).

(v) As in the previous case, we have $\Pi_m > \frac{1}{4}k^{1-9s_1-4\delta}$. The number of points in I_5 admits the estimate: $\#(I_5) < cv_d^{-1}k^{\frac{2}{3}+2s_1}$. Therefore,

$$\Sigma_5 = \sum_{m \in I_5} \frac{4k^{\beta}}{p_m\sqrt{\Pi_m}} \leq \frac{8k^{\beta}}{k \cdot k^{(1-9s_1-4\delta)/2}} \sum_{m \in I_5} 1.$$

Hence, $\Sigma_5 < cv_d^{-1}k^{\beta-\frac{5}{6}+7s_1+2\delta}$. Adding (8.23), (8.26), (8.32), and the last estimate, we obtain (8.20). ∎

8.2. **Proof of Geometric Lemma 2.16.** The set $\mathcal{O}^{(1)}(\vec{b})$ is defined by formula (2.83), where $\mathcal{O}_m^{\pm}(\vec{b})$ are open disks around $\varphi_m^{\pm}(\vec{b})$, the points $\varphi_m^{\pm}(\vec{b})$ being the two zeros in $\mathbb{C}$ of $|\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2 = k^2$. Radius $r_m(\vec{b})$ of $\mathcal{O}_m^{\pm}(\vec{b})$ is defined in Section 8.2.1. Lemma 2.16 is the combination of Lemma 8.9 proven in Section 8.2.2 and Lemma 8.10 proven in Section 8.2.3.

8.2.1. *Definition of the set* $\mathcal{O}^{(1)}(\vec{b})$. Let us recall that r_m are the radii of the discs of the first non-resonant set $\mathcal{O}^{(1)}$. They are defined by formula (8.4). The radii $r_m(\vec{b})$ of the discs in $\mathcal{O}^{(1)}(\vec{b})$ are, roughly speaking, defined as follows: $r_m(\vec{b}) = k^{-2+2\beta_1+22s_1+15\delta}$, when $m=0$; $r_m(\vec{b}) \approx r_m$, when $p_m(\vec{b}) \geq 3k^{s_1}, |2k-p_m(\vec{b})| > k^{1-8s_1-4\delta}$; $r_m(\vec{b}) << r_m$, when $|2k-p_m(\vec{b})| \leq k^{1-8s_1-4\delta}$ or $p_m(\vec{b}) < 3k^{s_1}$. We have to reduce the $r_m(b)$ for smaller $p_m(\vec{b})$ and for $p_m(\vec{b})$ close to $2k$ in order to ensure that each component of $\mathcal{O}^{(1)}(\vec{b})$ is sufficiently small (by (8.4), r_m becomes bigger, when p_m is small or tends to $2k$).

Definition 8.8. The radius $r_m(\vec{b})$ of the two open disks $\mathcal{O}_m^{\pm}(\vec{b})$ in $\mathbb{C}$ centered at $\varphi_m^{\pm}(\vec{b})$ are defined by

$$r_m(\vec{b}) := \begin{cases} k^{-2+2\beta_1+22s_1+15\delta} & \text{if } m=0, \\ \dfrac{k^{-2+2\beta_1+3s_1+2\delta}}{p_m(\vec{b})} & \text{if } m\neq 0, p_m(\vec{b}) < k^{1-8s_1-4\delta}, \\ \dfrac{k^{-1+2\beta_1}}{p_m(\vec{b})\Pi_m(\vec{b})\sqrt{\left|1-\frac{p_m^2(\vec{b})}{4k^2}\right|}} & \text{if } p_m(\vec{b}) \geq k^{1-8s_1-4\delta},\ |2k-p_m(\vec{b})| > k^{1-8s_1-4\delta}, \\ \dfrac{k^{-4+2\beta_1+10s_1+4\delta}}{\sqrt{\left|1-\frac{p_m^2(\vec{b})}{4k^2}\right|}} & \text{if } |2k-p_m(\vec{b})| \leq k^{1-8s_1-4\delta}, \left|1-\frac{p_m^2(\vec{b})}{4k^2}\right| \geq \frac{1}{4}k^{-4+2\beta_1+10s_1+4\delta}, \\ 4k^{-2+\beta_1+5s_1+2\delta} & \text{if } \left|1-\frac{p_m^2(\vec{b})}{4k^2}\right| < \frac{1}{4}k^{-4+2\beta_1+10s_1+4\delta}, \end{cases} \tag{8.33}$$

where $\Pi_m(\vec{b})$ is defined by (8.3), $\vec{p}_m(\vec{b})$ being substituted instead of $\vec{p}_m$.

Considering as in Lemma 8.3, we easily show $r_m(\vec{b}) = o(1)$ as $k \to \infty$.

8.2.2. *Lemma 8.9.*

Lemma 8.9. *Let* $100s_1 < \beta_1 < 1/12 - 28s_1 - 14\delta$. *Suppose* $\varphi \in \hat{\Phi}_0' \setminus \mathcal{O}^{(1)}(\vec{b})$. *Then for every* $m \in \mathbb{Z}^2$ *such that*

$$\left||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2 - k^2\right| < k^{\beta_1} \tag{8.34}$$

the following inequalities hold:

$$\min_{0<p_q<k^{s_1}} \big||\vec{k}(\varphi)+\vec{p}_{m+q}(\vec{b})|_*^2-k^2\big| > k^{\beta_1} \tag{8.35}$$

$$\big||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2-k^2\big|\big||\vec{k}(\varphi)+\vec{p}_{m+q_1}(\vec{b})|_*^2-k^2\big|\big||\vec{k}(\varphi)+\vec{p}_{m+q_2}(\vec{b})|_*^2-k^2\big| > k^{2\beta_1}, \tag{8.36}$$

when $0<p_{q_1}(0), p_{q_2}(0)<k^{s_1}$. *This property is preserved in the* $k^{-4+2\beta_1-2s_1-\delta}$ *neighborhood of* $\hat{\Phi}'_0\setminus\mathcal{O}^{(1)}(\vec{b})$.

Proof. First, we consider the case $m=0$. It is easy to see that $\big||\vec{k}(\varphi)+\vec{b}|_*^2-k^2\big| \geq kb_0r_0$, when $\varphi\notin\mathcal{O}_0^{\pm}(\vec{b})$. Using the estimate $b_0\geq k^{-1-16s_1-12\delta}$ and the formula for r_0, we get

$$\big||\vec{k}+b|_*^2-k^2\big| \geq k^{-2+2\beta_1+6s_1+3\delta}. \tag{8.37}$$

Next,

$$\big||\vec{k}(\varphi)+\vec{p}_{q_i}(\vec{b})|_*^2-k^2\big| = \big||\vec{k}(\varphi)+\vec{p}_{q_i}(\vec{b})|_*^2-|\vec{k}+b|_*^2\big|+O(k^{\beta_1}) =$$
$$2\left|\left\langle \vec{k}(\varphi),\vec{p}_{q_i}(\vec{0})\right\rangle\right|+O(k^{\beta_1}),\quad i=1,2.$$

Considering that $\varphi\in\hat{\Phi}'_0$, we obtain

$$2\left|\left\langle \vec{k}(\varphi),\vec{p}_{q_i}(\vec{0})\right\rangle\right| > k^{1-3s_1-\delta}. \tag{8.38}$$

Hence,

$$\big||\vec{k}(\varphi)+\vec{p}_{q_i}(\vec{b})|_*^2-k^2\big| > \frac{1}{4}k^{1-3s_1-\delta},\quad i=1,2. \tag{8.39}$$

Thus, (8.35) holds for $m=0$. Multiplying (8.37) and (8.39) for $i=1,2$, we arrive at (8.36) for $m=0$.

Let $p_m(\vec{b})<3k^{1-8s_1-4\delta}$, $m\neq 0$. It easily follows from the definition of $\mathcal{O}_m^{\pm}(\vec{b})$,

$$\left||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2-k^2\right| \geq kp_m(\vec{b})r_m \geq k^{-1+2\beta_1+3s_1+2\delta}. \tag{8.40}$$

Next,

$$|\vec{k}(\varphi)+\vec{p}_{m+q_i}(\vec{b})|_*^2-|\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2 = 2\left|\left\langle \vec{k}(\varphi),\vec{p}_{q_i}(\vec{0})\right\rangle\right|+O(k^{1-7s_1-4\delta}),\ i=1,2.$$

Considering that $\varphi\in\hat{\Phi}'_0$, we again obtain (8.38). Hence,

$$\left||\vec{k}(\varphi)+\vec{p}_{m+q_i}(\vec{b})|_*^2-|\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2\right| > \frac{1}{2}k^{1-3s_1-\delta}.$$

Taking into account (8.34), we obtain

$$\left||\vec{k}(\varphi)+\vec{p}_{m+q_i}(\vec{b})|_*^2-k^2\right| > \frac{1}{4}k^{1-3s_1-\delta}. \tag{8.41}$$

Thus, (8.35) holds for the case $p_m(\vec{b})<3k^{1-8s_1-4\delta}$. Multiplying (8.40) and (8.41) for i=1,2, we obtain (8.36).

If $p_m(\vec{b})\geq k^{1-8s_1-4\delta}$, $|2k-p_m(\vec{b})|>k^{1-8s_1-4\delta}$, we use the same considerations as in Lemma 8.4 to show that

$$\left||\vec{k}(\varphi)+\vec{p}_m(\vec{b})|_*^2-k^2\right|\left||\vec{k}(\varphi)+\vec{p}_{m+q_i}(\vec{b})|_*^2-k^2\right| > k^{2\beta_1},\ \text{for } i=1,2.$$

The estimates (8.35), (8.36) easily follow.

Let $|2k - p_m(\vec{b})| < k^{1-8s_1-4\delta}$. Using (8.11) and considering as in Lemma 8.4, we obtain that

$$\big||\vec{k}(\varphi) + \vec{p}_m(\vec{b})|_*^2 - k^2\big| > k^{-2+2\beta_1+10s_1+4\delta} \tag{8.42}$$

when $\varphi \notin O^{(1)}(\vec{b})$. Suppose at least one of $\varphi_m^{\pm}(\vec{b})$ is in the $(k^{-4s_1-2\delta})$-neighborhood of $\hat{\Phi}_0'$. Then, $\Pi_m(\vec{b}) > \frac{1}{4}k^{1-9s_1-4\delta}$, see formula (8.30) and the footnote there. Using the inequality (8.14), see the footnote there, we obtain:

$$\big||\vec{k}(\varphi) + \vec{p}_{m+q_i}(\vec{b})|_*^2 - k^2\big| > \frac{1}{2}\Pi_m(\vec{b}), \quad i = 1, 2. \tag{8.43}$$

Estimates (8.35), (8.36) easily follow. To finish the proof it remains to show that at least one $\varphi_m^{\pm}(\vec{b})$ is in the $k^{-4s_1-2\delta}$ neighborhood of $\hat{\Phi}_0'$. If both $\varphi_m^{\pm}(\vec{b})$ are not in the $k^{-4s_1-2\delta}$ neighborhood of $\hat{\Phi}_0'$, then each is inside a disc $O_q^{\pm}$, $0 < p_q < k^{s_1}$ and further then $2k^{-4s_1-2\delta}$ from its boundary. Therefore, the distance of φ to $\varphi_m^{\pm}$ is greater than $k^{-4s_1-2\delta}$. However, (8.34) and (8.16), yield $|\varphi - \varphi_m^{\pm}| \leq 10k^{\beta_1/2-1}$. This contradiction proves that this case is not possible. Since all estimates are stable with respect to the perturbation of φ of order $k^{-4+2\beta_1-2s_1-\delta}$, the statement of the lemma holds in the $k^{-4+2\beta_1-2s_1-\delta}$ neighborhood of $\hat{\Phi}_0' \setminus \mathcal{O}^{(1)}(\vec{b})$ too. ∎

8.2.3. *Lemma 8.10 .*

Lemma 8.10. *Suppose* $100s_1 < \beta_1 < 1/12 - 28s_1 - 14\delta$ *and* $\mathcal{O}_c^{(1)}(\vec{b})$ *is a connected component of* $\mathcal{O}^{(1)}(\vec{b})$. *Then, the size of* $\mathcal{O}_c^{(1)}(\vec{b})$ *does not exceed* $ck^{-\gamma}$, $\gamma = 11/6 - \beta_1 - 12s_1 - 4\delta$. *A component* $\mathcal{O}_c^{(1)}(\vec{b})$ *contains no more than* $c_1k^{2/3+2s_1}$ *discs. The total size of* $\mathcal{O}^{(1)}(\vec{b})$ *does not exceed* $2\pi c_1 k^{-5/6+\beta_1+12s_1+4\delta}$. *The total number of discs in* $\mathcal{O}^{(1)}(\vec{b})$ *is less than* ck^{2+2s_1}.

Proof. Let Δ_* be a rectangle in $\mathbb{C}$: $\Delta_* = \{\varphi : |\Re\varphi - \varphi_*| \leq k^{-1},\ |\Im\varphi| < k^{-\delta}\}$ for some $\varphi_* \in [0, 2\pi)$. Clearly, Φ_0 is the union of such rectangles. Let

$$I_* = \{m \in \mathbb{Z}^2,\ p_m(\vec{b}) < 4k,\ \varphi_m^{\pm}(\vec{b}) \in \Delta_*\},$$

where $\varphi_m^{\pm}(\vec{b}) \in \Delta_*$ means that either $\varphi_m^{+}(\vec{b}) \in \Delta_*$ or $\varphi_m^{-}(\vec{b}) \in \Delta_*$. We will show that the number of points in I_* does not exceed $6c_0k^{1+2s_1}$ and

$$\Sigma_* := \sum_{m \in I_*} r_m(\vec{b}) < ck^{-\gamma} \tag{8.44}$$

This means that the total size of $\mathcal{O}^{(1)}(\vec{b}) \cap \Delta_*$ is less than $k^{-\gamma}$. Considering that $\gamma > 1$, we easily obtain that the size of each $\mathcal{O}_c^{(1)}(\vec{b})$ is less then $ck^{-\gamma}$. Since Φ_0 consists of $2\pi k$ rectangles Δ_*, the total size of $\mathcal{O}^{(1)}(\vec{b})$ does not exceed $k^{-5/6+\beta_1+12s_1+4\delta}$. Let us prove (8.44). In accordance with the Definition 8.8 and by analogy with the proof of Lemma 8.6, we split I_* into six sets:

$$I_{*0} := \{m = 0\}, \qquad I_{*1} := \{m \in I_* \setminus \{0\} : p_m(\vec{b}) < k^{1-8s_1-4\delta}\},$$

$$I_{*2} := \{m \in I_* : p_m(\vec{b}) \geq k^{1-8s_1-4\delta},\ 2k - p_m(\vec{b}) > k^{1-8s_1-4\delta}\},$$

$$I_{*3} := \{m \in I_* :\ p_m(\vec{b}) - 2k > k^{1-8s_1-4\delta}, \},$$

$$I_{*4} := \left\{ m \in I_* : |2k - p_m(\vec{b})| \le k^{1-8s_1-4\delta}, \left|1 - \frac{p_m^2(\vec{b})}{4k^2}\right| \ge \frac{1}{4}k^{-4+2\beta_1+10s_1+4\delta} \right\},$$

$$I_{*5} := \left\{ m \in I_* : \left|1 - \frac{p_m^2(\vec{b})}{4k^2}\right| < \frac{1}{4}k^{-4+2\beta_1+10s_1+4\delta} \right\}.$$

We define $\Sigma_{*j} := \sum_{m\in I_{*j}} r_m(\vec{b}),\ j = 0,..,5$, and get an upper bound for each sum. Formula

$$\Sigma_{*0} = k^{-2+2\beta_1+22s_1+15\delta} \tag{8.45}$$

immediately follows from Definition 8.8. Obviously,

$$\Sigma_{*1} = k^{-2+2\beta_1+3s_1+2\delta} \sum_{m\in I_{*1}} \frac{1}{p_m(\vec{b})}. \tag{8.46}$$

We estimate the sum in the right-hand side by an integral:

$$\sum_{m\in I_{*1}} \frac{1}{p_m(\vec{b})} \le k^{2s_1} + k^{2s_1} \int_{I'_{*1}} \frac{1}{r} r dr d\varphi < k^{3s_1},$$

where I'_{*1} is a (ck^{-s_1}) neighborhood of I_{*1}. Substituting the last estimate into (8.46), we obtain:

$$\Sigma_{*1} < ck^{-2+2\beta_1+6s_1+2\delta}. \tag{8.47}$$

Let us estimate Σ_{*2}. First we note that $\sqrt{1 - \frac{p_m^2(\vec{b})}{4k^2}} > \frac{1}{4}k^{-4s_1-2\delta}$, since $2k - p_m > k^{1-8s_1-4\delta}$. For the moment we assume that

$$\sum_{m\in I_{*2}} \frac{1}{\Pi_m(\vec{b})} \le ck^{7s_1+3\delta}. \tag{8.48}$$

Then,

$$\Sigma_{*2} = \sum_{m\in I_{*2}} \frac{k^{-1+2\beta_1}}{p_m \Pi_m(\vec{b})\sqrt{1 - \frac{p_m^2}{4k^2}}} \le \frac{ck^{-1+2\beta_1}}{k^{1-8s_1-4\delta} \cdot k^{-4s_1-2\delta}} \sum_{m\in I_{*3}} \frac{1}{\Pi_m(\vec{b})}.$$

Thus,

$$\Sigma_{*2} < ck^{-2+2\beta_1+19s_1+9\delta}. \tag{8.49}$$

Let us prove (8.48). Definition of $\varphi_m^{\pm}(\vec{b})$ yields:

$$\left|\vec{k}\left(\varphi_m^{\pm}(\vec{b})\right)\right| = k, \quad \left|\vec{k}\left(\varphi_m^{\pm}(\vec{b})\right) + \vec{p}_m(\vec{b})\right| = k. \tag{8.50}$$

Considering that $|\varphi_m^{\pm}(\vec{b}) - \varphi_*| < k^{-1}$, we obtain: $|\vec{k}(\varphi_*)| = k$, $|\vec{k}(\varphi_*) + \vec{p}_m(\vec{b})| = k + O(1)$, $|O(1)| < 1$. This means that $\vec{p}_m(\vec{b})$ belongs to the ring $\mathcal{R}$ of radii $k \pm 1$ centered at $\vec{k}(\varphi_*)$. We split $\mathcal{R}$ into several components depending on the value of $\Pi_m(\vec{b})$. Indeed, let

$$\Omega_{*j} = \{m \in \mathbb{Z}^2 :\ \vec{p}_m(\vec{b}) \in \mathcal{R},\ k^{2s_1+j\delta} \le \Pi_m(\vec{b}) < k^{2s_1+(j+1)\delta}\},$$

$j = 0, \cdots, J,\ J = \lfloor \frac{1-s_1}{\delta} \rfloor$. Let us show that

$$\Sigma_{m\in\Omega_{*j}} \Pi_m^{-1}(\vec{b}) < 4k^{7s_1+2\delta}. \tag{8.51}$$

We estimate the number of points in Ω_{*j}. Indeed, by the definition of Π_m for every $m\in\Omega_{*j}$ there is a $\vec p_{q_m}$, such that $0<p_{q_m}<k^{s_1}$ and

$$\left|< \vec k\left(\varphi_m^{\pm}(\vec b)\right)+\vec p_m(\vec b),\vec p_{q_m}>\right|+k^{2s_1}<k^{2s_1+(j+1)\delta}.$$

Considering that $|\varphi_m^{\pm}(\vec b)-\varphi_*|<k^{-1}$, we obtain that

$$\left|< \vec k(\varphi_*)+\vec p_m(\vec b),\vec p_q>\right|<2k^{2s_1+(j+1)\delta}\ \text{for a } q:\ 0<p_q<k^{s_1},\ \text{and } \vec p_m\in\mathcal{R}.$$

The number of such points in $\mathcal{R}$, obviously, does not exceed $4k^{5s_1+(j+1)\delta}$ for a fixed q and $4k^{9s_1+(j+1)\delta}$ for all q. Using now the estimate $\Pi_m\geq k^{2s_1+j\delta}$, we arrive at (8.51). The estimate (8.48) easily follows.

Let us estimate Σ_{*3}. Using (8.30), we get $r_m<2k^{-3+2\beta_1+13s_1+6\delta}$. By (8.10), $\Re\varphi_m^{\pm}+\pi=\varphi_m \pmod{2\pi}$ when $m\in I_{*3}$. Therefore, for any $m\in I_{*3}$, we have $|\varphi_m-\pi-\varphi_*|<k^{-1}$ and $2k<p_m<4k$. Now it is obvious that the number of points in I_{*3} does not exceed ck. The estimate

$$\Sigma_{*3}<ck^{-2+2\beta_1+13s_1+6\delta} \tag{8.52}$$

easily follows. Next, we estimate Σ_{*4}. Suppose we have checked that

$$\sum_{m\in I_{*4}}\frac{1}{\sqrt{|2k-p_m(\vec b)|}}<ck^{5/3-\beta_1+2s_1}. \tag{8.53}$$

Then,

$$\Sigma_{*4}=\sum_{m\in I_{*4}}\frac{k^{-4+2\beta_1+10s_1+4\delta}}{\sqrt{\left|1-\frac{p_m^2}{4k^2}\right|}}\leq ck^{-\frac{7}{2}+2\beta_1+10s_1+4\delta}\sum_{m\in I_{*4}}\frac{1}{\sqrt{|2k-p_m|}}.$$

Therefore,

$$\Sigma_{*4}<ck^{-11/6+\beta_1+12s_1+4\delta}. \tag{8.54}$$

To show (8.53), we note that $|2k-p_m|\geq\frac12k^{-3+2\beta_1+10s_1+4\delta}$ follows from $1-\frac{p_m^2}{4k^2}\geq\frac14k^{-4+2\beta_1+10s_1+4\delta}$. We split I_{*4} into three regions:

$$\begin{aligned}
\omega_{*1}&:=\{m\in I_{*4},\ k^{\delta}\leq|2k-p_m|\leq k^{1-8s_1-4\delta}\},\\
\omega_{*2}&:=\{m\in I_{*4},\ k^{-1}\leq|2k-p_m|<k^{\delta}\},\\
\omega_{*3}&:=\{m\in I_{*4},\ \frac12k^{-3+2\beta_1+10s_1+4\delta}\leq|2k-p_m|<k^{-1}\}.
\end{aligned}$$

The corresponding sums we denote as σ_j, $j=1,2,3$. It is easy to estimate σ_1 by an integral:

$$\sigma_1:=\sum_{k^{\delta}\leq|2k-p_m|\leq k^{1-8s_1-4\delta}}\frac{2}{\sqrt{|2k-p_m|}}<ck\int_{k^{\delta}}^{k^{1-8s_1-4\delta}}\int_0^{2\pi}\frac{1}{\sqrt t}\,d\varphi dt\leq ck^{\frac32-4s_1-2\delta}.$$

Now we estimate σ_2. It is easy to see that the number of points in ω_{*2} does not exceed $c_0k^{1+\delta+2s_1}$. Using the estimate $2k-p_m>k^{-1}$, we obtain:

$$\sigma_2<ck^{3/2+2s_1+\delta}.$$

Let us estimate σ_3 . Definition of $\varphi_m^\pm(\vec b)$ yields $\left|\vec k\left(\varphi_m^\pm(\vec b)\right)\right|_*^2 = k^2$, $\left|\vec k\left(\varphi_m^\pm(\vec b)\right)+\vec p_m(\vec b)\right|_*^2 = k^2$. By (8.10), $|\Im\varphi_m^\pm| < k^{-1}$ when $m \in \omega_{*3}$. Therefore, $|\varphi_m^\pm - \varphi_*| < 2k^{-1}$. Hence, we obtain: $|\vec k(\varphi_*)| = k$, $|\vec k(\varphi_*) + \vec p_m(\vec b)| = k + O(1)$, $|O(1)| < 1$. This means that $\vec p_m(\vec b)$ belongs to the ring $\mathcal{R}$ of radii $k \pm 1$ centered at $\vec k(\varphi_*)$. We also introduce the ring $\mathcal{R}_* = \{\vec k : \left||\vec k| - 2k\right| < k^{-1}\}$. Obviously $\omega_{*3} \subset \mathcal{R}_*$. Thus, $\omega_{*3} \subset \mathcal{R} \cap \mathcal{R}_*$. It is not difficult to show that the ring $\mathcal{R}$ cuts from $\mathcal{R}_*$ an "arc" of the length $O(k^{1/2})$. Using well-known results from the theory of lattices, see [27], we obtain that $\mathcal{R} \cap \mathcal{R}_*$ contains no more than $ck^{1/6+s_1}$ points $\vec p_m(\vec b)$ for any $\vec b$. Considering the estimate $|2k - p_m| \geq \frac{1}{2}k^{-3+2\beta_1+10s_1+4\delta}$, we arrive at the inequality:

$$\sigma_3 < ck^{5/3-\beta_1-4s_1-2\delta}.$$

Adding the estimates for σ_1, σ_2 and σ_3 and considering that $\beta_1 < 1/6 - 10s_1 - 11\delta$, we obtain (8.53).

Let us estimate Σ_{*5}. The number of points in I_{*5} can be estimated the same way as the number of points in ω_{*3}, i.e., it is less than $ck^{1/6+s_1}$. Hence,

$$\Sigma_{*5} < ck^{-11/6+\beta_1+6s_1+2\delta}. \tag{8.55}$$

Adding the estimates (8.45), (8.47), (8.52), (8.49), (8.54) and (8.55) and considering that $\beta_1 < 1/6 - 10s_1 - 11\delta$, we obtain (8.44).

Let us show that the number of points in I_* does not exceed $c_0k^{1+2s_1}$. First, we count all real $\varphi_m^\pm(\vec b)$. Obviously, $|\varphi_* - \varphi_m^\pm(\vec b)| < k^{-1}$. Using (8.50), we obtain $|\vec k(\varphi_*) + \vec p_m(\vec b)| = k + O(1)$, $|O(1)| < 1$. The number of points $\vec p_m(\vec b)$ in this ring does not exceed $c_0k^{1+2s_1}$, s_1 appearing since the size of our lattice is of order k^{-s_1}. If $\Im\varphi_m^\pm(\vec b) \neq 0$, then, by (8.10), $\varphi_m = \Re\varphi_m^\pm(\vec b) + \pi$ modulo 2π. Hence, $|\varphi_m - \pi - \varphi_*| < k^{-1}$. Considering that $p_m(\vec b) < 4k$, we obtain that the number of such points does not exceed $4c_0k^{1+2s_1}$. Hence, the total number of points in I_* does not exceed $6c_0k^{1+2s_1}$.

It remains to show that a component $\mathcal{O}_c^{(1)}(\vec b)$ contains no more than $c_1k^{2/3+s_1}$ discs. Indeed, the length of $\mathcal{O}_c^{(1)}(\vec b)$ is less then $ck^{-\gamma}$. This means that all points $\varphi_m^\pm(\vec b)$ belonging to $\mathcal{O}_c^{(1)}(\vec b)$, are, in fact, in a square of the size $ck^{-\gamma}$ centered at a point $\varphi_{**} \in \Delta_*$. Let us consider all real $\varphi_m^\pm(\vec b)$ in the component. Obviously $|\Re\varphi_{**} - \varphi_m^\pm(\vec b)| < ck^{-\gamma}$. Using (8.50), we obtain $|\vec k(\Re\varphi_{**}) + \vec p_m(\vec b)| = k + O(k^{1-\gamma})$. By [27], the number of points $\vec p_m(\vec b)$ in this ring does not exceed $c_1k^{2/3+s_1}$, s_1 appearing since the size of our lattice is of order k^{-s_1}. If $\Im\varphi_m^\pm(\vec b) \neq 0$, then, by (8.10), $\varphi_m = \Re\varphi_m^\pm(\vec b) + \pi$ modulo 2π. and $\cosh\Im\varphi_m^\pm(\vec b) = p_m/2k$. Hence, $|\varphi_m - \pi - \Re\varphi_{**}| < ck^{-\gamma}$ and $|p_m - x_{**} < ck^{1-\gamma}$, $x_{**} = k\cosh\Im\varphi_{**}$. Obviously the number of points $\vec p_m(\vec b)$ is such a region is $O(1)$. Hence the total number of points $\varphi_m^\pm(\vec b)$ in $\mathcal{O}_c^{(1)}(\vec b)$ is less than $c_1k^{2/3+s_1}$. Considering that the number of points $\vec p_m(\vec b)$ satisfying the inequality $p_m(\vec b) < 4k$ is less than ck^{2+2s_1}, we obtain that the total number of discs in $\mathcal{O}^{(1)}(\vec b)$ is less than ck^{2+2s_1}. ■

9. Appendices.

Appendix 1. Proof of Lemma 2.30. By Lemma 2.10, Part 2, the function $\vec{\varkappa}_1(\varphi)$ is holomorphic in $\hat{\Phi}_1$ and $\lambda^{(1)}(\vec{\varkappa}_1(\varphi)) = k^2$. Hence, the equation (2.130) is equivalent to $\lambda^{(1)}(\vec{y}^{(1)}(\varphi)) = \lambda^{(1)}(\vec{y}^{(1)}(\varphi) - \vec{b}) + \epsilon_0$. We use perturbation formula (2.58): $|\vec{y}^{(1)}(\varphi)|_*^2 + f_1\big(\vec{y}^{(1)}(\varphi)\big) = |\vec{y}^{(1)}(\varphi) - \vec{b}|_*^2 + f_1\big(\vec{y}^{(1)}(\varphi) - \vec{b}\big) + \epsilon_0$. This equation can be rewritten as

$$2\langle \vec{y}^{(1)}(\varphi), \vec{b}\rangle_* - |\vec{b}|^2 + f_1\big(\vec{y}^{(1)}(\varphi)\big) - f_1\big(\vec{y}^{(1)}(\varphi) - \vec{b}\big) - \epsilon_0 = 0. \tag{9.1}$$

Using the notation $\vec{b} = b_0(\cos\varphi_b, \sin\varphi_b)$, dividing both sides of the equation (9.1) by $2b_0k$, and considering that $\vec{y}^{(1)}(\varphi) = \vec{\varkappa}_1(\varphi) + \vec{b} = (k + h_1)\vec{\nu} + \vec{b}$, we obtain:

$$\cos(\varphi - \varphi_b) - \epsilon_0 g_1(\varphi) + g_2(\varphi) = 0, \tag{9.2}$$

where $g_1(\varphi) = (2b_0k)^{-1}$ and

$$g_2(\varphi) = \frac{\langle \vec{h}_1(\varphi), \vec{b}\rangle_*}{b_0k} - \frac{b_0}{2k} + \Big(f_1\big(\vec{y}^{(1)}(\varphi)\big) - f_1\big(\vec{y}^{(1)}(\varphi) - \vec{b}\big)\Big)g_1(\varphi), \quad \vec{h}_1(\varphi) = h_1(\varphi)\vec{\nu}.$$

Let us estimate $g_2(\varphi)$. Using the inequality (2.70) for h_1, and considering that $b_0 < k^{-1-16s_1-12\delta}$, we easily obtain:

$$\left|\frac{\langle \vec{h}_1(\varphi), \vec{b}\rangle_*}{b_0k}\right| \le \frac{2h_1}{k} = O(k^{-4+30s_1+20\delta}), \quad \frac{b_0}{2k} \le \frac{1}{2}k^{-2-16s_1-12\delta}.$$

By (2.61), $\left|f_1(\vec{y}^{(1)}(\varphi)) - f_1(\vec{y}^{(1)}(\varphi) - \vec{b})\right| \le \sup|\nabla f_1|b_0 = O(k^{-2+33s_1+22\delta}b_0)$, and therefore, $\left|\big(f_1(\vec{y}^{(1)}(\varphi)) - f_1(\vec{y}^{(1)}(\varphi) - \vec{b})\big)g_1(\varphi)\right| = O(k^{-3+33s_1+22\delta})$. Thus, we have $g_2(\varphi) = O(k^{-2-16s_1-12\delta})$. Using $\epsilon_0 < b_0k^{-1-16s_1-12\delta}$, we obtain $\epsilon_0 g_1(\varphi) < k^{-2-16s_1-12\delta}$. Thus,

$$g_2(\varphi) - \epsilon_0 g_1(\varphi) = O(k^{-2-16s_1-12\delta}). \tag{9.3}$$

Suppose that $\varphi_b \pm \dfrac{\pi}{2}$ is in the $\big(\frac{1}{8}k^{-2-16s_1-11\delta}\big)$-neighborhood of $\hat{\Phi}_1$. We draw two circles $C_\pm$ centered at $\varphi_b \pm \dfrac{\pi}{2}$ with the radius $\frac{1}{8}k^{-2-16s_1-11\delta}$. They are both inside the complex $2k^{-2-16s_1-11\delta}$-neighborhood of Φ_1, the perturbation series converging and the estimate (9.3) holds. For any φ on $C_\pm$, $|\varphi - (\varphi_b \pm \pi/2)| = \frac{1}{8}k^{-2-16s_1-11\delta}$ and, therefore, $|\cos(\varphi - \varphi_b)| > \frac{1}{16}k^{-2-16s_1-11\delta} > |g_2(\varphi) - \epsilon_0 g_1(\varphi)|$ for any $\varphi \in C_\pm$. By Rouché's Theorem, there is only one solution of the equation (9.2) inside each $C_\pm$. If $\varphi_b + \pi/2$ is not in the $(\frac{1}{8}k^{-2-16s_1-11\delta})$−neighborhood of $\hat{\Phi}_1$, then $|\cos(\varphi - \varphi_b)| > \frac{1}{16}k^{-2-16s_1-11\delta}$ in $\hat{\Phi}_1$ and, hence, equation (9.2) has no solution. Thus, there are at most two solutions in $\hat{\Phi}_1$ and $|\varphi_{\epsilon_0}^\pm - (\varphi_b \pm \pi/2)| < \frac{1}{8}k^{-2-16s_1-11\delta}$.

Appendix 2. Proof of Lemma 2.31. Using the perturbation formula (2.58), we obtain:

$$\frac{\partial}{\partial\varphi}\lambda^{(1)}\big(\vec{y}^{(1)}(\varphi)\big) = \frac{\partial}{\partial\varphi}\Big[\lambda^{(1)}\big(\vec{y}^{(1)}(\varphi)\big) - k^2\Big] = \frac{\partial}{\partial\varphi}\Big[\lambda^{(1)}\big(\vec{y}^{(1)}(\varphi)\big) - \lambda^{(1)}\big(\vec{y}^{(1)}(\varphi) - \vec{b}\big)\Big] =$$

$$\Big\langle \nabla_{\vec{y}}\lambda^{(1)}\big(\vec{y}^{(1)}(\varphi)\big) - \nabla_{\vec{y}}\lambda^{(1)}\big(\vec{y}^{(1)}(\varphi) - \vec{b}\big), \frac{\partial}{\partial\varphi}\vec{y}^{(1)}(\varphi)\Big\rangle_* =$$

$$\left\langle \nabla\big|\vec{y}^{(1)}(\varphi)\big|_*^2 - \nabla\big|\vec{y}^{(1)}(\varphi) - \vec{b}\big|_*^2, (k+h_1)\vec{\mu} + h_1'\vec{\nu}\right\rangle_* + \left\langle \nabla f_1\big(\vec{y}^{(1)}(\varphi)\big) - \nabla f_1\big(\vec{y}^{(1)}(\varphi) - \vec{b}\big), (k+h_1)\vec{\mu} + h_1'\vec{\nu}\right\rangle_*, \tag{9.4}$$

where $\vec{\nu} = (\cos\varphi, \sin\varphi)$ and $\vec{\mu} = \vec{\nu}' = (-\sin\varphi, \cos\varphi)$. Note that

$$\nabla\big|\vec{y}^{(1)}(\varphi)\big|_*^2 - \nabla\big|\vec{y}^{(1)}(\varphi) - \vec{b}\big|_*^2 = \vec{y}^{(1)}(\varphi) - 2\big(\vec{y}^{(1)}(\varphi) - \vec{b}\big) = 2\vec{b} \tag{9.5}$$

Substituting (9.5) into (9.4), we get $\frac{\partial}{\partial\varphi}\lambda_j^{(1)}\big(\vec{y}^{(1)}(\varphi)\big) = T_1 + T_2$,

$$T_1 = 2\left\langle \vec{b}, (k+h_1)\vec{\mu} + h_1'\vec{\nu}\right\rangle_*,$$
$$T_2 = \left\langle \nabla f_1\big(\vec{y}^{(1)}(\varphi)\big) - \nabla f_1\big(\vec{y}^{(1)}(\varphi) - \vec{b}\big), (k+h_1)\vec{\mu} + h_1'\vec{\nu}\right\rangle_*.$$

Considering that φ is close to $\varphi_b \pm \pi/2$, we readily obtain: $\langle\vec{b}, \vec{\nu}\rangle_* = o(b_0)$, $\langle\vec{b}, \vec{\mu}\rangle_* = \pm b_0(1+o(1))$. Using also estimates (2.70) for h_1, we get $T_1 = \pm 2b_0k(1+o(1))$. By (2.63), $\left|\nabla f_1\big(\vec{y}^{(1)}(\varphi)\big) - \nabla f_1\big(\vec{y}^{(1)}(\varphi) - \vec{b}\big)\right| = O(b_0k^{-2+36s_1+24\delta})$. Hence, $T_2 = o\,(b_0k)$. Adding the estimates for T_1, T_2, we get (2.132).

Appendix 3. Proof of Corollary 3.6. Let $\tau_0 \in \chi_2$. Taking into account the relation $\lambda_j^{(1)}(\tau_0 + 2\pi p/N_1a) = k^2$ and the definition of C_2, we see that $|\lambda_j^{(1)}(\tau_0 + 2\pi p/N_1a) - z| = \epsilon_1/2$. Using (3.13) and the last equality, we easily obtain: $|\lambda_n^{(1)}(\tau_0 + 2\pi\hat{p}/N_1a) - z| \geq \epsilon_1/2$ for $\lambda_n^{(1)}(\tau_0 + 2\pi\hat{p}/N_1a) \neq \lambda_j^{(1)}(\tau_0 + 2\pi p/N_1a)$. Therefore, for any $z \in C_2$,

$$\|(\tilde{H}^{(1)}(\tau_0) - z)^{-1}\| \leq 2/\epsilon_1, \tag{9.6}$$

i.e., (3.18) is proved for $\tau_0 \in \chi_2$. Now we consider τ in the complex $(\epsilon_1k^{-1-\delta})-$neighborhood of χ_2. By Hilbert relation,

$$(\tilde{H}^{(1)}(\tau) - z)^{-1} = (\tilde{H}^{(1)}(\tau_0) - z)^{-1} + T_1T_2(\tilde{H}^{(1)}(\tau) - z)^{-1},$$
$$T_1 = (\tilde{H}(\tau_0) - z)^{-1}(\tilde{H}_0(\tau_0) + k^2), \quad T_2 = (\tilde{H}_0(\tau_0) + k^2)^{-1}(\tilde{H}_0(\tau_0) - \tilde{H}_0(\tau)).$$

Suppose we have checked that $\|T_1T_2\| < k^{-\delta}$. Then, using (9.6), we easily arrive at (3.18). The estimate $\|T_1\| < 4k^2/\epsilon_1$, easily follows from (9.6). The estimate $\|T_2\| < 2\epsilon_1k^{-2-\delta}$ easily follows from $|\tau - \tau_0| < \epsilon_1k^{-1-\delta}$. Thus, $\|T_1T_2\| < 8k^{-\delta}$ and, hence, (3.18) is proved.

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